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# A Quotient Subspace Algorithm for Testing Controllability and Computing Brunovsky's Outputs

Philippe Mullhaupt

***Abstract*— A novel algorithm is proposed for testing controllability and computing the Brunovsky output of a single input system. Rather than computing explicitly the controllability matrix, it operates by building successive images of the input matrix in specific quotient subspaces. The algorithm can be used to test controllability to a higher degree of numerical accuracy than standard controllability tests.**

## I. Introduction

The computation of canonical forms and their characterizations received a lot of attention from the control community [2], [6], [1], [3], [8], [10], [9], [5]. A common technique for single input systems is to compute the last row of the inverse of the controllability matrix. This can also be used for pole placement using Ackermann's formula, although from a numerical point of view this method tends to become quickly ill-conditioned as the size of the system increases.

A novel algorithm for computing the Brunovsky's output of a single input state space system is proposed. Rather than building explicitly the controllability matrix, the algorithm proceeds by computing residue classes in successive quotient subspaces. The algorithm can be used to test controllability to a higher order of numerical accuracy than the standard test based on the rank of the controllability matrix.

The paper is organised as follows: Section II describes the algorithm. Section II-A presents the details of it using a pivotal strategy. The pivot is however not unique and leads to different choices of coordinates. Therefore Section II-B gives a convergence proof using diagrams which has the main advantage of not depending on a specific choice of coordinates. Section II-C contains an example illustrating the algorithm. A full discussion on the choice of coordinates and numerical conditioning is done in Section III. Finaly conclusions and perspectives are given in Section IV.

## II. The Algorithm

A single input single outpur linear time-invariant system is considered and defined by,

$$\dot{x} = Ax + Bu, \tag{1}$$

where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

In the SISO setting a Brunovsky output is a combination of states corresponding to the output of a chain of $n$ integrators equivalent to the original system (for a formal defintion see [2]). Hence the output is unique up to scalar multiplication. Classically the computation of the Brunvosky's output is done based on the inverse of the controllability matrix $C = (\,B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B\,)$ and picking the last line of $C^{-1}$.

The following algorithm does not compute explicitly the controllability matrix but proceeds in establishing the output using successive images of the $B$ matrix in specific quotient spaces.

### A. Algorithm description

- $\mathsf{A}_{[k]}$, $\mathsf{Z}_{[k]}$ and $\Phi_{[k]}$ are $(n-k \times n)$ matrices.
- $\Upsilon_{[k]}$ is a $(n-k-1 \times n-k)$ matrix.
- Finally, $\mathsf{B}_{[k]}$ is a $(n-k \times 1)$ vector.

*Algorithm 1:*

1) *Initialisation*: set $\mathsf{Z}_{[0]} = I_{n\times n}$ the $(n \times n)$ identity matrix, $\mathsf{A}_{[0]} = A$, $\mathsf{B}_{[0]} = B$ and $k = 0$.
2) *Induction*: Pick a $j$ such that $b_{[k],j} \neq 0$ to play the role of the pivot. Define the $(n-k-1 \times n-k)$ elimination matrix $\Upsilon_{[k]}$ chosen such that $\Upsilon_{[k]}\mathsf{B}_{[k]} = 0$, i.e.

$$\Upsilon_{[k]} = \begin{pmatrix} b_{[k],j} & 0 & \dots & 0 & -b_{[k],1} & 0 & \dots & 0 \\ 0 & b_{[k],j} & \dots & 0 & -b_{[k],2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{[k],j} & -b_{[k],j-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -b_{[k],j+1} & b_{[k],j} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -b_{[k],n-k} & 0 & \dots & b_{[k],j} \end{pmatrix}. \tag{2}$$

Then set,

$$\Phi_{[k]} = \Upsilon_{[k]}\mathsf{A}_{[k]},$$

and perform the updates,

$$\begin{aligned} \mathsf{Z}_{[k+1]} &= \Upsilon_{[k]}\mathsf{Z}_{[k]} \\ \mathsf{A}_{[k+1]} &= \Phi_{[k]}A \\ \mathsf{B}_{[k+1]} &= \Phi_{[k]}B. \end{aligned}$$

Set $k = k + 1$.
3) *Termination*: Repeat Step 2) until $k = n - 1$.
4) $\mathsf{Z}_{[n-1]}x$ is the Brunovsky's output.

### B. Algorithm convergence

*Theorem 1:* If the system (1) is controllable, i.e.

$$\text{rank}\,(\,B \quad AB \quad \dots \quad A^{n-1}B\,) = n$$

then Algorithm 1 converges in $n-1$ steps to give the Brunvosky's output as $\mathsf{Z}_{[n-1]}x$.

Ph. Mullhaupt is with Laboratoire d'automatique, Faculté des Sciences et Techniques de l'Ingénieur, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland philippe.muellhaupt@epfl.ch

The convergence of the algorithm is shown using a diagram oriented proof. The map $\tilde{\Phi}$ is introduced to emphasize its unicity compared to $\Phi$ which depends on a choice of coordinates. The idea is to build successive isomorphisms between $\mathbf{R}^{n-k+1}/\mathrm{im}(\tilde{\Phi}_{[k-2]}B)$, which is isomorphic to the group onto which $\tilde{\Phi}_{[k-1]}$ maps, and $\cap_{i=0}^{k-1}\ker(B^T(A^T)^i)$, a base of which is given by the lines of $\mathsf{Z}_{[k]}$. At the end $\mathsf{Z}_{[n-1]}$ contains only one line and gives the Brunovsky's output. The isomorphism is very important since it justifies the elimination done in parallel on $\mathsf{A}_{[k]}$ and $\mathsf{Z}_{[k]}$ at Step 2 of the algorithm description.

*Proof:* *A. Initialisation step*:

Since $\mathbf{R}^n/\mathrm{im}B$ is isomorphic to $\mathbf{R}^{n-1}$ the map defined by $A$ induces a map $\tilde{\Phi}_{[0]}$ that makes the following diagram commutative (can stands for the canonical map):

$$\begin{tikzcd}
\mathbf{R}^n \arrow[r, "A"] \arrow[d, "\tilde{\Phi}_{[0]}"'] & \mathbf{R}^n \arrow[d, "\mathrm{can}"] \\
\mathbf{R}^{n-1} \arrow[r, leftrightarrow, "\cong"] & \mathbf{R}^n/\mathrm{im}B
\end{tikzcd}$$

Under the hypothesis of controllability $\mathrm{im}(AB)$ is not contained in $\mathrm{im}B$ and therefore by isomorphism $\mathrm{im}(\tilde{\Phi}_{[0]}B)$ is not zero in $\mathbf{R}^{n-1}$. Hence the quotient $\mathbf{R}^{n-1}/\mathrm{im}(\tilde{\Phi}_{[0]}B)$ can be performed:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[r, "A"] \arrow[d, "\tilde{\Phi}_{[0]}"'] & \mathbf{R}^n \arrow[d, "\mathrm{can}"] \\
\mathbf{R}^{n-1} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathrm{can}"'] & \mathbf{R}^n/\mathrm{im}B \\
\mathbf{R}^{n-1}/\mathrm{im}(\tilde{\Phi}_{[0]}B) &
\end{tikzcd}$$

Since by isomorphism $\mathrm{im}(\tilde{\Phi}_{[0]}B)$ corresponds to the equivalent class of $\mathrm{im}(AB)$ in $\mathbf{R}^n/\mathrm{imB}$, it pulls back to $\mathrm{im}B\oplus\mathrm{im}(AB)$ in $\mathbf{R}^n$, hence we can complete the diagram:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[rr, "A"] \arrow[d, "\tilde{\Phi}_{[0]}"'] & & \mathbf{R}^n \arrow[dl, "\mathrm{can}"'] \arrow[dd, "\mathrm{can}"] \\
\mathbf{R}^{n-1} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathrm{can}"'] & \mathbf{R}^n/\mathrm{im}B & \\
\mathbf{R}^{n-1}/\mathrm{im}(\tilde{\Phi}_{[0]}B) \arrow[rr, leftrightarrow, "\cong"] & & \mathbf{R}^n/[\mathrm{im}B\oplus\mathrm{im}(AB)]
\end{tikzcd}$$

The isomorphism theorem gives:

$$\begin{aligned}
\mathbf{R}^n/\mathrm{im}B &\cong \ker B^T \\
\mathbf{R}^n/[\mathrm{im}B\oplus\mathrm{im}(AB)] &\cong \ker B^T\cap\ker(B^TA^T)
\end{aligned}$$

*B. Induction step*. we suppose the following at step $k$:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[d, "\tilde{\Phi}_{[k-1]}"'] & \\
\mathbf{R}^{n-k} \arrow[d, "\mathrm{can}"'] & \\
\mathbf{R}^{n-k}/\mathrm{im}(\tilde{\Phi}_{[k-1]}B) \arrow[r, leftrightarrow, "\cong"] & \mathbf{R}^n/\oplus_{i=0}^{k}\mathrm{im}(A^iB)
\end{tikzcd}$$

and we will proove that this holds for $k+1$, until $k=n-2$. Since $\mathbf{R}^{n-k}/\mathrm{im}(\tilde{\Phi}_{[k-1]}B)$ is isomorphic to $\mathbf{R}^{n-k-1}$, the $A$ map makes the following diagram commutative:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[r, "A"] \arrow[dd, "\tilde{\Phi}_{[k]}"'] & \mathbf{R}^n \arrow[d, "\tilde{\Phi}_{[k-1]}"'] & \\
& \mathbf{R}^{n-k} \arrow[d, "\mathrm{can}"'] & \\
\mathbf{R}^{n-k-1} \arrow[r, leftrightarrow, "\cong"] & \mathbf{R}^{n-k}/\mathrm{im}(\tilde{\Phi}_{[k-1]}B) \arrow[r, leftrightarrow, "\cong"] & \mathbf{R}^n/\oplus_{i=0}^{k}\mathrm{im}(A^iB)
\end{tikzcd}$$

Due to controllability $\mathrm{im}(A^{k+1}B)\not\subseteq\oplus_{i=0}^{k}\mathrm{im}(A^iB)$ and therefore by isomorphism $\mathrm{im}(\tilde{\Phi}_{[k]}B)$ is not zero in $\mathbf{R}^{n-k-1}$. Hence the quotient $\mathbf{R}^{n-k-1}/\mathrm{im}(\tilde{\Phi}_{[k]}B)$ can be performed completing the diagram:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[r, "A"] \arrow[dd, "\tilde{\Phi}_{[k]}"'] & \mathbf{R}^n \arrow[d, "\tilde{\Phi}_{[k-1]}"'] \\
& \mathbf{R}^{n-k} \arrow[d, "\mathrm{can}"'] \\
\mathbf{R}^{n-k-1} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathrm{can}"'] & \mathbf{R}^{n-k}/\mathrm{im}(\tilde{\Phi}_{[k-1]}B) \arrow[d, leftrightarrow, "\cong"'] \\
\mathbf{R}^{n-k-1}/\mathrm{im}(\tilde{\Phi}_{[k]}B) & \mathbf{R}^n/\oplus_{i=0}^{k}\mathrm{im}(A^iB)
\end{tikzcd}$$

Since $\operatorname{im}(\tilde{\Phi}_{[k]}B)$ corresponds to the equivalent class of $\operatorname{im}(AB)$ in $\mathbf{R}^n/\oplus_{i=0}^{k}\operatorname{im}(A^iB)$ by isomorphism, it pulls back to $\oplus_{i=0}^{k+1}\operatorname{im}(A^iB)$ in $\mathbf{R}^n$ hence the diagram can be completed:

$$\begin{tikzcd}
\mathbf{R}^n \arrow[rr, "A"] \arrow[d, "\tilde{\Phi}_{[k]}"'] & & \mathbf{R}^n \arrow[dl, "\text{can}"'] \arrow[dd, "\text{can}"] \\
\mathbf{R}^{n-k-1} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\text{can}"'] & \mathbf{R}^n/\oplus_{i=0}^{k}\operatorname{im}(A^iB) & \\
\mathbf{R}^{n-k-1}/\operatorname{im}(\tilde{\Phi}_{[k]}B) \arrow[rr, leftrightarrow, "\cong"] & & \mathbf{R}^n/\oplus_{i=0}^{k+1}\operatorname{im}(A^iB)
\end{tikzcd}$$

Applying the isomorphism theorem one gets,

$$\mathbf{R}^n/\oplus_{i=0}^{k+1}\operatorname{im}(A^iB) \quad\cong\quad \cap_{i=0}^{k+1}\ker(B^T(A^T)^i)$$

Hence the algorithm builds inductively $\cap_{i=0}^{k}\ker(B^T(A^T)^i)$, for $k = 1,\dots,n-2$ by applying the same elimination process that occurs while computing the quotient group $\mathbf{R}^{n-k}/\operatorname{im}(\tilde{\Phi}_{[k]}B)$ due to isomorphism. In the algorithm description it is $\mathsf{Z}_{[k]}$ that is a base of $\cap_{j=0}^{k}\ker(B^T(A^T)^j)$. After $k = n-2$ steps the Brunovsky's output. ■

*Remark 1:* Notice the importance of $\operatorname{im}(A^{k+1}B) \not\subseteq \oplus_{i=0}^{k}\operatorname{im}(A^iB)$ for otherwise the system would be uncontrollable. Therefore as the algorithm proceeds $\mathsf{B}_{[k]}$ should never become zero. This leads to a controllability test that will be presented in Section III-B.

### C. Illustration

*Example 1:* To illustrate the algorithm we will consider the example in [5].

*Initialisation :*

$$\mathsf{A}_{[0]} = A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{M_B} & -\frac{r}{M_B} \end{pmatrix}$$

$$\mathsf{B}_{[0]} = B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M_B} \end{pmatrix} \mathsf{Z}_{[0]} = Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

*First iteration $k = 1$,* pivot $b_{[0],4}$:

$$\mathsf{\Phi}_{[0]} = \begin{pmatrix} 0 & -\frac{1}{M_B} & 0 & 0 \\ 0 & 0 & \frac{k}{MM_B} & \frac{r}{MM_B} \\ 0 & 0 & 0 & -\frac{1}{M_B} \end{pmatrix}$$

$$\mathsf{Z}_{[1]} = \begin{pmatrix} -\frac{1}{M_B} & 0 & 0 & 0 \\ 0 & -\frac{1}{M_B} & 0 & -\frac{1}{M} \\ 0 & 0 & -\frac{1}{M_B} & 0 \end{pmatrix}$$

$$\mathsf{A}_{[1]} = \mathsf{\Phi}_{[0]}\mathsf{A}_{[0]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{kr}{MM_B^2} & \frac{k}{MM_B} - \frac{r^2}{MM_B^2} \\ 0 & 0 & \frac{k}{M_B^2} & \frac{r}{M_B^2} \end{pmatrix}$$

$$\mathsf{B}_{[1]} = \mathsf{\Phi}_{[0]}\mathsf{B}_{[0]} = \begin{pmatrix} -\frac{1}{MM_B} \\ -\frac{r}{MM_B^2} \\ \frac{1}{M_B^2} \end{pmatrix}$$

*Second iteration $k = 2$,* pivot $b_{[1],3}$:

$$\mathsf{\Phi}_{[1]} = \begin{pmatrix} 0 & 0 & \frac{k}{MM_B^3} & \frac{r}{MM_B^3} \\ 0 & -\frac{1}{M_B^3} & -\frac{r}{MM_B^3} & -\frac{1}{MM_B^2} \end{pmatrix}$$

$$\mathsf{Z}_{[2]} = \begin{pmatrix} -\frac{1}{M_B^3} & 0 & -\frac{1}{MM_B^2} & 0 \\ 0 & 0 & 0 & \frac{k}{MM_B^3} \end{pmatrix}$$

$$\mathsf{B}_{[2]} = \mathsf{\Phi}_{[1]}B = \begin{pmatrix} -\frac{r}{MM_B^4} \\ \frac{-k}{MM_B^4} \end{pmatrix}$$

*Third iteration $k = 3$,* pivot $b_{[2],2}$:

$$\mathsf{Z}_{[3]} = \begin{pmatrix} \frac{k}{MM_B^7} & -\frac{r}{MM_B^7} & \frac{kM_B - r^2}{M^2M_B^7} & -\frac{r}{M^2M_B^2} \end{pmatrix}$$

*Brunovsky's output*:

$$\frac{1}{M^2M_B^7}\left(kMx_1 - rMx_2 + (kM_B - r^2)x_3 - rM_Bx_4\right).$$

This last expression can be checked to corresponds to what was found in [5].

## III. Choice of $\Upsilon_k$

There is some arbitrariness in the choice of the pivotal element. A solution to this problem is to use a QR decomposition of $\mathsf{B}_{[k]}$:

$$B_{[k]} \quad=\quad Q_{[k]}\begin{pmatrix} r_{[k]} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $Q_{[k]}$ is an orthogonal matrix. Then set

$$K_{[k]} = \begin{pmatrix} O_{n-k-1\times 1} & I_{n-k-1} \end{pmatrix} Q_{[k]}.$$

The rows of $K_{[k]}$ constitute a base of the annulator of $B_{[k]}$. $\Upsilon_{[k]}$ becomes defined once an invertible $(n-k-1\times n-k-1)$ matrix $W_{[k]}$ is made:

$$\Upsilon_{[k]} \quad=\quad W_{[k]}K_{[k]}.$$

The next sections discusses the choice of $W_{[k]}$ based on the singular values of $K_{[k]}\mathsf{A}_{[k]}$.

### A. *Conditioning of* $\mathsf{A}_{[k]}$ *and the choice of* $W_{[k]}$

When the initial data $A$ and $B$ induce an ill-conditioned controllability matrix, the algorithm will tend to produce poorer $\mathsf{B}_{[k]}$ as $k$ increases. To circumvent this difficulty $W_{[k]}$ is chosen so as to improve the conditioning of $\Phi_{[k]}$ over the one of $\mathsf{A}_{[k]}$. It is a simple scaling operation based on the singular value decomposition [4] of $K_{[k]}\mathsf{A}_{[k]}$:

$$K_{[k]}\mathsf{A}_{[k]} = U_{[k]}\Lambda_{[k]}V_{[k]}^T.$$

The weighting is then chosen as,

$$W_{[k]} \quad = \quad \bar{\Lambda}_{[k]}^{-1}U_{[k]}^T,$$

where $\bar{\Lambda}_{[k]}$ is the diagonal part of the $\Lambda_{[k]}$, i.e. $\Lambda_{[k]} = (\,\bar{\Lambda}_{[k]} \quad O_{(n-k-1\times k+1)}\,)$.

### B. *Controllability test based on the algorithm*

Controllability can be checked by verifying that the $\mathsf{B}_{[k]}$ are different from $0$ while the algorithm proceeds. Scaling is used at each iteration. The following controllability indicator is proposed:

*Definition 1:*

$$\gamma = \min_k \frac{\check{\lambda}_{[k]}}{\hat{\lambda}_{[k]}}\sqrt{\mathsf{B}_{[k]}^T\mathsf{B}_{[k]}},$$

where $\check{\lambda}_{[k]} = \min \bar{\Lambda}_{[k]}$ and $\hat{\lambda}_{[k]} = \max \bar{\Lambda}_{[k]}$.

As long as $\gamma > \epsilon$ where $\epsilon$ is the smallest possible floating point number representable, then the system is controllable.

*Example 2:* The following system is considered [7] where all computations are done using $\epsilon = 2.204\,10^{-16}$.

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2^{-1} & 0 & \dots & 0 \\ 0 & 0 & 2^{-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2^{-j} \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This is a classical example of a system which is well conditioned from a point of view of $A$ and $B$ but leads to an ill-conditioned controllability matrix very quickly as $j$ increases. To test controllability one normally uses one of the following:

1) rank $(\,B \quad AB \quad \dots \quad A^{n-1}B\,) = n$.
2) rank $(B, A - \lambda_i I) = n$ , $i = 1, \dots, n$. where $\lambda_i$ are the eigenvalues of $A$.

It is well known that the first criterion is bad from a numerical point of view when evaluating the rank using a singular value decomposition method. For this example the technique fails already at $j = 10$. On the other hand, Method 2) works well until $j = 43$. However, the method proposed outperforms both by assessing controllability until $j = 54$. Table I gives a summary of the results.

Note that it is hard to improve on this limit since the relative conditioning of $A$ is rcond$(A) = 2.22\,10^{-16}$ when $j = 52$. Hence it seems that $j = 54$ is the upper limit in this example with $\epsilon = 2.204\,10^{-16}$.

| Test | $j_{max}$ |
|---|---|
| rank $(\,B \quad AB \quad \dots \quad A^{n-1}B\,) = n$ | 10 |
| rank $(B, A - \lambda_i I) = n \; i = 1, \dots, n$ | 43 |
| $\gamma = \min_k \frac{\check{\lambda}_{[k]}}{\hat{\lambda}_{[k]}}\sqrt{\mathsf{B}_{[k]}^T\mathsf{B}_{[k]}} > \epsilon$ | 54 |

TABLE I

MAXIMUM INDICE $j_{max}$ FOR WHICH THE CONTROLLABILITY TEST INFERS CONTROLLABILITY

Another usefull remark is that the conditioning of $\bar{\Lambda}_{[k]}$ becomes bad starting from $j = 49$. This is illustrated in Table II. Even if the method is restricted to ensuring good conditioning of $\bar{\Lambda}_{[k]}$ the proposed methodology gives a conclusive answer as long as $j \leq 49$, which is superior to both methods 1) and 2).

| j | 49 | 50 | 54 | 55 |
|---|---|---|---|---|
| $\gamma$ | $1.96\,10^{-14}$ | $9.91\,10^{-15}$ | $6.43\,10^{-16}$ | $1.31\,10^{-16}$ |
| $\rho$ | $3.9\,10^{-16}$ | $1.93\,10^{-16}$ | $1.16\,10^{-17}$ | $5.75\,10^{-18}$ |

TABLE II

CONTROLLABILITY INDICATOR AND THE CONDITIONING OF THE SCALING MATRIX, $\rho = \min_k$ RCOND$(\bar{\Lambda}_{[k]})$

## IV. CONCLUSIONS AND FUTURE WORKS

The paper presented a quotient subspace algorithm for testing controllability and computing the Brunovsky output of a single input linear state space system.

A difficult numerical example illustrated the fact that the controllability test proposed could outperform classical ones.

Future research will attend the multivariable setting where it is hoped that the technique will shed some new light on the topics in [6], [1] and [5].

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