49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Controllability of Homogeneous Single-Leader Networks

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Abstract— **This paper addresses an aspect of controllability in a single-leader network when the agents are homogeneous. In such a network, indices are not assigned to the individual agents and controllability, which is typically a point to point property, now becomes a point to set property, where the set consists of all permutations of the target point. Agent homogeneity allows for choice of the optimal target point permutation that minimizes the distance to the system's reachable subspace, which we show is equivalent to finding a minimum sum-of-squares clustering with constraints on the cluster sizes. However, finding the optimal permutation is NP-hard. Methods are presented to find suboptimal permutations in the general case and the optimal permutation when the agent positions are 1-D.**

I. Introduction

Research in multi-agent systems has mainly focused on designing decentralized controllers that allow for agents to autonomously achieve global goals, such as reaching consensus (e.g. [1], [2], [3], [4], [5]) or achieving formations (e.g. [6], [7]). However, in many of the intended applications for multi-agent systems, such as search and rescue, it is more likely that agents will be working closely with humans as opposed to acting completely autonomously. The research problem that arises involves understanding how a human controller can affect and interact with an entire network of agents, without directly communicating with each of them.

In this paper, we focus on when a human takes control of a single agent within the network, while all other agents are executing a nearest neighbor averaging rule. The agents thus form a single-leader network where the positions of all follower agents within the network can be affected by controlling the leader agent (e.g. [8], [9], [10], [11]). We investigate the controllability problem when the network consists of homogeneous agents. Since agents are interchangeable, it does not matter which agent goes where, as long as there is an agent at each of the target locations. Controllability is now no longer a point to point property of the system, but instead becomes a point to set property, where the set consists of all permutations of the target point. Homogeneity in the system lets us choose a permutation of the target point that is closest to the system's reachable subspace. However, we show that finding such an optimal permutation is NP-hard.

The outline of this paper is as follows: Section 2 presents system dynamics for a single-leader network. Section 3 reviews previous work on the controllability of a single-leader network and discusses the problem of controllability in a homogeneous single-leader network, as well as the computational complexity of finding the optimal permutation of a target point that is closest to the system's reachable subspace. Finally, Section 4 explores methods to find the optimal permutation of a target point both in the general case and the special case of 1-D agents.

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II. System Dynamics

Consider a team of $N+1$ agents, numbered $1,\dots,N+1$, with positions $x_i \in \mathbb{R}^n$, for $i = 1,\dots,N+1$, respectively. Let the information flow amongst agents in the network be represented by an undirected static graph $G = (V,E)$, where $V = \{v_1,\dots,v_{N+1}\}$ and $(v_i,v_j) \in E$ if and only if information flows between agents i and j. The neighbor set $N_i = \{j | (v_i,v_j) \in E\}$ represents the set of indices of all agents that share an edge with agent i in E.

Suppose the agents form a single-leader network where all followers execute a nearest neighbor averaging rule, while the leader's position is the external input u. Without loss of generality, assume the $N+1$th agent is the leader while agents $1,\dots,N$ are followers. The agents' dynamics are:

$$\begin{cases} \dot{x}_i & = & -\sum_{j\in N_i} (x_i - x_j) \quad, \forall i = 1,\dots,N \\ x_{N+1} & = & u. \end{cases} \tag{1}$$

The adjacency matrix of G is the $(N+1)\times(N+1)$ symmetric matrix A where $a_{i,j}$, the element in the ith row and jth column, is given by

$$a_{i,j} = \begin{cases} 1 & (v_i,v_j) \in E \\ 0 & otherwise. \end{cases} \tag{2}$$

The degree matrix of the graph G is a $(N+1)\times(N+1)$ diagonal matrix Δ, where

$$\Delta_{i,j} = \begin{cases} |N_i| & i = j \\ \\ 0 & otherwise. \end{cases} \tag{3}$$

Finally, the graph Laplacian matrix L is given by

$$L = \Delta - A, \tag{4}$$

which can be decomposed into blocks

$$L = \left[\begin{array}{c:c} L_f & \ell \\ \hdashline \ell^T & \xi \end{array}\right], \tag{5}$$

where the dimension of L_f is $N\times N$, ℓ is $N\times 1$, and $\xi \in \mathbb{R}$.

Let $x = \left[x_1^T,\dots,x_N^T\right]^T \in \mathbb{R}^{Nn}$ be the concatenated positions of all follower agents, where $x_j = \left[x_{j,1},\dots,x_{j,n}\right]^T \in$

$\mathbb{R}^n$, for $j = 1, \ldots, N$. Define $d_i : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^N$, for $i = 1, \ldots, n$, as a function that returns the positions of the N follower agents along the ith dimension, i.e., $d_i(x) = [x_{1,i}, \ldots, x_{N,i}]^T$. The dynamics of the follower agents' positions along the ith dimension are given by

$$d_i(\dot{x}) = -L_f d_i(x) - \ell u_i, \tag{6}$$

where u_i is the ith element of u. Since the dynamics along each dimension are decoupled, the dynamics of x can be written using $\otimes$, the Kronecker product, as the linear system

$$\dot{x} = -(L_f \otimes I_n)x - (\ell \otimes I_n)u, \tag{7}$$

where I_n is the $n \times n$ identity matrix.

III. Controllability in Homogeneous Single-Leader Networks

Many applications of multi-agent systems, such as the deployment of mobile sensor networks, require a large number of homogeneous agents that are initially in close proximity to spread out and reach a desired target point (positions of the follower agents). Analyzing the controllability of single-leader networks allows us to understand, for a given network topology, the range of target points that can be achieved from a human user steering the network through the leader.

A. Controllability of Single-Leader Networks

In a single-leader network, the dynamics of the follower agents along each dimension are decoupled and given by the linear system (6). Treating each dimension separately, the reachable subspace is given by the range space of the controllability Grammian Γ, where

$$\Gamma = \left[-\ell \;\; L_f\ell \;\; \ldots \;\; (-L_f)^{N-1}(-\ell)\right]. \tag{8}$$

The reachable subspace of a single-leader network was found in [10] to have an interesting interpretation involving the graph topology. Before stating this result, we must first review some definitions from [10], [12], and [13].

Definition 3.1: Given a vertex set V, let $\Pi = \{C_1, \ldots, C_M\}$ be a partition of V, where $C_i \subset V$ for $i = 1, \ldots, M$, $C_1 \cup \ldots \cup C_M = V$, and $C_i \cap C_j = \emptyset$ when $i \neq j$. We will call each C_i a *cell*.

Definition 3.2: Given a vertex v and a cell C, the *node to cell degree* gives the number of vertices in cell C that share an edge with v, and is given by $deg(v, C) = card(\{v' \in C | (v, v') \in E\})$.

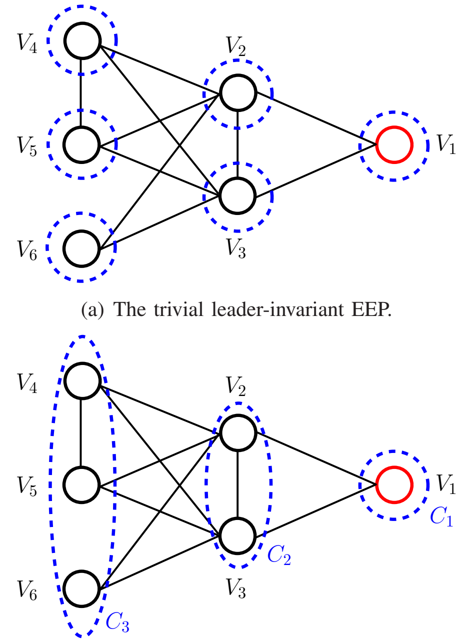
For example, in Figure 1(b), C_1, C_2, C_3 are cells that partition the vertices in the network and $deg(v_2, C_3) = 3$.

Definition 3.3: An *external equitable partition* (*EEP*) is a partition Π such that $\forall C \in \Pi, v \in C$ and $v' \in C \Rightarrow deg(v, C') = deg(v', C') \; \forall C' \in \Pi - \{C\}$.

Definition 3.4: An EEP is *leader-invariant* if the vertex corresponding to the leader agent belongs to its own cell.

Definition 3.5: A leader-invariant EEP is *maximal* if it has the fewest number of cells in any leader-invariant EEP.

For example, Figure 1(a) is a leader-invariant EEP, while Figure 1(b) is a maximal leader-invariant EEP.



(a) The trivial leader-invariant EEP.

(b) The maximal leader-invariant EEP.

Fig. 1. Two examples of leader-invariant EEPs of a single-leader network, where V_1 is the vertex for the leader agent. (a) shows the trivial leader-invariant EEP, while (b) gives the maximal leader-invariant EEP. Since the two partitions are different, the network is not completely controllable.

With these definitions, we now state a result relating controllability of a network to its maximal leader-invariant EEP. In [10] it was shown that follower agents within the same cell of the maximal leader-invariant EEP asymptotically approach the centroid of agents within that cell. That result is stated again below for easy reference.

Theorem 3.1: Assume a single-leader network has an information flow graph with a maximal leader-invariant EEP of k cells, numbered $1, \ldots, k$, that do not contain leader agents. In [10] it was shown that the range space of the controllability Grammian for (6), the follower agent dynamics along any dimension, is given by

$$R(\Gamma) = span\{w_1, \ldots, w_k\} \tag{9}$$

where $w_i \in \mathbb{R}^N$ and $w_{i,j}$, the jth element of w_i, is defined by

$$w_{i,j} = \begin{cases} 1 & v_j \in cell\ i \\ 0 & otherwise. \end{cases} \tag{10}$$

From the above theorem, it is seen that a network is completely controllable and can reach any target point only when the maximal leader-invariant EEP is trivial, i.e., each follower agent is contained within its own cell. When multiple agents are within the same cell, they asymptotically approach each other. Referring back to Figure 1, we see that the trivial leader-invariant EEP is not the same as the maximal leader-invariant EEP so the network is not completely controllable.

B. Optimally Reachable Target Points

The set of reachable target points in a network is restricted by the choice of leader agent and the network topology. In many situations, a tolerable margin of error may be allowed between where the agents are located and where the user desires them to be. Thus, we will investigate how close a network can get to a given target point.

We will model agents initially being at close proximity to one another by assuming zero initial conditions on the positions of the follower agents in the network. Such an assumption greatly simplifies the analysis of controllability, since agents within the same cell start and stay together.

Assumption 3.1: The agent positions, x, are initially 0.

With zero initial conditions on x, a target point of follower agents $x_T \in \mathbb{R}^{Nn}$ is reachable if and only if $d_i(x_T) \in R(\Gamma)$, for $i = 1, \ldots, n$. Depending on the network topology, the system of follower agents is not always completely controllable and so may not be able to reach a target point perfectly. Therefore, for a given x_T, it is useful to find the optimal reachable target point $x^*(x_T)$, which minimizes

$$J(x_T, x) = ||x_T - x||^2 = \sum_{i=1}^{n} ||d_i(x_T) - d_i(x)||^2, \quad (11)$$

such that $d_i(x) \in R(\Gamma)$, for $i = 1, \ldots, n$.

Theorem 3.2: For a given x_T, the optimal reachable $x^*(x_T)$, where $d_i(x^*(x_T)) \in R(\Gamma)$, for $i = 1, \ldots, n$, that minimizes (11) is

$$x^*(x_T) = \left(WW^T \otimes I_n\right) x_T, \quad (12)$$

where

$$W = \left[\frac{w_1}{||w_1||} \cdots \frac{w_k}{||w_k||}\right], \quad (13)$$

and $w_1, \ldots, w_k$ are as given in (9).

Proof: Minimizing $||x_T - x||^2$ is equivalent to minimizing $||d_i(x_T) - d_i(x)||$ individually for each i because the dynamics along each dimension are decoupled. The Hilbert Projection Theorem says that the optimal reachable $d_i(x^*(x_T)) \in R(\Gamma)$ that minimizes $||d_i(x_T) - d_i(x)||$ is the projection of $d_i(x_T)$ onto the subspace $R(\Gamma)$. The reachable subspace $R(\Gamma)$ is spanned by vectors $w_1, \ldots, w_k$ as given in (9). Therefore, the optimal choice of $d_i(x)$ is given by

$$d_i(x^*(x_T)) = \sum_{j=1}^{k} \frac{w_j^T d_i(x_T)}{||w_j||^2} w_j. \quad (14)$$

Letting W be as defined in (13), (14) can be rewritten as

$$d_i(x^*(x_T)) = WW^T d_i(x_T).$$

Since this holds for $i = 1, \ldots, n$, $x^*(x_T)$ is written as (12). ∎

The expression determined for $x^*(x_T)$ has an interesting and intuitive interpretation that will be useful later. Define $g_i : \mathbb{R}^{Nn} \to \mathbb{R}^n$, for $i = 1, \ldots, N$, as a function that returns the n dimensional coordinates of the ith agent, i.e., $g_i(x) = x_i$. Further, define $m : \{1, \ldots, N\} \to \{1, \ldots, k\}$ as a function that takes in an index of a follower agent and returns the index of the cell it belongs to in the maximal leader-invariant EEP. Let m^{-1} be the inverse image function that takes in a cell number and returns a set containing the indices of the follower agents that belong to that cell.

Corollary 3.1: For a given x_T and corresponding $x^*(x_T)$ that minimizes (11),

$$g_i(x^*(x_T)) = \frac{1}{|m^{-1}(m(i))|} \sum_{j \in m^{-1}(m(i))} g_j(x_T). \quad (15)$$

In other words, agent positions in cell j of $x^*(x_T)$ are all located at the centroid of the agent positions in cell j of x_T.

Proof: From the definition of vectors w_j given in (10), the expression for $d_i(x^*(x_T))$ in (14) can be interpreted. The numerator term of each summand $w_j^T d_i(x_T)$ is the sum along the ith dimension of the positions of all agents in cell j in the target point. That quantity is then divided by $||w_j||^2$, which is the number of agents in cell j, so the result is the centroid along the ith dimension of all agent positions in cell j of x_T. Finally, that value is multiplied to w_j, thereby assigning it to the ith dimensional component of the positions of all agents in cell j of $x^*(x_T)$. Since this is true for along all dimensions $i = 1, \ldots, n$, the result is that agent positions in cell j of $x^*(x_T)$ are all located at the centroid of the agent positions in cell j of x_T. ∎

With an expression for $x^*(x_T)$, it is now possible to compute the minimum cost associated with any given x_T.

Corollary 3.2: For a given x_T and corresponding $x^*(x_T)$, the minimum cost $J^*(x_T) = J(x_T, x^*(x_T))$ is

$$J^*(x_T) = x_T^T \left(I_{Nn} - WW^T \otimes I_n\right) x_T. \quad (16)$$

Proof: Plugging in the expression (12) for x^* into the cost (11), expanding the norm-squared, and noticing that the term $I_{Nn} - WW^T \otimes I_n$ is symmetric results in

$$J^*(x_T) = x_T^T \left(I_{Nn} - WW^T \otimes I_n\right)^2 x_T.$$

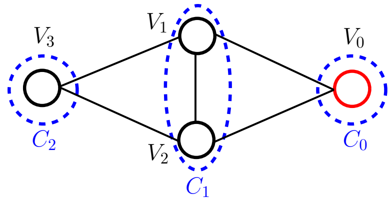
Expanding the squared term and making use of the fact that the columns of W are orthonormal results in (16). ∎

C. Homogeneous Networks

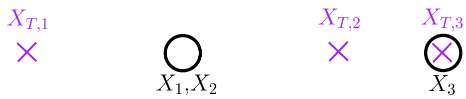
Equation (16) represents the cost associated with the closest that a particular single-leader network can reach a target point x_T. Notice that x_T represents the specification to have each agent i be located at $g_i(x_T)$, for $i = 1, \ldots, N$. However, in a network of homogeneous agents, the roles of agents are interchangeable and so it makes no difference if instead we ask agent i to go to $g_j(x_T)$ and agent j to go to $g_i(x_T)$. In fact, any permutation of the agent indices in x_T to some $(P \otimes I_n) x_T$, where P is a permutation matrix, ends up specifying the same target point if all we care about is the presence of an agent at each of the target positions. However, the new target point may be "more reachable" in the sense that $J^*((P \otimes I_n) x_T) < J^*(x_T)$.

Example 3.1: Consider a 1-D single-leader network with $N = 3$ follower agents as illustrated in Figure 2(a), where agents 1 and 2 are in cell 1 and agent 3 is in cell 2 of the maximal leader-invariant EEP of the network. The range space of the controllability Grammian is thus

$$R(\Gamma) = \{w_1, w_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$



(a) The single-leader network used in Example 3.1, where the leader agent's vertex is V_0.



(b) The closest the agents in the network (circles) can reach target point x_T (X's).

(c) The closest the agents in the network (circles) can reach the permuted target points Px_T (X's).

Fig. 2. The topology of the single-leader network in Example 3.1 is given in (a). (b) shows the closest the agents in the network can reach $x_T = [1\ 9\ 10]^T$, while (c) shows the closest the agents can reach $Px_T = [9\ 10\ 1]^T$. Notice that the Px_T results in an error less than x_T and so Px_T is the better specification of the target point.

Let $x_T = \left[\begin{array}{ccc} 1 & 9 & 10 \end{array}\right]^T$ and

$$P = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right],$$

then $J^*(x_T) = 32$, while $J^*(Px_T) = 0.5$. Therefore, as illustrated in Figures 2(b) and 2(c), Px_T is a better specification of the target point than x_T.

To exploit the advantage of homogeneity towards a network's controllability, it is necessary to solve the following problem that will be the focus of the rest of this paper:

Problem 3.1: Given a single-leader network and target point of follower agents x_T. Let $\mathcal{P}$ be the set of all $N \times N$ permutation matrices. Find P^* such that

$$P^* = \arg\min_{P \in \mathcal{P}} J^*((P \otimes I_n) x_T). \tag{17}$$

Calculating P^* can be viewed as finding the optimal specification of a target point. A more intuitive interpretation of finding P^* is to treat it as a constrained clustering problem on the N target positions $g_1(x_T), \ldots, g_N(x_T)$.

Definition 3.6: A *multiset* is a collection of objects in which order is ignored, but where multiplicity is significant.

For example, $M_1 = \{1,3,4\}$, $M_2 = \{1,3,4,4\}$, and $M_3 = \{1,4,3,4\}$ are all multisets. $M_2 = M_3$, but $M_1 \neq M_2$ and $M_1 \neq M_3$. Also, $|M_1| = 3$, while $|M_2| = |M_3| = 4$.

Definition 3.7: Given a multiset S, a *clustering* of S is a partitioning of the elements of S into multisets $c_1, \ldots, c_k$.

Now, let S be a multiset of agent positions. Within each cluster c_i, define the distortion measure of that cluster as

$$D(c_i) = \sum_{z \in c_i} ||z - \theta(c_i)||^2, \tag{18}$$

where $\theta(c_i)$ is the centroid of all positions in c_i. Define the cost of a clustering as the total distortion measure, given by

$$H(c_1, \ldots, c_k) = \sum_{i=1}^{k} D(c_i). \tag{19}$$

Problem 3.2: The *Euclidean minimum sum-of-squares clustering problem* is to find a clustering $c_1^*, \ldots, c_k^*$, given a multiset of positions S, so as to minimize (19).

Theorem 3.3: Suppose a single-leader network has a maximal leader-invariant EEP of exactly k cells containing follower agents, numbered $1, \ldots, k$. Finding the optimal permutation P^* for a target x_T in Problem 3.1 is equivalent to solving Problem 3.2 for the multiset of target positions, $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, under the constraint that $|c_i| = |m^{-1}(i)|$, the number of agents in cell i, for $i = 1, \ldots, k$.

Proof: Given a permutation matrix P, let $p : \{1, \ldots, N\} \to \{1, \ldots, N\}$ take in an agent index and returns the permuted index such that, for $j = 1, \ldots, N$, $g_j(x_T) = g_{p(j)}((P \otimes I_n) x_T)$. Let $c_i = \{g_j((P \otimes I_n) x_T) | m(j) = i\}$, for $i = 1, \ldots, k$, be a clustering of $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, where target positions in $(P \otimes I_n) x_T$ with indices in cell i are assigned to c_i.

Notice that $|c_i| = |m^{-1}(i)|$, the number of agents in each cell i, for $i = 1, \ldots, k$. Considering different permutations of agent indices for the target point x_T is equivalent to considering different cell assignments of the target positions, which is equivalent to considering clusterings $c_1, \ldots, c_k$ of S. The cost (16) associated with a chosen permutation P of target positions can be rewritten using (11) and (15) as

$$\begin{aligned}
& J^*((P \otimes I_n) x_T) \\
= & \sum_{i=1}^{n} ||d_i((P \otimes I_n) x_T) - d_i(x^*((P \otimes I_n) x_T))||^2 \\
= & \sum_{i=1}^{N} ||g_i((P \otimes I_n) x_T) - g_i(x^*((P \otimes I_n) x_T))||^2 \\
= & \sum_{i=1}^{N} ||g_i((P \otimes I_n) x_T) - \theta(c_{m(i)})||^2 \\
= & \sum_{i=1}^{k} \sum_{j | m(j) = i} ||g_j((P \otimes I_n) x_T) - \theta(c_i)||^2 \\
= & \sum_{i=1}^{k} \sum_{z \in c_i} ||z - \theta(c_i)||^2 = H(c_1, \ldots, c_k),
\end{aligned}$$

which shows that the cost is equivalent to (19).

Given the P^* that solves Problem 3.1, an optimal clustering $c_1^*, \ldots, c_k^*$ that solves Problem 3.2 under the constraint that $|c_i| = |m^{-1}(i)|$, for $i = 1, \ldots, k$, can be computed by the polynomial-time algorithm:

Let $c_1^*, \ldots, c_k^*$ be empty multisets;
for $i = 1, \ldots, N$ **do**
  Add $g_i((P^* \otimes I_n) x_T)$ to $c_{m(i)}^*$;
end

Alternatively, given an optimal clustering $c_1^*, \ldots, c_k^*$, the matrix P^* can be computed by the polynomial time algorithm:

Let $Q = R = \{1, \ldots, N\}$, and $P^* = 0$ ($N \times N$ matrix);
for $i = 1 \ldots, k$ **do**
  for *each* $z \in c_i^*$ **do**
    Find any $j \in Q$ such that $g_j(x_T) = z$;
    Remove j from Q;
    Find a $b \in m^{-1}(i)$ such that $b \in R$;
    Remove b from R;
    Set the element $P^*_{b,j}$ to 1;
  end
end

Thus, finding P^* in Problem 3.1 is equivalent to finding an optimal clustering $c_1^*, \ldots, c_k^*$ for $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, that minimizes (19) subject to $|c_i|$ equaling the number of agents in cell i, for $i = 1, \ldots, k$. ■

Viewing the problem of finding the optimal permutation for a target specification as a size-constrained version of the Euclidean minimum sum-of-squares clustering problem given in Problem 3.2 is very useful because it allows us find the computational complexity associated with the task.

Theorem 3.4: The problem of finding the optimal permutation matrix P^* in Problem 3.1 is NP-hard.

Proof: It was shown in [14] that the Euclidean minimum sum-of-squares clustering problem described in Problem 3.2 is NP-hard by using a reduction from the DENSEST CUT problem for the case of $k = 2$ clusters. Using almost the same procedure, we will show that the optimization version of the MAX BISECTION problem, which was shown in [15] to be NP-hard, reduces to the size-constrained Euclidean minimum sum-of-squares problem for $k = 2$ clusters.

Let $G = (V, E)$ be an undirected graph. Define B_1, B_2 as a partition of V such that $|B_1| = |B_2| = \frac{N}{2}$, where N is assumed to be even. The MAX BISECTION problem is to find B_1^* and B_2^* so as to maximize $|\mathcal{E}(B_1, B_2)|$, where $\mathcal{E}(B_1, B_2) = \{(v_i, v_j) \in E | v_i \in B_1 \text{ and } v_j \in B_2\}$.

Arbitrarily number and orient the edges in E as $e_1, \ldots, e_{|E|}$ so that each e_i is an ordered pair of vertices. Define the incidence matrix $\mathcal{I}$ as a $N \times |E|$ matrix such that for each $e_k = (v_i, v_j) \in E$, $\mathcal{I}_{i,k} = -1$ and $\mathcal{I}_{j,k} = 1$. Have $x_1, \ldots, x_N \in \mathbb{R}^{|E|}$ be such that x_i^T equals the ith row of $\mathcal{I}$. Define the multiset $S = \{x_1, \ldots, x_N\}$. Have c_1, c_2 be two clusters that partition S subject to the size constraint $|c_1| = |c_2| = \frac{N}{2}$. Let B_1 and B_2 be a partition of V, where $B_i = \{v_j | x_j \in c_i\}$, for $i = 1, 2$.

Let the function $\phi_j : \mathbb{R}^{|E|} \to \mathbb{R}$ take in a vector and return the jth element of its argument. Computing the total distortion of the cluster as in (19), we have

$$\begin{aligned} H(c_1, c_2) &= \sum_{i=1}^{2} \sum_{z \in c_i} ||z - \theta(c_i)||^2 \\ &= \sum_{j=1}^{|E|} \sum_{i=1}^{2} \sum_{z \in c_i} \left(\phi_j(z) - \phi_j(\theta(c_i))\right)^2 \end{aligned}$$

If $e_j \in \mathcal{E}(B_1, B_2)$, then either $\phi_j(z)$ equals 1 for exactly one $z \in c_1$ and equals -1 for exactly one $z \in c_2$ with all others equaling 0 and thus $\phi_j(\theta(c_1)) = \frac{2}{N}$ and $\phi_j(\theta(c_2)) = -\frac{2}{N}$, or the same statements above but with c_1 and c_2 switched. Furthermore, if $e_j \notin \mathcal{E}(B_1, B_2)$, then $\phi_j(\theta(c_1)) = \phi_j(\theta(c_2)) = 0$. Using these properties:

$$\begin{aligned} H(c_1, c_2) &= \sum_{e \in \mathcal{E}(B_1, B_2)} \sum_{i=1}^{2} \left(\left(\frac{N}{2} - 1\right)\left(\frac{2}{N}\right)^2 \right. \\ &\quad \left. + \left(1 - \frac{2}{N}\right)^2 \right) + \sum_{e \notin \mathcal{E}(B_1, B_2)} 2 \\ &= 2\left(1 - \frac{2}{N}\right)|\mathcal{E}(B_1, B_2| + 2\left(|\mathcal{E}(B_1, B_1)|\right. \\ &\quad \left. + |\mathcal{E}(B_2, B_2)|\right) \\ &= 2|E| - \frac{4}{N}|\mathcal{E}(B_1, B_2)|. \end{aligned}$$

Choice of B_1^* and B_2^*, or equivalently the choice of c_1^* and c_2^*, that minimizes $H(c_1, c_2)$ also maximizes $|\mathcal{E}(B_1, B_2)|$, since $|E|$ and N are constant. Therefore, the NP-hard MAX BISECTION problem reduces to size-constrained Euclidean minimum sum-of-squares, which is equivalent to finding P^* in Problem 3.1, and so finding P^* is also NP-hard. ■

IV. Finding the Optimal Specification of a Target Network

In order to exploit homogeneity in the controllability of a single-leader network, it is necessary to compute the optimal permutation matrix P^* in Problem 3.1 or equivalently, the optimal clustering $c_1^*, \ldots, c_k^*$ from Theorem 3.3. However, in Theorem 3.4 it was shown that the complexity of the problem in the general case is NP-hard. This section explores heuristic-based and approximation algorithms for solving the general problem, as well as the special case of the problem where agents positions are 1-D.

A. Heuristic and Approximation Algorithms

A popular method for finding locally optimal solutions to the Euclidean sum of squares problem is the k-means algorithm (e.g. [16]). However, Theorem 3.3 adds equality constraints on the size of individual clusters. In [17], a constrained k-means clustering algorithm is proposed that finds locally optimal clusterings which minimize (19), where the minimum size of individual cluster can be specified. Equality constraints on the cluster sizes are imposed when minimum cluster sizes are chosen to sum to N.

B. Special Case: 1-D Networks

Recall that we assumed a network of N agents with a maximal leader-invariant EEP of k cells containing follower agents. In the general multi-dimensional case, finding P^* in Problem 3.1 involves considering at most $N!$ possible permutation matrices, or equivalently $N!$ clusterings by Theorem 3.3. However, in the special case of 1-D networks, only $k!$ clusterings need to be considered by exploiting a special property of constrained clustering in 1-D networks.

Definition 4.1: Given a clustering $c_1, \ldots, c_k$ of a multiset S of 1-D points, a cluster c_i is *compact* if $\nexists x_{i1}, x_{i2} \in c_i$ and $\nexists x_j \in c_j$ such that $x_{i1} < x_j < x_{i2}$, $\forall j \neq i$.

Lemma 4.1: The optimal clustering $c_1^*, \dots, c_k^*$ of a multiset S of 1-D points, which minimizes (19) with $|c_i|$ predefined for $i = 1, \dots, k$, involves only compact clusters.

Proof: We start by showing that elements in every non-compact clustering can always be reassigned to decrease (19) without changing the cluster sizes. Assume $c_1, \dots, c_k$ are not all compact, then $\exists\, x_{a1}, x_{a2} \in c_a$ and $x_b \in c_b$ such that $x_{a1} < x_b < x_{a2}$, for some c_a and c_b where $a \neq b$. Furthermore, define the function

$$H_o\left(c_1, \dots, c_k, m_1, \dots, m_k\right) = \sum_{i=1}^{k} \sum_{z \in c_i} \left(z - m_i\right)^2,$$

where $H\left(c_1, \dots, c_k\right) \leq H_o\left(c_1, \dots, c_k, m_1, \dots, m_k\right)$ with equality if and only if $m_i = \theta\left(c_i\right)$, for $i = 1, \dots, k$. The total distortion cost of the clustering can be rewritten as

$$\begin{aligned} H\left(c_1, \dots, c_k\right) &= H_o\left(c_1, \dots, c_k, \theta\left(c_1\right), \dots, \theta\left(c_k\right)\right) \\ &= Q - 2R\left(c_a, c_b, \theta\left(c_a\right), \theta\left(c_b\right)\right), \end{aligned}$$

where

$$Q = \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{k} |c_i| \theta\left(c_i\right)^2 - 2 \sum_{i=1, i\neq a,b}^{k} \left(\theta\left(c_i\right) \sum_{z \in c_i} z\right)$$

and

$$R\left(c_a, c_b, m_a, m_b\right) = m_a \sum_{z \in c_a} z + m_b \sum_{z \in c_b} z.$$

If $\theta\left(c_a\right) \geq \theta\left(c_b\right)$, assign $\hat{c}_a$ the $|c_a|$ largest elements of $c_a \cup c_b$, while giving $\hat{c}_b$ the remaining elements. Otherwise, if $\theta\left(c_a\right) < \theta\left(c_b\right)$, then let $\hat{c}_a$ have the $|c_a|$ smallest elements of $c_a \cup c_b$, while $\hat{c}_b$ gets the rest. Furthermore, define $\hat{c}_i = c_i \,\forall\, i \neq a, b$. Notice that $|\hat{c}_i| = |c_i|$, for i=1, …, k. Then since after the reassignment, $\theta\left(\hat{c}_a\right) \neq \theta\left(c_a\right)$ and $\theta\left(\hat{c}_b\right) \neq \theta\left(c_b\right)$,

$$\begin{aligned} H\left(c_1, \dots, c_k\right) &= Q - 2R\left(c_a, c_b, \theta\left(c_a\right), \theta\left(c_b\right)\right) \\ &\geq Q - 2R\left(\hat{c}_a, \hat{c}_b, \theta\left(c_a\right), \theta\left(c_b\right)\right) \\ &= H_o\left(\hat{c_1}, \dots, \hat{c_k}, \theta\left(c_1\right), \dots, \theta\left(c_k\right)\right) \\ &> H\left(\hat{c}_1, \dots, \hat{c}_k\right). \end{aligned}$$

Thus, whenever a clustering $c_1, \dots, c_k$ is not all compact, it is always possible to obtain a new clustering $\hat{c}_1, \dots, \hat{c}_k$ with a lower total distortion. Since there are only a finite number of ways to cluster points in S, an optimal cluster must exist and it must involve only compact clusters. ■

Knowing the optimal clustering is compact reduces the number of clusterings that need to be searched.

Theorem 4.1: Finding $c_1^*, \dots, c_k^*$ to minimize (19) for a 1-D network requires considering at most $k!$ clusterings.

Proof: Since points are 1-D, only the ordering of the k compact clusterings matter in finding $c_1^*, \dots, c_k^*$. Thus, at most only $k!$ clusterings need to be considered. ■

V. Conclusions

This paper extended the notion of controllability in a single-leader network to the case of homogeneous agents. By taking advantage of the fact that agents are interchangeable, controllability in the homogeneous setting was phrased as a point to set property of the system, where the set corresponds to all permutations of a target point. It was shown that different permutations of a target point may be at different distances from the system's reachable subspace, and so is more reachable than others. Finding the optimal permutation of a target point that is closest to the network's reachable subspace was shown to be equivalent to solving an Euclidean minimum sum-of-squares clustering problem with constraints on the cluster sizes. However, the task was shown to be NP-hard. As a consequence, methods were presented for finding suboptimal permutations in the general case, as well as finding the optimal permutation in the special case when the network consists of 1-D agents.

Acknowledgments

This work is partially supported by the Office of Naval Research through the MURI Heterogeneous Unmanned Networked Teams (HUNT).

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