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Published in:
I E E E Conference on Decision and Control. Proceedings

DOI (link to publication from Publisher):
10.1109/CDC.2010.5717418

Publication date:
2010

Document Version
Accepted author manuscript, peer reviewed version

Robust Helicopter Stabilization in the Face of Wind Disturbance

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***Abstract*— When a helicopter is required to hover with minimum deviations from a desired position without measurements of an affecting persistent wind disturbance, a robustly stabilizing control action is vital. In this paper, the stabilization of the position and translational velocity of a nonlinear helicopter model affected by a wind disturbance is addressed. The wind disturbance is assumed to be a sum of a fixed number of sinusoids with unknown amplitudes, frequencies and phases. An estimate of the disturbance is introduced to be adapted using state measurements for control purposes. A nonlinear controller is then designed based on nonlinear adaptive output regulations and robust stabilization of a chain of integrators by a saturated feedback. Simulation results show the effectiveness of the control design in the stabilization of helicopter motion and the built-in robustness of the controller in handling parameter and model uncertainties.**

I. INTRODUCTION

Autonomous helicopters are highly agile and has six degree of freedom maneuverability making them a favourite candidate for a wide range of practical applications including agriculture, cinematography, inspection, surveillance, search and rescue, reconnaissance, etc. For certain tasks the ability of a helicopter to follow a given state reference is crucial for a successful outcome; for instance when hovering over a ship for rescue operations or when flying close to power lines or wind turbines for inspections. In windy conditions this becomes a significant challenge for any pilot and hence an autopilot capable of accounting effectively for the wind disturbance is a realistic alternative. In this work, the authors present a control design for longitudinal, lateral, and vertical helicopter stabilization in the presence of a wind disturbance, with intrinsic robustness property in handling parameter and model uncertainties.

Firstly, some previous works are reviewed. In [10], a feedback-feedforward proportional differential (PD) controller is developed for heave motion control. With the assistance of a gust estimator, the controller is reported to be able to handle the influence from horizontal gusts. The effects of rotary gusts in forward and downward velocity of a helicopter is addressed in [6]. It is shown that via a state feedback law, the rotary gust rejection problem is always solvable. In a work by Wang et al., a multi-mode linear control stategy is designed for unmanned helicopters in the presence of model uncertainties, atmospheric disturbances and handling qualities requirements [9].

This work is supported by Universiti Teknologi Malaysia and Ministry of Higher Education Malaysia.

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In the present work, assuming that all the state variables (position, attitude and derivatives hereof) are available for measurements, a control strategy combining nonlinear adaptive regulation and robust stabilization of a chain of integrators by a saturated feedback is carried out (see for instance, [1], [2] and [3]). Guided by a control solution for vertical trajectory tracking presented in [3] and [4], the design technique presented therein is extended here where a robust longitudinal, lateral and vertical stabilizer capable of compensating for parameter and model uncertainties is developed for helicopter stabilization in the presence of wind disturbance in all three axes.

In the next section, mathematical model governing a model helicopter and a problem statement are given. This is followed by Section III which cover the vertical, longitudinal and lateral stabilization. After the presentation of simulation results in Section IV, conclusion and future works are discussed in Section V.

II. PRELIMINARIES

In this section, mathematical model of a helicopter will be described after which the problem statement will be given.

A. Helicopter Model

The motion of the center of mass of a helicopter is expressed in an inertial coordinate frame F_i as

$$M\ddot{p}^i = Rf^b,$$

where M is the mass and $p^i = [x, y, z]^\top \in \mathbb{R}^3$ is the position of the center of mass of the helicopter with respect to the origin of F_i. The rotation matrix R is parametrized in terms of unit quaternions $\mathbf{q} = (q_0, q) \in S_4$ where q_0 and $q = [q_1, q_2, q_3]^\top$ denote the scalar and the vector parts of the quaternion respectively (see Appendix A.2 in [3]). With the overall control inputs given by main and tail rotor thrusts, T_M and T_T respectively, and longitudinal and lateral main rotor tip path plane tilt angles, a and b respectively, a simplified resultant external force f^b in a body-fixed coordinate frame F_b is taken as

$$f^b = \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} + R^T \begin{bmatrix} 0 \\ 0 \\ Mg \end{bmatrix} + R^T \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}, \tag{1}$$

where d_x, d_y, d_z are wind disturbances that affect the helicopter motion in x, y and z axis respectively. Assuming that the tilt angles a and b are small, the following equations of motion can be derived,

$$\ddot{x} = \frac{-(2q_1q_3 + 2q_0q_2)T_M}{M} + \frac{d_x}{M} \tag{2}$$

$$\ddot{y} = \frac{-(2q_2q_3 - 2q_0q_1)T_M}{M} + \frac{d_y}{M} \quad (3)$$

$$\ddot{z} = \frac{-(1 - 2q_1^2 - 2q_2^2)T_M}{M} + g + \frac{d_z}{M}. \quad (4)$$

Also,

$$J\dot{\omega}^b = -S(\omega^b)J\omega^b + \tau^b, \quad \dot{\mathbf{q}} = \frac{1}{2}\begin{bmatrix} -q^T \\ q_0 I + S(q) \end{bmatrix}\omega^b,$$

where $S(\cdot)$ is a skew symmetric matrix, $\omega^b \in \mathbb{R}^3$ represents the angular velocity in F_b and J is the inertia matrix. The external torques τ^b expressed in F_b are given by the following equation

$$\tau^b = \begin{bmatrix} \tau_{f_1} \\ \tau_{f_2} \\ \tau_{f_3} \end{bmatrix} + \begin{bmatrix} R_M \\ M_M + M_T \\ N_M \end{bmatrix}, \quad (5)$$

where τ_{f_1}, τ_{f_2}, τ_{f_3} are moments generated by the main and tail rotors and R_M, M_M, N_M, M_T are moments of the aerodynamic forces [5]. With some approximations a compact torque equation is obtained,

$$\tau^b = A(T_M)\mathbf{v} + B(T_M), \quad \mathbf{v} := [a, b, T_T]^\top, \quad (6)$$

where $A(T_M)$ and $B(T_M)$ are a matrix and a vector whose entries are functions of T_M and the above mentioned parameters. Since one of the objectives of the controller to be designed is to handle parameter uncertainties, all nominal values of the helicopter parameters are collected in μ_0 with its additive uncertainties μ_Δ ranging in a compact set. Also, $M = M_0 + M_\Delta$, $J = J_0 + J_\Delta$, $A(T_M) = A_0(T_M) + A_\Delta(T_M)$ and $B(T_M) = B_0(T_M) + B_\Delta(T_M)$.

B. Problem Statement

The objective of the controller is to stabilize a helicopter affected by a disturbance $d = [d_x, d_y, d_z]^\top$ in the presence of parameter and model uncertainties. The disturbance that affects the acceleration of the helicopter is modeled to be in the following form,

$$d_j = \sum_{i=1}^{N} A_{ji}\cos(\Omega_i t + \varphi_{ji}),$$

with unknown amplitude A_{ji}, phase φ_{ji} and frequency Ω_i, for $j = x, y, z$ and $i = 1, \ldots, N$. It can be shown that the disturbance is generated by the following linear time-invariant exosystem,

$$\begin{aligned} \dot{w}_j &= \mathrm{S}(\varrho)w_j \\ d_j &= R\mathrm{S}^2(\varrho)w_j, \quad j = x, y, z, \end{aligned} \quad (7)$$

where $w_j \in \mathbb{R}^{2N}$, $\varrho = [\Omega_1, \ldots, \Omega_N]^\top$, $\mathrm{S}(\varrho) = \mathrm{diag}\big(H(\Omega_1), \ldots, H(\Omega_N)\big)$ with

$$H(\Omega_i) = \begin{bmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{bmatrix}, \quad i = 1, \ldots, N,$$

and $R = \big[[1\ 0], \ldots, [1\ 0]\big]$ is a $1\times 2N$ matrix (see [3], pp. 89-90). It is assumed that the frequencies Ω_i belong to a compact set. To note, the initial conditions $w_j(0)$ of the exosystems represent the amplitudes and phases of the disturbance. In the next section, it will be clear how the representation of the disturbance in such a form can be advantageous in the development of an internal model for stabilizing control input generation.

III. CONTROLLER DESIGN

The control design is divided into two parts. In the first part, a stabilizing T_M is constructed for the vertical dynamics. This is then used together with virtual controls q_1 and q_2 for longitudinal and lateral stabilization.

A. Stabilization of Vertical Dynamics

With reference to the vertical dynamics (4) given above, to counter the nominal effect of the gravity, the following preliminary control law is chosen,

$$T_M = \frac{gM_0 + u}{1 - \mathrm{sat}_c(2q_1^2 + 2q_2^2)}, \quad (8)$$

where $\mathrm{sat}_c(s) := \mathrm{sign}(s)\min\{|s|, c\}$ is a saturation function with $0 < c < 1$. This yields

$$\ddot{z} = -\frac{\psi_c^z(q)u}{M} + g\big(1 - \frac{M_0}{M}\psi_c^z(q)\big) + \frac{d_z}{M},$$

where

$$\psi_c^z(q) = \frac{1 - 2q_1^2 - 2q_2^2}{1 - \mathrm{sat}_c(2q_1^2 + 2q_2^2)}$$

and u is an input to be designed.

If $q(t)$ is small enough such that $\psi_c^z(q) = 1$, then a *feedforward* u needed for vertical stabilization is

$$c_u(w_z, \mu, \varrho) = gM_\Delta + d_z.$$

Note that if in steady-state the control $c_u(w_z, \mu, \varrho)$ is achieved, zero z acceleration will be produced. Since $c_u(w_z, \mu, \varrho)$ depends on $M_\Delta \in \mu$ and d_z which is a function of ϱ and $w_z(0)$, it can be only implemented with an internal model. Hence, the desired control is rewritten as an output of a linear system

$$\begin{aligned} \frac{\partial \tau_z}{\partial w_z}\mathrm{S}(\varrho)w &= \Phi(\varrho)\tau_z(w_z, \mu) \\ c_u(w_z, \mu, \varrho) &= \Gamma(\varrho)\tau_z(w_z, \mu), \end{aligned} \quad (9)$$

where

$$\tau_z(w_z, \mu) = \begin{bmatrix} gM_\Delta \\ w_z \end{bmatrix}, \ \Phi(\varrho) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{S}(\varrho) \end{bmatrix},$$

$$\Gamma(\varrho) = \begin{bmatrix} 1 & \Gamma_2(\varrho) \end{bmatrix},$$

and

$$\Gamma_2(\varrho) = \begin{bmatrix} -\Omega_1^2 & 0 & -\Omega_2^2 & 0 & \ldots & -\Omega_N^2 & 0 \end{bmatrix}.$$

It has been shown in Lemma 3.3.1 in [3] that there exist controllable matrix pair (F, G) with dimensions $(2N+1)\times(2N+1)$ and $(2N+1)\times 1$ respectively, a unique $1\times(2N+1)$ vector $\Psi_\varrho = [1\ \Psi_{2,\varrho}]$ and a $(2N+1)\times(2N+1)$ nonsingular matrix T_ϱ such that $\Phi(\varrho) = T_\varrho^{-1}(F + G\Psi_\varrho)T_\varrho$, $\Gamma(\varrho) = \Psi_\varrho T_\varrho$

and

$$\begin{aligned}\frac{\partial \bar{\tau}_z}{\partial w_z}\mathrm{S}(\varrho)w &= (F+G\Psi_\varrho)\bar{\tau}_z(w_z,\mu,\varrho)\\ c_u(w_z,\mu,\varrho) &= \Psi_\varrho \bar{\tau}_z(w_z,\mu,\varrho),\end{aligned} \tag{10}$$

where $\bar{\tau}_z(w_z,\mu,\varrho) = T_\varrho \tau_z(w_z,\mu)$.

Because $c_u(w_z,\mu,\varrho)$ depends on unknown parameters and unmeasurable state w_z, the control is chosen as $u = u_{\text{im}} + u_{\text{st}}$ which consists of an internal model control u_{im} and a stabilizing control u_{st}. Thus the controller can be taken as

$$\begin{aligned}\dot{\xi}_z &= (F+G\hat{\Psi})\xi_z + g_{\text{stz}}\\ u &= \hat{\Psi}\xi_z + u_{\text{st}},\end{aligned} \tag{11}$$

with the update law

$$\dot{\hat{\Psi}}_2 = \gamma \xi_{z2}^T(\dot{z}+k_1 z) - [\text{tas}_d(\hat{\Psi}_2)]^\top$$

where $g_{\text{stz}} = Gu_{\text{st}} + FGM_0\dot{z}$, $u_{\text{st}} = k_2(\dot{z}+k_1 z)$ for $k_1, k_2 > 0$, $\xi_z = [\xi_{z1}\ \xi_{z2}]^\top$ with $\xi_{z1} \in \mathbb{R}$, $\xi_{z2} \in \mathbb{R}^{2N}$, $\hat{\Psi} = [1\ \hat{\Psi}_2]$ with a $1 \times 2N$ row vector $\hat{\Psi}_2$, $\gamma > 0$ and $\text{tas}_d(\cdot)$ is a dead-zone function with $d > \max_{i=1}^{2N}\left(|(\Psi_{2,\varrho})_i|\right)$. Defining

$$\eta = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}\eta_1\\ \tilde{\Psi}_2^T\end{bmatrix}, \text{ with } \eta_1 = \begin{bmatrix}\chi\\ z\\ \zeta\end{bmatrix},$$

where the change of coordinates are set as

$$\begin{aligned}\chi &:= \xi_z - \bar{\tau}_z(w_z,\mu,\varrho) + GM\dot{z}\\ \tilde{\Psi}_2 &:= \hat{\Psi}_2 - \Psi_{2,\varrho}\\ \zeta &:= \dot{z} + k_1 z,\end{aligned}$$

the following equations are obtained,

$$\begin{aligned}\dot{\eta}_1 &= \Big(A + A_1\big(\psi_c^z(q)-1\big)\Big)\eta_1\\ &\quad -\Big(\frac{1}{M}b + B\big(\psi_c^z(q)-1\big)\Big)\xi_{z2}^\top\eta_2 - B\rho\\ \dot{\eta}_2 &= \gamma\xi_{x2}b^T\eta_1 - \text{tas}_d(\eta_2 + \Psi_{2,\varrho}),\end{aligned} \tag{12}$$

where $\rho = \big(\psi_c^z(q)-1\big)\big(gM_0 + \Psi_\varrho\bar{\tau}_z(w,\mu,\varrho)\big)$,

$$A = \begin{bmatrix} F & k_1 FGM_\triangle & -FGM_\Delta\\ 0 & -k_1 & 1\\ -\frac{1}{M}\Psi_\varrho & -k_1(\Psi_\varrho G + k_1) & (\Psi_\varrho G + k_1 - \frac{1}{M}k_2)\end{bmatrix},$$

$$\begin{aligned}A_1 &= \begin{bmatrix} -G\Psi_\varrho & -G\Psi_\varrho GMk_1 & G\Psi_\varrho GM - Gk_2\\ 0 & 0 & 0\\ -\frac{1}{M}\Psi_\varrho & -k_1\Psi_\varrho G & (\Psi_\varrho G - \frac{1}{M}k_2)\end{bmatrix},\\ b &= \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ and } B = \begin{bmatrix}G\\0\\\frac{1}{M}\end{bmatrix}.\end{aligned}$$

Using the same arguments as in Proposition 3.3.2 in [3], it can be shown that there exists $k_2^* > 0$ such that if $k_2 > k_2^*$, A is a Hurwitz matrix. Subsequently when q is small enough such that $\psi_c^z(q) = 1$, Proposition 3.4.2 guarantees that if the initial states of the exosystems belongs to a compact set $\mathcal{W}$ defined therein, then system (12) is globally asymptotically and locally exponentially stable. Note that even though the system equations are different than that of in [3] due to the addition of a disturbance and a different vertical control, the two propositions still apply.

It is only natural now to ensure that the condition $\psi_c^z(q) = 1$ can be achieved in finite time. Setting

$$\mathbf{v} = A_0(T_M)^{-1}\big(\tilde{\mathbf{v}} - B_0(T_M)\big)$$

yields the torque equation

$$\tau^b(\tilde{\mathbf{v}}) = L(T_M)\tilde{\mathbf{v}} + \Delta(T_M),$$

where $\tilde{\mathbf{v}}$ is an additional control input to be determined, $L(T_M) = I + A_\Delta(T_M)A_0^{-1}(T_M)$ and $\Delta(T_M) = B_\Delta(T_M) - A_\Delta(T_M)A_0^{-1}(T_M)B_0(T_M)$. Dropping the superscript b in ω^b, choose

$$\tilde{\mathbf{v}} = K_P\big(\eta_1 - K_D(\omega - \omega_d)\big),$$

where $K_P, K_D > 0$ are design parameters,

$$\eta_1 := q_r - q \quad \text{and} \tag{13}$$

$$q_r = q^* + q_d. \tag{14}$$

The expressions for the desired angular velocity ω_d and desired quaternions q_r will be given in the next subsection. In Appendix A, the existence of a Lyapunov function for the attitude dynamical system is shown. This condition is a key element in the proof of Proposition 5.7.1 in [3]. Consequently for the problem in hand, it could be stated that for any $T^* > 0$, there exist $K_D^* > 0$, $\lambda_3^*(K_D) > 0$, $K_d^*(K_D) > 0$ and $K_P^*(K_D) > 0$, such that for all positive $K_D \le K_D^*$, $\lambda_3 \le \lambda_3^*(K_D)$, $K_d \le K_d^*(K_D)$ and $K_P \ge K_P^*(K_D)$, the trajectory of the attitude dynamical system is bounded, $q_0(t) > \varepsilon$ for arbitrary $0 < \varepsilon < 1$, $t > 0$ and $\psi_c^z(q(t)) = 1$ for all $t \ge T^*$, with initial conditions $(q(0),\omega(0)) \in \mathcal{Q}\times\Omega$, where $\mathcal{Q}$ and Ω are compact sets (cf. Prop. 5.7.1 and see definition of $\mathcal{Q}$ in [3]).

B. Stabilization of Longitudinal and Lateral Dynamics

Now, we will show that with an appropriate selection of design parameters, virtual controls q_1 and q_2 can be manipulated to produce horizontal stability. The control law (8) that has been designed for vertical stabilization also appears in the longitudinal (2) and lateral (3) dynamics. The numerator of (8) can be expanded as

$$gM_0 + u = gM + d_z + y_{\mathbf{z}}(\mathbf{z},w),$$

where $y_{\mathbf{z}}(\mathbf{z},w) = \tilde{\Psi}\bar{\tau}_z(w,\mu,\varrho) + (\tilde{\Psi} + \Psi_\varrho)(\chi - GM\dot{z}) + k_2(\dot{z} + k_1 z)$ and $\mathbf{z} = \eta$. Subsequently, (2) and (3) can be written as

$$\begin{aligned}\ddot{x} &= \frac{-\tilde{d}(\mathbf{q},t)q_2 + m(\mathbf{q},t)q_1q_3 + n_x(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w)}{M} + \frac{d_x}{M}\\ \ddot{y} &= \frac{\tilde{d}(\mathbf{q},t)q_1 + m(\mathbf{q},t)q_2q_3 + n_y(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w)}{M} + \frac{d_y}{M},\end{aligned}$$

where

$$\tilde{d}(\mathbf{q},t) = \frac{2(gM+d_z)q_0}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)},$$

$$
\begin{aligned}
m(\mathbf{q},t) &= -\frac{2(gM+d_z)}{1-\text{sat}_c(2q_1^2+2q_2^2)},\\
n_x(\mathbf{q}) &= \frac{-(2q_1q_3+2q_0q_2)}{1-\text{sat}_c(2q_1^2+2q_2^2)},\\
n_y(\mathbf{q}) &= \frac{-(2q_2q_3-2q_0q_1)}{1-\text{sat}_c(2q_1^2+2q_2^2)}.
\end{aligned}
$$

Introducing the bank of integrators $\dot{\eta}_x = x$, $\dot{\eta}_y = y$ and $\dot{\eta}_q = q_3$, the following new state variables are defined

$$
\begin{aligned}
\zeta_1 &:= \begin{bmatrix}\eta_y\\ \eta_x\end{bmatrix}\\
\zeta_2 &:= \begin{bmatrix}y\\ x\end{bmatrix} + \lambda_1\sigma(\frac{K_1}{\lambda_1}\zeta_1)\\
\zeta_3 &:= \begin{bmatrix}\dot{y}\\ \dot{x}\\ \eta_q\end{bmatrix} + P_1\lambda_2\sigma(\frac{K_2}{\lambda_2}\zeta_2),
\end{aligned}
$$

where

$$
P_1 = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \text{ and } P_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}.
$$

By adopting the following 'nested saturated' control law

$$
q^* = -P_2\lambda_3\sigma(\frac{K_3}{\lambda_3}\zeta_3), \tag{15}
$$

where the function $\sigma(\cdot)$ is a vector-valued saturation function of suitable dimension and, K_i and λ_i, $i = 1, 2, 3$ are design parameters, the time derivatives can be written as

$$
\begin{aligned}
\dot{\zeta}_1 &= -\lambda_1\sigma(\frac{K_1}{\lambda_1}\zeta_1) + \zeta_2\\
\dot{\zeta}_2 &= -\lambda_2\sigma(\frac{K_2}{\lambda_2}\zeta_2) + P_0\zeta_3 + K_1\sigma'(\frac{K_1}{\lambda_1}\zeta_1)\dot{\zeta}_1\\
M\dot{\zeta}_3 &= -\tilde{D}(t)P_2\lambda_3\sigma(\frac{K_3}{\lambda_3}\zeta_3) + \tilde{D}(t)q_d\\
&\quad + MK_2P_1\sigma'(\frac{K_2}{\lambda_2}\zeta_2)\dot{\zeta}_2 - \tilde{D}(t)\eta_1 + p + d,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\tilde{D}(t) &= \begin{bmatrix}\tilde{d}(\mathbf{q},t) & m(\mathbf{q},t)q_3 & 0\\ m(\mathbf{q},t)q_3 & -\tilde{d}(\mathbf{q},t) & 0\\ 0 & 0 & M\end{bmatrix}, d = \begin{bmatrix}d_y\\ d_x\\ 0\end{bmatrix}\\
\text{and } p &= \begin{bmatrix}n_y(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w)\\ n_x(\mathbf{q})y_{\mathbf{z}}(\mathbf{z},w)\\ 0\end{bmatrix}.
\end{aligned}
$$

Note that if one can set $q_d = -\tilde{D}^{-1}(t)d$, the disturbance d can be completely eliminated from subsystem (16) and shown to be *input-to-state stable* (ISS) with restrictions on the inputs (η_1, p) and linear asymptotic gains (see Lem. 5.7.4 in [3]). However, since $\tilde{D}$ is a function of uncertain M and unknown d, it is not entirely possible to have such a q_d. As a result, the following is proposed,

$$
q_d = -K_d\tilde{D}_0^{-1}(t)\hat{d}, \tag{17}
$$

where $K_d > 0$ is another design parameter, $\hat{d} = [\hat{d}_y, \hat{d}_x, 0]^\top$ is a disturbance estimate to be adapted and

$$
\tilde{D}_0^{-1}(t) = \begin{bmatrix}\tilde{d}_0(\mathbf{q},t) & m_0(\mathbf{q},t)q_3 & 0\\ m_0(\mathbf{q},t)q_3 & -\tilde{d}_0(\mathbf{q},t) & 0\\ 0 & 0 & \frac{1}{M_0}\end{bmatrix}, \tag{18}
$$

with

$$
\begin{aligned}
\tilde{d}_0(\mathbf{q},t) &= \frac{1-\text{sat}_c(2q_1^2+2q_2^2)}{2(q_0^2+q_3^2)(gM_0+\hat{\Psi}_2\xi_{z2})},\\
m_0(\mathbf{q},t) &= -\frac{1-\text{sat}_c(2q_1^2+2q_2^2)}{2(q_0^2+q_3^2)(gM_0+\hat{\Psi}_2\xi_{z2})}.
\end{aligned}
$$

It is important to notice at this point that a constraint on d_z should be imposed. To have $T_M > 0$ (see (8) and (9)) and to avoid singularities in (18), it is required that $|d_z(t)| < gM$ for all $t > 0$. Note that from (7), (9), (10) and (11), $\hat{d}$ can be taken as

$$
\hat{d} = K_dPe_\xi + P\bar{\tau} + \tilde{P}_2\xi_2, \tag{19}
$$

where

$$
P = \begin{bmatrix}\Psi_\varrho & 0\\ 0 & \Psi_\varrho\\ 0 & 0\end{bmatrix}, \tilde{P}_2 = \begin{bmatrix}\tilde{\Psi}_2 & 0\\ 0 & \tilde{\Psi}_2\\ 0 & 0\end{bmatrix}, e_\xi = \begin{bmatrix}e_{\xi y}\\ e_{\xi x}\end{bmatrix},
$$

$$
\bar{\tau} = \begin{bmatrix}\bar{\tau}_y\\ \bar{\tau}_x\end{bmatrix}, \xi_2 = \begin{bmatrix}\xi_{y2}\\ \xi_{x2}\end{bmatrix},
$$

and

$$
e_{\xi i} = \frac{\xi_i - \bar{\tau}_i}{K_d}, \tag{20}
$$

with $\xi_i = [\xi_{i1}, \xi_{i2}]^\top$, for $i = x, y$. To note, e_ξ is governed by

$$
\dot{e}_\xi = K_d\tilde{F}e_\xi + \tilde{G}\tilde{P}_2\xi_2 + g_{\text{st}}, \tag{21}
$$

where

$$
\tilde{F} = \begin{bmatrix}F+G\Psi_\varrho & 0\\ 0 & F+G\Psi_\varrho\end{bmatrix}, \tilde{G} = \begin{bmatrix}G & 0\\ 0 & G\end{bmatrix}
$$

and

$$
g_{\text{st}} = k_4\tilde{G}\begin{bmatrix}\dot{y}\\ \dot{x}\end{bmatrix} + k_4k_3\tilde{G}\begin{bmatrix}y\\ x\end{bmatrix}, k_3, k_4 > 0. \tag{22}
$$

Now in addition to (13), set

$$
\begin{aligned}
\eta_2 &:= \omega - \omega_d - \frac{1}{K_D}\eta_1\\
\eta_3 &:= P_3\eta_1 - e_\xi,
\end{aligned}
$$

where

$$
P_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ \vdots & \vdots & \vdots\end{bmatrix}
$$

is a $(4N+2)\times 3$ matrix and

$$\omega_d = Q_d \dot{q}_d \tag{23}$$

with

$$Q_d = \frac{2}{\varepsilon}\begin{bmatrix} \varepsilon^2+q_{d1}^2 & q_{d1}q_{d2} & -\varepsilon q_{d2} \\ q_{d1}q_{d2} & \varepsilon^2+q_{d2}^2 & \varepsilon q_{d1} \\ \varepsilon q_{d2} & -\varepsilon q_{d1} & \varepsilon^2 \end{bmatrix},$$

keeping in mind that $q_d = [q_{d1}, q_{d2}]^\top$. Taking the time derivatives,

$$\begin{aligned} \dot{\eta}_1 &= -\frac{1}{2}\big(q_0 I + S(q)\big)(\eta_2 + \frac{1}{K_D}\eta_1 + \omega_d) + \dot{q}_r \\ J\dot{\eta}_2 &= -S(\omega)J(\eta_2 + \frac{1}{K_D}\eta_1 + \omega_d) - K_P K_D L(T_M)\eta_2 \\ &\quad -\frac{1}{K_D}J\dot{\eta}_1 + \Delta(T_M) - J\dot{\omega}_d \\ \dot{\eta}_3 &= P_3\dot{\eta}_1 - K_d\tilde{F}e_\xi - \tilde{G}\tilde{P}_2\xi_2 - g_{\text{st}}. \end{aligned} \tag{24}$$

We will now study the stability of feedback interconnection of subsystems (16) and (24). Subsystem (16) is a system with state $(\zeta_1, \zeta_2, \zeta_3)$ and inputs $(\eta_1, \eta_3, p, d, \bar{\tau})$, while subsystem (24) is a system with state (η_1, η_2, η_3) and inputs $(y_{\zeta 0}, y_{\zeta 1}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$. Let M^L, M^U and d^L, d^U be such that for all $t > 0$, $M^L \le M \le M^U$ and $0 < d^L \le \tilde{d}(\mathbf{q}, t) \le d^U$ respectively. Additionally, assume that $q_0(t) > \varepsilon > 0$, $\|\omega\| \le m_\omega$, $\|\omega_d\| \le m_{\omega_d}$, $\|\tilde{D}_0^{-1}\hat{d}\| \le m_{q_d}$, $\|Q_d\| \le m_{Q_d}$ and $\|e_\xi\| \le m_{e_\xi}$ for all $t > 0$ and some positive m_ω, m_{ω_d}, m_{q_d}, m_{Q_d} and m_{e_ξ}. With that in mind, the following proposition is stated.

Proposition 1: Let K_D be fixed and let K_i^* and λ_i^*, $i = 1, 2, 3$, be such that the following inequalities are satisfied

$$\begin{aligned} &\frac{\lambda_2^*}{K_2^*} < \frac{\lambda_1^*}{4}, \quad \frac{\lambda_3^*}{K_3^*} < \frac{\lambda_2^*}{4}, \quad 4K_1^*\lambda_1^* < \frac{\lambda_2^*}{4}, \\ &4K_2^*\lambda_2^* < \frac{d^L}{M^U}\frac{\lambda_3^*}{8}, \quad 24\frac{K_1^*}{K_2^*} < \frac{1}{6} \\ &\text{and} \quad 24\frac{K_2^*}{K_3^*} < \frac{1}{6}\frac{d^L}{d^U}\frac{M^L}{M^U}. \end{aligned}$$

Moreover let $R_{\eta 1}$, $R_{\eta 2}$ and $R_{\eta 3}$ be arbitrary positive numbers. Then, there exist positive numbers K_P^*, K_d^*, ϵ^*, $R_{\zeta 1}$, $R_{\zeta 2}$, $R_{\zeta 3}$, $\gamma_{\zeta 1}$, $\gamma_{\zeta 2}$, $\gamma_{\zeta 3}$, $\gamma_{\eta 1}$, $\gamma_{\eta 2}$ and $\gamma_{\eta 3}$ such that, taking $\lambda_i = \epsilon^{i-1}\lambda_i^*$ and $K_i = \epsilon K_i^*$, $i = 1, 2, 3$, for all $K_P \ge K_P^*$, $K_d \le K_d^*$ and $0 < \epsilon \le \epsilon^*$, the feedback interconnection of subsystems (16) and (24) is ISS without restrictions on the initial state, restrictions $(\epsilon^2 R_{\zeta 1}, \epsilon^2 R_{\zeta 2}, \epsilon^2 R_{\zeta 3}, R_{\eta 2}, R_{\eta 1}, R_{\eta 3})$ on the inputs $(p, d, \bar{\tau}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$ and linear asymptotic gains; in particular, if $\|p\|_\infty < \epsilon^2 R_{\zeta 1}$, $\|d\|_\infty < \epsilon^2 R_{\zeta 2}$, $\|\bar{\tau}\|_\infty < \epsilon^2 R_{\zeta 3}$, $\|I_{\eta 1}\|_\infty < R_{\eta 1}$, $\|I_{\eta 2}\|_\infty < R_{\eta 2}$ and $\|I_{\eta 3}\|_\infty < R_{\eta 3}$, then $(\zeta_1(t), \zeta_2(t), \zeta_3(t), \eta_1(t), \eta_2(t), \eta_3(t))$ satisfies the following asymptotic bound

$$\begin{aligned} &\|(\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3)\|_a \le \\ &\max\big\{\gamma_{\zeta 1}\|p\|_a, \gamma_{\zeta 2}\|d\|_a, \gamma_{\zeta 3}\|\bar{\tau}\|_a, \\ &\gamma_{\eta 1}\|I_{\eta 1}\|_a, \frac{\gamma_{\eta 2}}{K_P}\|I_{\eta 2}\|_a, \gamma_{\eta 3}\|I_{\eta 3}\|_a\big\}, \end{aligned}$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_a$ denote the $\mathcal{L}_\infty$ norm and asymptotic $\mathcal{L}_\infty$ norm respectively [8].

Proof: The proof of Proposition 1 involves showing that the two subsystems are ISS separately, and that the composed system satisfies the *small gain theorem* (see for instance, [2]). In this paper however, only subsystem (24) is shown to be ISS and hence the following lemma is presented.

Lemma 1: Let K_D be fixed and assume that $q_0(t) > \varepsilon > 0$ and $L(T_M(t)) \ge l_1$, for all $t \ge 0$. There exist positive numbers $\bar{\lambda}_3^*$, $\bar{K}_3^*(K_D)$, $\bar{K}_P^*(K_D)$, $\bar{K}_d^*(K_D, K_3)$ and $r_{\zeta 0}$, $r_{\zeta 1}$, r_{I1}, r_{I2}, r_{I3} such that, for all $\lambda_3 \le \bar{\lambda}_3^*$, $K_3 \le \bar{K}_3^*(K_D)$, $K_P \ge \bar{K}_P^*(K_D)$ and $K_d \le \bar{K}_d^*(K_D, K_3)$, system (24) is ISS, without restrictions on the initial state and on the inputs $(y_{\zeta 0}, y_{\zeta 1}, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$ and with linear asymptotic gains; in particular, for all bounded inputs $y_{\zeta 0}$, $y_{\zeta 1}$, $I_{\eta 1}$, $I_{\eta 2}$ and $I_{\eta 3}$, the state (η_1, η_2, η_3) satisfies the asymptotic bound

$$\begin{aligned} \|(\eta_1, \eta_2, \eta_3)\|_a \le \max\big\{&r_{\zeta 0}\|y_{\zeta 0}\|_a, r_{\zeta 1}\|y_{\zeta 1}\|_a, \\ &r_{I1}\|I_{\eta 1}\|_a, \frac{r_{I2}}{K_P}\|I_{\eta 2}\|_a, r_{I3}\|I_{\eta 3}\|_a\big\}. \end{aligned}$$

Proof: See Appendix B.

Note that using the same arguments as in [3] and [4], subsystem (16) can be shown to be ISS without restrictions on the initial state, restrictions on the inputs and with linear asymptotic gains. Subsequently, also based on discussion therein, since the feedback interconnection of subsystems (16) and (24) can be proven to satisfy all the conditions of the small gain theorem, the proof of Proposition 1 is completed.

Recall that earlier in the vertical dynamics stabilization, K_P is required to be arbitrarily large while λ_3 and K_d are needed to be arbitrarily small, which is also part of the requirements for longitudinal and lateral dynamics stabilization as suggested by Proposition 1. To be noted, only restrictions on inputs $(p, I_{\eta 1}, I_{\eta 2}, I_{\eta 3})$ can be always fulfilled in finite time as $p(t)$ is asymptotically vanishing (since $y_{\mathbf{z}}(\mathbf{z}, w)$ is) and, $R_{\eta 1}$, $R_{\eta 2}$ and $R_{\eta 3}$ can be chosen arbitrarily large. Consequently, only for sufficiently small magnitude of disturbance, for any initial conditions $w(0) \in \mathcal{W}$, $\mathbf{z}(0) \in \mathcal{Z}$, $(x(0), \dot{x}(0), y(0), \dot{y}(0)) \in \mathbb{R}^4$, $(q(0), \omega(0)) \in \mathcal{Q} \times \Omega$ with $q_0(0) > 0$, by choosing K_P and K_d sufficiently large and small respectively,

$$\begin{aligned} &\lim_{t\to\infty}\|(x(t), \dot{x}(t), y(t), \dot{y}(t))\| \le R_S(\|d\|_a), \quad \text{and} \\ &\lim_{t\to\infty}\|(z(t), \dot{z}(t))\| = 0, \end{aligned}$$

where $\mathcal{Z}$ is a compact set and R_S is a class $\mathcal{K}$ function (see [2] for the definition of a class $\mathcal{K}$ function).

IV. SIMULATION RESULTS

Hover flight of an autonomous helicopter equipped with the proposed autopilot and influenced by a wind disturbance is simulated. The designed controller that makes a stable hover possible can be summarized as follows,

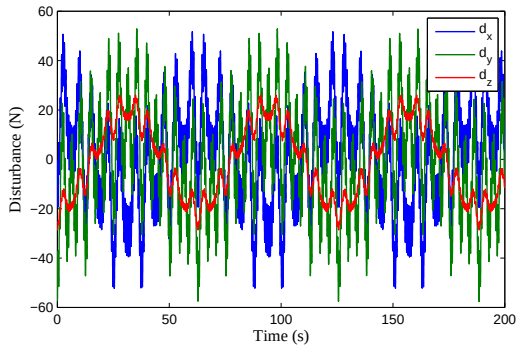

Fig. 1. Wind disturbance

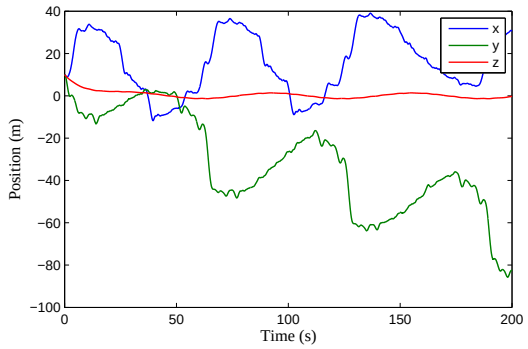

Fig. 2. Position when disturbance adaptation is turned off.

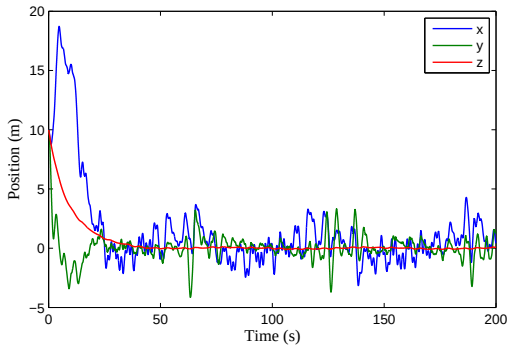

Fig. 3. Position when disturbance adaptation is turned on.

1) *Vertical dynamics stabilizer*

$$\dot{\xi}_z = (F + G\hat{\Psi})\xi_z + k_2 G(\dot{z} + k_1 z) + FGM_0\dot{z} \tag{25}$$

$$\begin{aligned} \dot{\hat{\Psi}}_2 &= \gamma \xi_{z2}^T (\dot{z} + k_1 z) \\ T_M &= \frac{gM_0 + \hat{\Psi}\xi_z + k_2(\dot{z} + k_1 z)}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)} \end{aligned} \tag{26}$$

2) *Longitudinal and lateral dynamics stabilizer*

$$\begin{aligned} \dot{\eta}_x &= x \\ \dot{\eta}_y &= y \\ \dot{\eta}_q &= q_3 \\ \dot{\xi}_x &= (F + G\hat{\Psi})\xi_x + k_4 G(\dot{x} + k_3 x) \quad (27) \\ \dot{\xi}_y &= (F + G\hat{\Psi})\xi_y + k_4 G(\dot{y} + k_3 y) \quad (28) \\ \hat{d}_x &= \hat{\Psi}\xi_x \\ \hat{d}_y &= \hat{\Psi}\xi_y \\ \mathbf{v} &= A_0(T_M)^{-1}\big(K_P(q_r - q) - K_P K_D(\omega - \omega_d) \\ &\quad - B_0(T_M)\big), \end{aligned}$$

where ω_d is given by (23) and q_r is as defined by (14), with q^* and q_d given by (15) and (17) respectively.

The simulation results presented here are based on a model of a small autonomous helicopter from [7]. To test the robustness property of the controller, parameter uncertainties are taken up to 30% of the nominal values. Even though the controller is designed based on simplified force and torque equations as described by (1) and (6) respectively, the helicopter model assumes full torque, (5) and full force equations. The wind disturbance shown in Fig. 1 is presented to the helicopter as a persistently acting external force generated by a 8-dimensional neutrally stable exosystem with $\varrho = (1, 1.5, 0.1, 10)$, $w_x(0) = (20, 1, 4, 0, -1800, 0, -0.1, -0.02)$, $w_y(0) = (10, 2, 10, 2, 1500, 0, 0.1, 0)$ and $w_z(0) = (5, 0, 1, 0, 2000, 0, 0.01, 0.01)$. To further challenge the controller, only 4-dimensional internal models (27), (28) and (25) are used. Positions of the helicopter in the face of the wind disturbance without $(\gamma = 0)$ and with disturbance adaptation are given in Fig. 2 and 3 respectively.

Without disturbance adaptation, while the controller fails to stabilize the x and y positions, z does converge fairly close to zero as could be seen in Fig. 2. Apparently, T_M is still capable of acting as a vertical stabilizer to a certain degree although the disturbance adaptation is turned off due to the presence of other terms in (26). The importance of information on the disturbance to the longitudinal/lateral stabilizer is demonstrated in Fig. 3. Now that the disturbance adaptation is turned on, z converges to zero and, x and y converge to a small neighbourhood of the origin as guaranteed by Proposition 1.

V. CONCLUSIONS AND FUTURE WORKS

A robust controller for helicopter stabilization to reject wind disturbance is presented. The wind disturbance affecting the helicopter is assumed to be a function of time of a fixed structure with unknown parameters. By designing an internal model that estimates the disturbance, a control design is carried out for longitudinal, lateral and vertical dynamics stabilization. Despite the presence of helicopter parameter and model uncertainties, simulation results clearly demonstrates the effectiveness of the control technique. As future works, indoor and outdoor flights are to be carried out to test the feasibility of the proposed controller. That gives an immediate challenge caused by the presence of servo dynamics and limitations on wind disturbance that could be handled.

APPENDIX

A. Appendix A

Defining $\tilde{\omega} := \omega + \bar{K}_D q$ where $\bar{K}_D := \frac{1}{K_D}$, the chosen control law becomes

$$\tilde{\mathbf{v}} \;=\; K_P q_r - \frac{K_P}{\bar{K}_D}\tilde{\omega} + \frac{K_P}{\bar{K}_D}\omega_d.$$

Consider a Lyapunov function candidate

$$V(q_0, \tilde{\omega}) = \frac{1-q_0}{q_0-\varepsilon} + \frac{1}{2}\tilde{\omega}^\top J\tilde{\omega}$$

defined on the open set $(\varepsilon, 1] \times \mathbb{R}^3$. Thus,

$$\begin{aligned}
\dot{V} \;=\; & -\frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2}\bar{K}_D\|q\|^2 \\
& +\tilde{\omega}^\top\Big(\frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2}I - \bar{K}_D^2 S(q)J - \frac{\bar{K}_D^2}{2}q_0 J\Big)q \\
& +\tilde{\omega}^\top\Big(\bar{K}_D S(q)J - \frac{K_P}{\bar{K}_D}L(T_M) + \frac{\bar{K}_D}{2}q_0 J \\
& +\frac{\bar{K}_D}{2}JS(q)\Big)\tilde{\omega} + K_P\tilde{\omega}^\top L(T_M)q_r \\
& +\frac{K_P}{\bar{K}_D}\tilde{\omega}^\top L(T_M)\omega_d + \tilde{\omega}^\top\Delta(T_M).
\end{aligned}$$

It is shown in [3], that a $T^* > 0$ can be chosen such that if $\psi_c^z(q(t)) = 1$ for all $t \geq T^*$, then $2l_1 I \leq L(T_M(t)) + L^T(T_M(t))$, $\|L(T_M(t))\| \leq l_2$ and $\|\Delta(T_M(t))\| \leq \delta$ for all $t > 0$. Since $\|q^*\| \leq \sqrt{3}\lambda_3$, $\|q_d\| \leq K_d m_{q_d}$ and $\|\omega_d\| \leq K_d m_{\omega_d}$,

$$\begin{aligned}
\dot{V} \;\leq\; & -\bar{K}_D\frac{(1-\varepsilon)}{2(q_0-\varepsilon)^2}\|q\|^2 + (2\bar{K}_D c_2 - \frac{K_P}{\bar{K}_D}l_1)\|\tilde{\omega}\|^2 \\
& +\Big(a_3 + \frac{3}{2}\bar{K}_D^2 c_2 + K_P l_2(\sqrt{3}\lambda_3 + K_d m_{q_d}) + \delta \\
& +\frac{K_P}{\bar{K}_D}K_d l_2 m_{\omega_d}\Big)\|\tilde{\omega}\|,
\end{aligned}$$

where $0 < c_1 \leq \|J\| \leq c_2$ and, a_1, a_2 and a_3 are as defined in [3]. To show that $\dot{V}$ can be made negative definite, it is desired to have

$$\begin{aligned}
& (2\bar{K}_D c_2 - \frac{K_P}{\bar{K}_D}l_1)\|\tilde{\omega}\|^2 + \Big(a_3 + \frac{3}{2}\bar{K}_D^2 c_2 \\
& +K_P l_2(\sqrt{3}\lambda_3 + K_d m_{q_d}) + \delta + \frac{K_P}{\bar{K}_D}K_d l_2 m_{\omega_d}\Big)\|\tilde{\omega}\| \\
& \leq -\bar{K}_D\frac{1-\varepsilon}{4}c_2\|\tilde{w}\|^2.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
& \frac{K_P}{\bar{K}_D}\big(l_1 a_1 - \sqrt{3}\lambda_3\bar{K}_D l_2 - K_d(\bar{K}_D m_{q_d} + m_{\omega_d})l_2\big) \\
\geq\; & \Big(a_3 + \frac{3}{2}\bar{K}_D^2 c_2 + \delta\Big) + \bar{K}_D c_2\Big(\frac{9-\varepsilon}{4}\Big)a_2.
\end{aligned}$$

If $\lambda_3 \leq \lambda_3^*(K_D)$, $K_d \leq K_d^*(K_D)$ and $K_P \geq K_P^*(K_D)$, where

$$\lambda_3^*(K_D) = \frac{l_1 a_1}{4\sqrt{3}\bar{K}_D l_2}, \quad K_d^*(K_D) = \frac{l_1 a_1}{4(\bar{K}_D m_{q_d} + m_{\omega_d})l_2},$$

$$K_P^*(K_D) = \frac{3c_2}{a_1 l_1}\bar{K}_D^3 + \frac{(9-\varepsilon)a_2 c_2}{2a_1 l_1}\bar{K}_D^2 + \frac{2(a_3+\delta)}{a_1 l_1}\bar{K}_D,$$

then

$$\dot{V}(q_0, \tilde{\omega}) < -\bar{K}_D\frac{1-\varepsilon}{2}\Big(\frac{\|q\|^2}{(q_0-\varepsilon)^2} + \frac{c_2}{2}\|\tilde{\omega}\|^2\Big)$$

for a fixed $\bar{K}_D \geq \bar{K}_D^*$ and all $(q_0, \tilde{\omega}) \in \mathcal{S}$, where $\bar{K}_D^*$ and $\mathcal{S}$ is a positive number and a compact set respectively as defined in [3].

B. Appendix B

Define an ISS-Lyapunov function candidate $V(\eta_1, \eta_2, \eta_3) = \eta_1^\top\eta_1 + \frac{1}{2}\eta_2^\top J\eta_2 + \eta_3^\top\eta_3$. From (19)-(24) and taking $g_{\text{st}} = k_4\tilde{G}(\dot{\zeta}_2 - K_1\sigma'(\frac{K_1}{\lambda_1}\zeta_1)\dot{\zeta}_1 + k_3\dot{\zeta}_1)$, the time derivative of $V(\eta_1, \eta_2, \eta_3)$ can be written as

$$\begin{aligned}
\dot{V} \;\leq\; & \big(\ell_1 + \delta_0(\frac{\ell_4+\ell_7}{2}+2) + \ell_5\frac{\delta_1}{2}\big)\|\eta_1\|^2 \\
& +\big(\ell_2 + \ell_4\frac{1}{2\delta_0} + \ell_6\frac{1}{2\delta_2} + \frac{\delta_3}{2}(\frac{1}{K_D}\|J\| + \ell_8 + \\
& 1)\big)\|\eta_2\|^2 + \big(\ell_3 + \ell_5\frac{1}{2\delta_1} + \delta_2(\frac{\ell_6+\ell_9}{2}+2)\big) \\
& \|\eta_3\|^2 + \big(\frac{1}{\delta_0} + \frac{1}{2K_D\delta_3}\|J\| + \frac{1}{\delta_2}\big)\|y_{\zeta 0}\|^2 \\
& +\frac{1}{2}\big(\frac{\ell_7}{\delta_0} + \frac{\ell_8}{\delta_3} + \frac{\ell_9}{\delta_2}\big)\|y_{\zeta 1}\|^2 + \frac{1}{\delta_0}\|I_{\eta 1}\|^2 \\
& +\frac{1}{2\delta_3}\|I_{\eta 2}\|^2 + \frac{1}{\delta_2}\|I_{\eta 3}\|^2, \text{ where} \\
y_{\zeta 0} \;=\; & -K_3 P_2\sigma'(\frac{K_3}{\lambda_3}\zeta_3)\big(-\frac{1}{M}\tilde{D}(t)P_2\lambda_3\sigma(\frac{K_3}{\lambda_3}\zeta_3) + \\
& K_2 P_1\sigma'(\frac{K_2}{\lambda_2}\zeta_2)\dot{\zeta}_2 + \frac{p}{M} + \frac{d}{M} - \frac{K_d}{M}L_d G_4\bar{\tau}\big), \\
y_{\zeta 1} \;=\; & \tilde{G}(\dot{\zeta}_2 - K_1\sigma'(\frac{K_1}{\lambda_1}\zeta_1)\dot{\zeta}_1 + k_3\dot{\zeta}_1), \\
I_{\eta 1} \;=\; & -K_d\Big(I - \frac{1}{2}\big(q_0 I + S(q)\big)Q_d\Big)(G_6\bar{\tau} + \tilde{D}_0^{-1}P\dot{\bar{\tau}}), \\
I_{\eta 2} \;=\; & K_d\Big(S(\omega)JQ_d + \frac{1}{K_D}J\Big(I - \frac{1}{2}\big(q_0 I + S(q)\big) \\
& Q_d\Big)\Big)(G_6\bar{\tau} + \tilde{D}_0^{-1}P\dot{\bar{\tau}}) + \Delta - J\dot{\omega}_d, \\
I_{\eta 3} \;=\; & K_d\Big(-\frac{1}{2K_D}P_3\big(q_0 I + S(q)\big) - K_d P_3\Big(I - \frac{1}{2} \\
& \big(q_0 I + S(q)\big)Q_d\Big)G_5 P_3 + K_3 P_3 Q_0 - K_d G_7 P_3\Big) \\
& P_4 e_\xi - K_d P_3\Big(I - \frac{1}{2}\big(q_0 I + S(q)\big)Q_d\Big)(G_6\bar{\tau} + \\
& \tilde{D}_0^{-1}P\dot{\bar{\tau}}) - K_d\tilde{G}\tilde{P}_2 P_6\bar{\tau} \quad \text{and} \\
Q_0(t) \;=\; & \frac{1}{M}P_2\sigma'(\frac{K_3}{\lambda_3}\zeta_3)\big(\tilde{D} + K_d L_d G_4 P_3\big), \\
Q_1(t) \;=\; & \frac{1}{M}P_2\sigma'(\frac{K_3}{\lambda_3}\zeta_3)L_d G_4, \\
L_d(t) \;=\; & I + \tilde{D}_\Delta(t)\tilde{D}_0^{-1}(t), \\
G_0(t) \;=\; & PG_7, \quad G_1(t) = \tilde{P}_2 P_6 G_7 + \dot{\tilde{P}}_2 P_6, \\
G_2(t) \;=\; & (K_d P\tilde{G}\tilde{P}_2 + \dot{\tilde{P}}_2)P_6 + \tilde{P}_2 P_6 G_7, \\
G_3(t) \;=\; & K_d P + \tilde{P}_2 P_6, \quad G_4(t) = P + \tilde{P}_2 P_6,
\end{aligned}$$

$$
\begin{aligned}
G_5(t) &= \dot{\tilde{D}}_0^{-1} G_4 + \tilde{D}_0^{-1}(K_d G_0 + G_1),\\
G_6(t) &= \dot{\tilde{D}}_0^{-1} G_4 + \tilde{D}_0^{-1} G_2, \quad G_7(t) = \tilde{F} + \tilde{G}\tilde{P}_2 P_6.
\end{aligned}
$$

Note that

$$
P_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \\ & & \vdots & & \ddots & \end{bmatrix} \text{ and } P_6 = \begin{bmatrix} P_5 & 0 \\ 0 & P_5 \end{bmatrix}
$$

is a $4N \times (4N+2)$ matrix and a $\mathbb{R}^{2N+1} \to \mathbb{R}^{2N}$ projection matrix respectively. In addition, for all $t \geq 0$, $\|Q_i(t)\| \leq m_i$ and $\|G_j(t)\| \leq g_j$ for some $m_i > 0$, $i = 0, 1$ and $g_j > 0$, $j = 0 \dots 7$ respectively. The arbitrary positive numbers δ_0, δ_1, δ_2 and δ_3 are obtained from Young's inequalities and

$$
\begin{aligned}
\ell_1 &:= -\frac{\varepsilon}{K_D} + \sqrt{3}\frac{\lambda_3}{K_D} + 2K_3 m_0 + \\
&\quad K_d\big(\frac{m_{q_d}}{K_D} + 2(1+m_{Q_d})g_5\big)\\
\ell_2 &:= -K_P K_D l_1 + (m_\omega + \frac{1}{K_D})\|J\|\\
\ell_3 &:= -\frac{\varepsilon}{K_D} + \sqrt{3}\frac{\lambda_3}{K_D} + 2K_3 m_0 + K_d\big(\frac{m_{q_d} + m_{e_\xi}}{K_D} + \\
&\quad 2K_3 m_1 + 4g_7 + 4(1+m_{Q_d})g_5\big)\\
\ell_4 &:= \Big(\frac{m_\omega}{K_D} + \frac{1}{K_D^2} + \frac{K_3}{K_D} m_0 + K_d\big(m_\omega m_{Q_d} + \\
&\quad \frac{1+m_{Q_d}}{K_D}\big)g_5\Big)\|J\| + 2\\
\ell_5 &:= 2K_d\big((1+m_{Q_d})g_5 + K_3 m_1\big)\\
\ell_6 &:= K_d\big((m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D})g_5 + \frac{K_3}{K_D} m_1\big)\|J\| + 2\\
\ell_7 &:= 2k_4 K_d (1+m_{Q_d}) m_{\tilde{D}_0} g_3\\
\ell_8 &:= k_4 K_d\big(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}\big) m_{\tilde{D}_0} g_3 \|J\|\\
\ell_9 &:= 2k_4 K_d\big(1 + (1+m_{Q_d}) m_{\tilde{D}_0} g_3\big).
\end{aligned}
$$

Set now

$$
\begin{aligned}
\bar{\lambda}_3^* &= \frac{\varepsilon}{5\sqrt{3}}, \quad \bar{K}_3^*(K_D) = \frac{\varepsilon}{10 K_D m_0},\\
\bar{K}_d^*(K_D, K_3) &= \min\big\{\bar{K}_{d1}^*(K_D, K_3), \bar{K}_{d2}^*(K_D, K_3)\big\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\bar{K}_{d1}^*(K_D, K_3) = \frac{\varepsilon}{5K_D\big(\frac{m_{q_d}}{K_D} + 3(1+m_{Q_d})g_5 + K_3 m_1\big)},\\
&\bar{K}_{d2}^*(K_D, K_3) =\\
&\frac{\varepsilon}{5K_D\big(\frac{m_{q_d} + m_{e_\xi}}{K_D} + 5(1+m_{Q_d})g_5 + 3K_3 m_1 + 4g_7\big)}.
\end{aligned}
$$

Next, fix

$$
\begin{aligned}
\delta_0 &= \frac{2\varepsilon}{5K_D(\ell_4 + \ell_7 + 4)}, \quad \delta_1 = 1,\\
\delta_2 &= \frac{2\varepsilon}{5K_D(\ell_6 + \ell_9 + 4)},\\
\delta_3 &= \frac{\ell_1 K_P K_D}{\frac{1}{K_D}\|J\| + \ell_8 + 1}.
\end{aligned}
$$

Moreover let,

$$
\begin{aligned}
\ell_4^*(K_D) &:= \Big(\frac{m_\omega}{K_D} + \frac{1}{K_D^2} + \frac{\bar{K}_3^*}{K_D} m_0 + \bar{K}_d^*\big(m_\omega m_{Q_d}\\
&\quad + \frac{1+m_{Q_d}}{K_D}\big)g_5\Big)\|J\| + 2\\
\ell_6^*(K_D) &:= \bar{K}_d^*\big((m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D})g_5 + \frac{\bar{K}_3^*}{K_D} m_1\big)\\
&\quad \|J\| + 2\\
\ell_7^*(k_4) &:= 2k_4 \bar{K}_d^*(1+m_{Q_d}) m_{\tilde{D}_0} g_3\\
\ell_8^*(k_4, K_D) &:= k_4 \bar{K}_d^*\big(m_\omega m_{Q_d} + \frac{1+m_{Q_d}}{K_D}\big) m_{\tilde{D}_0} g_3 \|J\|\\
\ell_9^*(k_4) &:= 2k_4 \bar{K}_d^*\big(1 + (1+m_{Q_d}) m_{\tilde{D}_0} g_3\big)\\
\delta_0^*(k_4, K_D) &:= \frac{2\varepsilon}{5K_D(\ell_4^* + \ell_7^* + 4)}\\
\delta_2^*(k_4, K_D) &:= \frac{2\varepsilon}{5K_D(\ell_6^* + \ell_9^* + 4)}\\
\delta_3^*(k_4, K_D) &:= \frac{\ell_1 K_P K_D}{\frac{1}{K_D}\|J\| + \ell_8^* + 1}.
\end{aligned}
$$

It can shown that there exist $\bar{K}_P^* > 0$ such that for all $K_P \geq \bar{K}_P^* \frac{1}{K_D}$ and for all $\lambda_3 \leq \bar{\lambda}_3^*$, $K_3 \leq \bar{K}_3^*$ and $K_d \leq \bar{K}_d^*$,

$$
\begin{aligned}
\dot{V} \leq\ & -\frac{\varepsilon}{5K_D}\big(\|\eta_1\|^2 + \|\eta_2\|^2 + \|\eta_3\|^2\big) + \big(\frac{1}{\delta_0^*} + \frac{1}{2K_D\delta_3^*}\\
&\|J\| + \frac{1}{\delta_2^*}\big)\|y_{\zeta 0}\|^2 + \frac{1}{2}\big(\frac{\ell_7^*}{\delta_0^*} + \frac{\ell_8^*}{\delta_3^*} + \frac{\ell_9^*}{\delta_2^*}\big)\|y_{\zeta 1}\|^2 +\\
&\frac{1}{\delta_0^*}\|I_{\eta 1}\|^2 + \frac{1}{2\delta_3^*}\|I_{\eta 2}\|^2 + \frac{1}{\delta_2^*}\|I_{\eta 3}\|^2.
\end{aligned}
$$

Hence, subsystem (24) is ISS (see for instance, [2]).

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