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Limitations of nonlinear stabilization over erasure channels

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Abstract—
In this paper, we study the problem of control of nonlinear systems over an erasure channel. The stability and performance metric are adopted from the ergodic theory of random dynamical systems to study the almost sure and second moment stabilization problem. The main result of this paper proves that, while there are no limitations for the almost sure stabilization, fundamental limitations arise for the second moment stabilization. In particular, we provide a necessary condition for the second moment stabilization of multi-state single input nonlinear systems expressed in terms of the probability of erasure and positive Lyapunov exponents of the open loop unstable system. The dependence of the limitation result on the Lyapunov exponents highlights, for the first time, the important role played by the global non-equilibrium dynamics of the nonlinear systems in obtaining the performance limitation. This result generalizes the existing results for the stabilization of linear time invariant systems over erasure channels and differs from the existing Bode-like fundamental limitation results for nonlinear systems, which are expressed in terms of the eigenvalues of the linearization.

I. INTRODUCTION

Networked controlled systems have been the focus of much research in recent years [1]. Among several relevant questions, one main problem is to characterize the limitations induced on closed loop stability and performance caused by the presence of unreliable communication channel(s) in the loop. The work of [2] has been the first to address noisy channel models and to show that different notions of closed loop stability (bounded moments) required a new notion of reliable communication of bits, and led to a new notion of capacity, the Anytime capacity. In particular, fading has been shown to affect the Anytime capacity of the channel and therefore its ability to stabilize the closed loop system. A fading channel model popular in the literature is the analog erasure channel, which is simply modeled as an on/off Bernoulli switch, and can be used as a simple model of a packet drop link with neglegible quantization effects. Most of the research on the estimation and control of network systems over noisy channels has considered LTI plants [3], [4]. There are no results that address this problem for nonlinear systems. The existing results for nonlinear systems without channel noise uncertainty have essentially reverted to the local linear analysis involving the eigenvalues of the linearized system [5], [6]. The problem of characterizing the limitations in the stabilization and estimation of network systems with nonlinear components over noisy, uncertain, channels is a timely research topic given the important role that nonlinear dynamics play in the applications such as network power systems and biological networks.

In this paper, we study the problem of characterizing the fundamental limitations in the stabilization of a nonlinear system which is controlled via an on/off fading channel with IID Bernoulli fading. The objective is to characterize the quality of service of the channel, in term of probability of successful transmission, to guarantee a certain stability notion of the closed loop system. We adopt the stability and performance metric, of almost sure and second moment stability, from the ergodic theory of random dynamical systems. The main result of this paper proves that, while there are no limitations for the almost sure stabilization, a fundamental limitation arises for the second moment stabilization. The limitation is expressed in terms of the probability of erasure and the positive Lyapunov exponents of the open loop unstable system. The dependence of the limitation result on the Lyapunov exponents highlights, for the first time, the important role played by the global non-equilibrium dynamics of the nonlinear systems in obtaining the performance limitation.

The paper presents two important innovations: 1) Extends the framework of random dynamical systems [7], [8] to controlled dynamical systems. 2) Connects the stability requirement with the Quality of Service of the channel and the positive Lyapunov exponents of the open loop system. In this sense, the results of the paper generalize those of [3], and the Lyapunov exponents therefore emerge as natural generalization of the linear system eigenvalues in capturing the limitations of nonlinear networked systems.

The organization of the paper is as follows. In section II, we present some preliminaries and definition from the theory of random dynamical systems. In section III, we present the main results on the almost sure and second moment exponential stabilization of the random control dynamical system. Simulation results are presented in section IV followed by conclusion in section V.

II. PRELIMINARIES AND DEFINITIONS

The set-up for the problem of stabilization of nonlinear systems over analog erasure channel is described by the following equation

$$x_{n+1} = f(x_n) + \xi_n b u_n, \tag{1}$$

where $x_n \in X \subset \mathbb{R}^N$ ia a compact state space, $u_n \in U \subset \mathbb{R}$ is a control input and $\{\xi_n\}_{n=0}^{\infty} \in \{0,1\}$ is a sequence of random

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variables assumed to be independent identically distributed (i.i.d.) with the following probability distribution

$$Prob(\xi_n = 1) = p, \quad Prob(\xi_n = 0) = 1 - p. \tag{2}$$

Throughout this paper, we assume that $(1-p)$ is bounded away from zero. The random variables $\{\xi_n\}_{n=0}^{\infty}$ model the erasure channel between the plant and the controller. We now make following assumptions on the system dynamics.

Assumption 1: The system mapping $f : X \to X$ is assumed to be smooth and the Jacobian $\frac{\partial f}{\partial x}(x)$ assumed to be invertible for almost all (w.r.t. Lebesgue measure) $x \in X$.

Assumption 2: The pair $\{f, b\}$ is assumed to be feedback linearizable i.e., there exists a feedback control input $\gamma : X \times U \to X$ and a coordinate transformation $\tilde{x}$ for X and $\tilde{u}$ for U (possibly locally defined) such that

$$f(x) + b\gamma(\tilde{x}, \tilde{u}) = A\tilde{x} + B\tilde{u}$$

where the matrix A and B are constant and in controllable canonical form.

The problem of feedback linearization for discrete time systems is extensively studied in the nonlinear control system literature. We refer the interested readers to references [9], [10], providing necessary and sufficient conditions for feedback linearization.

Assumption 3: We assume that there exists a $\beta > 1$ such that $\| \frac{\partial f(x)}{\partial x} v \| > \beta \| v \|$ for all $x \in X$ and $v \in T_x X$ (the tangent space of X at x). This assumption has following consequences on the system dynamics:

Cons.1. Let $\Lambda_{exp}^1, \ldots, \Lambda_{exp}^N$ be the Lyapunov exponents of the open loop system, then all the Lyapunov exponents are positive (Refer to definition 7 for Lyapunov exponents).

Cons.2. The open loop system has an unique ergodic invariant measure which is equivalent to Lebesgue (Refer to definition 6 for ergodic invariant measure) .

Cons.1 follows from the definition of Lyapunov exponents and for the proof of the Cons.2 refer to [11] (Theorem 1.3 Chapter III).

Assumption 4: We assume that the control input u_n for (1) is state feedback and deterministic (i.e., $u_n = k(x_n)$).

Few comments on the assumptions are necessary. While assumptions 1 and 2 are technical in nature, assumption 3 has important consequences on the system dynamics. Assumption 3 is analogous to the assumption made in the case of linear time invariant (LTI) systems that all the eigenvalues of system matrix are unstable. In the case of LTI systems, one can always separate the system dynamics into stable and unstable part by appropriate coordinate transformation. Similarly in the case of nonlinear system it should be possible to separate the stable and unstable part along the stable and unstable manifolds by performing appropriate change of coordinate [12]. However the mathematical sophistication that is required to do so is outside the scope of this paper and will be the topic of future research. Hence, for the simplicity of presentation of the main message of this paper, we make the assumption 3.

Remark 5: In the proof of the main result of this paper, we only require that Cons.1 and Cons.2 to be true. The assumption 3 is sufficient for Cons.1 and Cons.2 to be true but not necessary.

We next discuss some preliminaries from ergodic theory of deterministic dynamical systems. For more details on the preliminaries, the interested readers can refer to [13], [14], [15]. Consider a discrete time deterministic dynamical system of the form:

$$x_{n+1} = T(x_n) \tag{3}$$

where $x_n \in X \subset \mathbb{R}^N$ a compact set, $T : X \to X$ is assumed to be at least C^1 function of x and its Jacobian $\frac{\partial T}{\partial x}(x)$ is invertible for almost all w.r.t. Lebesgue measure $x \in X$. We denote by $\mathscr{M}(X)$ and $\mathscr{B}(X)$ the space of probability measures on X and the Borel $\sigma-$ algebra of sets on X respectively.

Definition 6 (Ergodic invariant measure): A probability measure $\mu \in \mathscr{M}(X)$ is said to be invariant for (3) if $\mu(B) = \mu(T^{-1}(B))$ for all sets $B \in \mathscr{B}(X)$. An invariant probability measure is said to be ergodic if any T-invariant set A i.e., $T^{-1}(A) = A$ has μ measure equal to one or zero.

Definition 7 (Lyapunov exonents): Consider the deterministic dynamical system (3) and let

$$L(x) = \lim_{n\to\infty} \left[\left(\prod_{k=0}^{n} \frac{\partial T}{\partial x}(x_k) \right)' \left(\prod_{k=0}^{n} \frac{\partial T}{\partial x}(x_k) \right) \right]^{\frac{1}{2n}}, \; x_0 = x \tag{4}$$

If λ_{exp}^i are the eigenvalues of $L(x_0)$ then the Lyapunov exponents Λ_{exp}^i are given by $\Lambda_{exp}^i = \log \lambda_{exp}^i$ for $i = 1, \ldots, N$. Furthermore if $L(x) \neq 0$ then

$$\lim_{n\to\infty} \frac{1}{n} \log |\det \left(\prod_{k=0}^{n} \frac{\partial T}{\partial x}(x_k) \right)| = \log \prod_{k=1}^{N} \lambda_{exp}^k(x). \tag{5}$$

Remark 8: The condition for the existence of limit in (4) is given by the Oseledec Multiplicative ergodic theorem [16]. These conditions are satisfied by the uncontrolled system (1) in the form of Assumption 1. Similarly the limit in (4) is independent of the initial condition x_0 if the system has unique ergodic invariant measure. Hence for the uncontrolled system (1) the Lyapunov exponents are independent of initial conditions due to Assumption 3. For the proof of equality (5), we refer the readers to [17] Proposition 1.3 and Theorem 1.6.

We now introduce stochastic notions of stabilities for a general random dynamical system of the form:

$$x_{n+1} = T(x_n, \gamma_n) \tag{6}$$

where $x \in X \subset \mathbb{R}^N$ is a compact set, $\gamma_n \in W$. The mapping $T : X \times W \to X$ is assume to be smooth with respect to $x \in X$ for every fixed $\gamma \in W$ and is measurable in γ for every fixed x. The γ_n is a sequence of i.i.d random variable taking finitely many values in the set $W = \{w_1, \ldots, w_r\}$ with following probability distribution

$$Prob(\gamma_n = w_k) = p_k, \quad k = 1, \ldots, r, \quad \forall n$$

Define a probability measure ρ on W by $\rho(\{w_k\}) = p_k$. Let $\Omega = \prod_{i=0}^{\infty} W_i$, $W_i = W$, and define the probability measure $\mathbb{P}$ on Ω by $\mathbb{P} = \prod_{i=0}^{\infty} \rho$.

The stochastic notions of stability that we use in this paper are derived from the linear derivative map associated with the RDS (6). The linear derivative map is defined as follows:

$$\eta_{n+1} = \frac{\partial T}{\partial x}(x_n, \gamma_n)\eta_n \tag{7}$$

where $\eta_n \in \mathbb{R}^N$ and $\frac{\partial T}{\partial x}(x,\gamma) : T_x X \to T_{T(x,\gamma)}$ is a Jacobian of T, mapping vectors from the tangent space at point $x \in X$ to the tangent space at point $T(x,\gamma)$. The linear derivative map has been used for incremental stability analysis of nonlinear systems. We use the linear derivative map to define two different notions of stochastic stabilities for random dynamical system (6) [8].

Definition 9 (Almost sure exponentially stable): The random dynamical system (6) is said to be almost sure exponentially stable if

$$\lim_{n\to\infty} \frac{1}{n} \log \parallel \eta_n \parallel < 0 \tag{8}$$

for almost all, w.r.t. Lebesgue measure, $x_0 \in X$ with $\parallel \eta_0 \parallel = 1$ and almost all, w.r.t. measure $\mathbb{P}$, random sequence $\{\gamma_n\}_{n=0}^{\infty} \in \Omega$.

Definition 10 (Second moment exponential stable): The random dynamical system (6) is said to be second moment exponential stable if there exists positive constants $K < \infty$ and $\beta < 1$ such that

$$E_{\xi_0^n}\left[\parallel \eta_{n+1} \parallel^2\right] < K\beta^n, \quad \forall n \geq 0 \tag{9}$$

for almost all w.r.t. Lebesgue measure $x_0 \in X$ with $\parallel \eta_0 \parallel = 1$ and where $\xi_0^n = \{\xi_0, \dots, \xi_n\}$.

The second moment exponential stability is stronger notion of stability than almost sure exponential stability.

III. MAIN RESULTS

The main results of this paper can be summarized as follows: *We prove that for almost sure stability (Definition 8) of the random dynamical system (1) we do not need any minimum quality of service i.e., almost sure stability is guaranteed for any non-zero probability of non-erasure. However for second moment stabilization (Definition 10) we require some minimum quality of service.* The following theorem is the first main result of this paper and the proof of it is omitted due to space constraints. The proof utilizes the fact that the system satisfies assumption 2.

Theorem 11: There exists a choice of feedback control input $u_n = k(x_n)$ such that the feedback control random dynamical system (1) is almost sure stable.

Now, we state the second main result of this paper on the second moment exponential stability of system (1).

Theorem 12: The necessary condition for the second moment exponential stability (Definition 10) of (1) is given by

$$(1-p)\left(\prod_{k=1}^{N} \lambda_{\exp}^{k}\right)^2 < 1 \tag{10}$$

where $\lambda_{exp}^k = \exp^{\Lambda_{exp}^k}$ and $\Lambda_{exp}^k > 0$ is the k^{th} positive Lyapunov exponent of the uncontrolled system $x_{n+1} = f(x_n)$ and $(1-p)$ is the probability of erasure i.e., $Prob(\xi_n = 0) = 1-p$.

We postpone the proof of this theorem till the end of this paper. Using the assumption 4 we write the feedback control random dynamical system (1) along with its derivative map as follows:

$$\begin{aligned} x_{n+1} &= f(x_n) + \xi_n b k(x_n) \\ \eta_{n+1} &= \left(\frac{\partial f}{\partial x}(x_n) + \xi_n b \frac{\partial k}{\partial x}(x_n)\right)\eta_n \end{aligned} \tag{11}$$

Given the complicated nature of the proof, we now outline some of the main steps that are involved in the proof of the Theorem 12.

1) We first define a relaxation of system equations (11). Doing so we derive a necessary condition for the second moment stability of relaxed system thereby providing necessary condition for the second moment stability of the actual system.

2) The necessary condition for the second moment stability of the relaxed system (Lemma 14) is based on Lyapunov analysis.

3) The Lyapunov function is used for deriving the optimal control for minimizing the second moment of the relaxed system (Lemma 16).

4) Finally we prove that the necessary condition for the second moment stability of the relaxed system is given by (10) (Lemma 17).

The relaxation of system equations (11) is motivated from the definition of the second moment exponential stability, (Definition 10) expressed in terms of the tangent space dynamics η. The relaxation of system equation (11) is defined as follows:

$$x_{n+1} = f(x_n) + \gamma_n b k(x_n) \tag{12a}$$

$$\eta_{n+1} = \left(\frac{\partial f}{\partial x}(x_n) + \xi_n b v(x_n)\right)\eta_n \tag{12b}$$

The system equations (12) are the relaxation of system equations (11) in the following sense.

1) The random variable $\gamma_n \in \{0,1\}$ appearing in the state space dynamics (12a) is assumed to be independent of $\xi_n \in \{0,1\}$. $\{\gamma_n\}_{n=0}^{\infty}$ is a sequence of i.i.d. random variables.

2) The input $v(x_n)$ applied to the tangent space dynamics is not constraint to be the derivative of $k(x_n)$, the input to the state space dynamics.

It is convenient to write the relaxed system as follows:

$$\begin{aligned} x_{n+1} &= f(x_n) + \gamma_n b k(x_n) =: F(x_n, \gamma_n) \\ \eta_{n+1} &= \left(\frac{\partial f}{\partial x} + \xi_n b v\right)(x_n)\eta_n =: \mathscr{A}(x_n, \xi_n)\eta_n \end{aligned} \tag{13}$$

The second moment exponential stability of (13) is defined as follows:

Definition 13 (Second moment exponential stability): The random dynamical system (13) is said to be second moment exponential stable if there exists positive constants $K < \infty$ and $\beta < 1$ such that

$$E_{[\xi_0^n, \gamma_0^n]}\left[\parallel \eta_{n+1} \parallel^2\right] < K\beta^n, \quad \forall n \geq 0 \tag{14}$$

for almost all w.r.t. Lebesgue initial conditions $x_0 \in X$ and $\| \eta_0 \| = 1$.

We now provide Lyapunov function-based characterization for the second moment stability of the relaxed system (13).

Lemma 14: If the system (13) is second moment exponentially stable (Definition 14) then there exists a matrix function of x, $P(x)$ such that $\alpha_1 I \le P(x) \le \alpha_2 I$ for some positive constants α_1, α_2 and satisfies

$$E_{[\xi_\ell, \gamma_\ell]}\left[\mathscr{A}'(x_\ell, \xi_\ell)P(x_{\ell+1})\mathscr{A}(x_\ell, \xi_\ell)\right] < P(x_\ell) \tag{15}$$

Proof: Define

$$P(x_\ell) := \sum_{n=\ell}^{\infty} E_{[\xi_\ell^n, \gamma_\ell^{n-1}]}\left[\left(\prod_{k=\ell}^{n}\mathscr{A}(x_k, \xi_k)\right)'\left(\prod_{k=\ell}^{n}\mathscr{A}(x_k, \xi_k)\right)\right] \tag{16}$$

With the above definition of $P(x_\ell)$ and using the fact $E_{\xi_n}[(\mathscr{A}(x_k, \xi_k))'\mathscr{A}(x_k, \xi_k)] > 0$, since $\mathscr{A}(x, \xi = 0) = \frac{\partial f}{\partial x}$ is assumed to be invertible (Assumption 1), it follows that $P(x_\ell)$ satisfies the inequality (15). Since the relaxed system is assumed to be second moment stable, there exists a constant $0 < K < \infty$ and $\lambda < 0$ such that

$$E_{[\xi_0^\ell, \gamma_0^{\ell-1}]}\left[\| \mathscr{A}(x_\ell, \xi_\ell)\cdots\mathscr{A}(x_0, \xi_0) \|^2\right] \le K e^{\lambda \ell} \quad \text{hence}$$

$$\| P(x) \| \le \frac{K}{1-e^{\lambda}}, \quad \text{or} \quad P(x) \le \alpha_2 I$$

The lower bound on $P(x)$ and the existence of α_1 follows from (15) and the assumption that $\mathscr{A}(x, \xi = 0)$ is invertible. ■

Definition 15 (Matrix Lyapunov function): We refer to the matrix function $P(x)$ satisfying the necessary condition (15) of Theorem 14 as matrix Lyapunov function.

We now use the matrix Lyapunov function to derive the optimal control that minimize the second moment for the relaxed system.

Lemma 16: Consider the second moment stabilization problem for the relaxed system (12). The optimal control input for the tangent space dynamics that minimize the second moment is given by

$$v(x_n) = -\frac{b'\bar{Q}(x_{n+1})A(x_n)}{b'\bar{Q}(x_{n+1})b}$$

where $A(x_n) := \frac{\partial f}{\partial x}(x_n)$ and $\bar{Q}(x_{n+1}) := E_{\gamma_n}[Q(x_{n+1})]$. The matrix function $\bar{Q}(x)$ can be obtained as the solution to the following Riccati equation corresponding to the minimum energy optimal control problem on the tangent space (Definition 18 and Theorem 19 in Appendix).

$$A'(x_n)\bar{Q}(x_{n+1})A(x_n) - \frac{A'(x_n)\bar{Q}(x_{n+1})bb'\bar{Q}(x_{n+1})A(x_n)}{\Delta + b'\bar{Q}(x_{n+1})b} = Q(x_n) \tag{17}$$

where $\Delta > 0$ is some constant. Furthermore the matrix function $\bar{Q}(x)$ will qualify as the valid Lyapunov function (Definition 15) provided

$$(1-p)\left(1 + b'\bar{Q}_0(x_{n+1})b\right) < 1,$$

where $Q_0(x_n) = \frac{Q(x_n)}{\Delta}$ satisfies (17) with Δ replace with one.

Proof: Let $P(x_n)$ be the matrix Lyapunov function satisfying the condition of the Theorem 14. To derive the optimal control for the tangent space dynamics, we write the control Lyapunov inequality as follows:

$$E_{[\xi_n, \gamma_n]}\left[(A(x_n) + \xi_n b v(x_n))' P(x_{n+1})(A(x_n) + \xi_n b v(x_n))\right] < P(x_n) \tag{18}$$

Taking expectation w.r.t. (ξ_n, γ_n) and using the fact that x_n is independent of γ_n and minimizing w.r.t. v, we get following expression for the optimal control $v(x_n)$

$$v(x_n) = -\frac{b'\bar{P}(x_{n+1})A(x_n)}{b'\bar{P}(x_{n+1})b}, \tag{19}$$

where $\bar{P}(x_{n+1}) := E_{\gamma_n}[P(x_{n+1})]$. Substituting (19) in (18), we get

$$A'(x_n)\bar{P}(x_{n+1})A(x_n) - \frac{A'(x_n)\bar{P}bb'\bar{P}A(x_n)}{\Delta_P + b'\bar{P}(x_{n+1})b} < P(x_n),$$

where $\Delta_P := b'\bar{P}(x_{n+1})b\frac{(1-p)}{p}$ and $\bar{P} = \bar{P}(x_{n+1})$. Since $P(x_{n+1})$ is the matrix Lyapunov function and hence bounded from below, we know that there exists some constant $\Delta > 0$ such that $\Delta_P \ge \Delta$. The above inequality necessarily implies

$$A'(x_n)\bar{P}(x_{n+1})A(x_n) - \frac{A'(x_n)\bar{P}bb'\bar{P}A(x_n)}{\Delta + b'\bar{P}(x_{n+1})b} < P(x_n)$$

and hence existence of matrix $R(x) \ge 0$ such that

$$A'(x_n)\bar{P}(x_{n+1})A(x_n) - \frac{A'(x_n)\bar{P}(x_{n+1})bb'\bar{P}(x_{n+1})A(x_n)}{\Delta + b'\bar{P}(x_{n+1})b} + R(x_n) = P(x_n)$$

The above equation resembles the Riccati like equation obtained from the optimal control problem on the tangent space (Definition 18 and Theorem 19 in Appendix). Hence one can show that there exists a matrix $Q(x) \le P(x)$ such that $Q(x)$ satisfies

$$A'(x_n)\bar{Q}(x_{n+1})A(x_n) - \frac{A'(x_n)\bar{Q}(x_{n+1})bb'\bar{Q}(x_{n+1})A(x_n)}{\Delta + b'\bar{Q}(x_{n+1})b} = Q(x_n) \tag{20}$$

Now if $b'\bar{Q}(x_{n+1})b\frac{(1-p)}{p} < \Delta$, then it is easy to see that $Q(x_n)$ satisfy

$$A'(x_n)\bar{Q}(x_{n+1})A(x_n) - p\frac{A'(x_n)\bar{Q}(x_{n+1})bb'\bar{Q}(x_{n+1})A(x_n)}{b'\bar{Q}(x_{n+1})b} < Q(x_n)$$

The condition $b'\bar{Q}(x_{n+1})b\frac{(1-p)}{p} < \Delta$ can be simplified to give

$$(1-p)(1 + b'\bar{Q}_0(x_{n+1}))b) < 1$$

where $Q(x) = \Delta Q_0(x)$ and matrix $Q_0(x)$ satisfies (20) with Δ replaced with one. ■

We now prove the result for the second moment exponential stability of the relaxed system (12).

Lemma 17: The necessary condition for the second moment exponential stability of the relaxed system (12) is given by

$$(1-p)\prod_{k=1}^{N}\left(\lambda_{\exp}^{k}\right)^2 < 1 \tag{21}$$

where $\lambda_{exp}^k = \exp^{\Lambda_{exp}^k}$ and $\Lambda_{exp}^k > 0$ is the k^{th} positive Lyapunov exponent of the uncontrolled system $x_{n+1} = f(x_n)$.

Proof: From Lemma 16, we know that the optimal solution $Q(x)$ of the Riccati equation will qualify as a Lyapunov function if

$$(1-p)\left(1+b^{'}\bar{Q}_0(x_{n+1})b\right) < 1 \tag{22}$$

where $Q_0(x)$ satisfies (20) with Δ replaced with one i.e.,

$$A^{'}(x_n)\bar{Q}_0(x_{n+1})A(x_n) - \frac{A^{'}(x_n)\bar{Q}_0(x_{n+1})bb^{'}\bar{Q}_0(x_{n+1})A(x_n)}{1+b^{'}\bar{Q}_0(x_{n+1})b} = Q_0(x_n) \tag{23}$$

Applying the determinant formula $\det(X+ac^{'}) = \det(X)(1+c^{'}X^{-1}a)$, where a and c are column vectors, to $A_{cl}(x_n) = A(x_n) - \frac{bb^{'}\bar{Q}_0(x_{n+1})A(x_n)}{1+b^{'}\bar{Q}_0(x_{n+1})b}$, we get

$$1+b^{'}\bar{Q}_0(x_{n+1})b = \det(A(x_n))\det(A_{cl}(x_n))^{-1} \tag{24}$$

Furthermore the Riccati equation (23) can be written in terms of A_{cl} as

$$A^{'}(x_n)\bar{Q}_0(x_{n+1})A_{cl}(x_n) = Q_0(x_n) \tag{25}$$

Hence the inequality (22) using (24) and (25) can be written as

$$(1-p)\det(A^2(x_n))\det(\bar{Q}_0(x_{n+1})\det(Q_0(x_n))^{-1} < 1 \tag{26}$$

where $\bar{Q}_0(x_{n+1}) = E_{\gamma_n}[Q_0(x_{n+1})]$. Now $E_{\gamma_n}[Q_0(x_{n+1})] \geq Q_0(\tilde{x}_{n+1})$, where $\tilde{x}_{n+1} = f(x_n)$ or $\tilde{x}_{n+1} = f(x_n) + bk(x_n)$. Hence we get following necessary condition for (26) to be true

$$(1-p)\det(A^2(x_n))\det(Q_0(\tilde{x}_{n+1})\det(Q_0(x_n))^{-1} < 1 \tag{27}$$

Let $x_n = f^n(x_0) := f \circ \ldots \circ f(x_0)$ (f composed n times). Since the inequality (27) needs to hold true at all points $x \in X$, computing the above inequality along the uncontrolled trajectory $x_n = f^n(x_0)$ we get

$$(1-p)^n \prod_{k=0}^{n-1} \det(A^2(f^k(x_0)))\det(Q_0(\tilde{x}_{k+1})Q_0(f^k(x_0))^{-1}) < 1$$

Taking log and average over n, we get

$$\frac{1}{n}\log q^n \prod_{k=0}^{n-1} \det(A^2(f^k(x_0)))\det(Q_0(\tilde{x}_{k+1})Q_0(f^k(x_0))^{-1}) < 0 \tag{28}$$

where $q := 1-p$ and $\tilde{x}_{k+1}$ is equal to $f(f^k(x_0)) + bk(f^k(x_0))$ or $f(f^k(x_0))$. Now we make use of the ergodic property of the uncontrolled system $x_{n+1} = f(x_n)$ to compute the quantity in the left hand side of the above inequality in the limit as $n \to \infty$. Using assumption 3, we know that system has a unique ergodic invariant measure μ which is equivalent to Lebesgue. Ergodictiy with respect to Lebesgue measure has the consequence that any positive Lebesgue measure set can be evolved forward to intersect any other positive measure set with the intersection having positive measure as well. Now consider any point $\tilde{x}_{k+1}$ in (28), by ergodicity we know that there exists an integer m such that $\| \tilde{x}_{k+1} - f^m(x_0) \| < \varepsilon$ and hence $|\det Q(\tilde{x}_{k+1})Q(f^m(x_0))^{-1})| \leq 1 \pm \delta$ with δ and ε arbitrary small. Hence for large enough n, we get from (28)

$$\frac{1}{n}\log\left[(1-p)^n(1\pm\delta)^n \prod_{k=0}^{n-1} \det(A^2(f^k(x_0)))\right] < 0$$

Since δ here is arbitrary, we get following necessary condition using the result from Definition 7 and Remark 8

$$(1-p)\prod_{k=1}^{N}\left(\lambda_{exp}^k\right)^2 < 1$$

This gives us the required necessary condition (21) for the second moment stability of the relaxed system. ■

We are now ready to prove the main result of this paper on second moment stability of system (1).

Proof: [Proof of Theorem 12] Since system equations (12) are relaxation of the actual system equations (11). The necessary condition for the second moment exponential stability of the relaxed system will form the necessary condition for the actual system. ■

IV. Examples

We consider a two dimensional piecewise linear map on $X = [-0.5, 0.5] \times [0,1]$, where the line $(-0.5, y)$ is identified with $(0.5, y)$ and similarly the line $(x, 0)$ is identified with $(x, 1)$. The system is described by following equation.

$$z_{n+1} = \begin{cases} A_1 z_n + B(u_n + w_n) & \text{for} \quad 0 \leq x_n \leq 0.5 \\ A_2 z_n + B(u_n + w_n) & \text{for} \quad -0.5 \leq x_n \leq 0 \end{cases} \tag{29}$$

where $z_n = (x_n, y_n)'$, $w_n \in [0, 0.01]$ is an uniform random variable, and

$$A_1 = \begin{pmatrix} 0.5 & 2 \\ -2 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -0.5 & 2 \\ -2 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

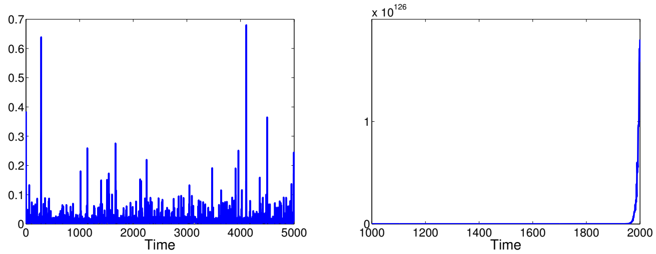
The Lyapunov exponents for the map are computed to be equal to $\Lambda_{exp}^1 = 0.6975$ and $\Lambda_{exp}^2 = 0.6965$. Since the map is expanding it satisfies the assumption 1 of the main theorem. The matrix A_1 and A_2 are chosen such that the system is continuous at the boundaries and along the lines where the phase space is identified. Based on the Lyapunov exponents, the critical transition probability is computed to be equal to $p^* = 1 - \frac{1}{\exp^{2(\Lambda_{exp}^1 + \Lambda_{exp}^2)}} = 0.93$. The feedback control gains K_1 and K_2 are chosen such that closed loop system with no erasure is stable.

$$K_1 = [1.2003 \quad 1.5645], \ K_2 = [0.3518 \quad 1.4791]$$

From Fig.1 and Fig. 2, we see that for value of non-erasure probability below critical value of p^*, the linearized state variance grows unbounded while for non-erasure probability above p^* the linearized state variance is bounded.

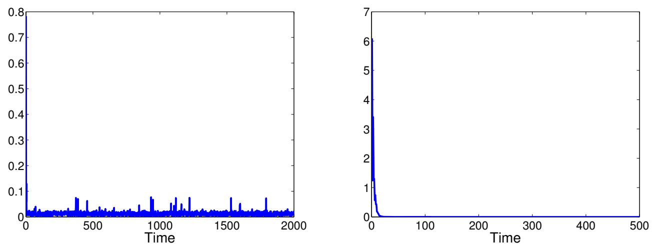
V. Conclusion

In this paper, we have considered the stabilization of a class of nonlinear systems by state feedback controller when the actuation command may be lost on a communication link with certain probability. We have adopted the notions of stability from the random theory of dynamical systems.



(a) (b)

Fig. 1. Plots for non-erasure probability of $p = 0.92$ below p^* (a) State trajectory; (b) Linearized state variance.



(a) (b)

Fig. 2. Plots for non-erasure probability of $p = 0.97$ above p^* (a) State trajectory; (b) Linearized state variance.

We have shown that the Laypunov exponents of the open loop plant characterize the stabilization limitations and the required quality of service of the communicaion link. Our preliminary results are quite encouraging and our novel approach, based on random dynamical system theory, is amenable to various extensions. One important feature of our main result is its global nature away from the equilibrium and the emergence of the open loop Lyapunov exponents as the natural generalization of the linear system eigenvalues in capturing the system limitations.

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VII. Appendix

Definition 18 (Optimal control on tangent space): Consider the system

$$x_{n+1} = f(x_n) + \gamma_n b k(x_n), \qquad \eta_{n+1} = \frac{\partial f}{\partial x}(x_n)\eta_n + b v_n \quad (30)$$

where $\{\gamma_n\}_{n=0}^{\infty} \in \{0,1\}$ be a sequence of i.i.d random variables. Let $x_0, x_1, \cdots$ be the trajectory of the state space dynamics and η_0 be the initial state for the tangent space dynamics. Consider the following infinite horizon cost criteria

$$\vartheta(x,\eta) = \min_{v_0, v_1, \cdots} \sum_{n=0}^{\infty} E_{\gamma_0^n}\left[\eta_n^{'} R(x_n)\eta_n + \frac{\Delta}{2} v_n^2\right] \quad (31)$$

where $R(x) \geq 0$ is a positive semi definite matrix, $\Delta > 0$ is some constant and $x = x_0$ and $\eta = \eta_0$. The objective is to find the sequence of control input $v_0, v_1, \cdots$ for the tangent space dynamics such that the cost function (31) is minimized.

Theorem 19: Consider the optimal control problem for the tangent space dynamics as defined in Definition 18, the optimal control input v_n^* for the tangent space dynamics that minimizes the infinite horizon cost criterion (31) is of the form: $v_n^* = -\frac{b^{'}\bar{P}(x_{n+1})A(x_n)}{\Delta + b^{'}\bar{P}(x_{n+1})b}\eta_n$ where $\bar{P}(x_{n+1}) := E_{\gamma_n}[P(x_{n+1})]$ and satisfies following Riccati like matrix equation

$$P(x_n) = A^{'}(x_n)\bar{P}(x_{n+1})A(x_n) - \frac{A^{'}(x_n)\bar{P}(x_{n+1})bb^{'}\bar{P}(x_{n+1})A(x_n)}{\Delta + b^{'}\bar{P}(x_{n+1})b} + R(x_n)$$

The optimal cost function is quadratic and is of the form $\vartheta^*(x,\eta) = \eta^{'} P(x)\eta$

The proof of this theorem follows along the lines of proof for the optimal control of linear time varying system. We omit the proof of this theorem due to space constraints.