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# A Projection Framework for Near-Potential Games

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**Abstract**—Potential games are a special class of games that admit tractable static and dynamic analysis. Intuitively, games that are “close” to a potential game should enjoy somewhat similar properties. This paper formalizes and develops this idea, by introducing a systematic framework for finding potential games that are close to a given arbitrary strategic-form finite game. We show that the sets of exact and weighted potential games (with fixed weights) are subspaces of the space of games, and that for a given game, the closest potential game in these subspaces (possibly subject to additional constraints) can be found using convex optimization. We provide closed-form solutions for the closest potential game in these subspaces, and extend our framework to more general classes of games.

We further investigate and quantify to what extent the static and dynamic features of potential games extend to “near-potential” games. In particular, we show that for a given strategic-form game, we can characterize the approximate equilibria and the sets to which better-response dynamics converges, as a function of the distance of the game to its potential approximation.

## I. INTRODUCTION

Potential games are a class of games with appealing static and dynamic properties. For instance, in such games pure-strategy Nash equilibria always exist, and many of the simple user dynamics (e.g., fictitious play) converge to a Nash equilibrium [1]–[3]. Because of these properties, potential games found numerous applications in various control and resource allocation problems, e.g., [1], [4]–[6]. However, many multi-agent strategic interactions in engineering and economics literatures cannot be directly modelled as a potential game.

Intuitively, games that are “close” to potential games, should inherit some of these static and dynamic properties. In this work, our goal is to provide a systematic framework for identifying a close potential game to some given game, and to study to which extent the properties of this potential game extend to the original game. For this purpose, we focus, in increased order of generality, on three well-known classes of potential games: exact potential games, weighted potential games, and ordinal potential games.

We first characterize the geometry of the problem, by formally defining the vector space of all games, and introducing a natural inner product structure on it. We show that the set of exact potential games, and the set of weighted potential games with fixed weights are *subspaces* of the space of

games (Theorem 2). Therefore, for any given game, the closest exact (or fixed-weight) potential game can be obtained by projection onto the relevant subspace. On the other hand, the set of all weighted potential games, and similarly the set of ordinal potential games, are nonconvex sets. Hence, finding the closest weighted and ordinal potential games to a given game requires solving nonconvex optimization problems.

For any finite game, we provide a *closed-form solution* for the closest fixed-weight potential game, by projecting the original game to the subspace of weighted potential games associated with these weights (Theorem 3). If the weights are unknown, the underlying set is no longer convex. Nevertheless, we present a related convex optimization formulation for finding close weighted potential games (Section IV-B). Although the game obtained through this approach will not necessarily be the closest weighted potential game to the original game, examples show that it often is a very good approximation and yields a weighted potential game whose distance (in terms of utility differences) to the original game is much smaller than that of the closest exact potential game.

Additionally, we show that the approximate equilibria and the better-response dynamics in arbitrary strategic-form finite games can be analyzed using the close weighted and ordinal potential games suggested by our framework (cf. Propositions 5 and 6). The main idea behind this approach is to use the “distance” of the original game from the set of potential games to approximately establish the properties of the original game. Our results indicate that the equilibria of the close potential game can be used to characterize the approximate equilibria of the original game, and the sets to which the update rules converge. Moreover, the closer the original game is to a potential game, the tighter our characterization becomes.

The remainder of this paper is organized as follows: In Section II we present relevant game-theoretic background. In Section III, we establish the geometric properties of the sets of potential games. We discuss different formulations for finding close weighted potential games to a given game in Section IV, followed by a numerical example in Section V. In Section VI, we show how this framework can be used to establish static and dynamic properties of a given near-potential game. We close in Section VII with concluding remarks and future directions.

## II. PRELIMINARIES

In this section we present the game theoretic background that is relevant to our work.

Throughout this paper we consider strategic-form finite games. A (noncooperative) finite game in strategic form

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consists of:

- A finite set of players, denoted by  $\mathcal{M} = \{1, \dots, M\}$ .
- Strategy spaces: A finite set of strategies (or actions)  $E^m$ , for every  $m \in \mathcal{M}$ .
- Utility functions:  $u^m : E \rightarrow \mathbb{R}$ , for every  $m \in \mathcal{M}$ .

A (strategic-form) game instance is accordingly given by the tuple  $\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$ .

We assume that each player in a game has at least two strategies:  $|E^m| \geq 2$ , for all  $m \in \mathcal{M}$ . The joint strategy space of a game is denoted by  $E = \prod_{m \in \mathcal{M}} E^m$ . We refer to a collection of strategies of all players as a *strategy profile* and denote it by  $\mathbf{p} = (p^1, \dots, p^M) \in E$ . The strategies of all players but the  $m$ th one is denoted by  $\mathbf{p}^{-m}$ .

The basic solution concept in a noncooperative game is that of a Nash Equilibrium (NE). A (pure) Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally, a strategy profile  $\mathbf{p} = (p^1, \dots, p^M)$  is a Nash equilibrium if

$$u^m(p^m, \mathbf{p}^{-m}) \geq u^m(q^m, \mathbf{p}^{-m}),$$

for every  $q^m \in E^m$  and  $m \in \mathcal{M}$ .

To address strategy profiles that are approximately a Nash equilibrium, we use the concept of  $\epsilon$ -equilibrium. A strategy profile  $\mathbf{p} = (p^1, \dots, p^M)$  is an  $\epsilon$ -equilibrium if

$$u^m(p^m, \mathbf{p}^{-m}) \geq u^m(q^m, \mathbf{p}^{-m}) - \epsilon$$

for every  $q^m \in E^m$  and  $m \in \mathcal{M}$ . Note that a Nash equilibrium is an  $\epsilon$ -equilibrium with  $\epsilon = 0$ .

We next define potential games [1], which are central to our discussion in the subsequent sections.

**Definition 2.1 (Potential Games):** Consider a noncooperative game  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$ . If there exists a function  $\phi : E \rightarrow \mathbb{R}$  such that for every  $m \in \mathcal{M}$ ,  $p^m, q^m \in E^m$ ,  $\mathbf{p}^{-m} \in E^{-m}$ ,

- 1)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m})$ , then  $\mathcal{G}$  is an *exact potential game*.
- 2)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) = w_m(u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}))$ , for some strictly positive weight  $w_m > 0$ , then  $\mathcal{G}$  is a *weighted potential game*.
- 3)  $\phi(p^m, \mathbf{p}^{-m}) - \phi(q^m, \mathbf{p}^{-m}) > 0 \Leftrightarrow u^m(p^m, \mathbf{p}^{-m}) - u^m(q^m, \mathbf{p}^{-m}) > 0$ , then  $\mathcal{G}$  is an *ordinal potential game*.

The function  $\phi$  is referred to as a *potential function* of the game. This definition suggests that potential games are games in which the interests of players are captured by a global potential function  $\phi$ .

Note that every exact potential game is a weighted potential game with  $w_m = 1$  for all  $m \in \mathcal{M}$ . From the definitions it also follows that every weighted potential game is an ordinal potential game. In other words, ordinal potential games generalize weighted potential games, and weighted potential games generalize exact potential games.

We denote the sets of exact, weighted, and ordinal potential games by  $\mathcal{P}$ ,  $\mathcal{WP}$ ,  $\mathcal{OP}$  respectively. In order to characterize  $\mathcal{WP}$ , it is sufficient to consider weights  $w_m \geq 1$ . This follows since in any weighted potential game, the

potential function and the weights can be jointly scaled by a positive scalar to obtain a different potential function and larger weights. Given a fixed set of weights  $w = \{w_m\}$ , we refer to the set of all weighted potential games with these weights as *fixed-weight* potential games, and denote this set by  $\mathcal{P}_w$ . In particular, the set of exact potential games is equivalent to  $\mathcal{P}_1$ . It can be seen that the set of weighted potential games can be expressed as a union of the sets of fixed-weight potential games over weights  $w_m \geq 1$  for all  $m \in \mathcal{M}$ , i.e.  $\mathcal{WP} = \cup_{w \geq 1} \mathcal{P}_w$ .

We conclude this section by providing necessary and sufficient conditions for a game to be an exact or ordinal potential game. We first introduce some key concepts used in establishing these conditions.

**Definition 2.2 (Path – Closed Path – Improvement Path):**

A path is a collection of strategy profiles  $\gamma = (\mathbf{p}_0, \dots, \mathbf{p}_N)$  such that  $\mathbf{p}_i$  and  $\mathbf{p}_{i+1}$  differ in the strategy of exactly one player. A path is a closed path (or a cycle) if  $\mathbf{p}_0 = \mathbf{p}_N$ . A path is an improvement path if  $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$  where  $m_i$  is the player who modifies its strategy when the strategy profile is updated from  $\mathbf{p}_{i-1}$  to  $\mathbf{p}_i$ .

The transition from strategy profile  $\mathbf{p}_{i-1}$  to  $\mathbf{p}_i$  is referred to as step  $i$  of the path. We refer to a closed improvement path such that the inequality  $u^{m_i}(\mathbf{p}_i) \geq u^{m_i}(\mathbf{p}_{i-1})$  is strict for at least a single step of the path, as a *weak improvement cycle*. We say that a closed path is *simple* if no strategy profile other than the first and the last strategy profiles is repeated along the path. For any path  $\gamma = (\mathbf{p}_0, \dots, \mathbf{p}_N)$ , let  $I(\gamma)$  represent the “utility improvement” along the path. Namely

$$I(\gamma) = \sum_{i=1}^N u^{m_i}(\mathbf{p}_i) - u^{m_i}(\mathbf{p}_{i-1}),$$

where  $m_i$  is the index of the player that modifies its strategy in the  $i$ th step of the path.

The following proposition provides an alternative characterization of exact and ordinal potential games. This characterization will be used in studying the geometry of sets of different classes of potential games (cf. Theorem 2).

**Proposition 1 ([1], [7]):** (i) A finite game  $\mathcal{G}$  is an exact potential game if and only if  $I(\gamma) = 0$  for all simple closed paths  $\gamma$ .

(ii) A finite game  $\mathcal{G}$  is an ordinal potential game if and only if it does not include weak improvement cycles.

### III. SETS OF POTENTIAL GAMES

In this section, we investigate the properties and the geometry of potential games. In particular, we show that the sets of weighted and ordinal potential games are nonconvex subsets of the space of games, but the set of exact potential games is a subspace.

We first provide a formal definition of the space of games. Consider games with set of players  $\mathcal{M}$ , set of strategy profiles  $E \triangleq E^1 \times \dots \times E^M$ . We denote by  $C_0 = \{f|f : E \rightarrow \mathbb{R}\}$ , the set of utility functions that can be present in a game with set of strategy profiles  $E$ . Since every function  $f \in C_0$  has a finite domain, it can be represented as a vector

in  $\mathbb{R}^{|E|}$ . Note that every utility function in a game with set of strategy profiles  $E$  and set of players  $\mathcal{M}$ , belongs to  $C_0$ , i.e.,  $u^m \in C_0$  for all  $m \in \mathcal{M}$ . We denote the set of all games with set of players  $\mathcal{M}$  and set of strategy profiles  $E$  by  $\mathcal{G}_{\mathcal{M},E}$ . Any two games in this space differ only in their payoff functions, thus this space can alternatively be identified by  $C_0^{|\mathcal{M}|} \approx \mathbb{R}^{|E||\mathcal{M}|}$ , i.e., the set of all payoff functions that define such games.

Before we present our result on the sets of potential games, we introduce the notion of convexity for sets of games.

**Definition 3.1:** Let  $B \subset \mathcal{G}_{\mathcal{M},E}$ . The set  $B$  is said to be convex if and only if for any two game instances  $\mathcal{G}_1, \mathcal{G}_2 \in B$  with collections of utilities  $u = \{u^m\}_{m \in \mathcal{M}}, v = \{v^m\}_{m \in \mathcal{M}}$  respectively, and for all  $\alpha \in [0, 1]$ ,

$$\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{\alpha u^m + (1 - \alpha)v^m\}_{m \in \mathcal{M}} \rangle \in B.$$

Below we show that the set of exact potential games is a subspace of the space of games, whereas, the sets of weighted and ordinal potential games are nonconvex.

**Theorem 2:** (i) The sets of exact potential games,  $\mathcal{P}$ , and fixed-weight potential games,  $\mathcal{P}_w$ , are subspaces of  $\mathcal{G}_{\mathcal{M},E}$ .

(ii) The sets of weighted potential games,  $\mathcal{WP}$ , and ordinal potential games,  $\mathcal{OP}$ , are nonconvex subsets of  $\mathcal{G}_{\mathcal{M},E}$ .

*Proof:* (i) Proposition 1 (i) implies that the set of exact potential games is the subset of space of games where the utility functions satisfy the condition  $I(\gamma) = 0$  for every simple closed path  $\gamma$ . Note that for each  $\gamma$ ,  $I(\gamma) = 0$  is a linear equality constraint on the utility functions  $\{u^m\}$ . Thus, the set of exact potential games is the intersection of the sets defined by these linear equality constraints. Therefore, it is a subspace of the space of games,  $\mathcal{G}_{\mathcal{M},E} \approx C_0^{|\mathcal{M}|}$ .

It can be seen from Definition 2.1 that similar to exact potential games, fixed-weight potential games are characterized by linear equality constraints. Thus, the proof for  $\mathcal{P}_w$  follows similarly.

(ii) We prove the claim by showing that the convex combination of two weighted potential games is not an ordinal potential game. This implies that the sets of both weighted and ordinal potential games are nonconvex since every weighted potential game is an ordinal potential game.

In Table I we present the payoffs and the potential in a two player game,  $\mathcal{G}_1$ , where each player has two strategies. Given strategies of both players the first table shows payoffs of players (the first number denotes the payoff of the first player), the second table shows the corresponding potential function. In both tables the first column stands for actions of the first player and the top row stands for actions of the second player. Note that this game is a weighted potential game with weights  $w_1 = 1, w_2 = 3$ .

$(u^1, u^2)$	A	B
A	0,0	0,4
B	2,0	8,6

$\phi$	A	B
A	0	12
B	2	20

TABLE I

PAYOFFS AND POTENTIAL IN  $\mathcal{G}_1$

Similarly, another game  $\mathcal{G}_2$  is defined in Table II. Note that this game is also a weighted potential game with weights

$(u^1, u^2)$	A	B
A	4,2	6,0
B	0,8	0,0

$\phi$	A	B
A	20	18
B	8	0

TABLE II

PAYOFFS AND POTENTIAL IN  $\mathcal{G}_2$

$(u^1, u^2)$	A	B
A	2,1	3,2
B	1,4	4,3

TABLE III

PAYOFFS IN  $\mathcal{G}_3$

$w_1 = 3, w_2 = 1$ . In Table III, we consider a game  $\mathcal{G}_3$ , in which the payoffs are averages (hence convex combinations) of payoffs of  $\mathcal{G}_1$  and  $\mathcal{G}_2$ .

Note that this game has a weak improvement cycle:

$$(A, A) - (A, B) - (B, B) - (B, A) - (A, A).$$

From Proposition 1 (ii), it follows that  $\mathcal{G}_3$  is not an ordinal potential game.

The above example shows that the sets of weighted and ordinal potential games with two players each of which has two strategies is nonconvex. For general  $n$  player games, the claim immediately follows by constructing two  $n$  player weighted potential games, and embedding  $\mathcal{G}_1$  and  $\mathcal{G}_2$  in these games. The details are omitted due to page constraints. ■

We next discuss the geometry of the sets of weighted and ordinal potential games. The above theorem together with  $\mathcal{WP} = \cup_{w \geq 1} \mathcal{P}_w$ , implies that the set of weighted potential games is an uncountable union of subspaces of  $\mathcal{G}_{\mathcal{M},E}$ . Given an ordinal potential game, the multiplication of the utilities by a positive scalar gives a game with the same weak improvement cycles. Since the original game is an ordinal potential game and does not have a weak improvement cycle, the scaled game cannot have a weak improvement cycle. Thus, the scaled game is an ordinal potential game as well. Therefore, we conclude that  $\mathcal{OP}$  is a cone. However, Theorem 2 implies that this is not a convex cone.

#### IV. PROJECTION TO THE SET OF POTENTIAL GAMES

In this section, we develop a framework for finding close weighted and ordinal potential games to a given game. It was shown in Section III that the fixed-weight potential games and exact potential games form subspaces. In Section IV-A, we develop a projection framework which gives closed form solution for the closest fixed-weight potential game to a given game. In Section IV-B, we discuss methods for choosing weights to obtain better weighted potential game approximations. Due to space constraints, in this section proofs are omitted, and they can be found in [8].

##### A. Fixed-Weight Potential Games

In this section, we obtain closed form solution of the closest fixed-weight potential game to a given game. Before presenting our results, we first introduce some necessary operators and notation.

For each player  $m \in \mathcal{M}$ , we define the difference operator  $D_m$  such that for all  $f : E \rightarrow \mathbb{R}$  and  $\mathbf{p}, \mathbf{q} \in E$  that differ in

the strategy of only player  $m$ ,

$$(D_m f)(\mathbf{p}, \mathbf{q}) = f(\mathbf{q}) - f(\mathbf{p}),$$

and  $(D_m f)(\mathbf{p}, \mathbf{q}) = 0$  otherwise. The difference operators allow for alternative definitions of potential games: A game is a weighted potential game with weights  $\{w_m\}$ , if there exists a potential function  $\phi$ , such that  $D_m u^m = w_m D_m \phi$  for all  $m \in \mathcal{M}$  (it is an exact potential game if all weights are equal to 1).

As mentioned earlier, the set of functions with domain  $E$ , is denoted by  $C_0$  and this set can equivalently be represented as a vector in  $\mathbb{R}^{|E|}$ . Motivated by this, we use the regular inner product of  $\mathbb{R}^{|E|}$  in  $C_0$ , i.e., the inner product of any  $f_1, f_2 \in C_0$  is given by  $\langle f_1, f_2 \rangle = \sum_{\mathbf{p} \in E} f_1(\mathbf{p}) f_2(\mathbf{p})$ .

For each player  $m$ , we define the set  $A_m = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p}, \mathbf{q} \in E \text{ differ in the strategy of only player } m\}$  and we denote the union of these sets by  $A = \cup_m A_m$ . We refer to functions  $X : E \times E \rightarrow \mathbb{R}$  such that  $X(\mathbf{p}, \mathbf{q}) = X(\mathbf{q}, \mathbf{p}) \in \mathbb{R}$  if  $(\mathbf{p}, \mathbf{q}) \in A$  and  $X(\mathbf{p}, \mathbf{q}) = 0$  otherwise as the pairwise ranking functions. We denote the space of all such functions by  $C_1$  and associate with it the inner product  $\langle X, Y \rangle = \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q})$ , where  $X, Y \in C_1$ . For all  $m \in \mathcal{M}$  and  $f \in C_0$ , it can be seen that  $(D_m f) \in C_1$ .

The adjoint of  $D_m$  is the unique operator  $D_m^*$  which satisfies,

$$\langle X, D_m f \rangle = \langle D_m^* X, f \rangle,$$

for all  $f : E \rightarrow \mathbb{R}$ ,  $X : E \times E \rightarrow \mathbb{R}$ . Using the definitions of the inner products and the difference operators, it follows that for all  $X \in C_1$  the operator  $D_m^*$  satisfies,

$$(D_m^* X)(\mathbf{p}) = - \sum_{\mathbf{q} \mid (\mathbf{p}, \mathbf{q}) \in A_m} X(\mathbf{p}, \mathbf{q}).$$

Three operators that are related to the difference operator and its adjoint are  $\Delta_{0,m} = D_m^* D_m$ ,  $\Delta_0 = \sum_m \Delta_{0,m}$  and  $\Pi_m = D_m^\dagger D_m$ , where  $\dagger$  denotes the pseudo inverse. The first two operators correspond to Laplacian operators on a graph associated with the game and the third operator is a projection operator to the orthogonal complement of the kernel of the difference operator  $D_m$  (for details see [9]).

We next present an optimization problem that finds the closest weighted potential game with weights  $\{w_m\}$  to a given game. We quantify the distance between two games with utility functions  $\{u^m\}$  and  $\{\hat{u}^m\}$  as  $\sum_{m \in \mathcal{M}} h_m \|u^m - \hat{u}^m\|^2$ , where we use the norm induced by the inner product in the space of payoff functions  $C_0$ , and  $h_m = |E^m|$ . It can be seen that  $\sum_{m \in \mathcal{M}} h_m \|u^m\|^2$  corresponds to a weighted  $l_2$  norm in the space of games  $\mathcal{G}_{\mathcal{M}, E}$  (see [9]). Given a game with payoff functions  $\{u^m\}$ , the closest weighted potential game with weights  $\{w_m\}$  and utility functions  $\{\hat{u}^m\}$  is the solution of the following least-squares optimization problem:

$$\begin{aligned} \min_{\phi, \{\hat{u}^m\}} \quad & \sum_m h_m \|u^m - \hat{u}^m\|^2 \\ \text{s.t.} \quad & D_m \phi = w_m D_m \hat{u}^m, \\ & \text{for all } m \in \mathcal{M}. \end{aligned} \quad (\text{P1 :})$$

Since all the constraints are linear, the feasible set of  $\{\hat{u}^m\}$  in this optimization problem is a subspace of  $C_0^{|\mathcal{M}|}$ . Thus, the optimal  $\{\hat{u}^m\}$  is unique and given by the projection of the original game to the feasible subspace with respect to the norm in the objective function. The next theorem characterizes the optimal solution of this problem.

**Theorem 3:** The optimal solution of P1 is given by  $\phi = \left( \sum_m \frac{1}{w_m^2} \Delta_{0,m} \right)^\dagger \sum_m \frac{1}{w_m} \Delta_{0,m} u^m$  and  $\hat{u}^m = \frac{1}{w_m} \Pi_m \phi + (I - \Pi_m) u^m$  for all players  $m \in \mathcal{M}$ .

If all weights are equal to 1, the above theorem provides a closed form solution for projection to the set of exact potential games.

### B. Choosing Weights for the Potential Game Approximation

In this section we discuss methods for choosing weights to obtain a close weighted potential game to a given game. As shown in Section III,  $\mathcal{WP}$  is nonconvex, implying that finding the closest weighted potential game to an arbitrary game involves solving nonconvex optimization problems. In particular, for any given game, the solution of problem P1 provides the closest fixed-weight potential game. The best weights (and the closest potential game to the original game) can be found by minimizing the optimal objective value of P1 over all weights. Using the closed form solution of P1, its optimal objective value as a function of the weights can be given as:

$$\sum_m h_m \left\| \Pi_m u^m - \Pi_m \left( \sum_k \frac{1}{w_k^2} \Delta_{0,k} \right)^\dagger \sum_k \frac{1}{w_k} \Delta_{0,k} u^k \right\|^2.$$

This is a nonconvex function of the weights, and hence solving for the optimal weights requires minimizing the above nonconvex function over the set of possible weights.

We next consider an alternative formulation for finding a close weighted potential game. For a given set of weights  $\{w_m\}$ , we first consider the problem

$$\begin{aligned} \min_{\phi, \{\hat{u}^m\}} \quad & \sum_m h_m \|w_m u^m - w_m \hat{u}^m\|^2 \\ \text{s.t.} \quad & D_m \phi = w_m D_m \hat{u}^m, \\ & \text{for all } m \in \mathcal{M}. \end{aligned} \quad (\text{P2 :})$$

It can be seen that P2 is a convex optimization problem, and it gives a weighted potential game approximation of the original game. To obtain a good approximation, we are interested in solving P2 with the weights that minimize its optimal objective value. Solving P2 with the best weights is equivalent to introducing a new variable  $\hat{v}^m = w_m \hat{u}^m$  and solving the following convex optimization problem:

$$\begin{aligned} \min_{\phi, \{w_m\}, \{\hat{v}^m\}} \quad & \sum_m h_m \|w_m u^m - \hat{v}^m\|^2 \\ \text{s.t.} \quad & D_m \phi = D_m \hat{v}^m, \\ & \text{for all } m \in \mathcal{M}. \end{aligned}$$

Using the solution of this problem, the optimal utilities corresponding to the best weights in P2 can be recovered from  $\hat{u}^m = \frac{1}{w_m} \hat{v}^m$ . Note that in the new formulation the

optimal weights and the utilities can be obtained by solving a convex optimization problem.

We next obtain a closed form solution for P2. Note that the optimization formulations P1 and P2 are related: Consider a set of utility functions  $\{u^m\}$  and assume that for a fixed set of weights  $\{w_m\}$ , the solution of P2 is given by  $\{\hat{u}^m\}$  and  $\phi$ . Comparing the optimization formulations P1 and P2, it can be seen that the closest exact potential game, in the sense of P1, to the game with utility functions  $\{w_m u^m\}$  is given by  $\{w_m \hat{u}^m\}$  and this game also has a potential function  $\phi$ . Hence, the solution of P2 can be obtained by: (i) scaling the utility functions of the original game by weights  $\{w_m\}$ , (ii) solving P1 with the scaled utility functions and unit weights, and (iii) scaling the utility functions that solve P1 by weights  $\left\{\frac{1}{w_m}\right\}$ . The following theorem, uses this observation to characterize the optimal solution of P2.

**Theorem 4:** The optimal solution of P2 is given by  $\phi = \Delta_0^\dagger \sum_m \Delta_{0,m} w_m u^m$  and  $\hat{u}^m = \frac{1}{w_m} \Pi_m \phi + (I - \Pi_m) u^m$  for all players  $m \in \mathcal{M}$ .

Using this theorem, it follows that for a fixed set of weights, the optimal cost of P2 is

$$\psi(w) \triangleq \sum_m h_m \left\| w_m \Pi_m u^m - \Pi_m \Delta_0^\dagger \sum_m \Delta_{0,m} w_m u^m \right\|^2,$$

where  $w$  denotes the vector of all weights  $\{w_m\}$ . Note that  $\psi$  is a convex function of the weights. Hence, weights that will lead to a better potential game approximation can be obtained by solving

$$\min_w \psi(w) \quad \text{s.t. } w_m \geq 1 \quad \text{for all } m \in \mathcal{M}. \quad (1)$$

We refer to the solution of P2, using the optimal weights with respect to (1), as the *best weighted potential game approximation in the sense of P2*.

If the original game is a weighted potential game, it can be shown that the best weighted potential game approximation of this game is itself (see [8] for a proof). Note that this also provides a way of checking whether a game is a weighted potential game or not: Given a game if the optimal objective value of (1) is equal to zero, then the original game is a weighted potential game, with weights that achieve this objective value.

We next present an alternative motivation for the formulation P2. Note that the strategic considerations in a game (such as the equilibria, and the best responses) do not change if utility functions in the game are scaled by different positive scalars. Hence, it makes sense to consider projections of scaled utility functions (with different weights) to the set of exact potential games, as formalized through problem P2. This approach provides a higher degree of freedom and leads potentially to a smaller distance than the distance from the closest exact potential game. We show in Section VI that tighter approximations of both static and dynamic properties of a game can be obtained through this approach.

As shown in Section III, the set of ordinal potential games is a nonconvex subset of  $\mathcal{G}_{\mathcal{M},E}$ . Because of this, finding the closest ordinal potential game to a given game

also requires solving a nonconvex optimization problem. However, it is possible to develop a convex optimization formulation, similar to the one we proposed for weighted potential games, in order to find a close ordinal potential game. Details are omitted due to page constraints and can be found in [8].

We conclude this section by mentioning that the framework presented here for finding close potential games to a given game is for finite games, and it does not immediately extend to continuous games. This can be seen as for continuous games, problems (P1) and (P2) become infinite dimensional optimization problems. We leave developing a similar framework for continuous games as a future goal.

## V. EXAMPLE

In this section, we compare the methods for finding close potential games proposed in the previous section on a specific example.

We focus on two-player games where each player has three strategies. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

For any  $\alpha \in [0, 1]$ , let  $\mathcal{G}_\alpha$  denote the game with payoff matrices  $(u^1, u^2) = (\alpha \cdot 3A + (1-\alpha) \cdot B, \alpha \cdot A + (1-\alpha) \cdot 3B)$ . It can be seen from this definition that  $\mathcal{G}_0$  is a weighted potential game with payoff matrices  $(u^1, u^2) = (B, 3B)$  and weights  $(3, 1)$  and similarly  $\mathcal{G}_1$  is a weighted potential game with payoff matrices  $(u^1, u^2) = (3A, A)$  and weights  $(1, 3)$ .

In our simulations, for different values of  $\alpha$ , we consider the game  $\mathcal{G}_\alpha$  and compute (i) the closest exact potential game (ii) the best weighted potential game approximation using the formulation P2 (iii) the closest weighted potential game to  $\mathcal{G}_\alpha$ . Here, we obtain the closest weighted potential game to  $\mathcal{G}_\alpha$  by exhaustive search over weights  $(w_1, w_2)$ , and the corresponding solution of P1. We calculate the distance from the original game to each of these games, by quantifying the distance between the games in terms of the norm defined in the space of games (for  $\mathcal{G}$  with payoffs  $\{u^m\}$ ,  $\|\mathcal{G}\|^2 = \sum_m h_m \|u^m\|^2$ ).

In Figure 1 we compare the distances between the original game and the close potential games obtained by different methods, as a function of  $\alpha$ . We see that the best weighted potential game approximation in the sense of P2, closely approximates the closest weighted potential game, and it improves significantly over the closest exact potential game.

## VI. STATIC AND DYNAMIC PROPERTIES OF NEAR-POTENTIAL GAMES

In this section we relate the static and dynamic properties of a given game with those of a nearby potential game. The proofs of the results in this section are omitted due to page constraints. Proofs as well as extensions of the presented results can be found in [8] and [10].

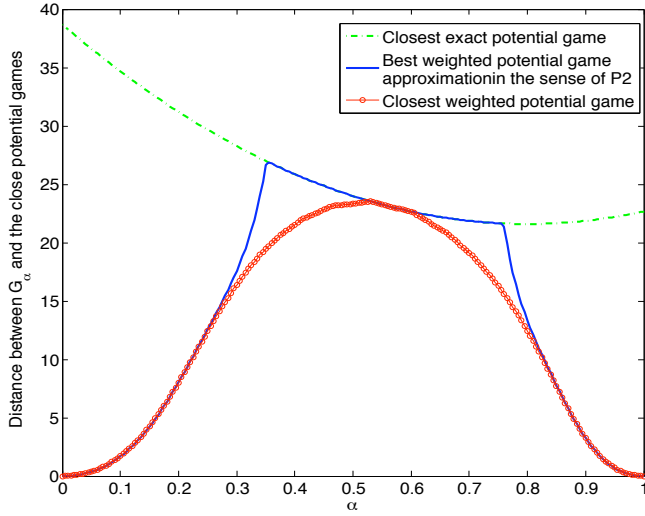


Fig. 1. Comparison of distances between the original game and the close potential games obtained by different methods.

### A. Static Properties

If two games have similar payoff functions, then their approximate equilibria are closely related. In this section, we use this property to study their sets of equilibria. The following lemma characterizes the  $\epsilon$ -equilibria of a game in terms of the  $\epsilon$ -equilibria of a nearby game.

**Lemma 1:** Let  $\mathcal{G}$  and  $\hat{\mathcal{G}}$  be two games with set of players  $\mathcal{M}$ , strategy profiles  $E$  and utility functions  $\{u^m\}$  and  $\{\hat{u}^m\}$  respectively. Assume that  $|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \epsilon_0$  for every  $m \in \mathcal{M}$  and  $\mathbf{p} \in E$ . Then,

(i) For any two strategy profiles  $\mathbf{p}$ , and  $\mathbf{q}$  that only differ in the strategy of player  $k$ , it follows that  $u^k(\mathbf{q}) - u^k(\mathbf{p}) \leq \hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p}) + 2\epsilon_0$ .

(ii) Every  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$  is an  $\epsilon$ -equilibrium of  $\mathcal{G}$  with  $\epsilon \leq 2\epsilon_0 + \epsilon_1$ .

The previous lemma can be used to characterize the approximate equilibrium set of an arbitrary game in terms of the approximate equilibrium set of a close potential game.

**Proposition 5:** Let a game  $\mathcal{G}$  and a set of weights  $\{w_m\}$  be given. For  $\mathcal{G}$  and these weights, denote the game which solves P2 by  $\hat{\mathcal{G}}$  and the corresponding objective value in P2 by  $\psi(w) = \alpha^2$ . Assume that  $h_m$  denotes the number of strategies player  $m$  has. Then, every  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$  is an  $\epsilon$ -equilibrium of  $\mathcal{G}$  with  $\epsilon \leq \epsilon_1 + 2 \max_m \frac{\alpha}{w_m \sqrt{h_m}}$ .

### B. Dynamic Properties

We next show that dynamics in an arbitrary game can be studied using a close potential game. Before stating our result, we introduce the approximate better-response dynamics:

**Definition 6.1 ( $\epsilon$ -Better-Response Dynamics):** At each time step  $t$ , a single player is chosen at random for updating its strategy, using a probability distribution with full support over the set of players. Let  $m$  be the player chosen at some time  $t$ , and let  $\mathbf{p} \in E$  denote the strategy profile at  $t$ . If  $u^m(\mathbf{p}) < \max_q u^m(q, \mathbf{p}^{-m}) - \epsilon$

then player  $m$  updates its strategy to a strategy  $q^m \in \{q \in E^m \mid u^m(q, \mathbf{p}^{-m}) > u^m(\mathbf{p}) + \epsilon\}$ , where the new strategy is chosen uniformly at random from this set; otherwise player  $m$  does not modify its strategy.

The following proposition shows that in an arbitrary game, approximate convergence of the better-response dynamics can be studied using the properties of a close potential game.

**Proposition 6:** Let a game  $\mathcal{G}$  and a set of weights  $\{w_m\}$  be given. For  $\mathcal{G}$  and these weights, denote the game which solves P2 by  $\hat{\mathcal{G}}$  and the corresponding objective value in P2 by  $\psi(w) = \alpha^2$ . Assume that  $\epsilon \geq 2 \max_m \frac{\alpha}{w_m \sqrt{h_m}}$ . In  $\mathcal{G}$ , the  $\epsilon$ -better-response dynamics is confined in the  $\epsilon$ -equilibrium set after finite time, with probability 1.

## VII. CONCLUSIONS

We introduced a geometric framework for the analysis of arbitrary games in terms of “close” potential games. We showed that the sets of weighted and ordinal potential games are nonconvex, as opposed to fixed-weight potential games, which form a subspace. Using this fact, we obtained closed-form solutions for the closest exact and fixed-weight potential games to a given game. Additionally, we introduced a convex optimization formulation which provides a close weighted potential game with arbitrary weights to any given strategic-form finite game.

Our results show that the static and dynamic properties of an arbitrary game can be analyzed by employing the properties of potential games that are close to it. Moreover, the proposed scheme for finding a close weighted potential game results in a tighter characterization of static and dynamic properties of a game, when compared to the results that can be obtained using exact potential games. We leave the application of our potential game approximation framework to analysis of various update rules and additional static properties, such as efficiency notions, as a future goal.

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