Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Filters for Linear Continuous-time Singular Systems

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Abstract— **This paper is concerned with designing filters for linear continuous-time singular systems. Two types of filters are developed; one is an infinite-time filter and the other is a finite-time one. For the former, the convergence of the filtering error to zero is asymptotic with time; but for the latter, it is finite time. Sufficient conditions for the solvability of these design problems are obtained in terms of rank conditions. It is noted that the designed filters are in linear state-space form and no information on the unknown input is required. A numerical simulation is used to illustrate the proposed method.**

I. INTRODUCTION

Mathematically speaking, a continuous-time singular system model is described by a set of coupled differential and algebraic equations, which include dynamic information of a real plant as well as algebraic constraints. Singular systems are also referred to as descriptor systems, implicit systems, generalized state-space, differential-algebraic systems or semi-state systems. They are often encountered in various applications such as power systems, social economic systems, circuits and robotics. Therefore, considerable attention has been drawn to singular systems, see, [2], [12], and references therein.

The problem of filter design has been a major area in optimal control and fault diagnosis. There are mainly two kinds of commonly used filters: Kalman filter and H_∞ filter, see [4], [9], [10], [11], [14], for instance. A Kalman filter generally ensures an optimal estimation of the state variables in the sense that the covariance of the estimation error is minimized, while an H_∞ filter guarantees the transfer function from the disturbance to the filtering error output satisfies a prescribed H_∞-norm bound constraint. Theses methods have been extended to singular systems. For example, Kalman filtering problems were tackled in [1] and [6] for discrete-time stochastic systems. In [7] and [8], H_∞ filters were designed, respectively for discrete- and continuous-time singular systems. Both full- and reduced-order filters were studied in [12] for continuous- and discrete-time singular systems. While in [13], the problem of H_∞ filtering was investigated for uncertain descriptor systems with discrete and distributed delays.

In this paper, we are concerned with the filter design for linear continuous-time singular systems. Two types of filters are developed; one converges asymptotically and the other in finite time. In the latter case, the filter structure comprises two infinite-time filters and a finite delay element. It is noted that the filters designed are in state-space form and no information on the unknown input is required.

A brief outline of this paper is as follows. In Section II, we first make an equivalent transformation of the original systems. A basic result is presented for the existence of filters for linear continuous-time singular systems on infinite time interval. In Section III, two sufficient conditions which guarantee the existence of the infinite-time filter and the finite-time filter are obtained in terms of rank relationships. Systematic design method for filters is also given in this part. Section IV provides an illustrative example and Section V concludes the paper.

Notation. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The Moore-Penrose inverse of a matrix M is denoted by $M^\dagger$. I is the identity matrix with compatible dimensions. $\mathbb{C}$ represents the complex plane. $\Re(\cdot)$ means the real part of a complex number. Notations rank$(\cdot)$, $\det(\cdot)$, $\lambda_j(\cdot)$ stand for the rank, the determinant, and the jth eigenvalue of a matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. PRELIMINARIES

Consider the following linear descriptor system:

$$\left\{ \begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bw(t), \\ y(t) & = & Cx(t) + Dw(t), \\ z(t) & = & Lx(t), \end{array} \right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $y(t) \in \mathbb{R}^m$ is the measurable output; $z(t) \in \mathbb{R}^q$ is the signal to be estimated; $w(t) \in \mathbb{R}^p$ is the disturbance input. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular; we shall assume that $\mathrm{rank}E = r \le n$. A, B, C, D and L are known real constant matrices with appropriate dimensions. And descriptor system (1) is assumed to be regular and impulse-free [2]. Without loss of generality, we assume that L is of full row rank. Denote

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \tilde{x}(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} C & D \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} L & 0 \end{bmatrix}.$$

Now we consider the following filter for the estimate of $z(t)$ on infinite horizon:

$$\left\{ \begin{array}{rcl} \dot{\zeta}(t) & = & \hat{A}\zeta(t) + \hat{B}y(t), \\ \bar{z}(t) & = & \zeta(t) + \hat{D}y(t), \end{array} \right. \tag{2}$$

where $\zeta \in \mathbb{R}^q$ and $\bar{z} \in \mathbb{R}^q$ are the state and output of the filter, respectively. $\hat{A} \in \mathbb{R}^{q\times q}$, $\hat{B} \in \mathbb{R}^{q\times m}$ and $\hat{D} \in \mathbb{R}^{q\times m}$

This work was partially supported by RGC Grant HKU 7031/06P

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are unknown matrices of appropriate dimensions, which are to be determined such that $\bar{z}$ asymptotically estimates z.
Remark 1 Although system (1) is a singular system, filter (2) is a standard qth order state-space system, where q is the dimension of the signal to be estimated. □

Lemma 1: The qth order filter (2) asymptotically estimates z; that is, $\bar{z} - z \to 0$ as $t \to \infty$, if there exists a matrix M such that the following conditions hold:

(c1) $\hat{A}M\tilde{E} + \hat{B}\tilde{C} = M\tilde{A}$,
(c2) $M\tilde{E} = \tilde{L} - \hat{D}\tilde{C}$,
(c3) $\hat{A}$ is Hurwitz.

Proof. Let $\xi(t) = \zeta(t) - M\tilde{E}\begin{bmatrix} x(t) \\ 0 \end{bmatrix}$, then from (c1) and (c2), we have

$$\begin{aligned}
\dot{\xi}(t) &= \dot{\zeta}(t) - M\tilde{E}\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \\
&= \hat{A}\zeta(t) + \hat{B}y(t) - M\tilde{A}\tilde{x}(t) \\
&= \hat{A}\xi(t) + (\hat{A}M\tilde{E} + \hat{B}\tilde{C} - M\tilde{A})\tilde{x}(t) \\
&= \hat{A}\xi(t).
\end{aligned}$$

From (c2), we obtain

$$\begin{aligned}
\bar{z}(t) - z(t) &= \zeta(t) + \hat{D}y(t) - \tilde{L}\tilde{x}(t) \\
&= (\xi(t) + M\tilde{E}\tilde{x}(t)) + \hat{D}\tilde{C}\tilde{x}(t) - \tilde{L}\tilde{x}(t) \\
&= \xi(t) + (M\tilde{E} + \hat{D}\tilde{C} - \tilde{L})\tilde{x}(t) \\
&= \xi(t).
\end{aligned}$$

Using (c3), we obtain $\lim_{t\to\infty}(\bar{z}(t) - z(t)) = 0$ and hence filter (2) estimates $z(t)$ asymptotically. □

III. Main Results

We first present a sufficient condition which ensures the existence of infinite-time filter (2) for system (1). Some of the techniques employed in the proof have been adapted from [3].

Theorem 1: There exists a filter of form (2) for system (1), if for any $\lambda \in \mathbb{C}, \Re(\lambda) \geq 0$ the following condition holds:

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} 0 & 0 & C & D \\ 0 & 0 & L & 0 \\ C & D & 0 & 0 \\ L & 0 & 0 & 0 \\ -A & -B & E & 0 \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} C & D & 0 & 0 \\ L & 0 & 0 & 0 \\ 0 & 0 & C & D \\ -A & -B & E & 0 \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} 0 & 0 & C & D \\ 0 & 0 & L & 0 \\ C & D & 0 & 0 \\ \lambda E - A & -B & E & 0 \end{bmatrix}.
\end{aligned} \tag{3}$$

Proof. Based on Lemma 1, the proof is separated into two steps.
Step (1). We prove the first equality in condition (3) is equivalent to the existence of a solution to nonlinear matrix equations (c1) and (c2). Denoting $\bar{E} = \tilde{E} - I$, notice that

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} 0 & 0 & C & D \\ 0 & 0 & L & 0 \\ C & D & 0 & 0 \\ L & 0 & 0 & 0 \\ -A & -B & E & 0 \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} 0 & \tilde{C} \\ 0 & \tilde{L} \\ \tilde{C} & 0 \\ \tilde{L} & 0 \\ -\tilde{A} & \tilde{E} \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} 0 & \tilde{C} \\ 0 & \tilde{L} \\ \tilde{C} & 0 \\ \tilde{L} & 0 \\ -\tilde{A} & \tilde{E} \end{bmatrix}\begin{bmatrix} I & 0 \\ \tilde{A} & I \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \tilde{L}\tilde{A} & \tilde{L} \\ \tilde{L} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} C & D & 0 & 0 \\ L & 0 & 0 & 0 \\ 0 & 0 & C & D \\ -A & -B & E & 0 \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} \tilde{C} & 0 \\ \tilde{L} & 0 \\ 0 & \tilde{C} \\ -\tilde{A} & \tilde{E} \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} \tilde{C} & 0 \\ \tilde{L} & 0 \\ 0 & \tilde{C} \\ -\tilde{A} & \tilde{E} \end{bmatrix}\begin{bmatrix} I & 0 \\ \tilde{A} & I \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \tilde{L} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix}.
\end{aligned}$$

On the other hand, from conditions (c1) and (c2) of Lemma 1, we obtain

$$\hat{A}\tilde{L} = \tilde{L}\tilde{A} - [\hat{D}\ T\ M]\begin{bmatrix} \tilde{C}\tilde{A} \\ \tilde{C} \\ \bar{E}\tilde{A} \end{bmatrix}, \tag{4}$$

where $T = \hat{B} - \hat{A}\hat{D}$. Since L is of full row rank, $\tilde{L}$ is also of full row rank. Therefore, there exists a matrix G such that $\begin{bmatrix} \tilde{L}^T & G^T \end{bmatrix}^T$ is nonsingular. Denote

$$\begin{bmatrix} H_1 & E_1 \end{bmatrix} = \begin{bmatrix} \tilde{L} \\ G \end{bmatrix}^{-1},$$

then $\tilde{L}H_1 = I_q$ and $\tilde{L}E_1 = 0$. Therefore, from (4) we obtain

$$\hat{A} = \tilde{L}\tilde{A}H_1 - [\hat{D}\ T\ M]\begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix}, \tag{5}$$

and

$$\hat{A}\tilde{L}E_1 = \tilde{L}\tilde{A}E_1 - [\hat{D}\ T\ M]\begin{bmatrix} \tilde{C}\tilde{A}E_1 \\ \tilde{C}E_1 \\ \bar{E}\tilde{A}E_1 \end{bmatrix} = 0. \tag{6}$$

From (5) and the definition of T, we can see that $\hat{A}$ and $\hat{B}$ can be obtained from the knowledge of $\hat{D}$, T and M. Now (6) and (c2) can be rewritten as

$$\begin{bmatrix} \hat{D} & T & M \end{bmatrix}\Sigma = \Theta, \tag{7}$$

where

$$\Sigma = \begin{bmatrix} \tilde{C}\tilde{A}E_1 & \tilde{C} \\ \tilde{C}E_1 & 0 \\ \bar{E}\tilde{A}E_1 & \tilde{E} \end{bmatrix},$$

and $\Theta = \begin{bmatrix} \tilde{L}\tilde{A}E_1 & \tilde{L} \end{bmatrix}$. There exists a solution to (7) if and only if

$$\operatorname{rank}\begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = \operatorname{rank}[\Sigma]. \tag{8}$$

Take

$$S_1 := \begin{bmatrix} [H_1\ \ E_1] & 0 \\ 0 & I \end{bmatrix},$$

then we have

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \tilde{L}\tilde{A} & \tilde{L} \\ \tilde{L} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix} S_1 \\
= &\operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A}H_1 & \tilde{C}\tilde{A}E_1 & \tilde{C} \\ \tilde{C}H_1 & \tilde{C}E_1 & 0 \\ \tilde{L}\tilde{A}H_1 & \tilde{L}\tilde{A}E_1 & \tilde{L} \\ I_q & 0 & 0 \\ \bar{E}\tilde{A}H_1 & \bar{E}\tilde{A}E_1 & \tilde{E} \end{bmatrix} \\
= &\ q + \operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A}E_1 & \tilde{C} \\ \tilde{C}E_1 & 0 \\ \tilde{L}\tilde{A}E_1 & \tilde{L} \\ \bar{E}\tilde{A}E_1 & \tilde{E} \end{bmatrix} \\
= &\ q + \operatorname{rank}\begin{bmatrix} \Sigma \\ \Theta \end{bmatrix}.
\end{aligned}$$

Similarly, we have

$$\operatorname{rank}\begin{bmatrix} \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \tilde{L} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix} = q + \operatorname{rank}[\Sigma].$$

Hence the first equality in condition (3) holds if and only if $\hat{D}$, T and M are solvable from (7) which in turn gives $\hat{A}$ and $\hat{B}$.

Step (2). We prove the second equality of condition (3) is equivalent to that $\hat{A}$ is Hurwtiz under the first equality of condition (3). Notice that

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} 0 & 0 & C & D \\ 0 & 0 & L & 0 \\ C & D & 0 & 0 \\ \lambda E - A & -B & E & 0 \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} 0 & \tilde{C} \\ 0 & \tilde{L} \\ \tilde{C} & 0 \\ \lambda\tilde{E} - \tilde{A} & \tilde{E} \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} 0 & I & \lambda I & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} 0 & \tilde{C} \\ 0 & \tilde{L} \\ \tilde{C} & 0 \\ \lambda\tilde{E} - \tilde{A} & \tilde{E} \end{bmatrix} \\
&\begin{bmatrix} I & 0 \\ \tilde{A} - \lambda I & I \end{bmatrix} \\
= &\operatorname{rank}\begin{bmatrix} \lambda\tilde{L} - \tilde{L}\tilde{A} & -\tilde{L} \\ \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix}.
\end{aligned}$$

On the other hand, we obtain the general solution of (7) as

$$\begin{bmatrix} \hat{D} & T & M \end{bmatrix} = \Theta\Sigma^\dagger + Z(I - \Sigma\Sigma^\dagger), \tag{9}$$

where Z is an arbitrary matrix with appropriate dimensions. Substituting (9) into (5) results in

$$\begin{aligned}
\hat{A} &= \tilde{L}\tilde{A}H_1 - \Theta\Sigma^\dagger\begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix} \\
&\quad -Z(I - \Sigma\Sigma^\dagger)\begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix} \\
&= \Lambda - Z\Gamma,
\end{aligned}$$

where

$$\Lambda = \tilde{L}\tilde{A}H_1 - \Theta\Sigma^\dagger\begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix},$$

$$\Gamma = (I - \Sigma\Sigma^\dagger)\begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix}.$$

There exists a matrix parameter Z such that all eigenvalues of matrix $\hat{A} = \Lambda - Z\Gamma$ be Hurwitz if and only if for $\forall\lambda \in \mathbb{C}, \Re(\lambda) \geq 0$,

$$\operatorname{rank}\begin{bmatrix} \lambda I - \Lambda \\ \Gamma \end{bmatrix} = q. \tag{10}$$

Consider the full column rank matrix

$$S_2 := \begin{bmatrix} I & \Theta\Sigma^\dagger \\ 0 & I - \Sigma\Sigma^\dagger \\ 0 & \Sigma\Sigma^\dagger \end{bmatrix},$$

and nonsingular matrix

$$S_3 := \begin{bmatrix} I & 0 \\ -\Sigma^\dagger \begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix} & I \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned}
& \text{rank}\begin{bmatrix} \lambda\tilde{L}-\tilde{L}\tilde{A} & -\tilde{L} \\ \tilde{C}\tilde{A} & \tilde{C} \\ \tilde{C} & 0 \\ \bar{E}\tilde{A} & \tilde{E} \end{bmatrix} S_1 \\
= \ & \text{rank}\begin{bmatrix} \lambda I_q-\tilde{L}\tilde{A}H_1 & -\tilde{L}\tilde{A}E_1 & -\tilde{L} \\ \tilde{C}\tilde{A}H_1 & \tilde{C}\tilde{A}E_1 & \tilde{C} \\ \tilde{C}H_1 & \tilde{C}E_1 & 0 \\ \bar{E}\tilde{A}H_1 & \bar{E}\tilde{A}E_1 & \tilde{E} \end{bmatrix} \\
= \ & \text{rank}\begin{bmatrix} \lambda I_q-\tilde{L}\tilde{A}H_1 & -\Theta \\ \begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix} & \Sigma \end{bmatrix} \\
= \ & \text{rank}S_2 \begin{bmatrix} \lambda I_q-\tilde{L}\tilde{A}H_1 & -\Theta \\ \begin{bmatrix} \tilde{C}\tilde{A}H_1 \\ \tilde{C}H_1 \\ \bar{E}\tilde{A}H_1 \end{bmatrix} & \Sigma \end{bmatrix} S_3 \\
= \ & \text{rank}\begin{bmatrix} \begin{bmatrix} \lambda I_q-\Lambda \\ \Gamma \end{bmatrix} & 0 \\ 0 & \Sigma \end{bmatrix} \\
= \ & \text{rank}\begin{bmatrix} \lambda I_q-\Lambda \\ \Gamma \end{bmatrix} + \text{rank}[\Sigma].
\end{aligned}$$

Furthermore, from the proof of Step (1), we establish the result in Step (2). □

Now we are in the position to construct a finite-time filter of the following form:

$$\left\{ \begin{array}{rcl} \dot{\check{\zeta}} & = & \check{A}\check{\zeta}+\check{B}y, \\ \hat{z} & = & \check{\zeta}+\check{D}y, \\ \check{z}(t) & = & K[\hat{z}(t)-e^{\check{A}d}\hat{z}(t-d)], \\ \hat{z}(t) & = & \phi(t),\ t\in[-d,0], \end{array}\right. \tag{11}$$

where $d > 0$, $\check{\zeta} \in \mathbb{R}^{2q}$ is the state vector, $\hat{z} \in \mathbb{R}^{2q}$ is the output, and $\check{z} \in \mathbb{R}^q$ is the estimate of z, and $\phi(\cdot)$ is a continuous function. $\check{A}$, $\check{B}$, $\check{D}$ and K are unknown matrices of appropriate dimensions, which are to be determined such that $\check{z}$ estimates z in finite time d, that is, $\check{z}(t) = z(t)$ for $t \geq d$.

Theorem 2: There exists a finite-time filter for system (1) if for any $\lambda \in \mathbb{C}$ the rank condition (3) holds.

Proof. From the proof of Step (2) in Theorem 1, if for any $\lambda \in \mathbb{C}$ the rank condition (3) holds, then there exists Z such that all eigenvalues of matrix $\hat{A}$ can be assigned arbitrarily. Take two different matrices denoted by Z_1 and Z_2, which satisfy that $\hat{A}_i = \Lambda - Z_i\Gamma$ are stable $i = 1, 2$. Then we obtain the corresponding matrices $\hat{D}_i$, T_i, M_i and $\hat{B}_i$. Denote

$$\check{A} = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \quad \check{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \check{D} = \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix},$$
$$N = \begin{bmatrix} I_q \\ I_q \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}, \quad \check{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}. \tag{12}$$

Since $\frac{d}{dt}(\bar{z}_i(t) - z(t)) = \hat{A}_i(\bar{z}_i(t) - z(t))$ for $i = 1, 2$, we have

$$\frac{d}{dt}(\hat{z}(t) - Nz(t)) = \check{A}(\hat{z}(t) - Nz(t)), \quad t \geq 0.$$

Therefore,

$$\hat{z}(t) - Nz(t) = e^{\check{A}d}[\hat{z}(t-d) - Nz(t-d)], \quad t \geq d. \tag{13}$$

Take $K = \begin{bmatrix} I_q & 0 \end{bmatrix}\begin{bmatrix} N & e^{\check{A}d}N \end{bmatrix}^{-1}$. The existence of d such that $\begin{bmatrix} N & e^{\check{A}d}N \end{bmatrix}$ is nonsingular and the left of the proof can be established along an argument similar to that in [5], hence omitted. □

Remark 2 From the proof of Theorem 2, it is known that the finite-time filter (11) is $2q$-th order. □

Remark 3 To construct the finite-time observer gain K, $\begin{bmatrix} N & e^{\check{A}d}N \end{bmatrix}$ is required to be nonsingular. Since $\begin{bmatrix} N & e^{\check{A}d}N \end{bmatrix}$ is row equivalent to

$$\begin{bmatrix} I & e^{\hat{A}_1 d} \\ 0 & e^{\hat{A}_1 d} - e^{\hat{A}_2 d} \end{bmatrix},$$

its nonsingularity is satisfied if and only if $e^{\hat{A}_2 d}e^{-\hat{A}_1 d}$ does not have unit eigenvalues. □

Remark 4 Since $\check{A}$ is Hurwitz and the initial function $\phi(t)$ is a given continuous function, $e^{\check{A}d}\hat{z}(t-d) \to 0$ when $d \to \infty$. On the other hand, since all eigenvalues of $\hat{A}_i = \Lambda - Z_i\Gamma$ can be assigned arbitrarily under condition of Theorem 2, Z_i can be chosen such that

$$\Re(\lambda_j(\hat{A}_2)) < \sigma < \Re(\lambda_j(\hat{A}_1)), \quad j = 1, 2, \ldots, q,$$

for some $\sigma < 0$. In this case, $e^{\hat{A}_2 d}e^{-\hat{A}_1 d} \to 0$ when $d \to \infty$. Therefore, we have

$$\begin{aligned}
& K \\
= \ & \begin{bmatrix} I_q & 0 \end{bmatrix}\begin{bmatrix} N, & e^{\check{A}d}N \end{bmatrix}^{-1} \\
= \ & \begin{bmatrix} I_q & 0 \end{bmatrix}\begin{bmatrix} I & e^{\hat{A}_1 d} \\ 0 & e^{\hat{A}_1 d} - e^{\hat{A}_2 d} \end{bmatrix}^{-1}\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}^{-1} \\
= \ & \begin{bmatrix} I_q - (I_q - e^{\hat{A}_2 d}e^{-\hat{A}_1 d})^{-1} & I_q - (e^{\hat{A}_2 d}e^{-\hat{A}_1 d})^{-1} \end{bmatrix} \\
\to \ & \begin{bmatrix} 0 & I_q \end{bmatrix}, \quad d \to \infty.
\end{aligned} \tag{14}$$

Therefore, it is obtained that

$$\check{z}(t) \to \begin{bmatrix} 0 & I_q \end{bmatrix}\hat{z}(t) = \bar{z}_2(t), \quad d \to \infty,$$

where $\bar{z}_2(t)$ is the output of one of the infinite-time filters, that is, $\bar{z}_2(t) \to z(t)$, $t \to \infty$. □

A schematic diagram illustrating the filter design procedures for a finite-time filter of form (11) or an infinite-time filter of form (2) for system (1) is shown in Figure 1. Detailed algorithms for the design of finite-time filters and infinite-time filters are given below.

Algorithm "Finite-time filter"

Step 1. Given $d > 0$. Choose Z_i such that $\hat{A}_i$ are stable, $i = 1, 2$, such that $e^{\hat{A}_2 d} e^{-\hat{A}_1 d}$ has no unit eigenvalues.
Step 2. Obtain $\hat{D}_i$, T_i and M_i from (9), and subsequently, obtain $\hat{B}_i = T_i + \hat{A}_i \hat{D}_i$ from $T_i = \hat{B}_i - \hat{A}_i \hat{D}_i$.
Step 3. Obtain $\check{A}$, $\check{B}$, $\check{D}$ from (12) and $K = \left[\begin{array}{cc} I_q & 0 \end{array}\right] \left[\begin{array}{cc} N, & e^{\check{A}d} N \end{array}\right]^{-1}$.
Step 4. Obtain a finite-time filter of form (11) with the coefficients $\check{A}$, $\check{B}$, $\check{D}$ and K obtained above.

Algorithm "Infinite-time filter"

Step 1. Choose Z such that $\hat{A}$ is stable.
Step 2. Obtain $\hat{D}$, T and M from (9), and subsequently, obtain $\hat{B} = T + \hat{A}\hat{D}$ from $T = \hat{B} - \hat{A}\hat{D}$.
Step 3. Obtain an infinite-time filter of form (2) with the coefficients $\hat{A}$, $\hat{B}$ and $\hat{D}$ obtained above.

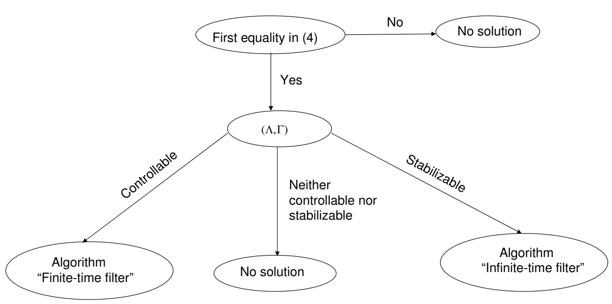

Fig. 1. Schematic Diagram for Filter Design

IV. Numerical Example

In this section, we present a numerical example to illustrate the developed method. Consider system (1) with the following data:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 6 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to check that first equality in (3) is satisfied. Solving (7) and substituting the solution into (5), we get $\Lambda = 0.0952$ and $\Gamma = \left[\begin{array}{cccccccccc} 0.0476 & 0 & 0.0476 & -0.1429 & -0.0476 & 0 & 0 & -0.1429 & 0 \end{array}\right]^T$. It is obvious that the rank condition in (10) holds for any $\lambda \in \mathbb{C}$. Taking

$$Z_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Z_2 = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$\hat{A}_1 = -0.1429, \quad \hat{B}_1 = \begin{bmatrix} -0.8776 & -0.4286 \end{bmatrix},$$

$$\hat{A}_2 = -0.3810, \quad \hat{B}_2 = \begin{bmatrix} -0.7642 & -1.1429 \end{bmatrix},$$

$$\hat{D}_1 = \begin{bmatrix} 0.1429 & 0 \end{bmatrix}, \quad \hat{D}_2 = \begin{bmatrix} 0.3810 & 0 \end{bmatrix}.$$

Take $d = 5$, it is easy to verify that

$$\det \begin{bmatrix} N, & e^{\check{A}d} N \end{bmatrix} = \det \begin{bmatrix} 1 & e^{-0.1429d} \\ 1 & e^{-0.3810d} \end{bmatrix} \neq 0.$$

Therefore, we obtain a finite-time observer (11) with the following coefficients:

$$\check{A} = \begin{bmatrix} -0.1429 & 0 \\ 0 & -0.3810 \end{bmatrix},$$

$$\check{B} = \begin{bmatrix} -0.8776 & -0.4286 \\ -0.7642 & -1.1429 \end{bmatrix},$$

$$\check{D} = \begin{bmatrix} 0.1429 & 0 \\ 0.3810 & 0 \end{bmatrix}, \quad \check{N} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $d = 5$, whose output $\check{z}(t)$ estimates exactly to the estimated output $z(t)$ in finite time $d = 5$. Figure 2 is drawn under $w(t) = 0.3\sin(t)$ and $x(0) = \begin{bmatrix} 1 & -2 & 2 & 0 \end{bmatrix}^T$, $\hat{z}(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $t \in [-5, 0]$, $\zeta(0) = 0$.

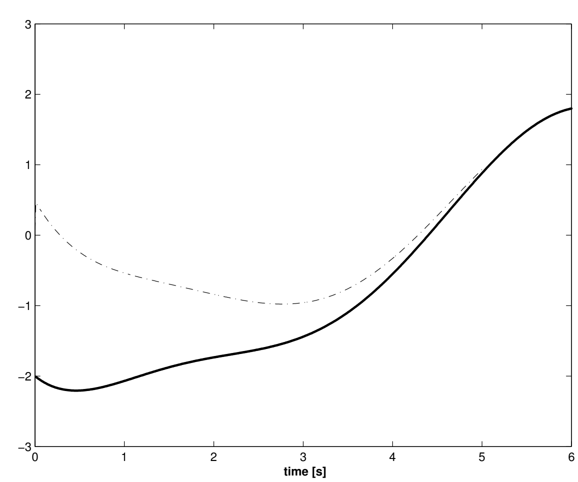

Fig. 2. $z(t)$ (solid) and its estimate $\check{z}(t)$ (dashed).

V. Conclusions

This paper has developed two types of filters for linear continuous-time descriptor systems, which include the infinite-time case and the finite-time case. Two sufficient conditions have been presented via rank relationships which ensure the existence of such filters. A systematic design algorithm for the filters has been provided. A numerical example has been used to demonstrate the proposed results. The method obtained in this paper can handle same problem for discrete-time singular systems.

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