Visibility maintenance via controlled invariance for leader-follower Dubins-like vehicles

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Abstract

The paper studies the *visibility maintenance problem* (VMP) for a leader-follower pair of Dubins-like vehicles with input constraints, and proposes an original solution based on the notion of controlled invariance. The nonlinear model describing the relative dynamics of the vehicles is interpreted as linear uncertain system, with the leader robot acting as an external disturbance. The VMP is then reformulated as a *linear constrained regulation problem with additive disturbances* (DLCRP). Positive $\mathcal{D}$-invariance conditions for linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the state, control input and disturbance are considered. The proposed design procedure is shown to be easily adaptable to more general working scenarios. Extensive simulation results are provided to illustrate the theory and show the effectiveness of our approach.

Key words: Autonomous mobile robots; Visibility maintenance; Leader-follower; Controlled invariance; Input constraints

1 Introduction

1.1 Problem description and motivation

This paper considers a number of visibility maintenance problems between autonomous vehicles. The simplest formulation is a leader-follower setup, in which both leader and follower are nonholonomic vehicles constrained to move along planar paths of bounded curvature, with limited positive forward speed. The follower vehicle's goal is to maintain the leader inside an appropriate sensing region. The theory of controlled invariance for uncertain linear systems is shown to be well suited for this objective as well as for achieving more involved tasks, such as, e.g., simultaneously reject unknown but bounded disturbances or preserve visibility in multi-vehicle chain formations.

We have two main motivations for the visibility maintenance problems studied in this article. First, we are interested in surveillance and patrolling problems with formations of robotic vehicles in aerial and ground environments. We envision scenarios where a robot equipped with sensors with limited sensing footprints (such as, e.g., panoramic cameras, laser range finders, or high resolution radars) moves in such a way as to maintain a second moving target within its field of view. Second, this work is motivated by the need to design network-wide visibility and connectivity maintenance algorithms for groups of robotic vehicles. In the multi-agent network domain, connectivity is indeed a classic requirement necessary to guarantee the correct completion of numerous distributed algorithms.

1.2 Literature review

In the context of *visibility maintenance* between pairs of vehicles, two distinct literature domains are relevant to this work. First, visibility-based pursuit-evasion problems for robots in complex environments have been investigated in continuous-time in (Guibas *et al.*, 1997; Gerkey *et al.*, 2006; Bhattacharya and Hutchinson, 2010) and in discrete-time in (Isler *et al.*, 2005): in these works

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the vehicles' dynamic models are elementary and the proposed solutions are not applicable to nonholonomic vehicles with limited positive forward speed. Second, the vast literature on aircraft pursuit-evasion has focused much attention to game theory, optimal control, and numerical algorithms. A hybrid-systems and game-theoretic approach to aircraft conflict resolution is pursued in (Tomlin *et al.*, 2000). Differential game problems between aircraft are discussed in (Merz and Hague, 1977; Jarmark and Hillberg, 1984; Shima and Shinar, 2002). In (Glizer, 1999), a planar pursuit-evasion problem in which the target set is defined by a capture radius and constraints on the angular state variables (line-of-sight angle) is analyzed, and a necessary and sufficient condition for capture of the evader from any initial state is established using a variational method. However, differently from the problem studied in this paper, the author considers constant positive forward speed for the nonholonomic vehicles and unlimited turning rate for the evader. More recently, in (Mazo Jr *et al.*, 2004), the problem of estimating and tracking the motion of a moving target by a team of unicycles equipped with directional sensors with limited range, is addressed using a hierarchical control scheme.

In the context of *connectivity maintenance* in multi-agent networks, the literature has experienced a recent spurt of growth (we refer to (Bullo *et al.*, 2009; Mesbahi and Egerstedt, 2010) for recent surveys on this topic): two typical multi-agent tasks requiring network connectivity are "consensus" (Moreau, 2005; Olfati-Saber *et al.*, 2007) and "rendezvous" (Cortés *et al.*, 2006; Lin *et al.*, 2007). In this active research area, robots with limited communication capability are often modeled as transmitters with disks of finite radius. As the vehicles move to achieve a goal, it is generally hard to guarantee the connectivity among the members of the group is preserved over time. In terms of design, it is then required to constrain robots' control inputs such that the resulting topology is always connected throughout its course of evolution. *Potential fields* and *geometric optimization methods* are the standard tools used in the literature to address the connectivity maintenance problem: a list of key references, yet far from being complete, is (Ando *et al.*, 1999; Spanos and Murray, 2005; Ji and Egerstedt, 2007; Zavlanos and Pappas, 2007; Dimarogonas and Kyriakopoulos, 2008; Savla *et al.*, 2009; Yang *et al.*, 2010). These works differ from the setup proposed in this paper in at least two important ways: first, the vehicle's dynamics is assumed to be locally controllable, and second, connections between two robots are bidirectional. In our problem, instead, we consider nonholonomic vehicles that are not locally controllable (they move forward with positive speed, along paths of bounded curvature) and we deal with sensor footprints such that the visibility links are not bidirectional. We finally point out that most of the works cited above do not explicitly account for robots' input constraints.

1.3 Original contributions

The basic setup considered in this paper consists of two nonholonomic agents: a *leader* (or evader) L and a *follower* (or pursuer) F. The robots can rotate with bounded angular velocity, but similarly to Dubins' vehicles (Dubins, 1957) can only move forward. The follower is equipped with a sensing device characterized by a *visibility set* $\mathcal{S}$, a compact and convex polyhedral region encoding both the position and angle information. The leader moves along a given trajectory and the follower aims at maintaining L always inside its visibility set $\mathcal{S}$, while respecting suitable bounds on the control inputs. Inspired by (Tiwari *et al.*, 2004), where the concept of cone invariance is used to solve the multiagent rendezvous problem and by the results in (Blanchini, 1990; Blanchini, 1991), this paper addresses the *visibility maintenance problem* (VMP) using the notion of controlled invariance. The key idea is to interpret the nonlinear model describing the relative dynamics of the leader and the follower, as a linear system with model parameter uncertainty, with the control input of the leader playing the role of an external disturbance. The VMP can then be easily reformulated as *linear constrained regulation problem with additive disturbances* (DLCRP). Positive $\mathcal{D}$-invariance conditions for general linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the state, control inputs and disturbances are considered. Analytical conditions for the solution of the VMP are obtained by symbolically solving with the *Fourier-Motzkin elimination* method, the set of linear inequalities defining the polytope of all the feasible state feedback matrices. The proposed design procedure can be easily adapted to provide the control with *unknown but bounded* (UBB) disturbances rejection capabilities. Other extensions are also discussed: we present conditions for the solution of the VMP when robots' desired displacement is defined through angular parameters instead of distances, and extend the results valid for a leader-follower pair of robots to chains of n vehicles. Extensive simulation results illustrate the theory in the different working scenarios. The present paper builds upon (Morbidi *et al.*, 2008), compared to which we provide herein a more detailed and extended theory, as well as a more accurate numerical validation.

1.4 Organization

The rest of the paper is organized as follows. In Sect. 2 the linear constrained regulation problem is reviewed and new positive $\mathcal{D}$-invariance conditions for linear systems with parameter uncertainty are presented. In Sect. 3 we introduce the VMP and prove the main result of the paper. In Sect. 4 we investigate some extensions of the basic setup of Sect. 3. In Sect. 5, simulation results are presented. In Sect. 6 the main contributions of the paper are summarized and possible avenues for future research are highlighted.

2 The linear constrained regulation problem

This section presents a series of results that are instrumental in addressing the visibility maintenance problem in Sect. 3. Our exposition will basically follow (Blanchini, 1991): Theorem 10, Corollary 11, Theorem 12 and Corollary 13 extend the corresponding results in (Blanchini, 1990; Blanchini, 1991) (see also (Blanchini and Miani, 2008, Ch. 4)), to linear uncertain systems with parametric disturbance matrix. Consider the following system,

$$\dot{s}(t) = A(q(t))\, s(t) + B(q(t))\, u(t), \tag{1}$$

where $s(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ are respectively the state and input vectors, $q(t) \in \mathcal{Q} \subset \mathbb{R}^p$ is the model parameter uncertainty vector, while $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Q}$ are assigned sets containing the origin, with $\mathcal{U}$ and $\mathcal{Q}$ compact. We assume that $A(q)$ and $B(q)$ are matrices of suitable dimensions whose entries are continuous functions of q. We will suppose $q(t)$ to be a piecewise continuous function of time.

Definition 1 (Positive invariance) *The set $\mathcal{S} \subset \mathbb{R}^n$ is positively invariant for system (1), if and only if, for every initial condition $s(0) \in \mathcal{S}$ and every admissible $q(t) \in \mathcal{Q}$, the solution obtained for $u(t) \equiv 0$, satisfies the condition $s(t) \in \mathcal{S}$ for $t > 0$.*

Definition 2 (Admissible region) *A region $\mathcal{S} \subset \mathbb{R}^n$ is said to be* admissible *for the feedback control law $u = Ks$, if and only if, for every $s \in \mathcal{S}$, the condition $u \in \mathcal{U}$ holds. If $\mathcal{U}$ and $\mathcal{S}$ are convex polyhedral sets containing the origin, the admissibility of $\mathcal{S}$ is simply equivalent to*

$$K v_i \in \mathcal{U},\;\; v_i \in vert(\mathcal{S}),\;\; i \in \{1, \ldots, \mu\}, \tag{2}$$

where $vert(\mathcal{S})$ denotes the set of vertices of $\mathcal{S}$.

We can now introduce the *linear constrained regulation problem* (LCRP), (Blanchini, 1991).

Problem 3 (LCRP) *Given a system in the form (1), find a linear feedback control law $u(t) = Ks(t)$ and a set $\mathcal{S} \subset \mathcal{X}$ such that, for every initial condition $s(0) \in \mathcal{S}$ and every admissible function $q(t) \in \mathcal{Q}$, the conditions $s(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.*

Theorem 4 *The LCRP has a solution if and only if there exists a feedback matrix K and a set $\mathcal{S} \subset \mathcal{X}$ that is positively invariant and admissible for the closed loop system*

$$\dot{s}(t) = F(q(t))\, s(t), \tag{3}$$

where $F(q(t)) = A(q(t)) + B(q(t))K$.

Theorem 5 (Sub-tangentiality condition) *Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact and convex set with nonempty interior. The positive invariance of $\mathcal{S}$ for (1) is equivalent to the following condition: for every $s_0 \in \partial\mathcal{S}$ and $q \in \mathcal{Q}$,*

$$A(q)\, s_0 \in T_{\mathcal{S}}(s_0), \tag{4}$$

where $T_{\mathcal{S}}(s_0)$ is the tangent cone to $\mathcal{S}$ at s_0 (see (Blanchini, 1999, Def. 3.1), and (Aubin and Frankowska, 1990, Ch. 4), (Aubin and Cellina, 1984, Ch. 5)) for more details).

The main difficulty in exploiting condition (4) to study the positive invariance of an assigned region $\mathcal{S}$ is that it has to be checked on the boundary of $\mathcal{S}$. However, if convex polyhedral sets are considered, only their vertices must be taken into account and easy algebraic conditions can be derived. In this respect, let us consider a system of the form (1), with

$$A(q(t)) = A_0 + \sum_{l=1}^{p} A_l\, q_l(t),\;\; B(q(t)) = B_0 + \sum_{l=1}^{p} B_l\, q_l(t), \tag{5}$$

where A_l and B_l, $l \in \{1, \ldots, p\}$, are constant matrices of appropriate dimension and $q(t)$ takes values in a compact and convex polyhedron $\mathcal{Q} \subset \mathbb{R}^p$, ($q_l(t)$ denotes the l-th component of vector $q(t)$). Let the set $\mathcal{U}$ be compact, convex and polyhedral as well. We consider a candidate compact and convex polyhedral set $\mathcal{S}$ containing the origin in its interior and we search for a feedback matrix K that assures the positive invariance of $\mathcal{S}$ for the closed loop system (3), (note that the previous assumptions on $\mathcal{S}$ will be retained throughout this section). Since $\mathcal{S}$ is polyhedral, then condition (4) is fulfilled on $\partial\mathcal{S}$ if and only if is fulfilled on every vertex of $\mathcal{S}$.

Theorem 6 *The set $\mathcal{S}$ is positively invariant for system (3) with feedback $u = Ks$, if and only if, for all $v_i \in vert(\mathcal{S})$ and $w_j \in vert(\mathcal{Q})$:*

$$F(w_j)\, v_i \in T_{\mathcal{S}}(v_i),\;\; i \in \{1, \ldots, \mu\},\, j \in \{1, \ldots, \nu\}.$$

The LCRP as formulated in Problem 3 does not require the stability. However, a desirable property is the global uniform stability of the closed loop system. The link between the stability property and the existence of positively invariant regions is established by Theorem 5.2 in (Blanchini, 1991).
Let us now turn our attention to systems in the form

$$\dot{s}(t) = A(q(t))\, s(t) + B(q(t))\, u(t) + E(q(t))\, \delta(t), \tag{6}$$

where the unknown external disturbance $\delta(t)$ is constrained in a compact and convex polyhedral set $\mathcal{D} \subset \mathbb{R}^l$ containing the origin. Note that with respect to the systems considered in (Blanchini, 1991), the structure in (6) is more general inasmuch as matrix E also depends on the uncertain parameter q. As an immediate extension of the positive invariance property introduced in Definition 1, we may require that the state s remains in $\mathcal{S}$ despite the presence of the disturbance $\delta(t)$.

Definition 7 (Positive $\mathcal{D}$-invariance) *The set $\mathcal{S} \subset \mathbb{R}^n$ is positively $\mathcal{D}$-invariant for system (6), if for every initial condition $s(0) \in \mathcal{S}$ and all admissible $q(t) \in \mathcal{Q}$ and $\delta(t) \in \mathcal{D}$, the solution obtained for $u(t) \equiv 0$, satisfies the condition $s(t) \in \mathcal{S}$ for $t > 0$.*

We can now introduce the *linear constrained regulation problem with additive disturbances* (DLCRP).

Problem 8 (DLCRP) *Given a system in the form (6), find a linear feedback control law $u(t) = Ks(t)$ and a set $\mathcal{S} \subset \mathcal{X}$ such that, for every initial condition $s(0) \in \mathcal{S}$ and every admissible $q(t) \in \mathcal{Q}$ and $\delta(t) \in \mathcal{D}$, the conditions $s(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.*

Theorem 9 *The DLCRP has a solution if and only if there exists a feedback matrix K and a set $\mathcal{S} \subset \mathcal{X}$ that is positively $\mathcal{D}$-invariant and admissible for the closed loop system $\dot{s}(t) = F(q(t))\, s(t) + E(q(t))\, \delta(t)$.*

Similarly to (5), we will henceforth suppose that $E(q(t)) = E_0 + \sum_{l=1}^{p} E_l\, q_l(t)$. We are now ready to state the *main theorem* of this section.

Theorem 10 (Main result) *The set $\mathcal{S}$ is positive $\mathcal{D}$-invariant for system (6) with feedback $u = Ks$, if and only if, for all $v_i \in vert(\mathcal{S})$, $\omega_j \in vert(\mathcal{Q})$ and $r_k \in vert(\mathcal{D})$,*

$$\begin{gathered} F(w_j)\, v_i + E(w_j)\, r_k \in T_{\mathcal{S}}(v_i), \\ i \in \{1, \dots, \mu\},\ j \in \{1, \dots, \nu\},\ k \in \{1, \dots, \eta\}. \end{gathered} \tag{7}$$

Proof: The proof is based on the same ideas as those in (Blanchini, 1990, Th. 2.1) and (Blanchini, 1991, Th. 4.1) (see also (Aubin and Cellina, 1984, Ch. 2, Sect. 4)) and the references therein). For the *necessity*, we have to prove that if $\mathcal{S}$ is a positive $\mathcal{D}$-invariant region for system (6), then condition (7) holds. The proof is straightforward, since for the sub-tangentiality condition the positive $\mathcal{D}$-invariance of $\mathcal{S}$ for system (6) is equivalent to

$$F(q)\, s + E(q)\, \delta \in T_{\mathcal{S}}(s),\ s \in \partial\mathcal{S},\, q \in \mathcal{Q},\, \delta \in \mathcal{D}, \tag{8}$$

that trivially implies condition (7). For *sufficiency*, let us consider s arbitrary in $\mathcal{S}$, q arbitrary in $\mathcal{Q}$ and δ arbitrary in $\mathcal{D}$. Supposing condition (7) holds, inclusion (8) has to be proved. We have that $s = \sum_{i=1}^{\mu} \alpha_i$, v_i, $q = \sum_{j=1}^{\nu} \beta_j\, w_j$, $\delta = \sum_{k=1}^{\eta} \rho_k\, r_k$ with $\sum_{i=1}^{\mu} \alpha_i = 1$, $\sum_{j=1}^{\nu} \beta_j = 1$, $\sum_{k=1}^{\eta} \rho_k = 1$, for some $0 \le \alpha_i \le 1$, $i \in \{1, \dots, \mu\}$, $0 \le \beta_j \le 1$, $j \in \{1, \dots, \nu\}$ and $0 \le \rho_k \le 1$, $k \in \{1, \dots, \eta\}$. We first prove that

$$\begin{gathered} F(q)\, \mathrm{v}_i + E(q)\, r_k \in T_{\mathcal{S}}(\mathrm{v}_i), \\ i \in \{1, \dots, \mu\},\ q \in \mathcal{Q},\ k \in \{1, \dots, \eta\}. \end{gathered} \tag{9}$$

Let w_{lj} be the l-th entry of w_j. We have that

$$\begin{aligned} & F(q)\, \mathrm{v}_i + E(q)\, r_k = \\ & = \big(F_0 + \sum_{l=1}^{p} F_l\, q_l\big)\mathrm{v}_i + \big(E_0 + \sum_{l=1}^{p} E_l\, q_l\big) r_k \\ & = \big(F_0 + \sum_{l=1}^{p} F_l \sum_{j=1}^{\nu} \beta_j\, w_{lj}\big)\mathrm{v}_i + \big(E_0 + \sum_{l=1}^{p} E_l \sum_{j=1}^{\nu} \beta_j\, w_{lj}\big) r_k \\ & = \sum_{j=1}^{\nu} \beta_j \big[\big(F_0 + \sum_{l=1}^{p} F_l\, w_{lj}\big)\mathrm{v}_i\big] + \sum_{j=1}^{\nu} \beta_j \big[\big(E_0 + \sum_{l=1}^{p} E_l\, w_{lj}\big) r_k\big] \\ & = \sum_{j=1}^{\nu} \beta_j \big[F(w_j)\, \mathrm{v}_i + E(w_j)\, r_k\big], \\ & \qquad\qquad i \in \{1, \dots, \mu\},\ k \in \{1, \dots, \eta\}. \end{aligned}$$

From (7) we have that $\sum_{j=1}^{\nu} \beta_j \big[F(w_j)\, \mathrm{v}_i + E(w_j)\, r_k\big] \in T_{\mathcal{S}}(\mathrm{v}_i)$, $i \in \{1, \dots, \mu\}$, $k \in \{1, \dots, \eta\}$, therefore (9) is proved. If π_i is a delimiting plane of $\mathcal{S}$ for s (i.e., such that $g_i^T s = \xi_i$), we may write s as a convex combination of the vertices of $\mathcal{S}$ that belong to π_i: $s = \sum_{h=1}^{\mu_i} \alpha_h\, \mathrm{v}_h$, with $g_i^T \mathrm{v}_h = \xi_i$ and $\sum_{h=1}^{\mu_i} \alpha_h = 1$, $0 \le \alpha_h \le 1$, $h \in \{1, \dots, \mu_i\}$. Then $g_i^T (F(q)s + E(q)\, \delta) = g_i^T (F(q) \sum_{h=1}^{\mu_i} \alpha_h\, \mathrm{v}_h + E(q) \sum_{k=1}^{\eta} \rho_k\, r_k)$. But from (9) and recalling the expression of the tangent cone when $\mathcal{S}$ is described in terms of its vertices, we have that $g_i^T (F(q)\, \mathrm{v}_h + E(q)\, r_k) \le 0$, that implies $g_i^T \big(F(q) \sum_{h=1}^{\mu_i} \alpha_h \mathrm{v}_h + E(q) \sum_{k=1}^{\eta} \rho_k\, r_k\big) \le 0$. By considering all the planes for s, condition (8) follows immediately. ∎

The application of Theorem 10 requires the knowledge of all cones $T_{\mathcal{S}}(\mathrm{v}_i)$, $i \in \{1, \dots, \mu\}$. An alternative solution is given by the following corollary (whose proof is analogous to that of (Blanchini, 1991, Corollary 4.1)) in which the Euler auxiliary system associated to (6) is involved (cf. (Blanchini and Miani, 2008, Sect. 12.1)).

Corollary 11 *The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (6), if and only if there exists $\tau > 0$ such that, for all $\mathrm{v}_i \in vert(\mathcal{S})$, $\omega_j \in vert(\mathcal{Q})$ and $r_k \in vert(\mathcal{D})$,*

$$\begin{gathered} v_i + \tau(F(w_j)\, v_i + E(w_j)\, r_k) \in \mathcal{S}, \\ i \in \{1, \dots, \mu\},\ j \in \{1, \dots, \nu\},\ k \in \{1, \dots, \eta\}. \end{gathered} \tag{10}$$

To overcome the problem of the choice of τ, we introduce Theorem 12 that provides a condition equivalent to (10). The proof is analogous to that of (Blanchini, 1990, Th. 2.3). Let $\mathcal{C}_i$ be the convex cone defined by the delimiting planes of $\mathcal{S}$ which contain v_i (see (Panik, 1993, Ch. 4)) :

$$\begin{aligned} \mathcal{C}_i = \{ & g_h^T s \le \xi_h,\ \xi_h > 0, \text{ for every } g_h^T \\ & \text{and } \xi_h : g_h^T \mathrm{v}_i = \xi_h,\ \mathrm{v}_i \in \mathrm{vert}(\mathcal{S})\}. \end{aligned}$$

Theorem 12 *The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (6), if and only if, for all $\tau > 0$, $v_i \in vert(\mathcal{S})$, $\omega_j \in vert(\mathcal{Q})$ and $r_k \in vert(\mathcal{D})$:*

$$v_i + \tau(F(w_j)\, v_i + E(w_j)\, r_k) \in \mathcal{C}_i,$$
$$i \in \{1, \ldots, \mu\},\ \ j \in \{1, \ldots, \nu\},\ \ k \in \{1, \ldots, \eta\}.$$

If the plane description of $\mathcal{S}$ is available, the next corollary, whose proof directly follows from that of Theorem 10, holds.

Corollary 13 *The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (6), if and only if, for every $\tau > 0$ and every $v_i \in vert(\mathcal{S})$, $\omega_j \in vert(\mathcal{Q})$,*

$$(I_n + \tau\, F(w_j))\, v_i \in \mathcal{C}_i^\star,\ \ i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, \nu\}, \tag{11}$$

where $\mathcal{C}_i^\star$ is the cone obtained by shifting the planes of $\mathcal{C}_i$ as follows :

$$\mathcal{C}_i^\star = \{g_h^T s \leq \xi_h \,-\, \max_{jk}\{\tau g_h^T E(w_j)\, r_k\},\, \omega_j \in vert(\mathcal{Q}),$$
$$r_k \in vert(\mathcal{D}),\, for\ every\ g_h^T : g_h^T v_i = \xi_h\}.$$

Remark 14 *According to Theorem 9, conditions (11) and (2) provide us with a set of inequalities in the unknown K defining the polytope $\mathcal{K}$ of all the state feedback matrices solving the DLCRP.*

3 The visibility maintenance problem

3.1 Modeling

Let $\Sigma_0 \equiv \{O_0\,; x_0,\, y_0\}$ be the fixed reference frame in $\mathbb{R}^2$, and $\Sigma_{\rm F} \equiv \{O_{\rm F}\,; x_{\rm F},\, y_{\rm F}\}$ and $\Sigma_{\rm L} \equiv \{O_{\rm L}\,; x_{\rm L},\, y_{\rm L}\}$ the reference frames attached to a *follower* robot F and a *leader* robot L (see Fig. 1). The robots are supposed to have single integrator dynamics,

$$\begin{aligned} \dot{p}_{\rm F}^{\rm F} &= \sigma_{\rm F}^{\rm F}\,, \quad & \dot{p}_{\rm L}^{\rm L} &= \sigma_{\rm L}^{\rm L}\,, \\ \dot{\theta}_{\rm F} &= \omega_{\rm F}\,, \quad & \dot{\theta}_{\rm L} &= \omega_{\rm L}\,, \end{aligned} \tag{12}$$

where $p_{\rm F}^{\rm F} = (x_{\rm F},\, y_{\rm F})^T$, $p_{\rm L}^{\rm L} = (x_{\rm L},\, y_{\rm L})^T$ are the positions, $\sigma_{\rm F}^{\rm F} = (\sigma_{\rm F}^{\rm F}[1], \sigma_{\rm F}^{\rm F}[2])^T$, $\sigma_{\rm L}^{\rm L} = (\sigma_{\rm L}^{\rm L}[1], \sigma_{\rm L}^{\rm L}[2])^T$ the linear velocities and $\omega_{\rm F}, \omega_{\rm L}$ the angular velocities of robots F and L in the frames $\Sigma_{\rm F}$ and $\Sigma_{\rm L}$, respectively. We are going to derive a dynamic model describing the relative dynamics of the robots F and L. Referring (12) to the frame Σ_0, we obtain (Siciliano *et al.*, 2008), $\dot{p}_{\rm F}^0 = R_{\rm F}^0(\theta_{\rm F})\, \sigma_{\rm F}^{\rm F}$, $\dot{p}_{\rm L}^0 = R_{\rm L}^0(\theta_{\rm L})\, \sigma_{\rm L}^{\rm L}$, where

$$R_{\rm F}^0(\theta_{\rm F}) = \begin{bmatrix} \cos\theta_{\rm F} & -\sin\theta_{\rm F} \\ \sin\theta_{\rm F} & \cos\theta_{\rm F} \end{bmatrix},\ R_{\rm L}^0(\theta_{\rm L}) = \begin{bmatrix} \cos\theta_{\rm L} & -\sin\theta_{\rm L} \\ \sin\theta_{\rm L} & \cos\theta_{\rm L} \end{bmatrix}.$$

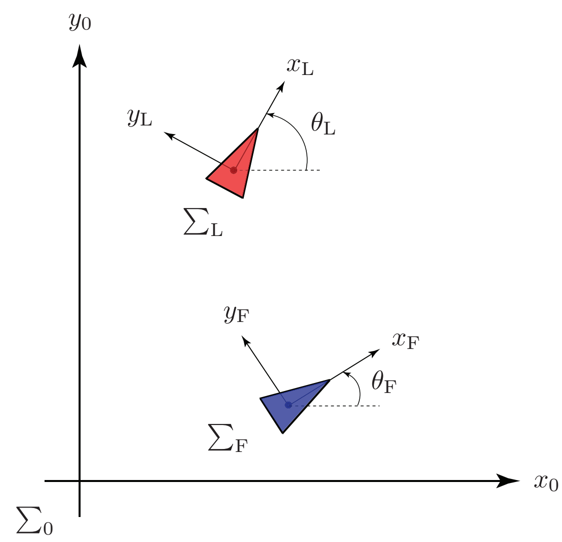

Fig. 1. Leader-follower setup.

The position of robot L with respect to $\Sigma_{\rm F}$ is then given by

$$p_{\rm L}^{\rm F} = R_0^{\rm F}(\theta_{\rm F})(p_{\rm L}^0 - p_{\rm F}^0), \tag{13}$$

where $R_0^{\rm F}(\theta_{\rm F}) = (R_{\rm F}^0(\theta_{\rm F}))^T$. Differentiating (13) with respect to time, we get

$$\dot{p}_{\rm L}^{\rm F} = \dot{R}_0^{\rm F}(\theta_{\rm F})(p_{\rm L}^0 - p_{\rm F}^0) + R_0^{\rm F}(\theta_{\rm F})\big(R_{\rm L}^0(\theta_{\rm L})\, \sigma_{\rm L}^{\rm L} - R_{\rm F}^0(\theta_{\rm F})\, \sigma_{\rm F}^{\rm F}\,\big). \tag{14}$$

Since

$$\dot{R}_0^{\rm F}(\theta_{\rm F}) = \left[\begin{smallmatrix} 0 & \omega_{\rm F} \\ -\omega_{\rm F} & 0 \end{smallmatrix}\right] R_0^{\rm F}(\theta_{\rm F}),$$

we can rewrite (14) as

$$\dot{p}_{\rm L}^{\rm F} = \begin{bmatrix} 0 & \omega_{\rm F} \\ -\omega_{\rm F} & 0 \end{bmatrix} p_{\rm L}^{\rm F} + R_{\rm L}^{\rm F}(\beta_{\rm L}^{\rm F})\, \sigma_{\rm L}^{\rm L} - \sigma_{\rm F}^{\rm F}, \tag{15}$$

where $\beta_{\rm L}^{\rm F} \triangleq \theta_{\rm L} - \theta_{\rm F}$. Collecting equation (15) and the relative angular dynamics of the robots together, we obtain the following system

$$\left[\begin{array}{c} \dot{p}_{\rm L}^{\rm F} \\ \hline \dot{\beta}_{\rm L}^{\rm F} \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{-I_2} & \begin{array}{c} p_{\rm L}^{\rm F}[2] \\ -p_{\rm L}^{\rm F}[1] \end{array} \\ \hline 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} \sigma_{\rm F}^{\rm F} \\ \hline \omega_{\rm F} \end{array}\right] + \left[\begin{array}{cc|c} \multicolumn{2}{c|}{R_{\rm L}^{\rm F}(\beta_{\rm L}^{\rm F})} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \sigma_{\rm L}^{\rm L} \\ \hline \omega_{\rm L} \end{array}\right], \tag{16}$$

where $p_{\rm L}^{\rm F} = (p_{\rm L}^{\rm F}[1],\, p_{\rm L}^{\rm F}[2])^T$. For the sake of simplicity, we will suppose that

$$\begin{aligned} \sigma_{\rm F}^{\rm F} &= (1 + v_{\rm F}\,,\ 0)^T, \\ \sigma_{\rm L}^{\rm L} &= (1 + v_{\rm L}\,,\ 0)^T, \end{aligned} \tag{17}$$

where $|v_{\rm F}(t)| < 1$, $|v_{\rm L}(t)| < 1$, for all $t \geq 0$. F and L will then behave in a way similar to *Dubins vehicles* since they can only move forward (however, differently from the standard Dubins model, $v_{\rm F}$ and $v_{\rm L}$ are not necessarily constant in our case). Substituting (17) in (16), we finally come up with the following system

$$\begin{bmatrix} \dot{p}^{\rm F}_{\rm L}[1] \\ \dot{p}^{\rm F}_{\rm L}[2] \\ \dot{\beta}^{\rm F}_{\rm L} \end{bmatrix} = \begin{bmatrix} \cos\beta^{\rm F}_{\rm L} - 1 \\ \sin\beta^{\rm F}_{\rm L} \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & p^{\rm F}_{\rm L}[2] \\ 0 & -p^{\rm F}_{\rm L}[1] \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{\rm F} \\ \omega_{\rm F} \end{bmatrix} + \begin{bmatrix} \cos\beta^{\rm F}_{\rm L} & 0 \\ \sin\beta^{\rm F}_{\rm L} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{\rm L} \\ \omega_{\rm L} \end{bmatrix}, \tag{18}$$

with state vector $s = (p^{\rm F}_{\rm L}[1], p^{\rm F}_{\rm L}[2], \beta^{\rm F}_{\rm L})^T \in \mathcal{X} \subset \mathrm{SE}(2)$, input vector $u = (v_{\rm F}, \omega_{\rm F})^T \in \mathcal{U} \subset (-1, 1) \times \mathbb{R}$ and disturbance vector $\delta = (v_{\rm L}, \omega_{\rm L})^T \in \mathcal{D} \subset (-1, 1) \times \mathbb{R}$. System (18) describes the relative dynamics of the Dubins-like vehicles F and L in the configuration space SE(2).

3.2 Problem statement

In the forthcoming analysis, we will suppose that robot F is equipped with a sensor (e.g., a panoramic camera, a laser range finder, etc.) with limited sensing range. We will call *visibility set* of robot F any compact and convex polyhedral set $\mathcal{S} \subset \mathcal{X}$ containing the origin in its interior. Please note that the visibility set generalizes the notion of *sensor footprint* since it encodes both the position and angle information.
Robot L moves along a given trajectory and robot F aims at keeping L always inside its visibility set $\mathcal{S}$, while respecting the control bounds. By referring to system (18), we can formalize this problem as follows:

Problem 15 (Visibility maintenance problem: VMP) *Let $\mathcal{S}$ be the visibility set of robot F and let $s(0) \in \mathcal{S}$. Find a control $u(t)$ such that for all $\delta(t) \in \mathcal{D}$, the conditions $s(t) \in \mathcal{S}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.*

In the following, we will refer to Problem 15 as to the VMP with candidate positively $\mathcal{D}$-invariant set $\mathcal{S}$, control set $\mathcal{U}$ and disturbance set $\mathcal{D}$.

3.3 Solution method

Next, we will transcribe system (18) into the linear parametric form (6): in this way, the VMP simply reduces to the DLCRP (Problem 8) introduced in Sect. 2 and suitable solvability conditions can be derived by means of (11) and (2).

After simple matrix manipulations in (18), we obtain

$$\begin{bmatrix} \Delta\dot{p}^{\rm F}_{\rm L}[1] \\ \dot{p}^{\rm F}_{\rm L}[2] \\ \dot{\beta}^{\rm F}_{\rm L} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\cos\beta^{\rm F}_{\rm L} - 1}{\beta^{\rm F}_{\rm L}} \\ 0 & 0 & \frac{\sin\beta^{\rm F}_{\rm L}}{\beta^{\rm F}_{\rm L}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p^{\rm F}_{\rm L}[1] \\ p^{\rm F}_{\rm L}[2] \\ \beta^{\rm F}_{\rm L} \end{bmatrix} + \begin{bmatrix} -1 & p^{\rm F}_{\rm L}[2] \\ 0 & -d - \Delta p^{\rm F}_{\rm L}[1] \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{\rm F} \\ \omega_{\rm F} \end{bmatrix} + \begin{bmatrix} \cos\beta^{\rm F}_{\rm L} & 0 \\ \sin\beta^{\rm F}_{\rm L} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{\rm L} \\ \omega_{\rm L} \end{bmatrix}, \tag{19}$$

which can be written in the form (6) with

$$A(q) = \begin{bmatrix} 0 & 0 & q_2 \\ 0 & 0 & 1 + q_1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B(q) = \begin{bmatrix} -1 & q_4 \\ 0 & -d - q_3 \\ 0 & -1 \end{bmatrix}, \tag{20}$$
$$E(q) = \begin{bmatrix} 1 + q_5 & 0 \\ q_6 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$q_1 = \frac{\sin\beta^{\rm F}_{\rm L}}{\beta^{\rm F}_{\rm L}} - 1, \;\; q_2 = \frac{\cos\beta^{\rm F}_{\rm L} - 1}{\beta^{\rm F}_{\rm L}}, \;\; q_3 = \Delta p^{\rm F}_{\rm L}[1], \tag{21}$$
$$q_4 = p^{\rm F}_{\rm L}[2], \qquad q_5 = \cos\beta^{\rm F}_{\rm L} - 1, \;\; q_6 = \sin\beta^{\rm F}_{\rm L}.$$

We made the following change of variables in (19),

$$(p^{\rm F}_{\rm L}[1], p^{\rm F}_{\rm L}[2], \beta^{\rm F}_{\rm L})^T \to (\Delta p^{\rm F}_{\rm L}[1], p^{\rm F}_{\rm L}[2], \beta^{\rm F}_{\rm L})^T,$$

where $\Delta p^{\rm F}_{\rm L}[1] = p^{\rm F}_{\rm L}[1] - d$ and d is a strictly positive constant. Two main reasons motivated this transformation: first of all, if robot F is able to keep L always inside a visibility set displaced of d with respect to its center (with $d > \max_{s_1, s_2 \in \mathrm{vert}(\mathcal{S})} \frac{1}{2} \|s_1 - s_2\|_2$), then this automatically guarantees robots' collision avoidance. Second, this choice simplifies the study of the VMP with chains of robots (see Sect. 4.3).
Notice that A_0, B_0 and E_0 in (20) (recall the notation used in Sect. 2) correspond to the constant matrices obtained by linearizing system (18) around the equilibrium $s_{eq} = (d, 0, 0)^T$, $u_{eq} = (0, 0)^T$, $\delta_{eq} = (0, 0)^T$.
For the sake of simplicity, we will henceforth make the following assumption:

Assumption 16 *Suppose that (see Fig. 2)*

$$\mathcal{U} = \big\{(v_{\rm F}, \omega_{\rm F})^T : v_{\rm F} \in [-V_{\rm F}, V_{\rm F}], \; \omega_{\rm F} \in [-\Omega_{\rm F}, \Omega_{\rm F}]\big\},$$
$$\mathcal{D} = \big\{(v_{\rm L}, \omega_{\rm L})^T : v_{\rm L} \in [-V_{\rm L}, V_{\rm L}], \; \omega_{\rm L} \in [-\Omega_{\rm L}, \Omega_{\rm L}]\big\}, \tag{22}$$

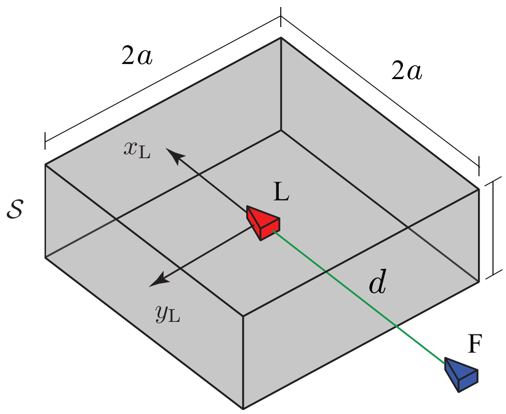

Fig. 2. The visibility set $\mathcal{S}$ in (23) and the pose of the robots L and F for $(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], p_{\mathrm{L}}^{\mathrm{F}}[2], \beta_{\mathrm{L}}^{\mathrm{F}})^T = (0,0,0)^T$, $d > a$.

$$\mathcal{S} = \big\{(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], p_{\mathrm{L}}^{\mathrm{F}}[2], \beta_{\mathrm{L}}^{\mathrm{F}})^T : \ \Delta p_{\mathrm{L}}^{\mathrm{F}}[1] \in [-a,\, a], \\ p_{\mathrm{L}}^{\mathrm{F}}[2] \in [-a,\, a],\ \beta_{\mathrm{L}}^{\mathrm{F}} \in [-b,\, b]\big\}, \tag{23}$$

where $V_{\mathrm{F}} < 1$, $V_{\mathrm{L}} < 1$, Ω_{F}, Ω_{L}, a, b, are strictly positive constants.

The definition (22) of control and disturbance sets is motivated by the presence of saturation bounds on the driving motors of physical robots. The candidate invariant set $\mathcal{S}$ is defined as in (23) because it is computationally simple to handle (this will allow us to provide concise solvability conditions for the VMP in Theorem 18), and because its horizontal section represents a reasonable good inner approximation of a disk sensor footprint e.g., due to an omnidirectional camera or a 360° laser scanner (the problem of precisely quantify the non-conservatism introduced by this approximation goes beyond the scope of this paper, and it is left as a subject of future research). Finally, it is worth emphasizing that is not unusual in the multi-agent systems literature to encounter rectangular footprints, that are typically used, for example, to model "push-broom" or line-scanner sensors (see, e.g., (Finke *et al.*, 2005)).
We now complete our transcription of system (18) into the linear parametric form (6), by defining an appropriate polyhedral set for the model parameter uncertainty:

$$\mathcal{Q} = \big\{(q_1, \ldots, q_6)^T : \ q_1 \in \big[\tfrac{\sin b}{b} - 1,\, 0\big], \\ q_2 \in \big[\tfrac{\cos b - 1}{b}, \tfrac{1-\cos b}{b}\big],\ q_3 \in [-a,\, a],\ q_4 \in [-a,\, a], \\ q_5 \in \big[\cos b - 1,\, 0\big],\ q_6 \in [-\sin b,\, \sin b]\big\}. \tag{24}$$

It is easy to show that, if $(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], p_{\mathrm{L}}^{\mathrm{F}}[2], \beta_{\mathrm{L}}^{\mathrm{F}})^T \in \mathcal{S}$, then definition (21) immediately implies that $q \in \mathcal{Q}$.

Remark 17 *In the previous passages, the nonlinear system (18) has been* absorbed *into a linear (controlled) differential inclusion (see (Blanchini and Miani, 2008, Sect. 2.1.2)). This is an approximate transformation: however, no matter how the input u is chosen, we have that any trajectory of the original system (18) is also a trajectory of the corresponding linear uncertain system (the opposite is clearly not true in general). As a consequence, if we are able to determine the qualitative behavior of the absorbing system, we can determine (in a conservative way) the behavior of the original system. Some tools are available in the robust control literature to quantify this conservativeness, such as, e.g., the recent nonlinear extensions of the* gap *and Vinnicombe's* ν-gap metrics *(see (Bian and French, 2005; James* et al.*, 2005) and the references therein). However, in the interest of brevity, we will not perform such an analysis in this paper.*

We are now ready to state the *main result* of this section.

Theorem 18 (Solvability of the VMP) *For the robots F and L, consider the VMP with candidate positive $\mathcal{D}$-invariant set $\mathcal{S}$, control set $\mathcal{U}$ and disturbance set $\mathcal{D}$ satisfying Assumption 16 with $d > a$, $0 < b \le \pi/2$. This VMP has a solution if the following conditions are satisfied*

$$V_{\mathrm{F}} \ge V_{\mathrm{L}} \left(1 + \frac{a \sin b}{d-a}\right) + 1 - \cos b + \frac{a\, b}{d-a}, \tag{25}$$

$$\Omega_{\mathrm{L}} \le \frac{(1 - V_{\mathrm{L}}) \sin b}{d+a}, \quad \frac{V_{\mathrm{L}} \sin b + b}{d-a} \le \Omega_{\mathrm{F}}. \tag{26}$$

The state feedback matrix has the form

$$K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \end{bmatrix}, \tag{27}$$

where k_{11}, k_{22} and k_{23} belong to the polytope $\mathcal{K} \subset \mathbb{R}^3$ defined by (30)-(31), (see the proof below).

Proof: Let us apply Corollary 13 to system (19). By selecting $\tau = 1$ in (11), we obtain

$$\begin{bmatrix} 1 - k_{11} + q_4 k_{21} & -k_{12} + q_4 k_{22} & q_2 - k_{13} + q_4 k_{23} \\ -(d+q_3) k_{21} & 1 - (d+q_3) k_{22} & 1 + q_1 - (d+q_3) k_{23} \\ -k_{21} & -k_{22} & 1 - k_{23} \end{bmatrix} v_i \in \mathcal{C}_i^\star. \tag{28}$$

Condition (28) must be evaluated only on the vertices $\mathrm{v}_1 = (a,a,b)^T, \mathrm{v}_2 = (a,a,-b)^T, \mathrm{v}_3 = (a,-a,b)^T, \mathrm{v}_4 = (a,-a,-b)^T$ since the set (23), whose plane representation is

$$\begin{bmatrix} 1/a & 0 & 0 \\ -1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & -1/a & 0 \\ 0 & 0 & 1/b \\ 0 & 0 & -1/b \end{bmatrix} \overline{s} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$\overline{s} = (\Delta p_{\mathrm{L}}^{\mathrm{F}}[1],\, p_{\mathrm{L}}^{\mathrm{F}}[2],\, \beta_{\mathrm{L}}^{\mathrm{F}})^T$, is symmetric with respect to the origin. The cones $\mathcal{C}_1^\star, \dots, \mathcal{C}_4^\star$ are given by

$$\mathcal{C}_1^\star = \{g_1^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}}}{a},\; g_3^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}} \sin b}{a},\; g_5^T\, \overline{s} \le 1 - \tfrac{\Omega_{\mathrm{L}}}{b}\},$$

$$\mathcal{C}_2^\star = \{g_1^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}}}{a},\; g_3^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}} \sin b}{a},\; g_6^T\, \overline{s} \le 1 - \tfrac{\Omega_{\mathrm{L}}}{b}\},$$

$$\mathcal{C}_3^\star = \{g_1^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}}}{a},\; g_4^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}} \sin b}{a},\; g_5^T\, \overline{s} \le 1 - \tfrac{\Omega_{\mathrm{L}}}{b}\},$$

$$\mathcal{C}_4^\star = \{g_1^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}}}{a},\; g_4^T\, \overline{s} \le 1 - \tfrac{V_{\mathrm{L}} \sin b}{a},\; g_6^T\, \overline{s} \le 1 - \tfrac{\Omega_{\mathrm{L}}}{b}\}.$$

Condition (28) can then be rewritten as:

$$\begin{bmatrix} a(1 - k_{11} + q_4 k_{21} - k_{12} + q_4 k_{22}) + b(q_2 - k_{13} + q_4 k_{23}) \\ a(1 - (k_{21} + k_{22})(d + q_3)) + b(1 + q_1 - k_{23}(d + q_3)) \\ -a(k_{21} + k_{22}) + b(1 - k_{23}) \end{bmatrix} \in \mathcal{C}_1^\star,$$

$$\begin{bmatrix} a(1 - k_{11} + q_4 k_{21} - k_{12} + q_4 k_{22}) - b(q_2 - k_{13} + q_4 k_{23}) \\ a(1 - (k_{21} + k_{22})(d + q_3)) - b(1 + q_1 - k_{23}(d + q_3)) \\ -a(k_{21} + k_{22}) - b(1 - k_{23}) \end{bmatrix} \in \mathcal{C}_2^\star,$$

$$\begin{bmatrix} a(1 - k_{11} + q_4 k_{21} + k_{12} - q_4 k_{22}) + b(q_2 - k_{13} + q_4 k_{23}) \\ a(-1 + (-k_{21} + k_{22})(d + q_3)) + b(1 + q_1 - k_{23}(d + q_3)) \\ -a(k_{21} - k_{22}) + b(1 - k_{23}) \end{bmatrix} \in \mathcal{C}_3^\star,$$

$$\begin{bmatrix} a(1 - k_{11} + q_4 k_{21} + k_{12} - q_4 k_{22}) - b(q_2 - k_{13} + q_4 k_{23}) \\ a(-1 + (-k_{21} + k_{22})(d + q_3)) - b(1 + q_1 - k_{23}(d + q_3)) \\ -a(k_{21} - k_{22}) - b(1 - k_{23}) \end{bmatrix} \in \mathcal{C}_4^\star. \tag{29}$$

Because of the special structure of $B(q)$ in (20), we can select a simplified state feedback matrix K of the form (27): this allows to the decouple the control inputs v_{F} and ω_{F} and visualize the polytope $\mathcal{K} \subset \mathbb{R}^3$ of all the feasible gain matrices. We can then rewrite (29) in the following simplified form:

$$\begin{aligned}
-k_{11} + q_4 k_{22} + \frac{b}{a}\, q_4\, k_{23} &\le -\frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a}\,, \\
-(d + q_3)\, k_{22} - \frac{b}{a}\,(d + q_3)\, k_{23} &\le -\frac{b}{a}\,(1 + q_1) - \frac{V_{\mathrm{L}} \sin b}{a}\,, \\
-k_{11} + q_4\, k_{22} - \frac{b}{a}\, q_4\, k_{23} &\le \frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a}\,, \\
-(d + q_3)\, k_{22} + \frac{b}{a}\,(d + q_3)\, k_{23} &\le \frac{b}{a}\,(1 + q_1) - \frac{V_{\mathrm{L}} \sin b}{a}\,, \\
-\frac{a}{b}\, k_{22} - k_{23} &\le -\frac{\Omega_{\mathrm{L}}}{b}\,, \\
\frac{a}{b}\, k_{22} - k_{23} &\le -\frac{\Omega_{\mathrm{L}}}{b}\,, \\
-k_{11} - q_4\, k_{22} + \frac{b}{a}\, q_4\, k_{23} &\le -\frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a}\,, \\
-k_{11} - q_4\, k_{22} - \frac{b}{a}\, q_4\, k_{23} &\le \frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a}\,.
\end{aligned} \tag{30}$$

The admissibility condition (2) leads to the additional constraints

$$\begin{aligned}
&k_{11} \le \frac{V_{\mathrm{F}}}{a}, && k_{11} \ge -\frac{V_{\mathrm{F}}}{a}, \\
&k_{22} + \frac{b}{a}\, k_{23} \le \frac{\Omega_{\mathrm{F}}}{a}, && k_{22} - \frac{b}{a}\, k_{23} \le \frac{\Omega_{\mathrm{F}}}{a}, \\
&k_{22} - \frac{b}{a}\, k_{23} \ge -\frac{\Omega_{\mathrm{F}}}{a}, && k_{22} + \frac{b}{a}\, k_{23} \ge -\frac{\Omega_{\mathrm{F}}}{a}.
\end{aligned} \tag{31}$$

The Fourier-Motzkin elimination is a mathematical algorithm for eliminating variables from a system of linear inequalities. Elimination of unknown k_{ij} from the system of inequalities, consists in creating another system of the same kind but without k_{ij}, such that both systems have the same solutions over the remaining variables. If one removes all variables from a system of inequalities with *numerical coefficients*, then one obtains a system of constant inequalities, which can be trivially decided to be true or false. This procedure can then be used to easily check whether a given system admits solutions or not (see Appendix A for more details).

Applying the Fourier-Motzkin elimination to the inequalities (30)-(31) with the assumption that $d > a$ (in order to fix the sign of the coefficients of k_{22} and k_{23} in the second and fourth inequality of (30)) we end up with the following conditions on the variables a, b, d, V_{F}, V_{L}, Ω_{F}, Ω_{L} and uncertain parameters $q_1, \dots, q_4$ (the detailed passages are reported in Appendix B),

$$\begin{aligned}
&\Omega_{\mathrm{L}} \le \tfrac{b(1+q_1) - V_{\mathrm{L}} \sin b}{d+q_3}\,, \quad \tfrac{b(1+q_1) + V_{\mathrm{L}} \sin b}{d+q_3} \le \Omega_{\mathrm{F}}\,, \\
&V_{\mathrm{F}} \ge V_{\mathrm{L}}\big(1 + \tfrac{q_4 \sin b}{d+q_3}\big) + b\big(q_2 + \tfrac{q_4(1+q_1)}{d+q_3}\big), && \text{for } q_4 > 0\,, \\
&V_{\mathrm{F}} \ge V_{\mathrm{L}}\big(1 + \tfrac{q_4 \sin b}{d+q_3}\big) - b\big(q_2 + \tfrac{q_4(1+q_1)}{d+q_3}\big), && \text{for } q_4 > 0\,, \\
&V_{\mathrm{F}} \ge V_{\mathrm{L}} + b\, q_2, && \text{for } q_4 = 0\,, \\
&V_{\mathrm{F}} \ge V_{\mathrm{L}}\big(1 - \tfrac{q_4 \sin b}{d+q_3}\big) + b\big(q_2 + \tfrac{q_4(1+q_1)}{d+q_3}\big), && \text{for } q_4 < 0\,, \\
&V_{\mathrm{F}} \ge V_{\mathrm{L}}\big(1 - \tfrac{q_4 \sin b}{d+q_3}\big) - b\big(q_2 + \tfrac{q_4(1+q_1)}{d+q_3}\big), && \text{for } q_4 < 0.
\end{aligned}$$

An appropriate selection of the parameters $q_1, \dots, q_4$ on the extremes of the intervals (24), (i.e., $q_1 = \frac{\sin b}{b} - 1$, $q_3 = a$, for the first inequality, $q_1 = 0$, $q_3 = -a$, for the second and $q_1 = 0$, $q_2 = \frac{1 - \cos b}{b}$, $q_3 = -a$, $q_4 = a$, for the third), leads to (25) and (26). ■

Some remarks are in order at this point.

The inequalities (25) and (26) (which are linear in V_{F}, V_{L}, Ω_{F}, Ω_{L} and nonlinear in a, b, d), specify the role played by each of the parameters introduced in Assumption 16, in the solvability of the VMP. In particular, they show how the bounds on the forward and angular velocity of the follower robot are affected by the size of the visibility set $\mathcal{S}$ and the velocity of the leader.

Note that conditions (25) and (26) are *necessary* and *sufficient* for the linear uncertain system (19). Note also that owing to (26), we have $\Omega_{\rm F} \geq \Omega_{\rm L}$. Once fixed the variables a, b, d, $V_{\rm F}$, $V_{\rm L}$, $\Omega_{\rm F}$, $\Omega_{\rm L}$ according to (25) and (26), the polytope of all the feasible state feedback matrices is given by (30)-(31): by evaluating (30)-(31) on the vertices of the polyhedron $\mathcal{Q}$, we can see that $\mathcal{K}$ is defined by a set of 392 inequalities, most of whom are redundant (see, e.g., the external (green) polytope in Fig. 3, below).

Remark 19 (Selection of the gain matrix K) *Since the polytope $\mathcal{K}$ contains infinite gain matrices, one needs an optimal criterion to select K, such as, e.g., minimizing any matrix norm. In the simulation experiments reported in Sect. 5, we have chosen the matrix*

$$K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \end{bmatrix},$$

with minimum 2-norm.

4 Extensions and applications

In this section, we propose various extensions of the basic setup considered in Theorem 18, and discuss a few applications. We study the VMP in the presence of unknown but bounded disturbances, and consider the case of the leader moving along a circular path around a stationary target. We also extend our results to chains of robots.

4.1 Rejection of unknown but bounded disturbances

Let us consider the following system

$$\begin{bmatrix} \Delta\dot{p}_{\rm L}^{\rm F}[1] \\ \dot{p}_{\rm L}^{\rm F}[2] \\ \dot{\beta}_{\rm L}^{\rm F} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\cos\beta_{\rm L}^{\rm F}-1}{\beta_{\rm L}^{\rm F}} \\ 0 & 0 & \frac{\sin\beta_{\rm L}^{\rm F}}{\beta_{\rm L}^{\rm F}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p_{\rm L}^{\rm F}[1] \\ p_{\rm L}^{\rm F}[2] \\ \beta_{\rm L}^{\rm F} \end{bmatrix} + \begin{bmatrix} -1 & 0 & p_{\rm L}^{\rm F}[2] \\ 0 & -1 & -\Delta p_{\rm L}^{\rm F}[1] - d \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{\rm F} \\ h_{\rm F} \\ \omega_{\rm F} \end{bmatrix} + \begin{bmatrix} \cos\beta_{\rm L}^{\rm F} & -\sin\beta_{\rm L}^{\rm F} & 0 \\ \sin\beta_{\rm L}^{\rm F} & \cos\beta_{\rm L}^{\rm F} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{\rm L} \\ h_{\rm L} \\ \omega_{\rm L} \end{bmatrix}. \tag{32}$$

With respect to (19), two new components, $h_{\rm F}$ and $h_{\rm L}$, are present in the input and disturbance vectors u and δ. They are *unknown but bounded* disturbances acting on the robots F and L (e.g., lateral wind in a real setup). Our purpose here is to solve the VMP in the presence of the disturbances $h_{\rm F}$, $h_{\rm L}$.
Collecting together all the perturbations acting on the nominal system (i.e., $v_{\rm L}$, $\omega_{\rm L}$, $h_{\rm F}$ and $h_{\rm L}$),

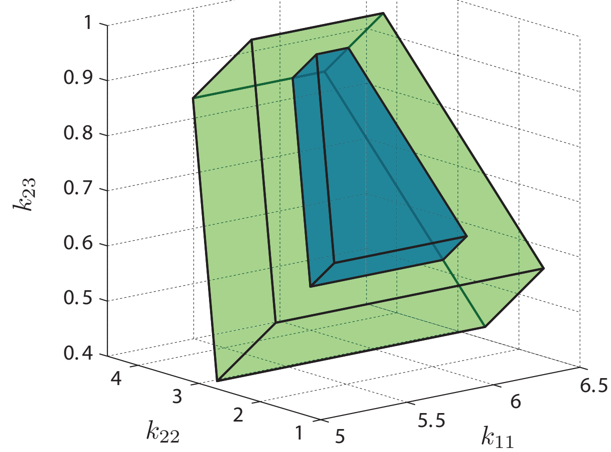

Fig. 3. *Polytopes $\mathcal{K}$ for a set of given parameters:* (*blue*, internal) with disturbances; (*green*, external) without disturbances.

we can rewrite (32) as

$$\begin{bmatrix} \Delta\dot{p}_{\rm L}^{\rm F}[1] \\ \dot{p}_{\rm L}^{\rm F}[2] \\ \dot{\beta}_{\rm L}^{\rm F} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\cos\beta_{\rm L}^{\rm F}-1}{\beta_{\rm L}^{\rm F}} \\ 0 & 0 & \frac{\sin\beta_{\rm L}^{\rm F}}{\beta_{\rm L}^{\rm F}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p_{\rm L}^{\rm F}[1] \\ p_{\rm L}^{\rm F}[2] \\ \beta_{\rm L}^{\rm F} \end{bmatrix} + \begin{bmatrix} -1 & p_{\rm L}^{\rm F}[2] \\ 0 & -\Delta p_{\rm L}^{\rm F}[1] - d \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{\rm F} \\ \omega_{\rm F} \end{bmatrix} + \left[\begin{array}{cc|cc} \cos\beta_{\rm L}^{\rm F} & 0 & 0 & -\sin\beta_{\rm L}^{\rm F} \\ \sin\beta_{\rm L}^{\rm F} & 0 & -1 & \cos\beta_{\rm L}^{\rm F} \\ 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_{\rm L} \\ \omega_{\rm L} \\ \hline h_{\rm F} \\ h_{\rm L} \end{array}\right]. \tag{33}$$

Let $\mathcal{U}$ be given in (22), and define

$$\mathcal{D} = \{(v_{\rm L}, \omega_{\rm L}, h_{\rm F}, h_{\rm L})^T : v_{\rm L} \in [-V_{\rm L}, V_{\rm L}], \omega_{\rm L} \in [-\Omega_{\rm L}, \Omega_{\rm L}], h_{\rm F} \in [-H_{\rm F}, H_{\rm F}], h_{\rm L} \in [-H_{\rm L}, H_{\rm L}]\}, \tag{34}$$

where $H_{\rm F}$, $H_{\rm L}$ are strictly positive constants. Using the same arguments as those in Theorem 18, we can prove the following corollary (note that the feedback matrix K is again of the form (27)).

Corollary 20 (VMP with disturbances) *Choose $\mathcal{U}$ and $\mathcal{S}$ as in Assumption 16, $\mathcal{D}$ as in (34), and let $d > a$, $0 < b \leq \pi/2$. The VMP for the robots* F *and* L *in the presence of the unknown but bounded disturbances $h_{\rm F}$, $h_{\rm L}$, has a solution if the following conditions are satisfied*

$$V_{\rm F} \geq V_{\rm L}\left(1 + \frac{a\sin b}{d-a}\right) + 1 - \cos b + \frac{a(H_{\rm F} + H_{\rm L} + b)}{d-a} + H_{\rm L}\sin b, \tag{35}$$

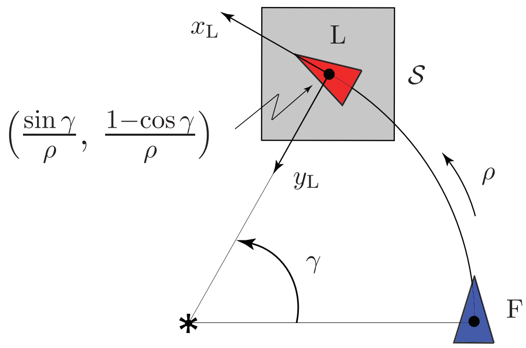

Fig. 4. *VMP on a circle:* pose of the robots L and F for $(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], \Delta p_{\mathrm{L}}^{\mathrm{F}}[2], \Delta\beta_{\mathrm{L}}^{\mathrm{F}})^T = (0,0,0)^T$, $v_{\mathrm{L}} = v_{\mathrm{F}}$ and $\omega_{\mathrm{L}} = \omega_{\mathrm{F}} = \rho$.

$$\begin{aligned}\Omega_{\mathrm{L}} &\le \frac{(1-V_{\mathrm{L}})\sin b - (H_{\mathrm{F}} + H_{\mathrm{L}})}{d+a},\\ \Omega_{\mathrm{F}} &\ge \frac{V_{\mathrm{L}}\sin b + b + (H_{\mathrm{F}} + H_{\mathrm{L}})}{d-a}.\end{aligned} \tag{36}$$

Remark 21 *Note that because of the additional terms H_{F} and H_{L}, conditions (35)-(36) are stricter than (25)-(26) and then the polytope $\mathcal{K}$ is smaller in this case. This is evident in Fig. 3, where the polytope $\mathcal{K}$ (*blue*, internal) obtained for $a = 0.15$ m, $b = \pi/3$ rad, $d = 1.6$ m, $V_{\mathrm{F}} = 0.95$ m/s, $V_{\mathrm{L}} = 0.1$ m/s, $\Omega_{\mathrm{F}} = \pi/2$ rad/s, $\Omega_{\mathrm{L}} = \pi/20$ rad/s and $H_{\mathrm{F}} = 0.2$ m/s, $H_{\mathrm{L}} = 0.1$ m/s is compared with the polytope (*green*, external) corresponding to $H_{\mathrm{L}} = H_{\mathrm{F}} = 0$ m/s.*

4.2 VMP on a circle

In this section, we will suppose that the leader robot moves along a circular path, around a static target. This scenario could be of interest in several real-world applications, such as, e.g., for environmental surveillance, patrolling or terrain and utilities inspection (Casbeer *et al.*, 2006; Susca *et al.*, 2008). Differently from Sect. 3.3, we will assume that the pose of robot L with respect to the frame of F is defined through the angle $0 < \gamma < \pi/2$ and the angular velocity $\rho > 0$, instead of the distance parameter d (see Fig. 4). Let us consider the following change of variables in system (18):

$$\begin{aligned}(p_{\mathrm{L}}^{\mathrm{F}}[1], p_{\mathrm{L}}^{\mathrm{F}}[2], \beta_{\mathrm{L}}^{\mathrm{F}})^T &\to (\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], \Delta p_{\mathrm{L}}^{\mathrm{F}}[2], \Delta\beta_{\mathrm{L}}^{\mathrm{F}})^T,\\ (v_{\mathrm{L}}, \omega_{\mathrm{L}})^T &\to (v_{\mathrm{L}}, \omega_{\mathrm{L}})^T,\\ (v_{\mathrm{F}}, \omega_{\mathrm{F}})^T &\to (v_{\mathrm{F}}, \Delta\omega_{\mathrm{F}})^T,\end{aligned}$$

where $\Delta p_{\mathrm{L}}^{\mathrm{F}}[1] = p_{\mathrm{L}}^{\mathrm{F}}[1] - \frac{\sin\gamma}{\rho}$, $\Delta p_{\mathrm{L}}^{\mathrm{F}}[2] = p_{\mathrm{L}}^{\mathrm{F}}[2] - \frac{1-\cos\gamma}{\rho}$, $\Delta\beta_{\mathrm{L}}^{\mathrm{F}} = \beta_{\mathrm{L}}^{\mathrm{F}} - \gamma$ and $\Delta\omega_{\mathrm{L}} = \omega_{\mathrm{L}} - \rho$, $\Delta\omega_{\mathrm{F}} = \omega_{\mathrm{F}} - \rho$. Following the same procedure detailed in the previous sections, we obtain the system

$$\begin{aligned}\begin{bmatrix}\Delta\dot{p}_{\mathrm{L}}^{\mathrm{F}}[1]\\ \Delta\dot{p}_{\mathrm{L}}^{\mathrm{F}}[2]\\ \Delta\dot{\beta}_{\mathrm{L}}^{\mathrm{F}}\end{bmatrix} &= \begin{bmatrix}0 & \rho & q_2 - \sin\gamma\\ -\rho & 0 & q_1 + \cos\gamma\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta p_{\mathrm{L}}^{\mathrm{F}}[1]\\ \Delta p_{\mathrm{L}}^{\mathrm{F}}[2]\\ \Delta\beta_{\mathrm{L}}^{\mathrm{F}}\end{bmatrix}\\ &+ \begin{bmatrix}-1 & q_4 + \frac{1-\cos\gamma}{\rho}\\ 0 & -q_3 - \frac{\sin\gamma}{\rho}\\ 0 & -1\end{bmatrix}\begin{bmatrix}v_{\mathrm{F}}\\ \Delta\omega_{\mathrm{F}}\end{bmatrix} + \begin{bmatrix}q_5 + \cos\gamma & 0\\ q_6 + \sin\gamma & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_{\mathrm{L}}\\ \Delta\omega_{\mathrm{L}}\end{bmatrix},\end{aligned} \tag{37}$$

where

$$\begin{aligned}q_1 &= \tfrac{\sin(\Delta\beta_{\mathrm{L}}^{\mathrm{F}}+\gamma)-\sin\gamma}{\Delta\beta_{\mathrm{L}}^{\mathrm{F}}} - \cos\gamma,\\ q_2 &= \tfrac{\cos(\Delta\beta_{\mathrm{L}}^{\mathrm{F}}+\gamma)-\cos\gamma}{\Delta\beta_{\mathrm{L}}^{\mathrm{F}}} + \sin\gamma,\end{aligned}$$

$$\begin{aligned}q_3 &= \Delta p_{\mathrm{L}}^{\mathrm{F}}[1], \quad q_5 = \cos(\Delta\beta_{\mathrm{L}}^{\mathrm{F}} + \gamma) - \cos\gamma,\\ q_4 &= \Delta p_{\mathrm{L}}^{\mathrm{F}}[2], \quad q_6 = \sin(\Delta\beta_{\mathrm{L}}^{\mathrm{F}} + \gamma) - \sin\gamma.\end{aligned}$$

Assumption 22 *Let us suppose that*

$$\mathcal{U} = \big\{(v_{\mathrm{F}}, \Delta\omega_{\mathrm{F}})^T : v_{\mathrm{F}} \in [-V_{\mathrm{F}}, V_{\mathrm{F}}],\ \Delta\omega_{\mathrm{F}} \in [-\Omega_{\mathrm{F}}, \Omega_{\mathrm{F}}]\big\},$$

$$\mathcal{D} = \big\{(v_{\mathrm{L}}, \Delta\omega_{\mathrm{L}})^T : v_{\mathrm{L}} \in [-V_{\mathrm{L}}, V_{\mathrm{L}}],\ \Delta\omega_{\mathrm{L}} \in [-\Omega_{\mathrm{L}}, \Omega_{\mathrm{L}}]\big\},$$

where $0 < V_{\mathrm{F}} < 1$, $0 < V_{\mathrm{L}} < 1$ and $0 < \Omega_{\mathrm{F}}$, $0 < \Omega_{\mathrm{L}} < \rho$. Let us also consider the following visibility set (see Fig. 4)

$$\begin{aligned}\mathcal{S} = \{(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1], \Delta p_{\mathrm{L}}^{\mathrm{F}}[2], \Delta\beta_{\mathrm{L}}^{\mathrm{F}})^T :\ &\Delta p_{\mathrm{L}}^{\mathrm{F}}[1] \in [-a, a],\\ &\Delta p_{\mathrm{L}}^{\mathrm{F}}[2] \in [-a, a],\ \Delta\beta_{\mathrm{L}}^{\mathrm{F}} \in [-b, b]\},\end{aligned} \tag{38}$$

where $a > 0$ and $b > 0$.

Since the state of system (37) is constrained in (38), the polyhedron $\mathcal{Q} \subset \mathbb{R}^6$ is defined as follows:

$$\begin{aligned}&q_1 \in \big[\tfrac{\sin(b+\gamma)-\sin\gamma}{b} - \cos\gamma, \tfrac{\sin(b-\gamma)+\sin\gamma}{b} - \cos\gamma\big],\\ &q_2 \in [\tfrac{\cos(b+\gamma)-\cos\gamma}{b} + \sin\gamma, \tfrac{-\cos(b-\gamma)+\cos\gamma}{b} + \sin\gamma],\\ &q_3 \in [-a, a],\ q_5 \in [\cos(b+\gamma) - \cos\gamma, \cos(b-\gamma) - \cos\gamma],\\ &q_4 \in [-a, a],\ q_6 \in [-\sin(b-\gamma) - \sin\gamma, \sin(b+\gamma) - \sin\gamma].\end{aligned}$$

The proof of the next theorem is analogous to that of Theorem 18 and it is omitted. The feedback matrix K has also in this case the form (27).

Theorem 23 (Solvability of the VMP on a circle) *Choose $\mathcal{U}$, $\mathcal{D}$ and $\mathcal{S}$ as in Assumption 22 and let $1 - \cos\gamma > \rho a$, $0 \le b \pm \gamma \le \pi/2$. The VMP on a circle*

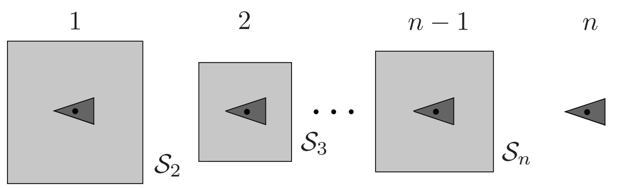

Fig. 5. Chain of n robots.

has a solution if the following conditions are satisfied

$$V_{\mathrm{F}} \geq V_{\mathrm{L}} \left(\cos(b-\gamma) + \tfrac{\sin(b+\gamma)(1-\cos\gamma-\rho\, a)}{\sin\gamma+\rho a}\right) + \cos\gamma + \rho a$$
$$- \tfrac{1-\cos\gamma-\rho\, a}{\sin\gamma+\rho a}\, (\sin(b+\gamma) - \sin\gamma + \rho a) - \cos(b+\gamma),$$

$$\Omega_{\mathrm{L}} \leq \rho \left(\frac{(1-V_{\mathrm{L}})\sin(b+\gamma)}{\sin\gamma+\rho a} - 1 \right),$$

$$\Omega_{\mathrm{F}} \geq \rho \left(\frac{V_{\mathrm{L}}\, \sin(b+\gamma) + \sin(b-\gamma) + \sin\gamma + \rho a}{\sin\gamma - \rho a} \right). \tag{39}$$

Note that differently from Theorem 18, in this case, owing to (39) is *not always* true (i.e., for all values of the parameters) that $\Omega_{\mathrm{F}} \geq \Omega_{\mathrm{L}}$.

4.3 Chain of robots

Next, we consider more complex robotic networks built by concatenating multiple leader-follower units (see Fig. 5). When equipped with wireless sensors, these arrays of robots could be used, for example, to efficiently monitor the temperature and/or salinity of the ocean or measure the average concentration of air pollutants (Lynch *et al.*, 2008). In what follows, the feasibility conditions of Theorem 18 will be extended to such robot chains in order to maintain network-wide visibility between the agents.

Because of the leader-follower hierarchy within the chain, vehicle $k+1$ (the "follower") will aim at keeping the vehicle ahead (robot k, the "leader"), in its visibility set. Let a_k, b_k, d_k, $d_k > a_k$, $b_k \leq \pi/2$, $k \in \{2,\ldots,n\}$, $n > 2$, be the strictly positive parameters defining the visibility set $\mathcal{S}_k$ of robot k-th and let $0 < V_k < 1$, $\Omega_k > 0$, $k \in \{1,\ldots,n\}$ be the bounds on k-th robot's linear and angular velocities (recall Assumption 16). By propagating conditions (25)-(26) of Theorem 18 starting from robot 1 (that guides the formation), we obtain the following set of inequalities (linear in V_k and Ω_k)

$$V_{k+1} \geq V_k \left(1 + \frac{a_{k+1}\sin b_{k+1}}{d_{k+1} - a_{k+1}}\right) + 1 - \cos b_{k+1}$$
$$+ \frac{a_{k+1}\, b_{k+1}}{d_{k+1} - a_{k+1}}, \quad k \in \{1,\ldots,n-1\}, \tag{40}$$

$$\Omega_1 \leq \frac{(1-V_1)\sin b_2}{d_2 + a_2}, \quad \Omega_n \geq \frac{V_{n-1}\sin b_n + b_n}{d_n - a_n}, \tag{41}$$

and for all $k \in \{2,\ldots,n-1\}$,

$$\frac{V_{k-1}\sin b_k + b_k}{d_k - a_k} \leq \Omega_k \leq \frac{(1-V_k)\sin b_{k+1}}{d_{k+1} + a_{k+1}}. \tag{42}$$

It is an easy task to verify that if $a_k = a$, $b_k = b$, $d_k = d$, $k \in \{2,\ldots,n\}$ and $V_k = V$, $k \in \{1,\ldots,n\}$, ($a > 0$, $d > a$, $0 < b \leq \pi/2$, $0 < V < 1$), condition (42) (and condition (40) as well) is *not* satisfied. Nevertheless, it can be proved that if V_k and at least one of the three parameters defining the visibility sets are left free to vary from robot to robot, then (40)-(42) can be always fulfilled. Fig. 6 shows the progression of a feasible set of parameters V_k, Ω_k, b_k for a chain of $n = 15$ robots, when $a_k = a = 0.1$ and $d_k = d = 7$, $k \in \{2,\ldots,15\}$.

We conclude this section with the following corollary and a few remarks:

Corollary 24 *The chain of robots can hold only a* finite *number of agents and cannot be* closed*, i.e., robot 1 cannot cyclically pursue robot n.*

Proof: Let us study the connection existing between parameters V_1 and V_n. After simple algebraic manipulations on (40), we obtain the following inequality:

$$V_1 \leq \frac{1}{\prod_{i=1}^{n}\left(1 + \dfrac{a_i\, \sin b_i}{d_i - a_i}\right)} V_n - \sum_{i=2}^{n} \frac{1 - \cos b_i + \dfrac{a_i\, \sin b_i}{d_i - a_i}}{\prod_{k=2}^{i}\left(1 + \dfrac{a_k\, \sin b_k}{d_k - a_k}\right)}. \tag{43}$$

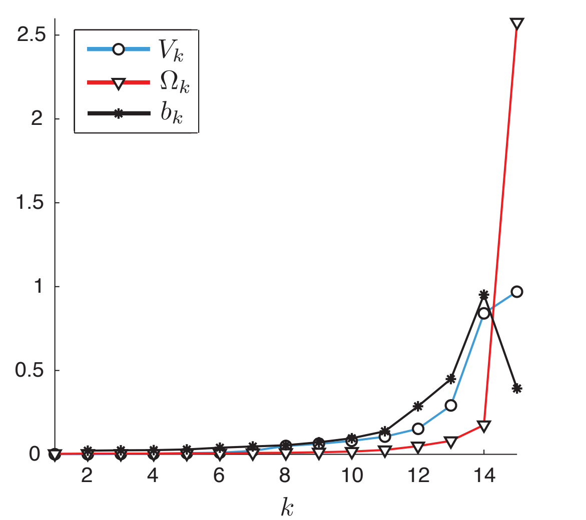

Fig. 6. Progression of a feasible set of parameters V_k, Ω_k, b_k for a chain of $n = 15$ robots ($a_k = 0.1$, $d_k = 7$, for all k). Angles are in radians.

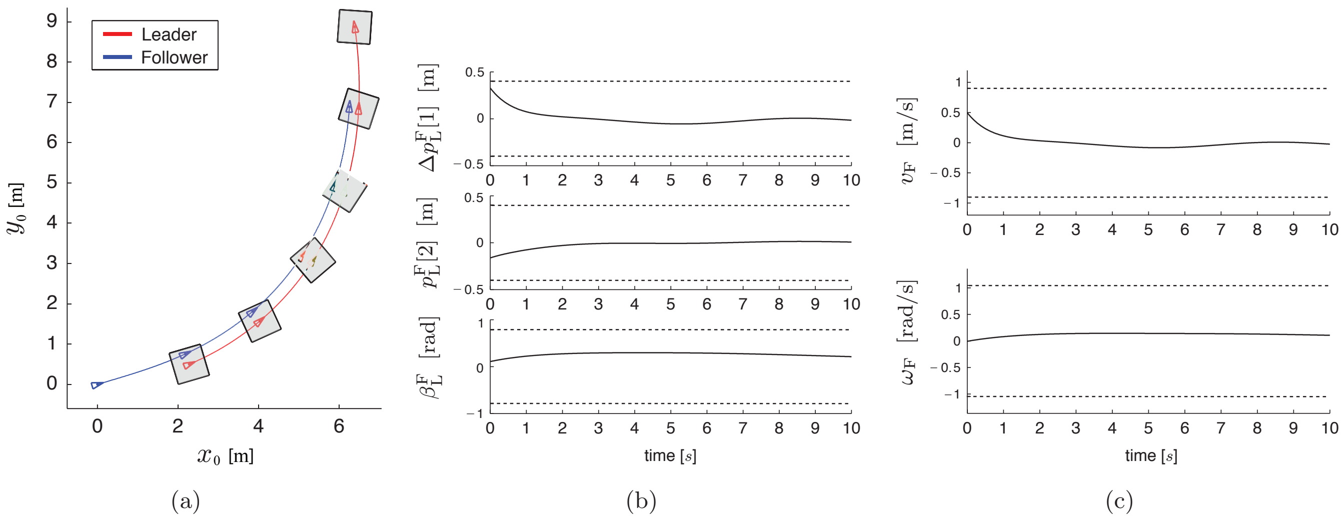

Fig. 7. *Basic scenario*: (a) Trajectory of the leader and follower, and visibility set $\mathcal{S}$; (b) $\Delta p_L^F[1]$, $p_L^F[2]$, β_L^F (solid) and bounds $\pm a$, $\pm a$, $\pm b$ (dash); (c) v_F, ω_F (solid) and bounds $\pm V_F$, $\pm \Omega_F$ (dash).

When the number of robots n tends towards infinity, the first term on the right-hand side of (43) converges to zero (for any choice of the parameters a_i, b_i, d_i satisfying the previous assumptions) asymptotically leading to the inequality $V_1 \leq 0$, which contradicts the initial hypothesis of a strictly positive V_1. On the other hand, if the chain of robots were closed, the following additional inequality should be satisfied

$$V_1 \geq V_n \left(1 + \frac{a_1 \sin b_1}{d_1 - a_1}\right) + 1 - \cos b_1 + \frac{a_1\, b_1}{d_1 - a_1},$$

but it is incompatible with condition (43). ■

It is easy to prove from condition (43), that once fixed a rule for the evolution of the parameters defining the visibility sets (i.e., $a_k = f_a(k)$, $b_k = f_b(k)$, $d_k = f_d(k)$, with f_a, f_b, $f_d : \mathbb{Z}_{>1} \to \mathbb{R}$ given discrete maps), an upper bound on the maximum number of robots the chain can hold is given by the maximum positive integer N that fulfills the following inequality:

$$\sum_{i=2}^{N} \left(1 - \cos b_i + \frac{a_i\, b_i}{d_i - a_i}\right) \prod_{k=i+1}^{N} \left(1 + \frac{a_k \sin b_k}{d_k - a_k}\right) < 1.$$

Note that once the state feedback matrices K_i, $i \in \{2, \ldots, n\}$ of the robots have been established, the implementation of the control laws is *totally distributed*: in fact, each agent only needs to know the relative position and orientation of the preceding vehicle in the chain, to execute its control action.

5 Simulation results

Extensive simulation experiments have been performed to illustrate the theory and assess the soundness of the proposed approach.

5.1 Basic scenario

The simulation results reported in Fig. 7 refer to the basic scenario studied at the beginning of Sect. 3. The leader robot moves with velocities $v_L(t) = 0.05 \sin(t)$, $\omega_L(t) = \frac{\pi}{20} \cos(0.1t)$. We set $V_L = 0.1$ m/s, $\Omega_L = \pi/15$ rad/s, $V_F = 0.9$ m/s, $\Omega_F = \pi/3$ rad/s, $a = 0.4$ m, $b = \pi/4$ rad and $d = 2$ m, according to the conditions of Theorem 18 and we chose the gain matrix in $\mathcal{K}$ with minimum 2-norm:

$$K = \begin{bmatrix} 1.5173 & 0 & 0 \\ 0 & 0.3707 & 0.4925 \end{bmatrix}.$$

Note that since K is in the interior of $\mathcal{K}$, the asymptotic stability is assured (cf. Theorem 5.2 in (Blanchini, 1991)). System (19) has been initialized with

$$\begin{aligned} &(\Delta p_L^F[1](0), p_L^F[2](0), \beta_L^F(0))^T \\ &\qquad = (0.3285,\, -0.1626,\, 0.1071)^T. \end{aligned} \tag{44}$$

Fig. 7(a) reports the trajectory of robot L and F and the visibility set $\mathcal{S}$, (in order to have a temporal reference the robots are drawn every two seconds). Fig. 7(b) shows that $\Delta p_L^F[1]$, $p_L^F[2]$, β_L^F (solid), keep inside the respective bounds $\pm a$, $\pm a$, $\pm b$ (dash), as expected. Finally, Fig. 7(c) exposes that the control inputs v_F, ω_F (solid), respect the corresponding bounds $\pm V_F$, $\pm \Omega_F$ (dash).

5.2 Rejection of unknown but bounded disturbances

In the simulation results reported in Fig. 8, the leader robot moves with velocities $v_L(t) = 0.01$,

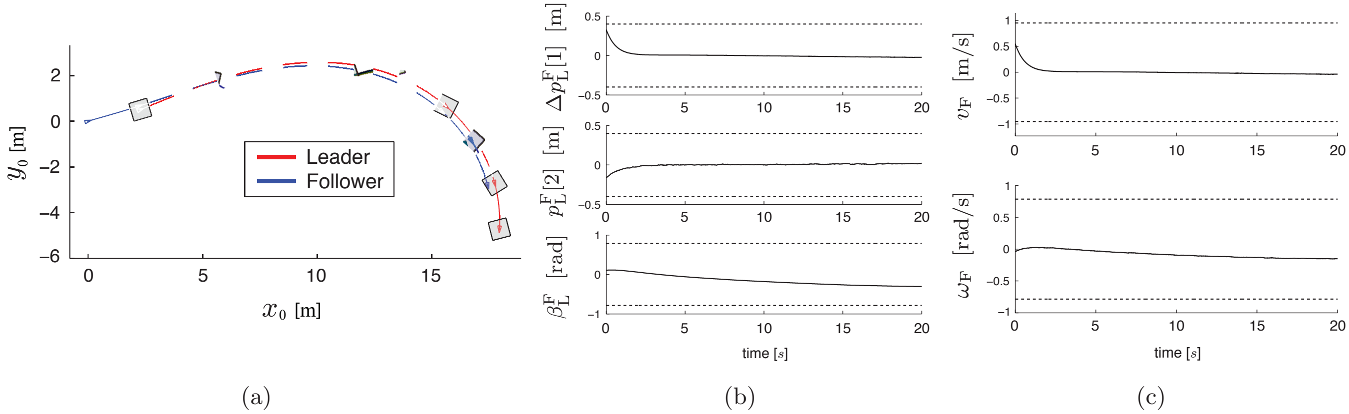

Fig. 8. *Rejection of unknown but bounded disturbances*: (a) Trajectory of the leader and follower, and visibility set $\mathcal{S}$; (b) $\Delta p_{\mathrm{L}}^{\mathrm{F}}[1]$, $p_{\mathrm{L}}^{\mathrm{F}}[2]$, $\beta_{\mathrm{L}}^{\mathrm{F}}$ (solid) and bounds $\pm a$, $\pm a$, $\pm b$ (dash); (c) v_{F}, ω_{F} (solid) and bounds $\pm V_{\mathrm{F}}$, $\pm \Omega_{\mathrm{F}}$ (dash).

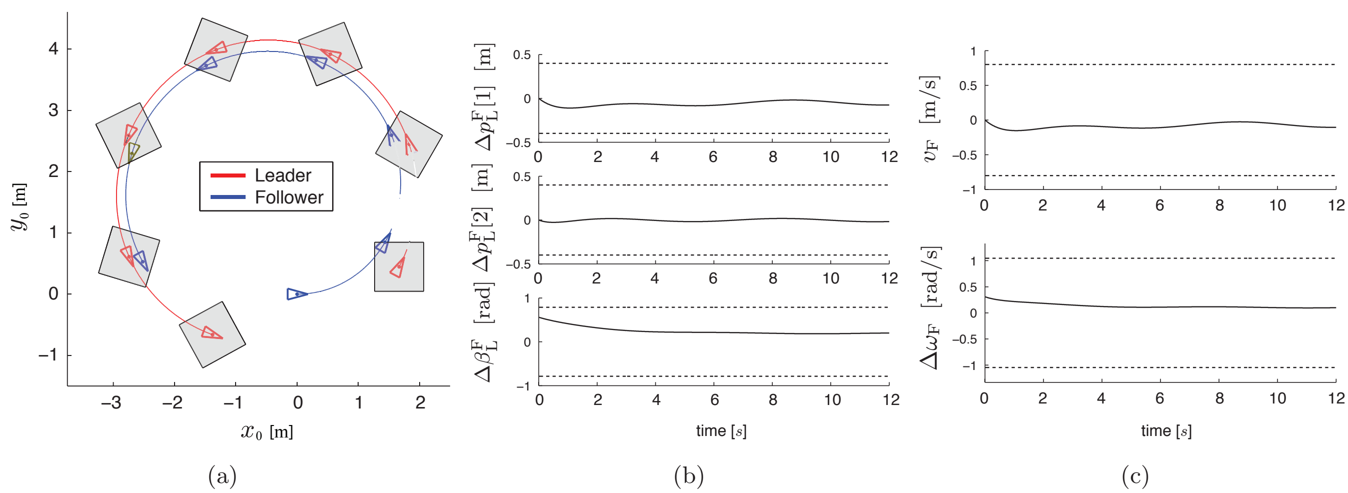

Fig. 9. *VMP on a circle*: (a) Trajectory of the leader and follower, and visibility set $\mathcal{S}$; (b) $\Delta p_{\mathrm{L}}^{\mathrm{F}}[1]$, $\Delta p_{\mathrm{L}}^{\mathrm{F}}[2]$, $\Delta \beta_{\mathrm{L}}^{\mathrm{F}}$ (solid) and bounds $\pm a$, $\pm a$, $\pm b$ (dash); (c) v_{F}, $\Delta\omega_{\mathrm{F}}$ (solid) and bounds $\pm V_{\mathrm{F}}$, $\pm \Omega_{\mathrm{F}}$ (dash).

$\omega_{\mathrm{L}}(t) = -\frac{\pi}{20}\sin(0.08t)$. Unknown but bounded disturbances $h_{\mathrm{L}}(t)$, $h_{\mathrm{F}}(t)$ (uniform random noises in the interval $(-0.1, 0.1)$) act on the leader and the follower (recall Sect. 4.1). Owing to the conditions of Corollary 20, we chose $V_{\mathrm{L}} = 0.03$ m/s, $\Omega_{\mathrm{L}} = \pi/18$ rad/s, $V_{\mathrm{F}} = 0.95$ m/s, $\Omega_{\mathrm{F}} = \pi/4$ rad/s, $H_{\mathrm{L}} = H_{\mathrm{F}} = 0.12$ m/s, $a = 0.4$ m, $b = \pi/4$ rad and $d = 2$ m. The initial condition of system (33) is (44) and, again, we selected the gain matrix in $\mathcal{K}$ with minimum 2-norm:

$$K = \begin{bmatrix} 1.6735 & 0 & 0 \\ 0 & 0.5896 & 0.5326 \end{bmatrix}.$$

Fig. 8(a) reports the trajectory of L and F and the visibility set $\mathcal{S}$. From Figs. 8(b) and 8(c), we note that despite the presence of the unknown but bounded disturbances, $\Delta p_{\mathrm{L}}^{\mathrm{F}}[1]$, $p_{\mathrm{L}}^{\mathrm{F}}[2]$, $\beta_{\mathrm{L}}^{\mathrm{F}}$ and v_{F}, ω_{F} respect the relative state and control bounds.

5.3 VMP on a circle

In Fig. 9, the leader robot moves with velocities $v_{\mathrm{L}}(t) = 0.05\sin(t)$, $\omega_{\mathrm{L}}(t) = \pi/30$, and the parameters $V_{\mathrm{L}} = 0.06$ m/s, $\Omega_{\mathrm{L}} = \pi/25$ rad/s, $V_{\mathrm{F}} = 0.8$ m/s, $\Omega_{\mathrm{F}} = \pi/3$ rad/s, $a = 0.4$ m, $b = \pi/4$ rad, $\rho = 0.3$ rad/s and $\gamma = \pi/6$ rad, have been chosen according to the conditions of Theorem 23 (recall Sect. 4.2). The minimum 2-norm gain matrix in $\mathcal{K}$ is, in this case

$$K = \begin{bmatrix} 1.3812 & 0 & 0 \\ 0 & 0.6051 & 0.5508 \end{bmatrix},$$

and the initial condition of system (37) is $(\Delta p_{\mathrm{L}}^{\mathrm{F}}[1](0), \Delta p_{\mathrm{L}}^{\mathrm{F}}[2](0), \Delta \beta_{\mathrm{L}}^{\mathrm{F}}(0))^T = (0, 0, 0.5597)^T$. Fig. 9(a) reports the trajectory of L and F and the visibility set $\mathcal{S}$. Figs. 9(b) and 9(c) show the time history of $\Delta p_{\mathrm{L}}^{\mathrm{F}}[1]$, $\Delta p_{\mathrm{L}}^{\mathrm{F}}[2]$, $\Delta \beta_{\mathrm{L}}^{\mathrm{F}}$ and v_{F}, $\Delta\omega_{\mathrm{F}}$, and the corresponding bounds.

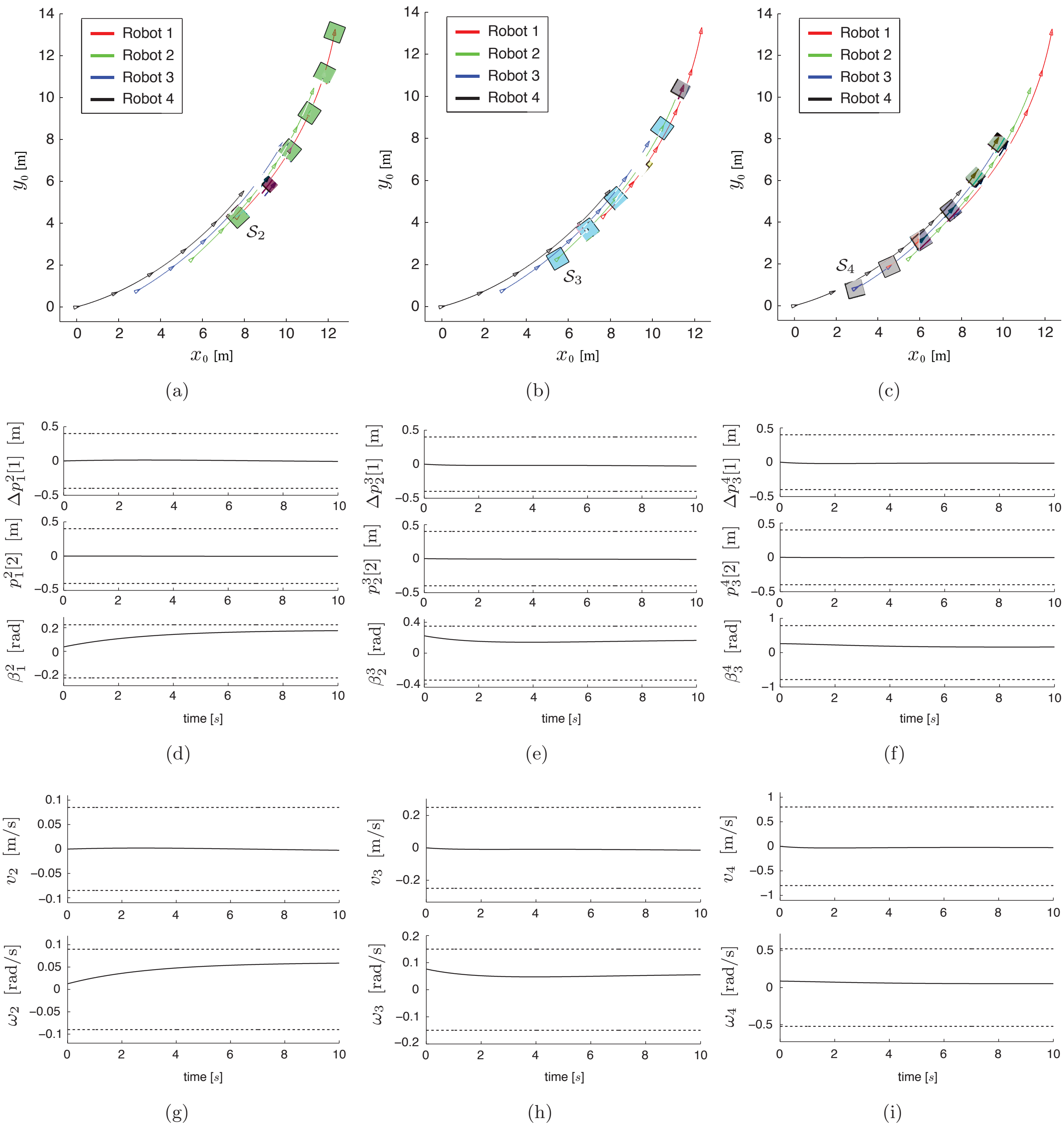

Fig. 10. *Chain of robots*: Trajectory of the 4 robots and (a) visibility set $\mathcal{S}_2$, (b) visibility set $\mathcal{S}_3$, (c) visibility set $\mathcal{S}_4$; (d) $\Delta p_1^2[1]$, $p_1^2[2]$, β_1^2 (solid) and bounds $\pm a_2$, $\pm a_2$, $\pm b_2$ (dash); (e) $\Delta p_2^3[1]$, $p_2^3[2]$, β_2^3 and bounds $\pm a_3$, $\pm a_3$, $\pm b_3$; (f) $\Delta p_3^4[1]$, $p_3^4[2]$, β_3^4 and bounds $\pm a_4$, $\pm a_4$, $\pm b_4$; (g) v_2, ω_2 (solid) and bounds $\pm V_2$, $\pm \Omega_2$ (dash); (h) v_3, ω_3 and bounds $\pm V_3$, $\pm \Omega_3$; (i) v_4, ω_4 and bounds $\pm V_4$, $\pm \Omega_4$.

5.4 *Chain of robots*

In Fig. 10 a chain of 4 robots is considered (recall Sect. 4.3). Robot 1 guides the formation with velocities, $v_1(t) = 0.01$, $\omega_1(t) = \pi/52$. The following set of parameters, satisfying conditions (40)-(42), has been used in the simulation: $V_1 = 0.02$ m/s, $V_2 = 0.085$ m/s, $V_3 = 0.25$ m/s, $V_4 = 0.8$ m/s, $\Omega_1 = \pi/50$ rad/s, $\Omega_2 = \pi/35$ rad/s, $\Omega_3 = \pi/21$ rad/s, $\Omega_4 = \pi/6$ rad/s, and $a_2 = a_3 = a_4 = 0.4$ m, $b_2 = \pi/14$ rad, $b_3 = \pi/9$ rad,

$b_4 = \pi/4$ rad and $d_2 = d_3 = d_4 = 3$ m. The minimum 2-norm gain matrices in the polytopes $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ of robots $2, 3$ and 4, are respectively

$$K_2 = \begin{bmatrix} 0.2066 & 0 & 0 \\ 0 & 0.0315 & 0.3361 \end{bmatrix},\ K_3 = \begin{bmatrix} 0.5087 & 0 & 0 \\ 0 & 0.0669 & 0.3400 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 1.7273 & 0 & 0 \\ 0 & 0.2678 & 0.3348 \end{bmatrix},$$

and the initial conditions for the three dynamic systems in the form (19) are, $(\Delta p_1^2[1](0), p_1^2[2](0), \beta_1^2(0))^T = (0,\ 0,\ 0.0374)^T$, $(\Delta p_2^3[1](0), p_2^3[2](0), \beta_2^3(0))^T = (0,\ 0,\ 0.2244)^T$, $(\Delta p_3^4[1](0), p_3^4[2](0), \beta_3^4(0))^T = (0,\ 0,\ 0.2618)^T$, where p_k^{k+1} denotes the position of robot k with respect to the reference frame attached to robot $k+1$ and $\beta_k^{k+1} \triangleq \theta_k - \theta_{k+1}$, $k \in \{1, 2, 3\}$. Figs. 10(a)-(c) report the trajectory of the 4 robots and the visibility sets $\mathcal{S}_2$ (green), $\mathcal{S}_3$ (cyan) and $\mathcal{S}_4$ (gray), respectively. Figs. 10(d)-(f) and 10(g)-(i) show the time history of $\Delta p_k^{k+1}[1]$, $p_k^{k+1}[2]$, β_k^{k+1}, $k \in \{1, 2, 3\}$ and v_k, ω_k, $k \in \{2, 3, 4\}$, and the relative state and control bounds.

6 Conclusions and future work

The paper proposes an original solution to the *visibility maintenance problem* (VMP) for a leader-follower pair of Dubins-like vehicles with input constraints. By interpreting the nonlinear model describing the relative dynamics of the robots as a linear system with parameter uncertainty, the VMP is reformulated as a *linear constrained regulation problem with additive disturbances* (DLCRP). General conditions for the positive $\mathcal{D}$-invariance of linear uncertain systems with parametric disturbance matrix are derived and used to study the feasibility of the VMP when box bounds on the state, input and disturbance are considered. The proposed design procedure can be easily adapted to provide the control with unknown but bounded disturbances rejection capabilities. Conditions for the solution of the VMP when robots' desired displacement is defined through angular parameters are also presented, and the extension to chains of n robots is discussed.

A drawback of the approach proposed in this paper is that it requires a great amount of computational work off-line, because the feedback matrix K is obtained as a solution of a large set of inequalities. In addition all the vertices of the visibility set $\mathcal{S}$ are required. In this respect, the solution to the LCRP proposed in (Vassilaki and Bitsoris, 1989) appears to be preferable to the one in (Blanchini, 1990), even though neither disturbances nor model parametric uncertainty are considered therein.

Future research lines include the extension of our results to robotic networks with arbitrary topologies and the application of the proposed approach to the study of consensus, rendezvous and coverage problems in the presence of visibility constraints. The use of polar coordinates to describe conic-like visibility sets $\mathcal{S}$, is also a subject of on-going research.

Appendix A: The Fourier-Motzkin elimination method

The Fourier-Motzkin elimination (FME), a generalization of Gauss elimination, is a computational method for solving a system

$$A\,x \leq b, \quad A \in \mathbb{R}^{m\times n}, \quad b \in \mathbb{R}^m,$$

of m linear inequalities in n variables (Motzkin, 1951; Schrijver, 1986). The key idea of the FME method is to eliminate one variable of the system $A\,x \leq b$ at each iteration and rewrite the resulting equations accordingly. Even though the number of variables decreases at each step, the number of inequalities in the remaining variables grows exponentially fast: in fact, at iteration j the number of inequalities to be evaluated is at most $\lfloor \frac{m}{2} \rfloor^{2^j}$. Because of its double-exponential computational complexity, the FME method can be applied efficiently only to problems with a small number of inequalities and it is not competitive with standard LP solvers However, differently from this numerical approach, the method can handle *symbolic inequalities*. We illustrate this idea with a simple example. Consider the following set of symbolic inequalities

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 &\leq b_1, && (45)\\ -a_{21}\,x_1 - a_{22}\,x_2 &\leq b_2, && (46)\\ a_{32}\,x_2 &\leq b_3, && (47) \end{aligned}$$

where a_{11}, a_{12}, a_{21}, a_{22}, a_{32}, b_1, b_2, $b_3 \in \mathbb{R}_{>0}$ are unknown parameters. We wish to determine under which conditions on these parameters, system (45)-(47) admits solutions. The first step is to eliminate the variable x_1. To this end, solving (45)-(46) in x_1, we get:

$$x_1 \leq \frac{b_1 - a_{12}\,x_2}{a_{11}}, \quad x_1 \geq \frac{-b_2 - a_{22}\,x_2}{a_{21}}.$$

Joining the two inequalities, after a few computations, we obtain the following condition on the variable x_2 (assuming that $a_{11}\,a_{22} - a_{12}\,a_{21} \neq 0$)

$$x_2 \geq \frac{-a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{12}\,a_{21}}. \qquad (48)$$

If we finally combine inequalities (47) and (48), we end up with the sought solvability condition of system (45)-(47), in terms of the symbolic parameters a_{11}, a_{12}, a_{21}, a_{22}, a_{32}, b_1, b_2, b_3:

$$\frac{b_3}{a_{32}} \geq \frac{-a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{12}\,a_{21}}.$$

Appendix B: Application of the Fourier-Motzkin elimination to the inequalities (30)-(31)

Our first step to solve the system of inequalities (30)-(31), is to eliminate variable k_{11}. Solving in k_{11} (under the assumption of $d > a$), we obtain the following set of conditions:

$$
\begin{aligned}
&k_{11} \geq q_4\, k_{22} + \frac{b}{a}\, q_4\, k_{23} + \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a},\\
&k_{11} \geq q_4\, k_{22} - \frac{b}{a}\, q_4\, k_{23} - \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a},\\
&k_{11} \geq -q_4\, k_{22} + \frac{b}{a}\, q_4\, k_{23} + \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a},\\
&k_{11} \geq -q_4\, k_{22} - \frac{b}{a}\, q_4\, k_{23} - \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a},\\
&k_{11} \geq -\frac{V_{\mathrm{F}}}{a}, \;\; k_{11} \leq \frac{V_{\mathrm{F}}}{a}.
\end{aligned}
$$

Combining these inequalities, we get:

$$
\begin{aligned}
&q_4\, k_{22} \leq \frac{V_{\mathrm{F}}}{a} - \frac{b}{a}\, q_4\, k_{23} - \frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a},\\
&q_4\, k_{22} \leq \frac{V_{\mathrm{F}}}{a} + \frac{b}{a}\, q_4\, k_{23} + \frac{b}{a}\, q_2 - \frac{V_{\mathrm{L}}}{a},\\
&q_4\, k_{22} \geq -\frac{V_{\mathrm{F}}}{a} + \frac{b}{a}\, q_4\, k_{23} + \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a},\\
&q_4\, k_{22} \geq -\frac{V_{\mathrm{F}}}{a} - \frac{b}{a}\, q_4\, k_{23} - \frac{b}{a}\, q_2 + \frac{V_{\mathrm{L}}}{a}.
\end{aligned}
$$

In order to eliminate the second variable, k_{22}, we should consider three cases, according to the sign of q_4. If we assume that $q_4 > 0$, we then obtain the following set of inequalities:

$$
\begin{aligned}
&k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{b\,(1+q_1) + V_{\mathrm{L}}\, \sin b}{a(d+q_3)}, && (49)\\
&k_{22} \geq \frac{b}{a}\, k_{23} + \frac{-b\,(1+q_1) + V_{\mathrm{L}}\, \sin b}{a(d+q_3)}, && (50)\\
&k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{L}}}{a}, && (51)\\
&k_{22} \geq \frac{b}{a}\, k_{23} + \frac{-V_{\mathrm{F}} + b\, q_2 + V_{\mathrm{L}}}{a q_4}, && (52)\\
&k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{-V_{\mathrm{F}} - b\, q_2 + V_{\mathrm{L}}}{a q_4}, && (53)\\
&k_{22} \geq \frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{F}}}{a}, \;\; k_{22} \geq -\frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{F}}}{a}, && (54)\\
&k_{22} \leq \frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{L}}}{a}, && (55)\\
&k_{22} \leq -\frac{b}{a}\, k_{23} + \frac{V_{\mathrm{F}} - b\, q_2 - V_{\mathrm{L}}}{a q_4}, && (56)\\
&k_{22} \leq \frac{b}{a}\, k_{23} + \frac{V_{\mathrm{F}} + b\, q_2 - V_{\mathrm{L}}}{a q_4}, && (57)\\
&k_{22} \leq -\frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{F}}}{a}, \;\; k_{22} \leq \frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{F}}}{a}. && (58)
\end{aligned}
$$

Combining conditions (49)-(54) and (55)-(58), we end up with a total set of 35 inequalities, of whom only 4 are non-trivial:

$$
\begin{aligned}
&\Omega_{\mathrm{L}} \leq \frac{b\,(1+q_1) - V_{\mathrm{L}}\, \sin b}{d+q_3}, \;\; \Omega_{\mathrm{F}} \geq \frac{b\,(1+q_1) + V_{\mathrm{L}}\, \sin b}{d+q_3},\\
&V_{\mathrm{F}} \geq V_{\mathrm{L}}\Big(1 + \frac{q_4 \sin b}{d+q_3}\Big) + b\Big(q_2 + \frac{q_4(1+q_1)}{d+q_3}\Big),\\
&V_{\mathrm{F}} \geq V_{\mathrm{L}}\Big(1 + \frac{q_4 \sin b}{d+q_3}\Big) - b\Big(q_2 + \frac{q_4(1+q_1)}{d+q_3}\Big).
\end{aligned}
$$

If we now assume that $q_4 = 0$, we obtain the unique condition: $V_{\mathrm{F}} \geq V_{\mathrm{L}} + b\, q_2$. Finally, if we assume that $q_4 < 0$, we come up with the following inequalities

$$
\begin{aligned}
&k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{b\,(1+q_1) + V_{\mathrm{L}}\, \sin b}{a(d+q_3)},\\
&k_{22} \geq \frac{b}{a}\, k_{23} + \frac{-b\,(1+q_1) + V_{\mathrm{L}}\, \sin b}{a(d+q_3)},\\
&k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{L}}}{a}, \quad k_{22} \geq -\frac{b}{a}\, k_{23} + \frac{V_{\mathrm{F}} - b\, q_2 - V_{\mathrm{L}}}{a q_4},\\
&k_{22} \geq \frac{b}{a}\, k_{23} + \frac{V_{\mathrm{F}} + b\, q_2 - V_{\mathrm{L}}}{a q_4}, \quad k_{22} \geq \frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{F}}}{a},\\
&k_{22} \geq -\frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{F}}}{a}, \quad k_{22} \leq \frac{b}{a}\, k_{23} - \frac{\Omega_{\mathrm{L}}}{a},\\
&k_{22} \leq \frac{b}{a}\, k_{23} + \frac{-V_{\mathrm{F}} + b\, q_2 + V_{\mathrm{L}}}{a q_4},\\
&k_{22} \leq -\frac{b}{a}\, k_{23} + \frac{-V_{\mathrm{F}} - b\, q_2 + V_{\mathrm{L}}}{a q_4},\\
&k_{22} \leq -\frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{F}}}{a}, \quad k_{22} \leq \frac{b}{a}\, k_{23} + \frac{\Omega_{\mathrm{F}}}{a},
\end{aligned}
$$

from which we deduce the following two new conditions:

$$
\begin{aligned}
&V_{\mathrm{F}} \geq V_{\mathrm{L}}\Big(1 - \frac{q_4\, \sin b}{d+q_3}\Big) + b\Big(q_2 + \frac{q_4\,(1+q_1)}{d+q_3}\Big),\\
&V_{\mathrm{F}} \geq V_{\mathrm{L}}\Big(1 - \frac{q_4\, \sin b}{d+q_3}\Big) - b\Big(q_2 + \frac{q_4\,(1+q_1)}{d+q_3}\Big).
\end{aligned}
$$

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