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Opinion dynamics for agents with opinion-dependent connections

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Opinion dynamics for agents with opinion-dependent connections

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Abstract— We study a simple continuous-time multi-agent system related to Krause's model of opinion dynamics: each agent holds a real value, and this value is continuously attracted by every other value differing from it by less than 1, with an intensity proportional to the difference.

We prove convergence to a set of clusters, with the agents in each cluster sharing a common value, and provide a lower bound on the distance between clusters at a stable equilibrium, under a suitable notion of multi-agent system stability.

To better understand the behavior of the system for a large number of agents, we introduce a variant involving a continuum of agents. We show, under some conditions, the existence of a solution to the system dynamics, convergence to clusters, and a non-trivial lower bound on the distance between clusters. Finally, we establish that the continuum model accurately represents the asymptotic behavior of a system with a finite but large number of agents.

I. INTRODUCTION

We study a continuous-time multi-agent model: each of n agents, labeled $1, \ldots, n$, maintains a real number ("opinion") $x_i(t)$, which is a continuous function of time and evolves according to the integral equation version of

$$\dot{x}_i(t) = \sum_{j: |x_i(t)-x_j(t)|<1} (x_j(t) - x_i(t)) . \qquad (1)$$

This model has an interpretation in terms of opinion dynamics: an agent considers another agent to be a neighbor if their opinions differ by less than 1, and agent opinions are continuously attracted by their neighbors' opinions. Numerical simulations show that the system converges to clusters inside which all agents share a common value. Different clusters lie at a distance of at least 1 from each other, and often approximately 2, as shown in Figure 1. The minimal distance of 1 between clusters is easily explained by the fact that clusters separated by a distance smaller than 1 would be attracting each other. The observation that the typical inter-cluster distance is close to 2 is however more surprising. We focus on understanding these convergence properties and
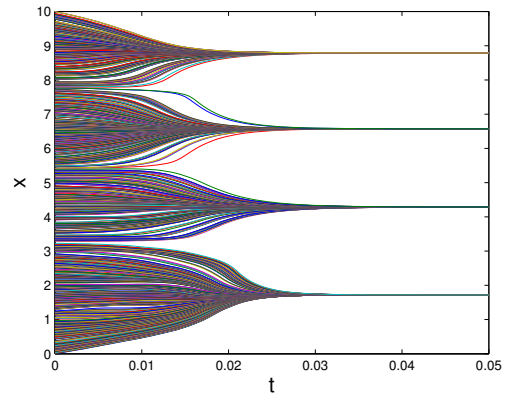
Fig. 1. Evolution with time of the values $x_i(t)$ for 1000 agents, with initial values randomly and uniformly distributed on $[0, 10]$. Observe the convergence to 4 clusters separated by slightly more than 2.

the structure of the set of clusters, including the asymptotic behavior for large n.

Observe that the network of interactions between agents in (1) *explicitly depends on the agent states*, as $x_j(t)$ influences $x_i(t + 1)$ only if $|x_i(t) - x_j(t)| < 1$. Many multi-agent systems involve a changing interaction topology; see e.g. [1], [10], [11], [16], [19], [20], and [17], [18] for surveys. In some cases, the interaction topology evolves randomly or according to some exogenous scheme, but in other cases it is modeled as a function of the agent states. The latter is typically the case for models of animals or robots with limited visibility. With some exceptions [7], [8], [12], however, this state-dependence is not taken into account in the analysis, probably due to the technical difficulties that it presents.

To address this issue, we have recently analyzed [2], [3] one of the simplest discrete-time multi-agent systems with state-dependent interaction topologies, namely, Krause's model[1] of opinion dynamics [13]: n agents maintain real numbers ("opinions") $x_i(t)$, $i = 1, \ldots, n$, and synchronously update them as follows:

$$x_i(t + 1) = \frac{\sum_{j: |x_i(t)-x_j(t)|<1} x_j(t)}{\sum_{j: |x_i(t)-x_j(t)|<1} 1} .$$

This model was particularly appealing due to its simple formulation, and due to some peculiar behaviors that it exhibits, which cannot be explained without taking into account the explicit dynamics of the interaction topology. Indeed, a first analysis using results on infinite inhomogeneous matrix products, as in [10], [14], shows convergence to clusters

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[1]The model is sometimes referred to as the Hegselmann-Krause model.

in which all agents share the same opinion, and that the distance between any two clusters is at least 1. Numerical simulations, however, show a qualitative behavior similar to the one shown in Figure 1 for the model (1): the distance between consecutive clusters is usually significantly larger than 1, and typically close to 2 when n is sufficiently large, a phenomenon for which no explanation was available.

Our goal in [2], [3] was thus to develop a deeper understanding of Krause's model and of these observed phenomena, by using explicitly the dynamics of the interaction topology. To this effect, we introduced a new notion of stability, tailored to such multi-agent systems, which provided an explanation for the observed inter-cluster distances when the number of agents is large. Furthermore, to understand the asymptotic behavior as the number of agents increases, we also studied a model involving a continuum of agents. We obtained partial convergence results for this continuum model, and proved nontrivial lower bounds on the inter-cluster distances, under some conditions.

Our results in [2], [3] were however incomplete in certain respects. In particular, the question of convergence of the continuum model remains open, and some of the results involve assumptions that are not easy to check a priori. We see two main reasons for these difficulties. First, the system is asymmetric, in the sense that the influence of $x_j(t)$ on $x_i(t+1)$ can be very different from that on $x_i(t)$ on $x_j(t+1)$, when i and j do not have the same number of neighbors. Second, the discrete time nature of the system allows, for the continuum model, buildup of an infinite concentration of agents with the same opinion, thus breaking the continuity of the agent distribution.

For the above reasons, we have chosen to analyze here the system (1), a continuous-time symmetric variant of Krause's model, for which we provide crisper and more complete results. One reason is that, thanks to the symmetry, the average value $\frac{1}{n}\sum_i x_i(t)$ is preserved, and the average value of a group of agents evolves independent of the interactions taking place within the group, unlike Krause's model. In addition, when two agent values approach each other, their relative velocity decays to zero, preventing the formation of infinite concentration in finite time. The continuous-time nature of the system brings up however some new mathematical challenges, related for example to the existence and uniqueness of solutions.

To summarize, the objective of the present paper is twofold. First, to advance our understanding of multi-agent systems with state-dependent interactions, by analyzing in full detail one simple but nontrivial such system. Second, to explain the convergence of agents to clusters separated by approximately twice the interaction distance, a phenomenon that often arises in such opinion dynamics models.

A. Outline and contributions

In Section II, we give some basic properties of the model (1), and prove convergence to clusters in which all agents share the same value. We then analyze the distance between consecutive clusters building on an appropriate notion of stability with respect to perturbing agents, introduced in [2], [3]. This analysis leads to a necessary and sufficient condition for stability that is consistent with the experimentally observed inter-cluster distances, and to a conjecture that the probability of convergence to a stable equilibrium tends to one as the number of agents increases. In Section III, we introduce a variant involving a continuum of agents, to approximate the model for the case of a finite but large number of agents. Under some smoothness assumptions on the initial conditions, we show the existence of a unique solution, convergence to clusters, and nontrivial lower bounds on the inter-cluster distances, consistent with the necessary and sufficient for stability in the discrete-agent model. Finally, in Section IV, we explore the relation between the two models, and establish that the behavior of the discrete model approaches that of the continuum model over finite but arbitrarily long time intervals, provided that the number of agents is sufficiently large.

The results summarized above differ from those those obtained in [2], [3] for Krause's model, in three respects: (i) we obtain the convergence of the continuum model, in contrast to the partial results obtained for Krause's model; (ii) all of our stability and approximation results are valid under some simple and easily checkable smoothness assumptions on the initial conditions, unlike the corresponding results in [3] which require, for example, the distance between the largest and smallest opinions to remain larger than 2 at all times; (iii) finally we settle the problem of existence and uniqueness of a solution to our equations, a problem that did not arise for Krause's discrete-time model. These stronger results were obtained by using proof techniques relying on the continuous evolution of the opinions, on the preservation of their average, and on the the symmetry of the interactions.

Most proofs are omitted here for space reasons. We refer the reader to [5] for a complete version of our results.

B. Related work

Our model (1) is closely related to that treated by Canuto et al. [6] who consider a continuum of multi-dimensional opinions, while treating discrete agents as a special case. In the case of discrete agents with one-dimensional opinions, the evolution is described by

$$\dot{x}_i(t) = \sum_j \xi\left(x_i(t) - x_j(t)\right)\left(x_i(t) - x_j(t)\right),$$

where ξ is a continuous nonnegative symmetric and decaying function, taking positive values only for arguments with amplitude smaller than a certain constant R. They also consider a discrete-time approximation of their model, described in the case of discrete agents with one-dimensional opinion by $x_i(t + \delta t) = x_i(t) + \delta t \dot{x}_i(t)$. Our model is therefore a particular case of their continuous-time model in one dimension, except that a step function does not satisfy their continuity assumption.

The authors of [6] prove convergence of the opinions, in distribution, to clusters separated by at least R for both dis-

crete and continuum-time models. Their convergence proof relies on the decrease of the measured variance of the opinion distribution, and is based on an Eulerian representation that follows the density of agent opinions, in contrast to the Lagrangian representation used in this paper, which follows the opinion x of each agent. It is interesting to note that despite the difference between these two methods for proving convergence, they both appear to fail in the absence of symmetry, and cannot be used to prove convergence for the continuum-agent variant of Krause's model.

Finally, the models in this paper are also related to other classes of rendezvous methods and opinion dynamics models, as described in [3], [15] and the references therein. Several more complex decentralized control laws are built on such rendezvous methods.

II. DISCRETE AGENTS

The differential equation (1) usually has no differentiable solutions. Indeed, observe that the right-hand side of the equation can be discontinuous when the interaction topology changes, which can prevent x from being differentiable. To avoid this difficulty, we consider functions $x : \Re^+ \to \Re^n$ that are solutions of the integral version of (1), namely

$$x_i(t) = x_i(0) + \int_0^t \sum_{j: |x_i(\tau) - x_j(\tau)| < 1} (x_j(\tau) - x_i(\tau)) \, d\tau.$$
(2)

Observe however that for all t at which $\dot{x}_i(t)$ exists, it can be computed using (1).

A. Existence and convergence

Time-switched linear systems are of the form $x(t) = x(0) + \int_0^t A_\tau x(\tau) \, d\tau$, where A_t is a piecewise constant function of t. They always admit a unique solution provided that the number of switches taking place during any finite time interval is finite. Position-switched systems of the form $\dot{x}(t) = x(0) + \int_0^t A_{x(\tau)} x(\tau) \, d\tau$ may on the other hand admit none or multiple solutions. Our model (2) belongs to the latter class, and indeed admits multiple solutions for some initial conditions. Observe for example that the two-agent system with initial condition $\tilde{x} = \left(-\frac{1}{2}, \frac{1}{2}\right)$ admits a first solution $x(t) = \tilde{x}$, and a second solution $x(t) = \tilde{x}e^{-t}$. The latter solution satisfies indeed the differential equation (1) at every time except 0, and thus satisfies (2). We will see however that such cases are exceptional.

We say that $\tilde{x} \in \Re^n$ is a *proper initial condition of (2)* if:
(a) There exists a unique $x : \Re^+ \to \Re^n : t \to x(t)$ satisfying (2), and such that $x(0) = \tilde{x}$.
(b) The subset of $\Re^+$ on which x is not differentiable is at most countable, and has no accumulation points.
(c) If $x_i(t) = x_j(t)$ holds for some t, then $x_i(t') = x_j(t')$, for every $t' \geq t$.

We then say that the solution x is a *proper solution* of (2).

Theorem 1: Almost all $\tilde{x} \in \Re^n$ (in the sense of Lebesgue measure) are proper initial conditions.

It follows from condition (c) and from the continuity of proper solutions that if $x_i(t) \geq x_j(t)$ holds for some t, then

this inequality holds for all subsequent times. For the sake of clarity, we assume thus in the sequel that the components of proper initial conditions are sorted, that is, if $i > j$, then $\tilde{x}_i \geq \tilde{x}_j$, which also implies that $x_i(t) \geq x_j(t)$ for all t. Moreover, an explicit computation, which we perform in Section III for a more complex system, shows that $|x_i(t) - x_j(t)| \geq |\tilde{x}_i - \tilde{x}_j| e^{-nt}$. Observe finally that if $x_{i+1}(t^*) - x_i(t^*) > 1$ holds for some t^* for a proper solution x, then $\dot{x}_{i+1}(t) \geq 0$ and $\dot{x}_i(t) \leq 0$ hold for almost all subsequent t, so that $x_{i+1}(t) - x_i(t)$ remains larger than 1. The system can then be decomposed into two independent subsystems, consisting of agents $1, \ldots, i$, and $i + 1, \ldots, n$, respectively.

There are several convergence proofs for the system (2). We present here a simple one, which highlights the importance of the average preservation and symmetry properties, and extends nicely to the continuum model [5]. Let F be the set of vectors $\tilde{s} \in \Re^n$ such that for all $i, j \in \{1, \ldots, n\}$, either $\tilde{s}_i = \tilde{s}_j$, or $|\tilde{s}_i - \tilde{s}_j| \geq 1$. We refer to vectors in F as *equilibria*.

Theorem 2: Every proper solution x of (2) converges to a limit $x^* \in F$; i.e., for any i, j, if $x_i^* \neq x_j^*$, then $|x_i^* - x_j^*| \geq 1$.

Proof: Observe that by symmetry, the equality

$$\sum_{i=1}^{k} \sum_{j \leq k: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)) = 0$$

holds for any k and any t. Therefore, it follows from (2) that for all t but possibly countably many,

$$\frac{d}{dt} \sum_{i=1}^{k} x_i(t) = \sum_{i=1}^{k} \sum_{j > k: |x_i(t) - x_j(t)| < 1} (x_j(t) - x_i(t)), \quad (3)$$

which is nonnegative because $j > k > i$ implies $x_j(t) - x_i(t) \geq 0$. Since $x_i(t) \leq \max_j x_j(0)$, for all i and $t \geq 0$, $\sum_{i=1}^{k} x_i(t)$ is bounded and therefore converges monotonically, for any k. It then follows that every $x_i(t)$ converges to a limit x_i^*. We assume that $x_k^* \neq x_{k+1}^*$ and suppose, to obtain a contradiction, that $x_{k+1}^* - x_k^* < 1$. Then, since every term $x_j(t) - x_i(t)$ on the right-hand side of (3) is nonnegative, the derivative on the left-hand side is asymptotically positive and bounded away from 0, preventing the convergence of $\sum_{i=1}^{k} x_i(t)$. Therefore, $x_{k+1}^* - x_k^* \geq 1$. ∎

B. Stable equilibria and inter-cluster distances

By the term *clusters*, we will mean the limiting values to which the agent opinions converge. With some abuse of terminology, we also refer to a set of agents whose opinions converge to the same value as a cluster. Theorem 2 implies that clusters are separated by at least 1. On the other hand, extensive numerical experiments indicate that the distance between adjacent clusters is typically significantly larger than one, and if the clusters contain the same number of agents, usually close to 2. We believe that this phenomenon can, at least partially, be explained by the fact that clusters that are too close to each other can be forced to merge by the presence of a small number of agents between them, as in Figure 2. To formalize this idea we introduce a generalization
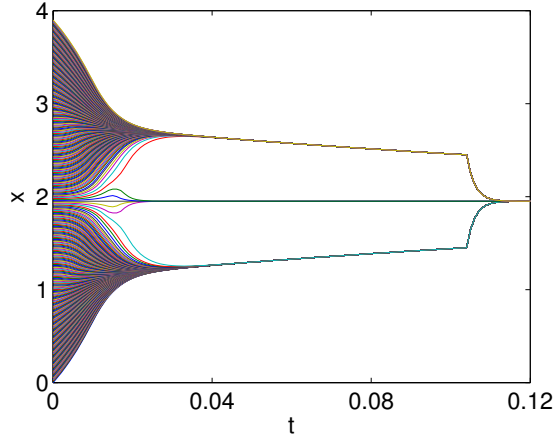
Fig. 2. Example of a temporary, "meta-stable," equilibrium. Initially, two clusters are formed and do not interact with each other, but they both interact with a small number of agents in between. As a result, the distance separating them eventually becomes smaller than 1. The clusters then attract each other directly and merge into a single, larger cluster.

of the system (2) in which each agent i has a weight w_i, and its opinion evolves according to

$$x_i(t) = x_i(0) + \int_0^t \sum_{j:\,|x_i(\tau)-x_j(\tau)|<1} w_j\left(x_j(\tau) - x_i(\tau)\right) d\tau.$$

(4)

The convergence result of Section II-A carries over to the weighted case (the proof is the same). We will refer to the sum of the weights of all agents in a cluster, as its *weight*. If all the agents in a cluster have exactly the same opinion, the cluster behaves as a single agent with this particular weight[2].

Let $\tilde{s} \in F$ be an equilibrium vector. Suppose that we add a new agent of weight δ and initial opinion x_0, consider the resulting configuration as an initial condition, and let the system evolve according to some solution $x(t)$ (we do not require uniqueness). We define $\Delta(\delta, \tilde{s})$ as the supremum of $|x_i(t) - \tilde{s}_i|$, where the supremum is taken over all possible initial opinions x_0 of the perturbing agent, all i, all times t, and all possible solutions $x(t)$ of the system (2). We say that $\tilde{s}$ is *stable* if $\lim_{\delta \downarrow 0} \Delta(\delta, \tilde{s}) = 0$. An equilibrium is thus unstable if some modification of fixed size can be achieved by adding an agent of arbitrarily small weight. This notion of stability is almost the same as the one that we introduced for Krause's model in [2], [3].

Theorem 3: An equilibrium is stable if and only if for any two clusters A and B with weights W_A and W_B, respectively, their distance is greater than $d = 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}$.

Proof: The proof is very similar to the proof of Theorem 2 in [3]. The main idea is the following. A perturbing agent can initially be connected to at most two clusters, and cannot perturb the equilibrium substantially if it is connected to none or one. If it is connected to two

[2] In the case of non-proper initial conditions leading to multiple solutions, there exists at least one solution in which each cluster behaves as a single agent with the corresponding weight.

clusters A, B, it moves in the direction of their center of mass $\frac{W_A \tilde{s}_A + W_B \tilde{s}_B}{W_A + W_B}$, while the two clusters move at a much slower pace, proportional to the perturbing agent's weight. We note that, by a simple algebraic calculation, the center of mass of two clusters is within unit distance from both clusters if and only if their distance is no more than d.

If the distance between the two clusters is more than d, then the center of mass of the two clusters is more than unit distance away from one of the clusters, say from B. Therefore, eventually the perturbing agent is no longer connected to B, and rapidly joins cluster A, having modified the cluster positions only proportionally to its weight. Thus, the equilibrium is stable.

On the other hand, if the distance between the two clusters is less than d, then the center of mass is less than unit distance away from both clusters. We can place the perturbing agent at the center of mass. Then, the perturbing agent does not move, but keeps attracting the two clusters, until eventually they become connected and then rapidly merge. Thus, the equilibrium is not stable.

If the distance between clusters is exactly equal to d, the center of mass is at exactly unit distance from one of the two clusters. Placing a perturbing agent at the center of mass results in nonunique solutions. In one of these solutions, the clusters start moving towards their center of mass, and the subsequent behavior is the same as in the case where the distance between clusters is smaller than d, thus again showing instability. Such a solution violates the differential version of (2) only at time $t = 0$ and thus satisfies (2). ∎

Theorem 3 characterizes stable equilibria in terms of a lower bound on inter-cluster distances. It allows for inter-cluster distances at a stable equilibrium that are smaller than 2, provided that the clusters have different weights. This is consistent with experimental observations for certain initial opinion distributions (see [9] for example). On the other hand, for the frequently observed case of clusters with equal weights, stability requires inter-cluster distances of at least 2. Thus, this result comes close to a full explanation of the observed inter-cluster distances of about 2.2. Of course, there is no guarantee that our system will converge to a stable equilibrium. (A trivial example is obtained by initializing the system at an unstable equilibrium.) However, we have observed that for a given distribution of initial opinions, and as the number of agents increases, we almost always obtain convergence to a stable equilibrium. This leads us to the following conjecture.

Conjecture 1: Suppose that the initial opinions are chosen randomly and independently according to a bounded probability density function with connected support, which is also bounded below by a positive number on its support. Then, the probability of convergence to a stable equilibrium tends to 1, as the number of agents increases to infinity.

In addition to extensive numerical evidence (see [9]), this conjecture is supported by the intuitive idea that if the number of agents is sufficiently large, convergence to an unstable equilibrium is made impossible by the presence of

at least one agent connected to the two clusters. It is also supported by results obtained in the next sections. A similar conjecture has been made for Krause's model [2], [3].

III. AGENT CONTINUUM

To further analyze the properties of (2) and its behavior as the number of agents increases, we now consider a variant involving a continuum of agents. We use the interval $I = [0, 1]$ to index the agents, and denote by Y the set of bounded measurable functions $\tilde{x} : I \to \Re$, attributing an opinion $\tilde{x}(\alpha) \in \Re$ to every agent in I. As an example, a uniform distribution of opinions is given by $\tilde{x}(\alpha) = \alpha$. We use the function $x : I \times \Re^+ \to \Re : (\alpha, t) \to x_t(\alpha)$ to describe the collection of all opinions at different times. [3] We denote by x_t the function in Y obtained by restricting x to a certain value of t. For a given initial opinion function $\tilde{x}_0 \in Y$, we are interested in functions x satisfying

$$\frac{d}{dt} x_t(\alpha) = \int_{\beta : |x_t(\alpha) - x_t(\beta)| < 1} (x_t(\beta) - x_t(\alpha)) \, d\beta, \quad (5)$$

Note that x_0, the restriction of x to $t = 0$, should not be confused with $\tilde{x}_0$, an arbitrary function in Y intended as an initial condition, but for which they may possibly exist none or several corresponding functions x.

The existence or uniqueness of a solution to (5) is not guaranteed, and there may moreover exist functions that satisfy this equation in a weaker sense, without being differentiable in t. For this reason, it is more convenient to formally define the model through an integral equation. For an initial opinion function $\tilde{x}_0 \in Y$, we are interested in measurable functions $x : I \times \Re^+ \to \Re : (\alpha, t) \to x_t(\alpha)$ such that

$$x_t(\alpha) = \tilde{x}_0(\alpha) + \int_0^t \int_{\beta : |x_\tau(\alpha) - x_\tau(\beta)| < 1} (x_\tau(\beta) - x_\tau(\alpha)) \, d\beta d\tau \tag{6}$$

for every t and for every $\alpha \in I$. One can easily prove that for any solution x of (6), $\bar{x}_t := \int_0^1 x_t(\alpha) \, d\alpha$ is constant, and $\int_0^1 (x_t(\alpha) - \bar{x}_t)^2 \, d\alpha$ is nonincreasing in t.

For the sake of simplicity, we will restrict attention to nondecreasing opinion functions, and define X as the set of nondecreasing bounded functions $\tilde{x} : I \to \Re$. This is no essential loss of generality, because the only quantities of interest relate to the distribution of opinions; furthermore, monotonicity of initial opinion functions can be enforced using a measure-preserving reindexing of the agents; finally, monotonicity is preserved by the dynamics under mild conditions. We will refer to element of X as nondecreasing functions. if $x : I \times [0, \infty) \to \Re$ is such that $x_t \in X$ for all t, we will also say that x is nondecreasing.

A. Existence and uniqueness of solutions

The existence of a unique solution to (6) is in general not guaranteed, as there exist initial conditions allowing for multiple solutions. Consider for example $\tilde{x}_0(\alpha) = -1/2$ if $\alpha \in [0, \frac{1}{2}]$, and $\tilde{x}_0(\alpha) = 1/2$ otherwise. Similar to our

[3]Note the reversal of notational conventions: the subscript now indicates time rather than an agent's index.

discrete-agent example, $x_t = \tilde{x}_0$ and $x_t(\alpha) = \tilde{x}_0(\alpha)e^{-t}$ are two possible solutions of (6). Nevertheless, we will see that a unique solution exists when the initial condition, as a function of α, has a positive and bounded increase rate; this is equivalent to assuming that the density of initial opinions is bounded from above and from below on its support, which is connected. It is convenient to introduce some additional notation. For positive real numbers m, M, we call $X_m^M \subset X$ the set of nondecreasing functions $\tilde{x} : I \to \Re$ such that

$$M \geq \frac{\tilde{x}(\beta) - \tilde{x}(\alpha)}{\beta - \alpha} \geq m$$

holds for every $\beta \neq \alpha$, and say that a function $\tilde{x} \in X$ is *regular* if it belongs to X_m^M for some $m, M > 0$. The following existence and uniqueness result, proved in [5], relies on the continuity at every regular function of the operator defining the integral equation (6).

Theorem 4: Suppose that the initial opinion function satisfies $\tilde{x}_0 \in X_m^M$, for some $m, M > 0$. Then the models (5) and (6) admit a unique and common solution x, and x satisfies

$$me^{-t} \leq \frac{x_t(\beta) - x_t(\alpha)}{\beta - \alpha} \leq M e^{4t/m}, \tag{7}$$

for every t and $\beta \neq \alpha$. x_t is regular at all time.

B. Convergence to clusters, and inter-cluster distances

In this section, we analyze the convergence of the opinions to clusters and characterize the fixed points and the possible limit points of the system, exhibiting the importance of the distances between clusters. In particular, we show that for regular initial conditions, the limit to which the system converges satisfies a condition on the inter-cluster distance similar to the one in Theorem 3, and give a necessary condition for the the stability of a fixed point.

Let $F \subset X$ be the set of nondecreasing functions $\tilde{s}$ such that for every $\alpha, \beta \in I$, either $\tilde{s}(\alpha) = \tilde{s}(\beta)$ or $|\tilde{s}(\alpha) - s(\beta)| > 1$. Similarly, let $\overline{F}$ be the set of nondecreasing functions $\tilde{s}$ such that for almost every pair $(\alpha, \beta) \in I^2$, either $\tilde{s}(\alpha) = \tilde{s}(\beta)$ or $|\tilde{s}(\alpha) - s(\beta)| \geq 1$. Finally, we say that $\tilde{s} \in X$ is a *fixed point* if the integral equation (6) with initial condition $\tilde{s}$ admits a unique solution $x_t = \tilde{s}$ for all t. The idea behind the proof of the next convergence theorem presents many similarities with that of Theorem 2.

Theorem 5: Let x be a solution of the integral equation (6) such that x_0 is regular. There exists a function $\tilde{y} \in \overline{F}$ such that $\lim_{t \to \infty} x_t(\alpha) = \tilde{y}(\alpha)$ holds for almost all α. Moreover, the set of nondecreasing fixed points contains F and is contained in $\overline{F}$.

As in the discrete case, we call *clusters* the discrete opinion values held by a positive measure set of agents at a fixed point $\tilde{s}$. For a cluster A, we denote by W_A, referred to as the *weight of the cluster*, the length of the interval $\tilde{s}^{-1}(A)$. By an abuse of language, we also call a cluster the interval $\tilde{s}^{-1}(A)$ of indices of the associated agents. The following result states that, for regular initial conditions, the limit to

which the system converges satisfies a condition on the inter-cluster distance similar to the one in Theorem 3. Its proof uses the continuity of x_t at each t to garantee the presence of perturbing agents between any two emerging clusters [5].

Theorem 6: Let $\tilde{x}_0 \in X$ be an initial opinion function, x the solution of the integral equation (6), and $\tilde{s} = \lim_{t\to\infty} x_t$ the fixed point to which x converges. If $\tilde{x}_0$ is regular, then

$$|B - A| \geq 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}} \tag{8}$$

holds for any two clusters A and B of $\tilde{s}$.

We now analyze the stability of the fixed points, under a classical definition of stability (in contrast to the nonstandard stability notion introduced for the discrete-agent system. Let $\tilde{s}$ be a fixed point of (6). We say that $\tilde{s}$ is stable, if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\tilde{s} - \tilde{x}_0\|_1 \leq \delta$, then $\|\tilde{s} - x_t\|_1 \leq \epsilon$ for every t and every solution x of the integral equation (6) with $\tilde{x}_0$ as initial condition. It can be shown that this classical notion of stability is stronger than the stability with respect to the addition of a perturbing agent used in Section II-B. More precisely, if we view the discrete-agent system as a special case of the continuum model, stability under the current definition implies stability with respect to the definition used in Section II-B.

Proposition 1: Let $\tilde{s}$ be a fixed point of (6). If $\tilde{s}$ is stable, then for any two clusters A and B,

$$|B - A| > 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}. \tag{9}$$

The proof relies on modifying the positions of an appropriate set of agents and "creating" some perturbing agents at the weighted average of the two clusters, inducing dynamics similar to those described in the proof of Theorem 3. See Chapter 10 of [9] or Theorem 6 in [3] for the same proof applied to Krause's model. We conjecture that the necessary condition in Proposition 1 is also sufficient.

Conjecture 2: A fixed point $\tilde{s}$ of (6) is stable according to the norm $\|\cdot\|_1$ if and only if, for any two clusters A, B,

$$|B - A| > 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}},$$

Conjecture 2 is a fairly strong statement. It implies, for example, that multiple clusters are indeed possible starting from regular initial conditions, which is an open question at present.

IV. RELATION BETWEEN THE DISCRETE AND CONTINUUM-AGENT MODELS

We now formally establish a connection between the discrete-agent and the continuum-agent models, and use this connection to argue that the validity of Conjecture 2 implies the validity of Conjecture 1.

The following result shows that continuum-agent model can be interpreted as the limit when $n \to \infty$ of the discrete-agent model, on any time interval of finite length. Its proof relies on the continuity of the opinion evolution with respect to the initial conditions. To avoid any risk of ambiguity, we use ξ to denote discrete vectors in the sequel. Moreover, we assume that such vectors are always sorted (i.e., $j > i \Rightarrow \xi_j \geq \xi_i$). We define the operator G that maps a discrete (nondecreasing) vector to a function by $G(\xi)(\alpha) = \xi_i$ if $\alpha \in [\frac{i-1}{n}, \frac{i}{n})$, and $G(\xi)(1) = \xi(n)$, where n is the dimension of the vector ξ. Let ξ be a solution of the discrete-agent model (2) with initial condition $\xi(0)$. One can verify that $G(\xi(t))$ is a solution to the continuum-agent integral equation for a (6) with $G(\xi(0))$ as initial condition. The discrete-agent model can thus be simulated by the continuum-agent model.

Theorem 7: Consider a regular initial opinion function $\tilde{x}_0$, and let $(\xi^{\langle n \rangle})_{n>0}$ be a sequence of (nondecreasing) vectors in $\Re^n$ such that $\lim_{n\to\infty} \left\|G(\xi^{\langle n \rangle}(0)) - \tilde{x}_0\right\|_\infty = 0$, and such that for each n, $\xi^{\langle n \rangle}(0)$ is a proper initial condition, admitting a unique solution $\xi^{\langle n \rangle}(t)$. Then, for every T and every $\epsilon > 0$, there exists n' such that

$$\left\|G(\xi^{\langle n \rangle}(t)) - x_t\right\|_\infty \leq \epsilon$$

holds for all $t \in [0, T]$ and $n \geq n'$.

When $\tilde{x}$ is regular, a simple way of building such a sequence $(\xi^{\langle n \rangle}(0))_{n>0}$ is to take $\xi_i^{\langle n \rangle}(0) = \tilde{x}_0(i/n)$. Theorem 7 implies that the discrete-agent model approximates arbitrarily well the continuum model for arbitrarily large periods of time, provided that the initial distribution of discrete opinions approximates sufficiently well the initial conditions of the continuum model. Now recall that according to Theorem 6, and for regular initial conditions, the continuum-agent model converges to a fixed point satisfying the inter-cluster distance condition (8). The conjunction of these two results seems thus to support our Conjecture 1, that the discrete-agent model converges to an equilibrium satisfying this same condition, provided that the number of agents is sufficiently large and that their initial opinions approximate some regular function. This argument, however, is incomplete because the approximation result in Theorem 7 is only valid over finite, not infinite, time intervals. Nevertheless, we will see that this reasoning would be valid, with some exceptions, if Conjecture 2 holds.

Proposition 2: Suppose that $\tilde{x}_0$ is regular, and suppose that the limit $\tilde{s}$ of the resulting solution x of (6) is stable and its clusters satisfy

$$|B - A| > 1 + \frac{\min\{W_A, W_B\}}{\max\{W_A, W_B\}}. \tag{10}$$

Let $\xi(0) \in \Re^n$ be a vector whose n entries are randomly and independently selected according to a probability density function corresponding to $\tilde{x}_0$. Then, the clusters of the limit of the corresponding solution of (2) satisfy (10), with probability that tends to 1 as $n \to \infty$.

We can now establish the connection between our two conjectures. Suppose that Conjecture 2 holds. Let $\tilde{x}_0$ be a regular initial condition. By Theorem 6, the resulting trajectory converges to a fixed point $\tilde{s}$ that satisfies the nonstrict inequality (8). We expect that generically the inequality will actually be strict, in which case, according to Conjecture 2,

$\tilde{s}$ is stable. Therefore, subject to the genericity qualification above, Proposition 2 implies the validity of Conjecture 1.

V. CONCLUSIONS

We have analyzed a simple continuous-time multi-agent system for which the interaction topology depends on the agent states. We worked with the explicit dynamics of the interaction topology, which raised a number of difficulties, as the resulting system is highly nonlinear and discontinuous. This is in contrast to the case of exogenously determined topology dynamics, which result into time-varying but linear dynamics.

After establishing convergence to a set of clusters in which agents share the same opinion, we focused on the inter-cluster distances. We proposed an explanation for the experimentally observed distances based on a notion of stability that is tailored to our context. This also led us to conjecture that the probability of convergence to a stable equilibrium (in which certain minimal inter-cluster distances are respected), tends to 1 as the number of agents increases.

We then introduced a variant of the model, involving a continuum of agents. For regular initial conditions, we proved the existence and uniqueness of solutions, the convergence of the solution to a set of clusters, and a nontrivial bound on the inter-cluster distances, of the same form as the necessary and sufficient condition stability for the discrete-agent model. Finally, we established a link between the discrete and continuum models, and proved that our first conjecture was implied by a seemingly simpler conjecture.

The results presented here are parallel to, but much stronger than those that we obtained for Krause's model of opinion dynamics [3]. Indeed, we have provided here a full analysis of the continuum model, under the mild and easily checkable assumption of regular initial conditions.

The tractability of the model in this paper can be attributed to (i) the inherent symmetry of the model, and (ii) the fact that it runs in continuous time, although the latter aspect also raised nontrivial questions related to the existence and uniqueness of solutions. We note however that similar behaviors have also been observed for systems without such symmetry. One can therefore wonder whether the symmetry is really necessary, or just allows for comparatively simpler proofs. One can similarly wonder whether our results admit counterparts in models involving high-dimensional opinion vectors, where one can no longer rely on monotonic opinion functions and order-preservation results.

As in our work on Krause's model, our study of the system on a continuum and the distances between the resulting clusters uses the fact that the density of agents between the clusters that are being formed is positive at any finite time. This however implies that, unlike the discrete-agent case, the clusters always remain indirectly connected, and it is not clear whether this permanent connection can eventually force clusters to merge. In fact, it is an open question whether there exists a regular initial condition that leads to multiple clusters, although we strongly suspect this to be the case. A simple proof would consist of an example of regular initial conditions that admit a closed-form formula for x_t. However, this is difficult because of the discontinuous dynamics. The only available examples of this type converge to a single cluster, as for example, in the case of any two dimensional distribution of opinions with circular symmetry (see [6]).

REFERENCES

[1] V.D. Blondel, J.M. Hendrickx, A. Olshevsky, and J.N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *Proceedings of the 44th IEEE Conference on Decision and Control (CDC'2005)*, pages 2996–3000, Seville, Spain, December 2005.

[2] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. On the 2R conjecture for multi-agent systems. In *Proceedings of the European Control Conference 2007 (ECC'2007)*, pages 874–881, Kos, Greece, July 2007.

[3] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. On Krause's multi-agent consensus model with state-dependent connectivity. *IEEE transactions on Automatic Control*, 54(11), pages 2586–2597, 2009.

[4] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. Existence and uniqueness of solutions for a continuous-time opinion dynamics model with state-dependent connectivity. *Technical report* http://www.inma.ucl.ac.be/~hendrickx/availablepublications/solutions-d-a.pdf, 2009.

[5] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis. Continuous-time average-preserving opinion dynamics with opinion-dependent communications To appear in *SIAM Journal on Control and Optimization*, arXiv:0907.4662v1 [math.OC], 2010.

[6] C. Canuto, F. Fagnani, and P. Tilli. A Eulerian approach to the analysis of rendez-vous algorithms. In *Proceedings of the 17th IFAC world congress (IFAC'08)*, pages 9039–9044, July 2008.

[7] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Transactions on Automatic Control*, 52:852–862, May 2007.

[8] F. Cucker and S. Smale. On the mathematics of emergence. *Japanese Journal of Mathematics*, 2:119–227, March 2007.

[9] J.M. Hendrickx. *Graphs and Networks for the Analysis of Autonomous Agent Systems*. PhD thesis, Universit catholique de Louvain, http://www.inma.ucl.ac.be/~hendrickx/availablepublications/Thesis_Julien_Hendrickx.pdf, 2008.

[10] J.M. Hendrickx and V.D. Blondel. Convergence of different linear and non-linear Vicsek models. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS'2006)*, pages 1229–1240, Kyoto, Japan, July 2006.

[11] A.Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[12] E.W. Justh and P. S. Krishnaprasad. Equilibria and steering laws for planar formations. *Systems and Control Letters*, 52(1):25–38, 2004.

[13] U. Krause. Soziale Dynamiken mit vielen Interakteuren. Eine Problemskizze. In *Modellierung und Simulation von Dynamiken mit vielen interagierenden Akteuren*, pages 37–51. 1997.

[14] J. Lorenz. A stabilization theorem for continuous opinion dynamics. *Physica A*, 355(1):217–223, 2005.

[15] J. Lorenz. Continuous opinion dynamics under bounded confidence: A survey. *International Journal of Modern Physics C*, 18(12):1819–1838, 2007.

[16] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[17] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, January 2007.

[18] W. Ren, R.W. Beard, and E.M. Atkins. *IEEE Control and Systems Magazine*, 27(2):71–82, April 2007.

[19] J.N. Tsitsiklis. *Problems in decentralized decision making and computation*. PhD thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, http://web.mit.edu/jnt/www/PhD-84-jnt.pdf, 1984.

[20] T. Vicsek, A. Czirok, I. Ben Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75:1226–1229, 1995.