A Nonsmooth IQC Method for Robust Synthesis.

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***Abstract*— Synthesis of robust H_∞ output feedback controllers for plants subject to parametric uncertainties is discussed. We use the formalism of integral quadratic constraints (IQCs) to model structured parametric uncertainties and present a nonsmooth optimization method for IQCs which reduces conservatism by way of dynamic multipliers. Numerical experiments indicate that the method is promising and expected to improve over currently available heuristics.**

NOTATION

Let $\mathbb{R}^{n\times m}$ be the space of $n\times m$ matrices, equipped with the scalar product $X \bullet Y = \mathrm{Tr}(X^\top Y)$, where $X^\top$ is the transpose of X, $\mathrm{Tr}(X)$ its trace. For complex matrices X^{H} is the conjugate transpose of X. For Hermitian or symmetric matrices, $X \succ Y$ means that $X-Y$ is positive definite, $X \succeq Y$ that $X-Y$ is positive semi-definite. We use λ_1 to denote the maximum eigenvalue of a symmetric or Hermitian matrix. We use concepts from nonsmooth analysis covered by [Cla83]. For a locally Lipschitz function $f:\mathbb{R}^n \to \mathbb{R}$, $\partial f(x)$ denotes its Clarke subdifferential at x.

I. INTRODUCTION

In this paper we present a numerical method to compute robust controller for plants with parametric uncertainties. Based on the integral quadratic constraints (IQCs) theory [MR97], [Jön96], our synthesis method differs from the standard BMI approach: robust controller are synthesized from the frequential point of view, using non-smooth optimization.

II. PARAMETRIC ROBUST SYNTHESIS

A. Model of a plant with uncertainties

We consider the synthesis of a robust output feedback controller for a plant subject to parametric uncertainties. The uncertain plant is described in the usual LFT (Linear Fractional Transform) format as

$$\begin{aligned}\begin{bmatrix}\dot x\\ z_\Delta\\ z\\ y\end{bmatrix} &= \begin{bmatrix}A & B_\Delta & B_1 & B_2\\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2}\\ C_1 & D_{1\Delta} & D_{11} & D_{12}\\ C_2 & D_{2\Delta} & D_{21} & 0\end{bmatrix}\begin{bmatrix}x\\ w_\Delta\\ w\\ u\end{bmatrix}\\ w_\Delta &= \Delta z_\Delta\,,\end{aligned} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state of the system, $u\in\mathbb{R}^{m_2}$ the control, $w\in\mathbb{R}^{m_1}$ the exogenous input, $z\in\mathbb{R}^{p_1}$ the performance variable, $y\in\mathbb{R}^{p_2}$ the measured output, and $(w_\Delta, z_\Delta)\in\mathbb{R}^{m_\Delta}\times\mathbb{R}^{p_\Delta}$ the uncertainty channel. The unknown operator $\Delta\in\mathbb{R}^{p_\Delta\times m_\Delta}$ is a structured block-diagonal matrix describing the model uncertainties and satisfying $\Delta^\top\Delta \preceq I$. We let $\mathbf{\Delta}$ denote the convex and compact set of all such uncertainties Δ.

We are seeking an output feedback controller K with state-space representation

$$\begin{cases}\dot\varsigma &= A_K\varsigma + B_K y\\ u &= C_K\varsigma + D_K y\end{cases},$$

of order $k\in\mathbb{N}$ such that the following conditions are satisfied:

$$\mathcal{A}(K,\Delta) \text{ is Hurwitz for all } \Delta\in\mathbf{\Delta}, \tag{2}$$

$$\|z\|_{L^2} \le \gamma\,\|w\|_{L^2} \text{ for all } \Delta\in\mathbf{\Delta}. \tag{3}$$

Here $\mathcal{A}(K,\Delta), \mathcal{B}(K,\Delta), \mathcal{C}(K,\Delta)$, and $\mathcal{D}(K,\Delta)$ are the state-space data of the closed-loop system with the uncertainty loop also closed, and (w,z) is the performance channel in (1).

We investigate a nonsmooth mathematical programming technique, based on IQC theory, which allows to synthesize a controller K satisfying these two conditions.

B. Integral Quadratic Constraints

Every IQC is determined by a multiplier $\Pi : j\mathbb{R}\to\mathbb{C}^{(n_o+n_i)\times(n_o+n_i)}$, which is a measurable operator with values in the space of complex Hermitian matrices.

Definition 1: We say that $w\in L^2(\mathbb{R})^{n_i}$ and $z\in L^2(\mathbb{R})^{n_o}$ satisfy the IQC defined by $\Pi(\cdot)$ if

$$\int_{-\infty}^{+\infty}\begin{bmatrix}\hat z(j\omega)\\ \hat w(j\omega)\end{bmatrix}^{\mathsf{H}}\Pi(j\omega)\begin{bmatrix}\hat z(j\omega)\\ \hat w(j\omega)\end{bmatrix}d\omega \ge 0.$$

Clearly, the performance condition (3) can be viewed as an IQC using the constant multiplier

$$\Pi_\gamma := \begin{bmatrix}-I & 0\\ 0 & \gamma^2 I\end{bmatrix}.$$

For robust stability (2) we let $\mathbf{\Pi}_{\Delta,0}$ the convex set of static multipliers Π_Δ on $(0,\infty]$ of the form

$$\Pi_\Delta := \begin{bmatrix}S & (jG)^{\mathsf{H}}\\ jG & -S\end{bmatrix}, \tag{4}$$

where $S\succ 0$, $S^{\mathsf{H}} = S, G^{\mathsf{H}} = G$ and where S, G both commute with all $\Delta\in\mathbf{\Delta}$. Clearly, $S = I$ and $G = 0$ is a particular candidate hence $\mathbf{\Pi}_{\Delta,0}$ is non-empty. Moreover, simple calculations show that $\forall\Pi_\Delta\in\mathbf{\Pi}_{\Delta,0}, \forall\Delta\in\mathbf{\Delta}$, z_Δ and $w_\Delta = \Delta z_\Delta$ satisfy the IQC defined by Π_Δ. Dynamic multipliers with the same property can now be constructed as

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piece-wise constant multipliers with structure (4). For $N \in \mathbb{N}$ we define the set $\mathbf{\Pi}_{\Delta,N}$ of multipliers

$$\Pi(j\omega) = \sum_{i=0}^{N} \chi_{[\omega_i,\omega_{i+1}]}(\omega)\Pi_i \tag{5}$$

where $0 = \omega_0 < \omega_1 < \cdots < \omega_{N+1} = \infty$, and for all $i = 0,\ldots,N$, $\Pi_i \in \mathbf{\Pi}_{\Delta,0}$. Finally, we let $\mathbf{\Pi}_\Delta := \cup_{N\in\mathbb{N}}\mathbf{\Pi}_{\Delta,N}$.

Let us introduce the closed-loop transfer matrix

$$T(s,K) = \begin{bmatrix} T_{\Delta\Delta}(s,K) & T_{\Delta w}(s,K) \\ T_{z\Delta}(s,K) & T_{zw}(s,K) \end{bmatrix} \tag{6}$$

of the closed-loop plant with state vector $x_{c\ell} = \begin{bmatrix} x & \varsigma \end{bmatrix}^\top$

$$\left\{ \begin{array}{rcl} \dot{x}_{c\ell} & = & \mathcal{A}(K)x_{c\ell} + \mathcal{B}(K)\begin{bmatrix} w_\Delta \\ w \end{bmatrix} \\ \begin{bmatrix} z_\Delta \\ z \end{bmatrix} & = & \mathcal{C}(K)x_{c\ell} + \mathcal{D}(K)\begin{bmatrix} w_\Delta \\ w \end{bmatrix} \end{array} \right. ,$$

where the state-space data $\mathcal{A}(K)$, $\mathcal{B}(K)$, $\mathcal{C}(K)$ and $\mathcal{D}(K)$ represent the closed-loop system with the Δ-loop $w_\Delta = \Delta z_\Delta$ still open. Then we have the following fundamental fact, see [Jön96], [MR97], [ANP07] for details.

Theorem 1: Suppose K is nominally closed-loop stabilizing, i.e. $\mathcal{A}(K)$ is Hurwitz. Then robust stability (2) and performance (3) hold for all $\Delta \in \mathbf{\Delta}$ provided there exists $\Pi_\Delta \in \mathbf{\Pi}_\Delta$ such that the following frequency domain inequality (FDI) is satisfied: $\forall \omega \in [0,\infty]$:

$$F(K,\Pi,\omega) := \begin{bmatrix} T(j\omega,K) \\ I \end{bmatrix}^{\mathsf{H}} \Pi(j\omega) \begin{bmatrix} T(j\omega,K) \\ I \end{bmatrix} \prec 0, \tag{7}$$

where

$$\Pi := \left[\begin{array}{cc|cc} \Pi_{\Delta,11} & 0 & \Pi_{\Delta,12} & 0 \\ 0 & I & 0 & 0 \\ \hline \Pi_{\Delta,12}^{\mathsf{H}} & 0 & \Pi_{\Delta,22} & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{array} \right]. \tag{8}$$

Remark 1: We introduce dynamic piece-wise constant complex multipliers to reduce conservatism of the IQC analysis tool. We refer the reader to [ANP07] for a comprehensive discussion. Also, the L^2-gain condition (3) can be extended to other performance IQCs.

III. SYNTHESIS OF ROBUST CONTROLLER

Inequality (7) is known as the robust performance FDI, see [MR97]. It strongly suggests introducing the nonsmooth function

$$f(K,\Pi) = \lambda_{1,\infty}\left(\begin{bmatrix} T(\cdot,K) \\ I \end{bmatrix}^{\mathsf{H}} \Pi(\cdot) \begin{bmatrix} T(\cdot,K) \\ I \end{bmatrix} \right), \tag{9}$$

where $\lambda_{1,\infty}(\mathcal{X}) := \max_{\omega\in[0,\infty]} \lambda_1(\mathcal{X}(\omega))$ for $\mathcal{X}$ a bounded operator $[0,\infty] \to \mathbb{S}^m$. Then, with γ and $N \in \mathbb{N}$ fixed, we consider the optimization program

$$\left\{ \begin{array}{ll} \text{minimize} & f(K,\Pi) \\ \text{subject to} & \Pi_\Delta \in \mathbf{\Pi}_{\Delta,N},\ \mathcal{A}(K) \text{ Hurwitz} \end{array} \right. \tag{10}$$

which is minimized until a value $f(K,\Pi) < 0$ is found. Once this is the case, K is known to be robustly stabilizing with robust performance no worse than γ, and $\Pi_\Delta(\cdot) \in \mathbf{\Pi}_\Delta$ is the corresponding multiplier certificate. In our tests we first run (10) with static multipliers $N = 0$, then we switch to dynamic multipliers with $N = 3$. In the sequel we describe methods to compute function values and subgradients of f, and explain how descent steps are generated.

A. Computing function values

Solving (10) requires computation of function values $f(K,\Pi)$, and this is where the restriction to (5) is needed. Indeed, for constant Π a natural extension of the quadratically convergent algorithm of [BBK89] allows to compute $f(K,\Pi)$ efficiently. It uses the fact that $\lambda_1(F(K,\Pi,\omega)) = \lambda$ iff $j\omega$ is an eigenvalue of a suitable Hamiltonian matrix $\mathcal{H}[\lambda]$, defined by means of the state-space data, the IQC multiplier and λ (see [Par99]). We refer to [ANP07] for a detailed description of this algorithm in the IQC context. The method naturally extends to dynamic multipliers $\Pi(j\omega)$ of the form (5) by computing N different values.

B. Subgradient computation

There are two sources of nonsmoothness in the objective function f: maximization over an infinite set of frequencies, and the inherent nonsmoothness of the maximum eigenvalue function λ_1. We define the set

$$\Omega(K,\Pi) := \{\omega \in \mathbb{R}^+ \ :\ f(K,\Pi) = \lambda_1(F(K,\Pi,\omega))\},$$

called the set of active frequencies, or peaks. As shown in [BBK89], [BNA06], [ANP07] we have

Lemma 1: Let $\Pi(s)$ be rational and K closed-loop stabilizing. Then the set $\Omega(K,\Pi)$ of active frequencies is either finite, or $\Omega(K,\Pi) = [0,\infty]$.

In the following, we assume that the set $\Omega(K,\Pi)$ is finite, which appears to be the rule in practice.

Nonsmoothness of the maximum eigenvalue function λ_1 occurs when the multiplicity of the maximal eigenvalue is greater than 1. The objective function f is a composite of $\lambda_{1,\infty}$, a semi-infinite nonsmooth but convex function, and a smooth nonlinear operator F. In consequence we have [ANP07]:

Lemma 2: $f = \lambda_{1,\infty} \circ F$ is regular in the sense of Clarke [Cla83].

This means that a suitable chain rule replacing the one of classical calculus is available, which makes these functions amenable to analysis. The complete formulas for the subgradients are given in [ANP07] and follow the lines of [AN06a], [AN06b].

Motivated by practical considerations, we will in the following specialize to the case where

$$\lambda_1(F(K,\Pi,\omega)) \text{ has multiplicity one } \forall\omega \in \Omega(K,\Pi). \tag{11}$$

This assumption leads to a simpler subgradients expression and consequently to simpler descent directions computations. Moreover, the multiple eigenvalue case is rarely observed in practice.

We let Q_ω the associated normalized (right) eigenvector. Here every $(K,\Pi) \to \lambda(F(K,\Pi,\omega))$ is smooth at (K,Π).

For $\omega \in \Omega(K,\Pi)$, using the chain rule under hypothesis (11), the gradients ϕ_ω of $(K,\Pi) \to \lambda(F(K,\Pi,\omega))$ are

$$\begin{aligned}\phi_\omega &= \nabla\lambda_1(F(K,\Pi,\omega))^*\nabla_{(K,\Pi)}F(K,\Pi,\omega)\\ &= Q_\omega Q_\omega^\top \bullet \nabla_{(K,\Pi)}F(K,\Pi,\omega).\end{aligned}$$

Combining the chain rule with the convex hull rule for the maximization over $[0,\infty]$ yields the subgradients $\Phi_{\tau_\omega} \in \partial f(K,\Pi)$

$$\begin{aligned}\Phi_\tau \quad &:= \sum_{\omega\in\Omega(K,\Pi)} \tau_\omega \phi_\omega \\ \text{with} \quad &\tau = (\tau_\omega), \tau_\omega \geq 0, \sum_{\omega\in\Omega(K,\Pi)} \tau_\omega = 1.\end{aligned}$$

We refer to [ANP07] for a complete calculus without hypothesis (11).

Equations (8) and (4) show the specific structure of the multiplier Π, which must be taken into account in the expression of the subgradients. With γ fixed, Π can be parameterized using matrices S and G in (4). To model the constraint $S \succ 0$, we define $S = \Sigma\Sigma^{\mathsf{H}}$, where Σ is a lower triangular Choleski factor of S. Subgradients with respect to Σ and G are then derived from the subgradient with respect to an unstructured Π in tandem with the chain rule applied to the smooth parametrization

$$\pi(\Sigma,G,\gamma) := \left[\begin{array}{cc|cc} \Sigma\Sigma^{\mathsf{H}} & 0 & (jG)^{\mathsf{H}} & 0 \\ 0 & I & 0 & 0 \\ \hline jG & 0 & -\Sigma\Sigma^{\mathsf{H}} & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{array}\right] \in \mathbf{\Pi}_{\Delta,N}\,.$$

C. Computing the descent step

With the computation of function values and subgradients at hand, we can now discuss how to generate descent steps for the objective $f(K,\Pi)$. Here we use the idea of Polak's optimality function [Pol97]. Given a finite extension $\Omega_e(K,\Pi) \supset \Omega(K,\Pi)$, a descent direction for f at (K,Π) is $H^* = (H_K^*, H_\Pi^*)$ obtained as the minimizer of

$$\begin{aligned}\theta_e(K,\Pi) \quad &:= \inf_H \sup_{\omega\in\Omega_e(K,\Pi)} \lambda_1(F(K,\Pi,\omega)) - f(K,\Pi) \\ &+ \quad \phi_\omega \bullet H + \tfrac{\delta}{2}\|H\|^2 \qquad (12)\end{aligned}$$

where $\delta > 0$ is fixed. Then a backtracking linesearch (using for instance an Armijo condition [DS96]) is performed in the descent direction H^* to compute the next iterate. Using Polak's optimality function (12) has several advantages. Firstly, H^* can be computed by solving a quadratic program (12), derived from the dual formulation [AN06b]. Secondly, H^* gives qualified descent for f in the sense that if $\theta_e(K,\Pi) < 0$, then

$$f'(K,\Pi;H^*) \leq \theta_e(K,\Pi) - \tfrac{1}{2\delta}\|H^*\|^2 < 0,$$

while $\theta_e(K,\Pi) = 0$ implies $0 \in \partial f(K,\Pi)$. As a consequence, $\theta_e(K,\Pi)$ is a measure of criticality of the iterate (K,Π) and will serve as a stopping criterion.

The choice of the extended set of frequencies $\Omega_e(K,\Pi)$ is of practical importance. Note that if the number of frequencies added to $\Omega_e(K,\Pi)$ is too small, then the descent direction is close to the direction of steepest-descent, which may lead to zigzagging. On the other hand, the number of added frequencies must not be too large for reason of efficiency in the step computation. In our numerical experiments $\Omega_e(K,\Pi)$ has been chosen by adding some equi-spaced points around each active frequencies [ANP07].

D. Multiband optimization

The FDI in Theorem 1 is a sufficient condition to ensure stability and performance of the closed-loop plant, for all $\Delta \in \mathbf{\Delta}$. This condition is potentially conservative [MR97] and a refinement can be obtained by using piece-wise constant multipliers (5). The semi-infinite program (10) is extended to dynamic multipliers $\Pi(j\omega)$ with 3 frequency bands, i.e. $\Pi_\Delta \in \mathbf{\Pi}_{\Delta,3}$. The objective function in (10) then takes the form

$$f(K,\Pi) = \max_{i=0,1,2} \max_{\omega\in[\omega_i,\omega_{i+1}]} \lambda_1\left(F(K,\Pi_i,\omega)\right).$$

Computation of f is thus performed on each frequency band $[\omega_i,\omega_{i+1}]$, using the Hamiltonian technique in section III-A.

IV. NUMERICAL EXAMPLES

A. Implementation

Our prototype algorithm was implemented in Matlab. Controllers are synthesized in three phases. In phase I, a closed loop stabilizing K_0 for the nominal plant ($\Delta = 0$) is computed, and as a rule we use the optimal H_∞-controller K_∞. In the full order case, $k = n$, this is of course standard and can be obtained via AREs or LMIs as available in the MATLAB control toolbox. In the reduced-order case, $k < n$, or when controller structures are imposed, this can be more involved. Here we use our H_∞ synthesis software described in [AN06a], [AN06b], [ABN07]. Any value γ_0 greater than the performance γ_∞ of the nominal closed-loop plant with initial controller K_0 can be used to initialize the performance parameter in (10). We choose $S_0 = I$ and $G_0 = 0$ to initialize the multiplier Π_Δ in (4). Once our algorithm is initialized successfully, phase II starts and the problem is solved for a static multiplier certificate $\Pi_\Delta \in \mathbf{\Pi}_{\mathbf{\Delta},\mathbf{0}}$, that is, $N = 0$ in (4). The following iterative procedure is used. At stage k, minimization of $(K,\Sigma,G) \to f(K,\pi(\Sigma,G,\gamma_k))$ over (K,Σ,G) is performed. As soon as a new feasible point (K_k,Σ_k,G_k) has been reached, that is

$$f(K_k,\pi(\Sigma_k,G_k,\gamma_k)) < 0,$$

we have a certificate $\Pi_k = \pi(\Sigma_k,G_k,\gamma_k)$ that a robust controller K_k with robust performance at most γ_k has been found. This procedure is now repeated a few times in order to improve over the latest γ_k. The value of γ is gradually reduced using a simple extrapolation scheme to replace γ_k by a smaller γ_{k+1}, until an optimal value γ^* with corresponding optimal (K^*,Π^*) is reached.

During phase III, optimization of a dynamic multiplier with $N = 3$ frequency bands is performed. This phase is initialized with $K_0 = K^*$ and $\Pi(j\omega) = \Pi^*$ for all ω. The optimization procedure is exactly the same as in the static case, and converges to the optimal triplet $(\bar{K}, \bar{\Pi}(\cdot), \bar{\gamma})$. Two intermediate frequencies ω_1, ω_2 have to be specified in order to determine 3 frequency bands. For the time being this is done by trial and error. In fact, we use the plot of the FDI at the optimal (K^*, Π^*) from the constant multiplier case to select these two values in such way that each frequency band contains at least one active or nearly active peak. This choice is tricky in so far as one of these peaks may later vanish during the optimization process.

All the numerical experiments presented here have been performed on a Linux laptop, with an AMD Turion 1.8GHz processor. Let's focus on Phase III, which is the slowest part of the method due to multiband optimization. For the 4 states CD player example, average iteration time is 0.4 seconds per iteration, and controller is synthesized after 406 iterations. In the other hand, for the larger CSE1 example (20 states) average iteration time is 3,1 seconds and synthesis is achieved after 6101 iterations.

B. Mass spring model

Our first example is a mass-spring system described by the following second-order differential equations

$$\begin{cases} m_1\ddot{x}_1 = -kx_1 + kx_2 - f\dot{x}_1 + f\dot{x}_2 + u \\ m_2\ddot{x}_2 = \ kx_1 - kx_2 + f\dot{x}_1 - f\dot{x}_2 \end{cases},$$

with $m_1 = m_2 = 0.5$kg, $k = 1$N/m, and $f = 0.0025$Ns/m. The interconnection structure used for synthesis is shown in Figure 1. Our goal is to synthesize a robust feedback controller K which stabilizes the position x_2 of mass m_2 in the presence of parameter uncertainties of $p\%$ in the parameters k and m_2,

$$|1\mathrm{Nm}^{-1} - k|/1\mathrm{Nm}^{-1} \leq p\%, \ |0.5\mathrm{kg} - \mathrm{m}_2|/0.5\mathrm{kg} \leq p\%$$

where $p \in \{5, 10, 15, 20, 25, 30\}$.

The controller variable is initialized with the full-order optimal H_∞ controller K_∞, so phase I is simple here. A lower bound for the optimal robust gain is then $\gamma_\infty = \|T_{wz}(K_\infty)\|_\infty$ which serves to initialize γ. Results are presented in Table I. For each value of p, we have computed: (a) the optimal robust performance bound γ^* with optimal robust controller K^*, obtained via (10) with $N = 0$, vs. the variation $p\%$, (b) the H_∞ performance of the nominal plant $\gamma^*_{\mathrm{nom}} = \|T_{wz}(0, K^*)\|_\infty$, (c) the value of the spectral abscissa α_{nom} of the nominal plant, and (d) the criticality measure θ_e, our optimality function.

As expected γ^* increases with the level of uncertainty p. We see in Table I that for small values of $p\%$, γ^* is still close to the optimal H_∞ performance γ_∞. As expected, improvement of performance is also observed when switching from static to dynamic 3-bands multipliers, $\gamma^* \to \bar{\gamma}$.

Figure 2 shows the contour plot of the spectral abscissa, for the mass-spring model with $p = 25\%$, for controllers K^* obtained with static (left hand side) and $\bar{K}$ with dynamic (right hand side) multipliers. The spectral abscissa is computed for each value of the uncertainties δm_2 and δk of m_2 and k. Figure 2, left plot, illustrates the potential conservatism of IQC formulations. Indeed, for static Π,

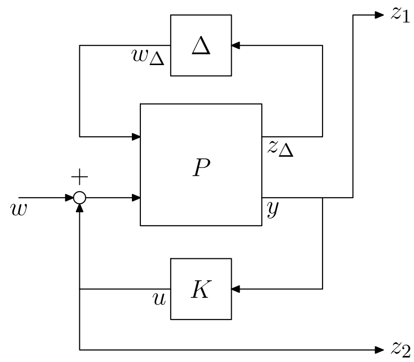

Fig. 1. Interconnection structure for the controller design of mass-spring and CD-player model

the boundary of the unstable region is very far from the square of admissible uncertainties. The right part of Figure 2 shows that the boundary of unstable region gets closer to the uncertainty square when dynamic multipliers $\bar{\Pi}(j\omega)$ are used. Hence conservatism has been reduced.

C. CD player

Our second example is a compact disc actuator described in [STBS92]. State-space data of the nominal plant are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1.77 & 0 \\ 0 & 0 \\ 0 & 1.97 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D = 0_{2\times 2}$$

The uncertainty channel describes a nonlinear perturbation of B with uncertain parameter $-0.63 \leq z \leq 0.63$ in such a way that the perturbed B^* is

$$B^* = \begin{bmatrix} 0 & 0 \\ 1.77 - 0.24z^2 & -1.27z \\ 0 & 0 \\ 5.34z & 1.97 - 1.77z^2 \end{bmatrix}.$$

The interconnection structure for synthesis is the same as before, see Figure 1. In phase I a locally optimal initial controller K_∞ of reduced-order $k = 2$ is computed using our nonsmooth H_∞ synthesis code. K_∞ is not robust as seen from the spectral abscissa plot $z \mapsto \alpha(\mathcal{A}(K_\infty, z))$ in figure 3 near the value $z = -0.5$ (line $-.-.-.$). Robust synthesis is shown in the same figure for both static (K^* with line $----$) and dynamic multipliers ($\bar{K}$ with line ——). In the dynamic case, we observe that the worst case performance decreases, while it slightly increases for $-0.5 \leq z \leq 0.5$.

D. CSE1

The third example is a model from [Lei03] and consists of coupled springs with dash-pots and masses. Input forces act on the left and on the right ends of the spring system.

	Mass Spring - 1 Band				Mass Spring - 3 Band					
$p\%$	γ^*	γ^*_{nom}	α^*_{nom}	θ_e	$\bar{\gamma}$	$\bar{\gamma}_{\text{nom}}$	$\bar{\alpha}_{\text{nom}}$	θ_e	ω_1	ω_2
5	1.3610	1.3551	-0.6172	-2.16e-04	1.3594	1.3553	-0.6206	-2.44e-03	0.76	1
10	1.6963	1.6807	-0.7869	-9.38e-04	1.6561	1.6270	-0.4136	-1.44e-04	0.76	2.30
15	2.0648	2.0106	-0.5615	-2.67e-03	1.6169	1.5168	-0.6114	-2.11e-01	2.40	4.10
20	2.5079	2.4064	-0.4501	-4.64e-03	1.7729	1.6544	-0.6835	-1.30e-01	2.40	4.10
25	3.0609	2.8979	-0.3646	-6.24e-03	1.9626	1.7751	-0.8070	-5.39e-02	2.40	4.10
30	3.7540	3.4569	-0.3146	-3.58e-05	3.6205	3.5405	-0.2272	-2.80e-04	0.76	45

TABLE I

MASS-SPRING SYSTEM. OPTIMAL FULL H_∞ CONTROLLER $\gamma_\infty = 1.3261$ AND $\alpha_\infty = -0.2129$. $\gamma^*_{\text{nom}} = \|T_{wz}(K^*, 0)\|_\infty$, $\bar{\gamma}_{\text{nom}} = \|T_{wz}(\bar{K}, 0)\|_\infty$. γ^* AND $\bar{\gamma}$ ARE OPTIMAL VALUES OBTAINED BY (10) WITH $N = 0$ AND $N = 3$. $\alpha^*_{\text{nom}} = \alpha(\mathcal{A}(K^*, 0))$, $\bar{\alpha}_{\text{nom}} = \alpha(\mathcal{A}(\bar{K}, 0))$.

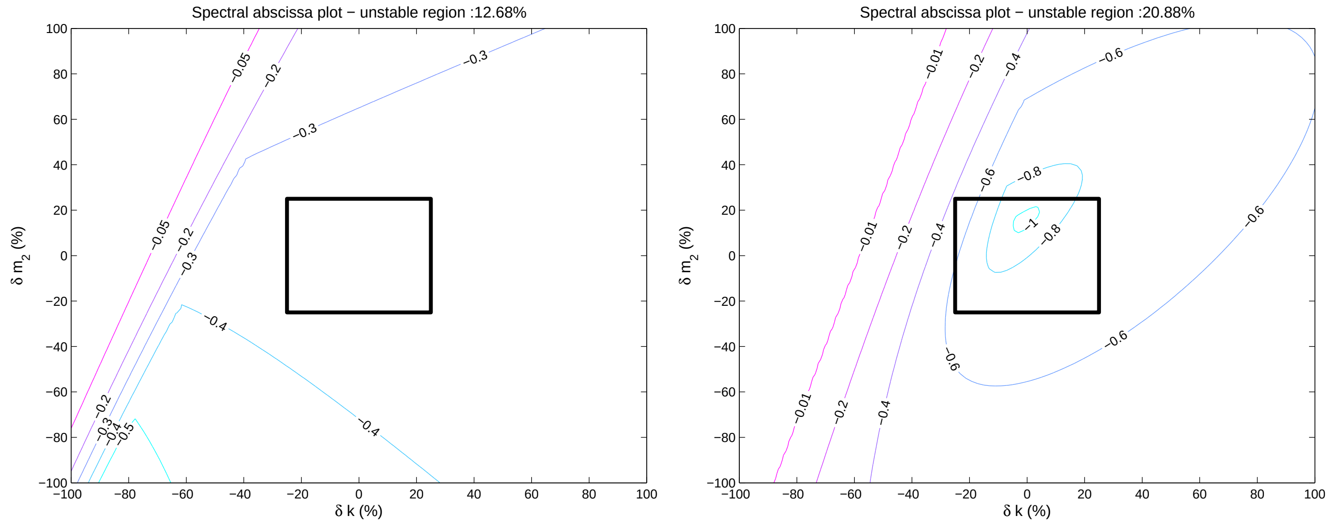

Fig. 2. Mass-spring system. Reduction of conservatism with dynamic $\Pi(j\omega)$. Contour plot of the perturbed closed-loop spectral abcsissa $\Delta \mapsto \alpha(\mathcal{A}(K^*, \Delta))$ (left), $\Delta \mapsto \alpha\left(\mathcal{A}(\bar{K}, \Delta)\right)$ (right). The black square shows the $\pm p = 25\%$ robustness region about nominal values. Left with controller K^* computed using a static multiplier Π^*. Right with controller $\bar{K}$ synthesized using a dynamic three-band multiplier $\bar{\Pi}(s)$. The percentage on top of the figures measures area of unstable region.

State space data are

$$\ell = 10, \mu = 4, \delta = 4, \kappa = 1,$$
$$M = \mu I_\ell, L = \delta I_\ell, N = P = I_\ell$$
$$K = \kappa \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & -1 \end{bmatrix},$$
$$A = \begin{bmatrix} 0_{\ell\times\ell} & I_\ell \\ -M^{-1}K & -M^{-1}L \end{bmatrix}, B = \begin{bmatrix} 0_{\ell\times 2} \\ -M^{-1}R \end{bmatrix},$$
$$C = \begin{bmatrix} N & P \end{bmatrix}, D = 0_{\ell\times 2}.$$

See [Lei03] for a description of the performance channel. A robustness channel has been defined to reflect an uncertainty of 75% on each of the diagonal elements of matrix M. The set $\mathbf{\Delta}$ contains all diagonal matrices verifying $\Delta^\top \Delta \preceq I$.

An initial controller of order $k = 7$ has been computed using our H_∞ synthesis code. Figure 4 displays the spectral abscissa and the performance of the initial and robust controllers on the subset of uncertainties of the form $\Delta = \tau I \in \mathbf{\Delta}$, with $-1 \le \tau \le 1$, abscissa variable p is defined by $p = 0.75\tau$. The initial (locally optimal) H_∞ controller K_∞ is not robust as can be seen in this Figure for p near -0.75. The Figure also shows robustness of the controllers K^* and $\bar{K}$ synthesized by our method using static and dynamic multiplier. Surprisingly the performance of $\bar{K}$ is even better than that of the nominal K_∞, which shows that K_∞ was only locally optimal. This phenomenon may arise for reduced order control.

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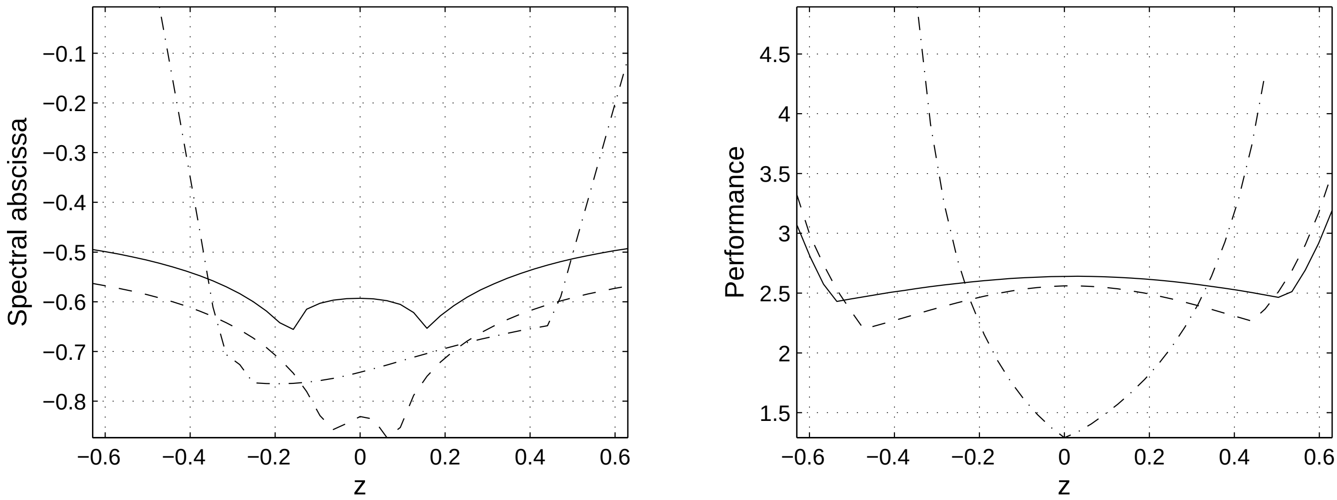

Fig. 3. Robust control of CD-player. Left plot shows spectral abscissa and right plot performance for initial nominal H_∞ controller (K_∞ dash-dotted line), robust controller with static (K^* dashed line) and dynamic ($\bar{K}$ plain line) multiplier. The nominal controller is not robust.

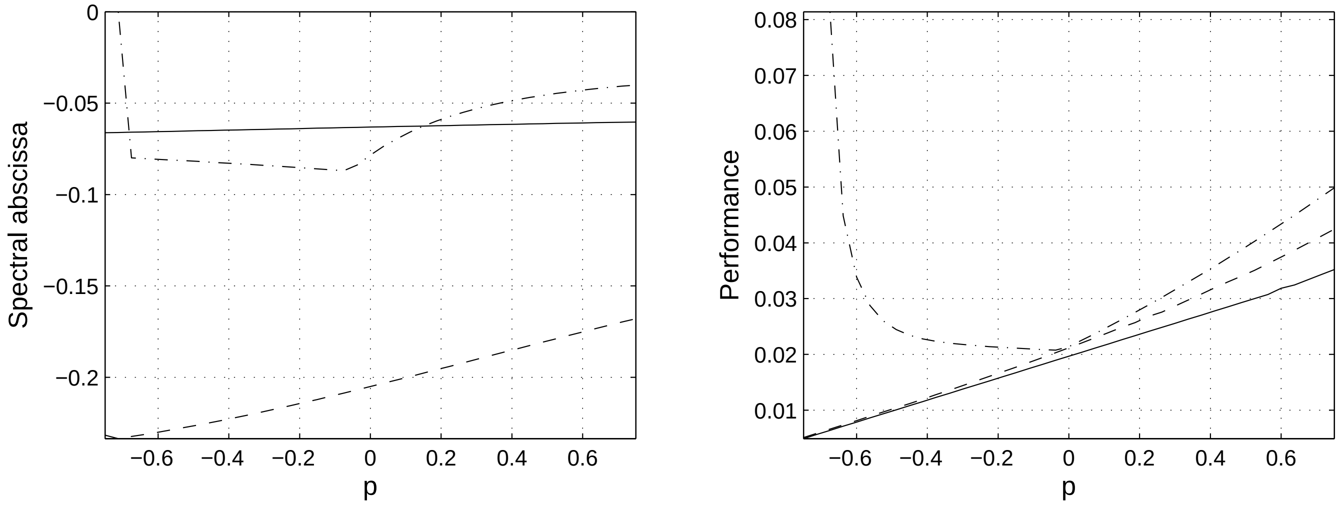

Fig. 4. Robust control of CSE1. Left plot shows spectral abscissa, right plot performance. Initial (locally optimal H_∞) controller K_∞: dash-dotted line, robust controller K^* with static multiplier: dashed line, and robust controller $\bar{K}$ with dynamic multiplier: plain line. K_∞ is not robust ($p \approx -.75$).

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