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Consensus in the network with uniform constant communication delay

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Abstract— **This paper studies the consensus among identical agents that are at most critically unstable and coupled through networks with uniform constant communication delay. An achievable upper bound of delay tolerance is obtained which explicitly depends on agent dynamics and network topology. The dependence on network topology disappears in the case of undirected networks. For any delay satisfying this upper bound, a controller design methodology without exact knowledge of the network topology is proposed so that the multi-agent consensus in a set of unknown networks can be achieved. Moreover, when the network topology is known, a larger delay tolerance is possible via a topology-dependent consensus controller. The results are illustrated by simulations.**

I. Introduction

The consensus problem in a network has received substantial attention in recent years, partly due to the wide applications in areas such as sensor networks and autonomous vehicle control. A relatively complete coverage of earlier work can be found in the survey paper [8], the recent books [15], [11] and references therein.

Consensus in the network with time delay has been extensively studied in the literature. Most results consider the agent model as described by single-integrator dynamics ([1], [12], [9]), or double-integrator dynamics ([13], [5], [2]). Specifically, it is shown in [9] that a network of single-integrator agents subject to uniform constant communication delay can achieve consensus with a particular linear local control protocol if and only if the delay is bounded by a maximum that is inversely proportional to the largest eigenvalue of the graph Laplacian associated with the network. This result was later on generalized in [1] to non-uniform constant or time-varying delays. Sufficient conditions for consensus among agents with first order dynamics were also obtained in [12]. The results in [9] were extended in [5], [2] to double integrator dynamics. An upper bound on the maximum network delay tolerance for second-order consensus of multi-agent systems with any given linear control protocol was obtained.

In this paper, we study the multi-agent consensus problem with uniform constant communication delay. The agents are assumed to be at most critically unstable, i.e. each agent has all its eigenvalues in the closed left half plane. The contribution of this paper with respect to [9], [1], [5], [2] is twofold: first, we find a sufficient condition on the tolerable communication delay for agents with high-order dynamics, which has an explicit dependence on the agent dynamics and network topology. For undirected network, this upper bound can be independent of the network topology provided that the network is connected. Moreover, in a special case where the agents only have zero eigenvalues, such as single- and double-integrator dynamics, arbitrarily large but bounded delay can be tolerated. Another layer of contribution is that for delays satisfying the proposed upper bound, we present a controller design methodology without exact knowledge of network topology so that multi-agent consensus in a set of unknown networks can be achieved. When the network topology is precisely known, the controller design can be modified to be topology-dependent and a larger delay tolerance is attainable. Some additional work exists on nonuniform communication delays, see [6], [7]. These address only very specific cases and no general design methodology exists at this moment.

The rest of the paper is organized as follows: notations and some preliminary results are declared in the remainder of Section I. System and network configuration and consensus problem formulations are given in Section II. The consensus problem with full-state coupling is solved in Section III. The corresponding problem with partial-state coupling is dealt with in Section IV. In Section V, we discuss the special of neutrally stable systems. Some technical lemmas are appended at the end of this paper.

A. Notations and Preliminaries

For a vector d, we denote a diagonal matrix by D=diag$\{d\}$ where the diagonal is specified by d. For column vectors $x_1, \ldots, x_n$, the stacking column vector of $x_1, \ldots, x_n$ is denoted by $[x_1; \ldots; x_n]$.

A graph G is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *arc*. G is *undirected* if $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$. Otherwise, G is *directed*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

The graph G is *weighted* if each arc (i, j) is assigned with a real number a_{ij}. For a weighted graph G, a matrix

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$L = \{\ell_{ij}\}$ with

$$\ell_{ij} = \begin{cases} \sum_{j=1}^{N} a_{ij}, & i = j \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with graph G. In the case where G has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (see [4]). If G has a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues have strictly positive real parts (see e.g. [10]).

II. Problem formulation

Consider a network of N identical agents

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \quad i = 1, \ldots, N, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} x^j(t-\tau). \end{cases} \tag{1}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $z^i \in \mathbb{R}^n$, $\tau > 0$ is an unknown constant satisfying $\tau \in [0, \bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^{N} \ell_{ij}$. In (1), each agent collects a delayed measurement z^i of the state of neighboring agents through the network, which we refer to as *full-state coupling*.

It is also common that z_i may consist of the output of neighboring agents instead of the complete state which can be formulated as follows:

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \\ y^i(t) = Cx^i(t), \qquad\qquad i = 1, \ldots, N, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} y^j(t-\tau), \end{cases} \tag{2}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $y^i, z^i \in \mathbb{R}^p$. We refer to the agents in this case as having *partial-state coupling*.

The matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N \times N}$ defines the *communication topology* which can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E})$ where $(j,i) \in \mathcal{E} \Leftrightarrow \ell_{ij} < 0$ and $a_{ii} = 0$ and $a_{ij} = -\ell_{ij}$ for $i \neq j$. The G is directed in general. However, in a special case where L is symmetric, G is undirected. This L is the the Laplacian matrix associated with G.

Assumption 1: The following assumptions are made throughout the paper:

1) The agents are at most critically unstable, that is, A has all its eigenvalues in the closed left half plane;
2) (A, B) is stabilizable and (A, C) is detectable;
3) The communication topology described by the graph G contains a directed spanning tree.

Remark 1: As noted before, under Assumption 1, L has one simple eigenvalue in zero and the others lie in the open left half plane.

It should be noted that in practice, perfect information of the communication topology is usually not available for controller design and that only some rough characterization of the network can be obtained. Using the non-zero eigenvalues of L as a "measure" for the graph, we can introduce the following definition to characterize a set of unknown communication topologies. Let $\lambda_1, \cdots, \lambda_N$ denoted the eigenvalues of L and assume $\lambda_1 = 0$.

Definition 1: For any $\gamma \geq \beta \geq 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of graphs satisfying Assumption 1 and whose associated Laplacian satisfies

$$|\lambda_i| \in (\beta, \gamma) \text{ and } \arg \lambda_i \in [-\varphi, \varphi]$$

for $i = 2, \ldots, N$.

Definition 2: The agents in the network achieve consensus if

$$\lim_{t \to \infty} (x^i(t) - x^j(t)) = 0, \quad \forall i, j \in \{1, \ldots, N\}.$$

Two consensus problems for agents with full-state coupling (1) and partial-state coupling (2) respectively can be formulated for this set of networks as follows:

Problem 1: Consider a network of agents (1) with full state coupling. The consensus problem given a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear static controllers $u^i = Fz^i$ for $i = 1, \ldots, N$ such that the agents (1) with $u^i = Fz^i$ achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

Problem 2: Consider a network of agents (2) with partial state coupling. The consensus problem with a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear dynamic control protocols of the form:

$$\begin{cases} \dot{\chi}^i = A_c \chi^i + B_c z^i \\ u^i = C_c \chi^i, \end{cases} \tag{3}$$

for $i = 1, \ldots, N$ such that the agents (2) with controller (3) achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

III. Consensus with full-state coupling

In this section, we consider agents with full-state coupling as given in (1) and solve Problems 1.

For a given set of networks $\mathcal{G}_{\beta,\gamma,\varphi}$, we design a decentralized local consensus controller for any network in $\mathcal{G}_{\beta,\gamma,\varphi}$ as follows:

$$u^i = \alpha F_\varepsilon z^i, \tag{4}$$

where $F_\varepsilon = B' P_\varepsilon$. Here P_ε is the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A' P_\varepsilon + P_\varepsilon A - P_\varepsilon B B' P_\varepsilon + \varepsilon I = 0. \tag{5}$$

and ε, as well as α, are design parameters which will be chosen according to β, γ and φ so that the multi-agent consensus can be achieved with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. Let

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Hurwitz.} \\ \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0\}, & \text{otherwise.} \end{cases}$$

The first main result of this paper is stated in the next theorem which solves the network consensus problem with respect to $\mathcal{G}_{\beta,\gamma,\varphi}$.

Theorem 1: For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (1) and any coupling network belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case Problem 1 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{6}$$

Moreover, it can be solved by the consensus controller (4) if (6) holds. Specifically, for given $\mathcal{G}_{\beta,\gamma,\varphi}$ and given $\bar{\tau}$ satisfying (6), there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for this α and any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (4) achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma 1 in the Appendix that Theorem 1 holds if for any $\gamma \geq \beta > 0$, $\bar{\tau} > 0$ and φ satisfying (6), there exist $\alpha > 0$ and ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\dot{x} = Ax - \lambda\alpha e^{j\psi} BF_\varepsilon x(t-\tau). \tag{7}$$

is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.

Since $\bar{\tau}$ and φ satisfy condition (6), there exists an $\alpha > 0$ such that

$$\arccos(\tfrac{1}{\alpha\beta}) > \varphi + \omega_{\max}\bar{\tau}. \tag{8}$$

Let this α be fixed. By Lemma 4, (8) implies that $\alpha\lambda\cos(\varphi) > 1$, and hence, $A - \alpha\lambda e^{j\psi} BF_\varepsilon$ is Hurwitz stable for $\psi \in [-\varphi, \varphi]$. Then it follows from Lemma 3 that system (7) is asymptotically stable if

$$\det\left[j\omega I - A + \alpha\lambda e^{j(\psi-\omega\tau)} BF_\varepsilon \right] \neq 0, \tag{9}$$

for $\omega \in \mathbb{R}$, $\tau \in [0, \bar{\tau}]$ and $\psi \in [-\varphi, \varphi]$.

First, we note that given (8), there exists a $\delta > 0$ such that

$$\arccos(\tfrac{1}{2\alpha\beta}) > \varphi + \omega\bar{\tau}, \quad \forall |\omega| < \omega_{\max} + \delta. \tag{10}$$

Next we will split the proof of (9) in two cases where $|\omega| < \omega_{\max} + \delta$ and $|\omega| \geq \omega_{\max} + \delta$ respectively.

If $|\omega| \geq \omega_{\max} + \delta$, we have $\det(j\omega I - A) \neq 0$, in another word, $\underline{\sigma}(j\omega I - A) > 0$. Hence, there exists $\mu > 0$ such that

$$\underline{\sigma}(j\omega I - A) > \mu, \quad \forall \omega, \text{ s.t. } |\omega| \geq \omega_{\max} + \delta.$$

To see this, note that for ω satisfying $|\omega| > \bar{\omega} := \max\{\|A\| + 1, \omega_{\max} + \delta\}$,

$$\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1.$$

But for ω with $|\omega| \in [\omega_{\max} + \delta, \bar{\omega}]$, there exists $\mu \in (0, 1]$ such that

$$\underline{\sigma}(j\omega I - A) \geq \mu,$$

which is due to the fact that $\underline{\sigma}(j\omega I - A)$ depends continuously on ω. Given α and $\lambda \in (\beta, \gamma)$, there exists $\varepsilon^* > 0$ such that $\|\lambda\alpha BF_\varepsilon\| \leq \mu/2$ for $\varepsilon < \varepsilon^*$. Then

$$\underline{\sigma}(j\omega I - A + \alpha\lambda e^{j(\psi-\omega\tau)} BF_\varepsilon) \geq \mu - \mu/2 \geq \mu/2.$$

Therefore, condition (9) holds for $|\omega| \geq \omega_{\max} + \delta$.

It remains to verify (9) with $|\omega| < \omega_{\max} + \delta$. By the definition of δ, we find that

$$\alpha\lambda\cos(\psi - \omega\tau) > \alpha\beta\cos(\varphi + |\omega|\bar{\tau}) > \tfrac{1}{2},$$

and hence by Lemma 4, $A - \alpha\lambda e^{j(\psi-\omega\tau)} BF_\varepsilon$ is Hurwitz stable, for $\omega \in (-\omega_{\max} - \delta, \omega_{\max} + \delta)$, $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. Therefore, (9) also holds with $|\omega| < \omega_{\max} + \delta$. ■

Remark 2: Some comments on implementation of the consensus controller (4) are worthwhile. Four parameters are chosen sequentially in the consensus design and analysis, namely α, δ, μ and ε. First, we select the scaling parameter α in (8) using the given data β, φ and $\omega_{\max}$. Then, δ is chosen based on network data and the choice of α and such a δ will yield corresponding value of μ. Eventually, ε is determined by μ and γ.

Remark 3: The consensus controller design depends only on the agent model and parameters $\bar{\tau}, \beta, \gamma$ and φ and is independent of specific network topology provided that the network satisfies Assumption 1.

In the special case where $\omega_{\max} = 0$, i.e. the eigenvalues of A are either zero or in the open left half plane, then arbitrarily bounded communication delay can be tolerated as formulated in the following corollary:

Corollary 1: For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (1) and any communication topology belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 1 is always solvable via the consensus controller (4). Specifically, for given $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, there exist α and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (4) achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Remark 4: In the previous study of network consensus problem, agents are normally assumed to have single- or double-integrator type dynamics. Based on Corollary 1, we find that the delay tolerance in such cases is independent of network topology and can be made arbitrarily large. This result in no way contradicts that in [9], [5], [2] since the goal here is to find the maximal achievable delay tolerance by controller design whereas obtained in [9], [5], [2] are the conditions on delay for which the consensus with certain given controller is not spoiled.

Remark 5: Note that for undirected and connected networks, the Laplacian associated with G is symmetric and hence has only real eigenvalues, i.e. we can set $\varphi = 0$. In this case, the upper bound of tolerable delay is independent of network topology. However, in directed networks, we have to sacrifice some robustness in the delay tolerance in order to cope with the complex part of Laplacian eigenvalues.

IV. Consensus with partial-state coupling

Next, we proceed to the case of partial-state coupling and design a dynamic consensus controller (3) which solves Problem 2.

For $\varepsilon > 0$, let P_ε be the positive definite solution of the ARE (5). A dynamic low-gain consensus controller can be constructed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \alpha B' P_\varepsilon \chi^i, \end{cases} \tag{11}$$

where K is such that $A + KC$ is Hurwitz stable. α and ε are design parameters to be chosen later. We shall prove that the consensus controller solves Problem 2.

Theorem 2: For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (2) with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case, Problem 2 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{12}$$

Moreover, it can be solved by the consensus controller (11) if (12) holds. Specifically, for given β and γ and given φ and $\bar{\tau}$ satisfying (12), there exist $\alpha > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (2) with controller (11) achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma 2 in the Appendix that Theorem 2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \alpha\lambda e^{j\psi} BB' P_\varepsilon \chi(t-\tau) \\ \dot{\chi}(t) = (A + KC)\chi(t) - KCx(t) \end{cases} \tag{13}$$

is asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A + KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_\varepsilon = \begin{bmatrix} 0 & -B' P_\varepsilon \end{bmatrix}.$$

First of all, for $\lambda \in (\beta, \gamma)$ and $\psi \in (-\varphi, \varphi)$, there exists α such that

$$\lambda\alpha \cos(\varphi + \omega_{\max}\bar{\tau}) > 2. \tag{14}$$

Let this α be fixed. By Lemma 5 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathcal{A}} + \alpha\lambda e^{j\psi} \bar{\mathcal{B}} \bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable for $\lambda \in (\beta, \gamma)$ and $\psi \in (-\varphi, \varphi)$. It follows from Lemma 3 that (13) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathcal{A}} - \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{B}} \bar{\mathcal{F}}_\varepsilon \right] \neq 0, \ \forall \omega \in \mathbb{R}, \\ \forall \lambda \in (\beta, \gamma), \ \forall \psi \in (-\varphi, \varphi), \ \forall \tau \in [0, \bar{\tau}]. \tag{15}$$

Given (14), there exists $\delta > 0$ such that

$$\lambda\alpha \cos(\varphi + \omega\bar{\tau}) > 1, \quad \forall |\omega| < \omega_{\max} + \delta. \tag{16}$$

We can show, as in the proof of Theorem 1, that there exists $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (15) holds for $|\omega| \geq \omega_{\max} + \delta$.

For $|\omega| < \omega_{\max} + \delta$, it follows from (16) and Lemma 5 that $\bar{\mathcal{A}} + \alpha\lambda e^{j(\psi - \omega\tau)} \bar{\mathcal{B}} \bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (15) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

Remark 6: The low-gain compensator (11) is constructed based on the agent model and the network characteristics β, γ and φ. The four parameters α, δ, μ and ε used in the design of controller (11) are chosen with the same order and relation as in the proof of Theorem 1.

The next corollary is concerned with the case $\omega_{\max} = 0$ where the eigenvalues of A are either zero or in the open left half plane.

Corollary 2: For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (2) with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\pi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 2 is solvable by the consensus controller (11). Specifically, for given β, γ, φ and $\bar{\tau} > 0$, there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (2) with controller (11) achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

V. Special case: Neutrally stable agents

We observe that the consensus controller design in Theorem 1 and Theorem 2 for general critically unstable agents depends on β which is related to the algebraic connectivity of the graph. We next consider a special case where the agent dynamics are neutrally stable, that is, the eigenvalues of A on the imaginary axis, if any, are semi-simple. Without loss of generality, we assume that $A' + A \leq 0$ which can be obtained after a suitable basis transformation. In this case, we shall show that the consensus controller design no longer requires the knowledge of β and hence allows us to deal with a larger set of unknown communication topologies that can be denoted as $\mathcal{G}_{0,\gamma,\varphi}$.

Consider the agents (1). Assume $A' + A \leq 0$. A local consensus controller can be constructed as

$$u^i = \varepsilon B' z^i. \tag{17}$$

We have the following theorem:

Theorem 3: For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (1) and any communication topology belonging to the set $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A' + A \leq 0$. In that case, Problem 1 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}, \tag{18}$$

Moreover, it can be solved by the consensus controller (17) if (18) holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying (18), there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (17) achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma 1 that Theorem 3 holds if the system

$$\dot{x} = Ax - \lambda\varepsilon e^{j\psi} BB' x(t-\tau) \tag{19}$$

is asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$, which, by Lemma 3, is true if and only if

$$\det\left[j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB' \right] \neq 0, \\ \forall \omega \in \mathbb{R}, \ \lambda \in (0, \gamma), \ \psi \in [-\varphi, \varphi], \ \tau \in [0, \bar{\tau}]. \tag{20}$$

There exists $\delta > 0$ such that

$$\omega\bar{\tau} + \varphi < \tfrac{\pi}{2}, \quad \forall \omega \text{ s.t. } |\omega| < \omega_{\max} + \delta.$$

For given $\lambda \in (0, \gamma)$, we can show with a similar argument as in the proof of Theorem 1 that there exists a $\mu > 0$ and a ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$ and $\lambda \in (0, \gamma)$

$$\underline{\sigma}(j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB') > \mu, \forall \omega \text{ s.t. } |\omega| \geq \omega_{\max} + \delta.$$

Hence, (20) is satisfied with $|\omega| \geq \omega_{\max} + \delta$.

It remains to show (20) for $|\omega| < \omega_{\max} + \delta$. Note that $\psi - \omega\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by definition of δ and hence $\cos(\psi - \omega\tau) > 0$. Then

$$[A - \varepsilon\lambda e^{j\psi - j\omega\tau} BB']^* + [A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'] = -2\lambda\varepsilon\cos(\psi - \omega\tau) BB' \leq 0$$

Since (A, B) is controllable, we conclude that $A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'$ is Hurwitz, and hence (20) also holds, with $|\omega| < \omega_{\max} + \delta$. ∎

The next theorem addresses the consensus problem for networks with partial state coupling. In this case, a low-gain consensus controller can be designed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \varepsilon B'\chi^i, \end{cases} \tag{21}$$

where K is such that $A + KC$ is Hurwitz.

Theorem 4: For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (2) with any communication topology belonging to $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A + A' \leq 0$. In that case, Problem 2 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{22}$$

Moreover, it can be solved by the consensus controller (21) if (22) holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying (22), there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (2) with controller (21) achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof: It follows from Lemma 2 in the Appendix that Theorem 2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \varepsilon\lambda e^{j\psi} BB'\chi(t - \tau) \\ \dot{\chi}(t) = (A + KC)\chi(t) - KCx(t) \end{cases} \tag{23}$$

is asymptotically stable for any $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A + KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_\varepsilon = \begin{bmatrix} 0 & -\varepsilon B' \end{bmatrix}.$$

By Lemma 6 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathcal{A}} + \alpha\lambda e^{j\psi}\bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. It follows from Lemma 3 that (23) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathcal{A}} - \alpha\lambda e^{j(\psi - \omega\tau)}\bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon\right] \neq 0, \ \forall\omega \in \mathbb{R}, \quad \forall\lambda \in (\beta, \gamma), \ \forall\psi \in (-\varphi, \varphi), \ \forall\tau \in [0, \bar{\tau}]. \tag{24}$$

Similarly as before, there exist $\delta > 0$ and $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (24) holds for $|\omega| \geq \omega_{\max} + \delta$.

On the other hand, $|\omega| < \omega_{\max} + \delta$, it follows from Lemma 6 that $\bar{\mathcal{A}} + \alpha\lambda e^{j(\psi - \omega\tau)}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (24) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

VI. Concluding remarks

In this paper, we study the multi-agent consensus with uniform constant communication delay for agents with high-order dynamics. A sufficient condition on delay is derived under which the multi-agent consensus is attainable. Whenever this condition is satisfied, a controller without the exact knowledge of network topology can be constructed such that consensus can be achieved in a set of networks.

Although this paper focuses on unknown communication topologies, when the perfect information about the topology is in fact available, the design procedure can be easily modified to achieve a stronger result. In this case, input u_i to each agents can be first scaled as $u_i = d_i\bar{u}_i$ where these d_i are such that $\mathrm{diag}\{d_i\}L$ has a simple eigenvalue at zero and the rest are real and strictly positive. The existence of such d_is is proved by [3]. Then we can design $\bar{u}_i$ following the procedure proposed in this paper.

Future research will continuous in two directions: 1. extend the results to non-identical agents; 2. consider non-uniform and time-varying delay.

Appendix

Due to space limitation, all the proofs have been deleted, which can be found in the full version [14].

A. Connection of network consensus to robust stabilization

Lemma 1: Problem 1 is solvable via consensus controller $u^i = Fz^i$ if the following $N - 1$ systems

$$\dot{\xi}^i(t) = A\xi^i(t) - \lambda_i BF\xi^i(t - \tau) \tag{A.1}$$

are asymptotically stable where λ_i, $i = 2, ..., N$ are the non-zero eigenvalues of the Laplacian associated with the communication topology.

Lemma 2: Problem 2 is solvable via consensus controller (3) if the following $N - 1$ systems

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) - \lambda_i BC_c\xi^i(t - \tau) \\ \dot{\chi}^i(t) = A_c\chi^i(t) + B_c z^i(t) \end{cases} \tag{A.2}$$

are asymptotically stable where λ_i for $i = 2, \ldots, N$ are the non-zero eigenvalues of the Laplacian matrix L.

The following lemma is adapted from [16].

Lemma 3: Consider a linear time-delay system

$$\dot{x} = Ax + A_d x(t - \tau). \tag{A.3}$$

Assume $A + A_d$ is Hurwitz. We have that (A.3) is globally asymptotically stable for $\tau \in [0, \bar{\tau}]$ if

$$\det\left[j\omega I - A - e^{-j\omega\tau}A_d\right] \neq 0, \quad \forall\omega \in \mathbb{R}, \ \forall\tau \in [0, \bar{\tau}],$$

for all $\omega \in \mathbb{R}$ and $\tau \in [0, \bar{\tau}]$.

In this subsection, we recall some classical robust properties of low-gain feedback and compensator. Consider an uncertain system

$$\begin{cases} \dot{x} = Ax + \mu Bu \\ y = Cx, \end{cases} \tag{A.4}$$

where (A, B) is stabilizable, (A, C) is detectable and A has all its eigenvalues in the closed left half plane. The $\mu \in \mathbb{C}$ is

input uncertainty. For $\varepsilon > 0$, let P_ε be the positive definite solution of ARE

$$A'P_\varepsilon + AP_\varepsilon - P_\varepsilon B'BP_\varepsilon + \varepsilon I = 0.$$

The robustness of a low-gain state feedback $u = -B'P_\varepsilon x$ is inherited from that of a classical LQR.

Lemma 4: $A - \mu BB'P_\varepsilon$ is Hurwitz stable for any $\mu \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \frac{1}{2}\}$.

The next lemma proves similar property of a low-gain compensator.

Lemma 5: For any *a priori* given bounded set

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\},$$

there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (A.4) and the low-gain compensator

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -B'P_\varepsilon \chi \end{cases} \tag{A.5}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

Lemma 6: Consider system (A.4). Suppose $A' + A \leq 0$. For any *a priori* given $\varphi \in (0, \frac{\pi}{2})$ and a bounded set

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid s \neq 0, \ \ \arg(s) \in [-\varphi, \varphi]\},$$

there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (A.4) and the low-gain compensator

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -\varepsilon B'\chi \end{cases} \tag{A.6}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

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