Physical Realizability Conditions for Mixed Bilinear-Linear Quantum Cascades with Pure Field Coupling*

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***Abstract*— This paper aims to provide conditions under which a quantum stochastic differential equation can serve as a model for interconnection of a bilinear system evolving on an operator group $SU(2)$ and a linear quantum system representing a quantum harmonic oscillator. To answer this question we derive algebraic conditions for the preservation of canonical commutation relations (CCRs) of quantum stochastic differential equations (QSDE) having a subset of system variables satisfying the harmonic oscillator CCRs, and the remaining variables obeying the CCRs of $SU(2)$. Then, it is shown that from the physical realizability point of view such QSDEs correspond to bilinear-linear quantum cascades.**

I. Introduction

In many applications, systems are interconnected in order to form more complex systems. Open quantum systems are not the exception. For instance, non-classical propagating electromagnetic fields, as now experimentally realizable, are an important resource in linear optics quantum information processing [3]. They can be constructed by cascading a two-level quantum system, as a source, with a cavity (quantum harmonic operator system) which filters the signals from the two-level system. In this case, the two-level system and the oscillator are separated by a transmission line such that there is no direct interaction between their system variables [7] (Figure 1). From a control perspective, such apparatus are of great importance. For instance, a natural question is whether it is possible to estimate the states of a source system via a simpler oscillator system, the latter playing a role of a Luenberger observer. The answer to such question is by no means obvious, and it primarily depends on how one choses to describe the quantum nature of the comprising systems and the interconnection itself.

It has been established that the framework of QSDEs provides an alternative description for studying quantum systems, in which it allows the translation of standard control techniques into a quantum mechanical framework [1], [6], [9], [15], [17], [18], [21]–[24]. The QSDE description is in agreement with the *Heisenberg picture* of quantum systems [20]. Not every QSDE describes a quantum system (for instance, CCRs are not satisfied necessarily), however there exist conditions under which linear and bilinear QSDEs obey quantum mechanical laws, namely *physical realizability conditions* [10], [11], [15]. Physical realizability conditions provide simple testable matrix conditions containing the essentials for a system to be considered quantum. In this context, quantum oscillators are described by linear QSDEs and two-level systems are described by bilinear QSDEs. However, the the task of, for example, observing a physically realizable two level system with a physically realizable linear QSDE by cascading requires first of all to ensure the physical realizability of the composite system. Such cascade system goes beyond the realm in which the physical realizability of linear and bilinear QSDEs has been studied so far. Therefore, it is important to consider *mixed* physical realizability conditions. That is to say, it is required a testable condition for the physical realizability of cascade bilinear-linear systems having a subset of system variables satisfying the harmonic oscillator CCRs, and the remaining variables obeying the CCRs of a two level system (i.e., the CCRs of $SU(2)$ [10], [19]). An analysis of this type also provides a glimpse of the full characterization of bilinear QSDEs with additive and multiplicative quantum noise as open quantum systems.

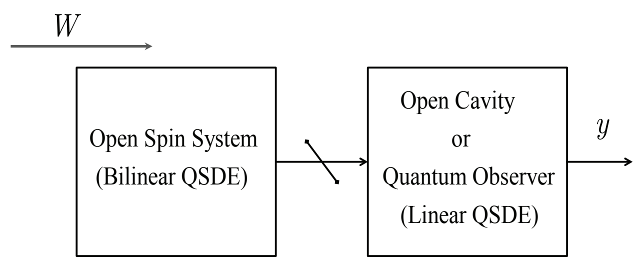

Fig. 1: Non interacting bilinear-linear quantum cascade open to a field W.

The earliest work on a systematic approach to cascade quantum systems can be trace to [4], [12]. In [13], the treatment of the quantum cascading problem was extended in a manner that completely characterizes the dynamics of the composite system from a network point of view. This setting is natural from the engineering point of view where the decomposition of systems plays a fundamental role in systems analysis and synthesis. This approach has been proved valuable since it shows explicitly the interacting field channels, and hence interconnections via those channels can be constructed in a natural manner. In contrast, the more

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standard way of describing quantum systems via evolution of a density operator does not allow a network methodology explicitly, because the interacting channels are averaged out and therefore the interconnection cannot be described directly. One way to keep track of the information about the coupling channels is through the Belavkin filter [1], but this approach requires measurements such as homodyne or heterodyne detection [22]. Using such measurements is precluded when the objective is coherent control, i.e., when the controller or observer is itself a quantum system [17]. Still the approach in [13] starts from a purely quantum description to then using QSDEs to give the description of the cascade in terms of quantum operators, which is the opposite to what physical realizability conditions provide. In other words, it is desired for control applications to find conditions under which a cascaded QSDE preserves the physical realizability conditions of the composite systems (quantum coherent cascades, in our case), and therefore allow to identify the underlying quantum operators, when they exist, governing the dynamics of the cascade. In this regard, the goal of this paper is twofold. First, the aim is to obtain conditions for the preservation of physical realizability of bilinear QSDEs having both additive and multiplicative quantum noise inputs, and having initial conditions satisfying mixed CCRs (a combination between the harmonic oscillator and finite level systems CCRs). The second goal is to provide necessary and sufficient conditions for the physical realizability of the bilinear-linear cascade of QSDEs.

The paper is organized as follows. Section II presents the basic preliminaries on open quantum systems, in particular, harmonic oscillator systems, two level systems and cascade of systems. In Section III, the algebraic machinery is given. This is followed by Section IV, in which the result on the preservation of mixed CCRs for bilinear QSDE with additive and multiplicative noise is developed. In Section V, the physical realizability of bilinear-linear QSDE cascades is analyzed. Finally, Section VI gives the conclusions and future research directions to follow.

II. Open Quantum Systems and their Cascade

A. Notation

Let $\mathbb{R}$ denote the real numbers and $\mathbb{C}$ the complex numbers with imaginary unit $\boldsymbol{i}$. The set of real and complex n-dimensional vectors are denoted $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. The set of real and complex n by m matrices are denoted $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$. The n-dimensional identity matrix is denoted by I_n, and the $n\times m$ dimensional zero matrix is $0_{n\times m}$. A separable Hilbert space is denoted by $\mathfrak{H}$. The set of operators in $\mathfrak{H}$ is denoted by $\mathfrak{T}(\mathfrak{H})$, the set of n dimensional vectors of operators in $\mathfrak{T}(\mathfrak{H})$ is denoted by $\mathfrak{T}(\mathfrak{H})^n$ and the set of $n\times m$ dimensional arrays of operators in $\mathfrak{T}(\mathfrak{H})$ is denoted by $\mathfrak{T}(\mathfrak{H})^{n\times m}$. The operator $\hat{I}$ denotes the identity in $\mathfrak{T}(\mathfrak{H})$. The operation $[\cdot,\cdot]:\mathfrak{T}(\mathfrak{H})\times\mathfrak{T}(\mathfrak{H})\to\mathfrak{T}(\mathfrak{H})$ is known as the commutator, and it is defined as $[x,y]=xy-yx$. For vectors $x\in\mathfrak{T}(\mathfrak{H})^n$ and $y\in\mathfrak{T}(\mathfrak{H})^m$ the commutator is given as

$$[x,y^T]\triangleq xy^T-(yx^T)^T\in\mathfrak{T}(\mathfrak{H})^{n\times m},$$

$x^{\#}\triangleq(x_1^*\,x_2^*\,\ldots\,x_n^*)^T$, $x^{\dagger}=(x^{\#})^T$, $(\cdot)^T$ denotes the transpose operation and $(\cdot)^*$ denotes the adjoint (or the complex conjugate in the case of complex vectors or matrices). On a quantum mechanical framework, it is common to multiply either vectors or matrices by arrays of operators. For example, let $A\in\mathbb{C}^{m\times n}$ and $X\in\mathfrak{T}(\mathfrak{H})^{n\times m}$, the (i,j) element of the multiplication of a matrix by an operator matrix is

$$(AX)_{ij}=\sum_{k=1}^{n}a_{ik}x_{kj}\in\mathfrak{T}(\mathfrak{H}).$$

obeys the usual matrix multiplication rules. These considerations allow to treat operators as system variables since in quantum mechanics they play the role of states, and therefeore allow us to use state space systems notation.

Remark: The operations between complex matrices and operators follow the guidelines of the standard *canonical quantization* [5], which in simple words is a recipe that promotes the system variables from a classical mechanical framework into an operator framework in order to obtain a quantum mechanical description of the system.

B. Open quantum systems

Quantum systems interacting with an external environment are known as *open quantum systems*. *Observables* in a Hilbert space $\mathfrak{H}$ represent physical quantities that can be measured, while quantum states give the current status of the system. Here open quantum systems are treated in the context of quantum stochastic processes [2], [20]. The non-commutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain CCRs, which lead to the *Heisenberg uncertainty principle* [16]. The environment consists of a collection of oscillator systems, each with the annihilation field operator $w(t)$ and the creation field operator $w^*(t)$ used for annihilation and creation of quanta at point t, and commonly known as the *boson quantum field* (a quantum version of a Wiener process). Here it is assumed that t is a real time parameter. These operators generate three interacting signals in the evolution of the system: the annihilation processes $W(t)$, the creation process $W^{\dagger}(t)$, and the counting process $\Lambda(t)$.

The unitary evolution of an observable $X\in\mathfrak{T}(\mathfrak{H})$ in the *Heisenberg picture* is described by the operator equation

$$X(t)=U^{\dagger}(t)(X\otimes\hat{I})\,U(t),\qquad(1)$$

where $U(t)$ is unitary for all t, and is the solution of the operator stochastic differential equation

$$dU(t)=\bigg((S-\hat{I})\,d\Lambda(t)+L\,dW^{\dagger}(t)-L^{\dagger}S\,dW(t)\\ -\frac{1}{2}(L^{\dagger}L+\boldsymbol{i}\mathcal{H})\,dt\bigg)\,U(t),$$

with initial condition $U(0)=\hat{I}$. $\mathcal{H}$ denotes the system *Hamiltonian* of the system, and L and S (unitary) determine the *coupling* of the system to the field and the interaction between fields, respectively. For simplicity, this paper will consider only one interactiong field W. Using the *quantum Itô formula* for $X_1, X_2 \in \mathfrak{T}(\mathfrak{H})$ [14], i.e.

$$d(X_1X_2) = (dX_1)X_2 + X_1(dX_2) + (dX_1)(dX_2), \tag{2}$$

the dynamics of (1) is expressed as

$$\begin{aligned} dX = &(S^\dagger XS - X)\, d\Lambda + \mathcal{L}(X)\, dt + S^\dagger[X, L]\, dW^\dagger \\ &+ [L^\dagger, X]S\, dW, \end{aligned} \tag{3}$$

where $\mathcal{L}(X)$ is the Lindblad operator defined as

$$\mathcal{L}(X) = -\boldsymbol{i}[X, \mathcal{H}] + \frac{1}{2}\left(L^\dagger[X, L] + [L^\dagger, X]L\right). \tag{4}$$

The output field is given by $Y(t) = U(t)^\dagger W(t)U(t)$, which amount to

$$dY = Ldt + SdW. \tag{5}$$

The dynamics of an open quantum systems is usually parametrized by the triple $(S, L, \mathcal{H})$. Henceforth assume that $S = \hat{I}$.

It is often convenient to express QSDEs in terms of quadrature fields, which make all system matrices real. This is provided by the following linear transformation of the interacting fields

$$\begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\boldsymbol{i} & \boldsymbol{i} \end{pmatrix} \begin{pmatrix} W \\ W^\dagger \end{pmatrix}, \tag{6}$$

where the operators $\bar{W}_1$ and $\bar{W}_2$ are now self-adjoint. Moreover, the Itô table (see [14]) for these quadrature fields is

$$\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 & d\bar{W}_2 \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{i} \\ -\boldsymbol{i} & 1 \end{pmatrix} dt. \tag{7}$$

Similarly, the quadrature form of the output fields can be obtained from the same quadrature transformation. Thus,

$$\begin{pmatrix} dY_1 \\ dY_2 \end{pmatrix} = \begin{pmatrix} L + L^\# \\ \boldsymbol{i}(L^\# - L) \end{pmatrix} dt + \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix}. \tag{8}$$

C. Linear open quantum systems

The Hilbert space for this class of systems is $\mathfrak{H}_1 = \ell^2(\mathbb{C})$ (the space of square integrable complex sequences) [9], and the vector of system variables is

$$x_1 \in \mathfrak{T}(\mathfrak{H}_1)^{2n}, \tag{9}$$

For instance, a single harmonic oscillator system variables in terms of the annihilator operator a and creation operator $a^\dagger$ is written in self-adjoint form $x_1 \in \mathfrak{T}(\mathfrak{H}_1)^2$ by using the transformation

$$x_1 = \begin{pmatrix} 1 & 1 \\ -\boldsymbol{i} & \boldsymbol{i} \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{10}$$

The CCRs for a and $a^\dagger$ are $[a, a] = [a^\dagger, a^\dagger] = 0$ and $[a, a^\dagger] = 1$. For a vector of n creation and n annihilator operators, one has that

$$[x_1, x_1^T] = \left[\begin{pmatrix} a_1 \\ a_1^\dagger \\ \vdots \\ a_n \\ a_n^\dagger \end{pmatrix}, (a_1\ a_1^\dagger\ \ldots\ a_n\ a_n^\dagger)\right] = (I_n \otimes J),$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In self-adjoint form, by applying (10), the CCRs are

$$[x_1, x_1^T] = 2\boldsymbol{i}\underbrace{(I_n \otimes J)}_{\triangleq\, \Theta}. \tag{11}$$

The Hamiltonian for this class of systems is the quadratic form $\mathcal{H}_1 = x_1^T R x_1$ with R real symmetric, and the coupling operator is considered to be linear, i.e., $L_1 = \Gamma_1 x_1$. The general form for the QSDE having these Hamiltonian an coupling operator is

$$dx_1 = Ax_1\, dt + B\, d\bar{W} \tag{12a}$$
$$dy_1 = Cx_1\, dt + d\bar{W}, \tag{12b}$$

where $A \in \mathbb{R}^{3\times3}$, $B \in \mathbb{R}^{n\times2}$ and $C \in \mathbb{R}^{2\times n}$, and $\bar{W} = (\bar{W}_1\ \bar{W}_2)^T$.

For system (12) to have any hope of being quantum mechanical, it is fundamental that system (12) preserves (11) over time. The next theorem gives conditions for the preservation of CCRs of x_1 over time.

Theorem 1: (See [9], [15].) QSDE (12a) with system variables as in (9) satisfying $[x_1(0), x_1(0)^T] = 2\boldsymbol{i}\Theta$ implies $[x_1(t), x_1(t)^T] = 2\boldsymbol{i}\Theta$ for all $t \geq 0$ if and only if

$$A\Theta + \Theta A^T + BJB^T = 0, \tag{13}$$

D. Two level open quantum system

For an open two-level quantum system interacting with one boson quantum field, the Hilbert space is $\mathfrak{H}_2 = \mathbb{C}^2$ and the vector of system variables is

$$x_2 \in \mathfrak{T}(\mathfrak{H}_2)^3, \tag{14}$$

Note that operators in $\mathfrak{T}(\mathfrak{H}_2)$ are simply matrices in $\mathbb{C}^{2\times2}$. These operators are chosen to be self-adjoint, so that x_2 satisfies $x_2 = x_2^\#$. In particular, an operator $\hat{\sigma} \in \mathfrak{T}(\mathfrak{H}_2)$ is spanned by the Pauli matrices [19], i.e., $\hat{\sigma} = \frac{1}{2}\sum_{i=0}^3 \kappa_i\sigma_i$, where $\kappa_0 = \mathrm{Tr}(\hat{\sigma})$, $\kappa_i = \mathrm{Tr}(\hat{\sigma}\sigma_i)$, and

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -\boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli matrices. Thus, $\kappa_0, \kappa_1, \kappa_2$ and κ_3 determine uniquely the operator $\hat{\sigma}$. The product of Pauli matrices satisfy

$$\sigma_i\sigma_j = \delta_{ij}I_3 + \boldsymbol{i}\sum_k \epsilon_{ijk}\sigma_k, \tag{15}$$

and therefore its CCRs are

$$[\sigma_i, \sigma_j] = 2\boldsymbol{i}\sum_k \epsilon_{ijk}\sigma_k, \tag{16}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} denotes the Levi-Civita tensor. Given that (15) allows to write any product Pauli operators as linear forms, a large class of polynomial quantum systems can be characterized by considering linear Hamiltonian and coupling operators, i.e., $\mathcal{H}_2 = \alpha_2 x_2$ and $L_2 = \Gamma_2 x_2$, where $\alpha_2^T \in \mathbb{R}^3$ and $\Gamma_2^T \in \mathbb{C}^3$.

Observe that, in general, the evolution of x_2 is a bilinear QSDEs with only multiplicative quantum noise expressed as

$$dx_2 = A_0\, dt + Ax_2\, dt + B_1 x_2\, d\bar{W}_1 + B_2 x_2\, d\bar{W}_2, \quad (17a)$$
$$dy_2 = Cx_2\, dt + d\bar{W}, \quad (17b)$$

where $A_0 \in \mathbb{R}^3$, $A, B_1, B_2 \in \mathbb{R}^{3\times3}$ and $C \in \mathbb{R}^{2\times n}$. Conditions for CCR preservation of x_2 are given in the next theorem.

Theorem 2: (See [10], [11].) QSDE (17a) with system variables as in (14) satisfying $[x_2(0), x_2(0)^T] = 2\boldsymbol{i}\Theta^-(x_2(0))$ implies $[x_2(t), x_2(t)^T] = 2\boldsymbol{i}\Theta^-(x_2)$ for all $t \geq 0$ if and only if

$$B_1 + B_1^T = B_2 + B_2^T = 0 \quad (18a)$$
$$B_1 B_2^T - B_2 B_1^T - \Theta(A_0) = 0 \quad (18b)$$
$$A^T + A + B_1 B_1{}^T + B_2 B_2{}^T = 0. \quad (18c)$$

The fact that all matrices in systems (12) and (17) are real is due to the quadrature transformation (6).

E. Cascades of open quantum systems

If the cascade connection of a two level system and a linear quantum system is considered, the composite system lives in $\mathfrak{H}_{12} = \mathfrak{H}_1 \otimes \mathfrak{H}_2 = \ell^2(\mathbb{C}) \otimes \mathbb{C}^2$, which is the completion of the direct product of $\ell^2(\mathbb{C})$ and $\mathbb{C}^2$. In this construction the system variables in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ when embedded in $\mathfrak{H}_{12}$ commute between each other. The cascade of open quantum systems is described by an algebraic operation on the $(S, L, \mathcal{H})$ parametrization. Such operation is defined next.

Definition 1: (See [13].) Given two open quantum systems parametrized by $\mathbf{G_1} = (S_1, L_1, \mathcal{H}_1)$ and $\mathbf{G_2} = (S_2, L_2, \mathcal{H}_2)$ having the same number of field channels, the *series product* $\mathbf{G_1} \triangleleft \mathbf{G_2}$ is defined as

$$\mathbf{G_1} \triangleleft \mathbf{G_2} = \Big(S_2 S_1, L_2 + L_1, \mathcal{H}_1 + \mathcal{H}_2 + \frac{1}{2\boldsymbol{i}}\left(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2\right)\Big) \quad (19)$$

Since we assume both S_1 and S_2 to be the identity operators in the corresponding spaces, the QSDE describing the cascade of systems (system 2 drives system 1) can then be written for $x^T = (x_1^T\ x_2^T)$ as

$$dx = \left(\mathcal{L}_1(x) + \mathcal{L}_2(x) + L_2^\dagger[x, L_1] + [L_1^\dagger, x]L_2\right) dt + [x, L_2 + L_1]\, dW + [L_2^\dagger + L_1^\dagger, x]\, dW^\dagger. \quad (20)$$

III. Some Algebraic Relations

Let $\beta = (\beta_1, \beta_2, \beta_3)^T \in \mathbb{C}^3$, and define the linear mapping $\Theta^- : \mathbb{C}^3 \to \mathbb{C}^{3\times3}$ such that

$$\Theta^-(\beta) = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}.$$

This mapping is understood for vector of operators by associating with β the vector of operators $\hat{\beta} = (\beta_1 \hat{I}, \beta_2 \hat{I}, \beta_3 \hat{I})^T \in \mathfrak{T}(\mathfrak{H}_2)^3$ such that

$$\Theta^-(\hat{\beta}) = \begin{pmatrix} 0 & \beta_3 \hat{I} & -\beta_2 \hat{I} \\ -\beta_3 \hat{I} & 0 & \beta_1 \hat{I} \\ \beta_2 \hat{I} & -\beta_1 \hat{I} & 0 \end{pmatrix} \in \mathfrak{T}(\mathfrak{H}_2)^{3\times3},$$

Abusing the notation, $\hat{I}$ will be omitted hereafter, and the fact that β is either a vector of numbers or a vectors of operators will be understood from the context. As an example, the product of Pauli operators can be expressed in a compact matrix form thanks to the mapping $\Theta^-(\cdot)$. That is,

$$x_2 x_2^T = I_3 + \boldsymbol{i}\Theta^-(x_2) \in \mathfrak{T}(\mathfrak{H}_2)^{3\times3}.$$

Observe here that the identity matrix I_3, under our convention, is strictly speaking denoting a three dimensional diagonal matrix of the identity operator in $\mathfrak{T}(\mathfrak{H}_2)$. Similarly, the CCRs for Pauli operators are written as

$$[x_2, x_2^T] = 2\boldsymbol{i}\Theta^-(x_2) \in \mathfrak{T}(\mathfrak{H}_2)^{3\times3}.$$

Considering the *stacking operator*, denoted vec, whose action on an $m\times n$ dimensional array creates a mn dimensional column vector by stacking its columns below one another. Applying vec to $\Theta^-(\beta)$ gives $\mathrm{vec}(\Theta^-(\beta)) = F\beta$, where $m = n = 3$, $F \triangleq (F_1, F_2, F_3)^T$, the (j,k) component of F_i is $(F_i)_{jk} = \epsilon_{ijk}$, and ϵ_{ijk} is the Levi-Civita tensor. Some properties of $\Theta^-(\cdot)$ are summarized in the next lemma (see [11] for more identities).

Lemma 1: (See [10], [11].) The mapping $\Theta^-(\cdot)$ satisfies

i. $\Theta^-(\beta)\gamma = -\Theta^-(\gamma)\beta$,

ii. $\Theta^-(\beta)\beta = 0$,

iii. $\Theta^-\left(\Theta^-(\beta)\gamma\right) = [\Theta^-(\beta), \Theta^-(\gamma)]$.

This properties hold when β and γ are either $\mathbb{C}^3$ vectors or $\mathfrak{T}(\mathfrak{H}_2)^3$ vectors.

The explicit computation of the vector fields in (3) and (20) for x_1 and x_2 is given in the next lemma.

Lemma 2: The nonzero coefficients of equations (3) and (4) for the dynamics of x_1, x_2 and the cascade $G_1 \triangleleft G_2$ are

$$\begin{aligned}
[x_1, \mathcal{H}_1] &= 2\boldsymbol{i}\Theta R x_1, \\
[x_1, L_1] &= 2\boldsymbol{i}\Theta \Gamma_1^T, \\
[x_1, L_1^\dagger] &= 2\boldsymbol{i}\Theta \Gamma_1^\dagger, \\
L_1^\dagger[x_1, L_1] &= 2\boldsymbol{i}\Theta \Gamma_1^T \Gamma^\# x_1, \\
[x_1, L_1^\dagger]L_1 &= -2\boldsymbol{i}\Theta \Gamma_1^\dagger \Gamma x_1, \\
[x_2, \mathcal{H}_2] &= -2\boldsymbol{i}\Theta^-(\alpha_2^T)x_2, \\
[x_2, L_2] &= -2\boldsymbol{i}\Theta^-(\Gamma_2^T)x_2, \\
[x_2, L_2^\dagger] &= -2\boldsymbol{i}\Theta^-(\Gamma_2^\dagger)x_2, \\
L_2^\dagger[x_2, L_2] &= -2\boldsymbol{i}\Theta^-(\Gamma_2^T)\Gamma_2^\dagger + 2\Theta^-(\Gamma_2^T)\Theta^-(\Gamma_2^\dagger)x_2, \\
[x_2, L_2^\dagger]L_2 &= 2\boldsymbol{i}\Theta^-(\Gamma_2^T)\Gamma_2^\dagger - 2\Theta^-(\Gamma_2^\dagger)\Theta^-(\Gamma_2^T)x_2, \\
L_2^\dagger[x_1, L_1] &= 2\boldsymbol{i}\Theta \Gamma_1^T \Gamma_2^\# x_2, \\
[L_1^\dagger, x_1]L_2 &= -2\boldsymbol{i}\Theta \Gamma_1^\dagger \Gamma_2 x_2.
\end{aligned}$$

From this lemma, system (12) is written as

$$\begin{aligned}
dx_1 &= 2\Theta\left(R + \mathfrak{F}(\Gamma_1^\dagger \Gamma_1)\right) x_1\, dt \\
&\quad + 2\boldsymbol{i}\Theta\left(\left(-\Gamma_1^\dagger + \Gamma_1^T\right) \ -\boldsymbol{i}\left(\Gamma_1^\dagger + \Gamma_1^T\right)\right) d\bar{W}, \\
dy_1 &= \begin{pmatrix} \Gamma_1 + \Gamma_1^\# \\ i(\Gamma_1^\# - \Gamma_1) \end{pmatrix} x_1\, dt + d\bar{W},
\end{aligned} \quad (21)$$

where $\mathfrak{F}(z) \triangleq \frac{1}{2\boldsymbol{i}}(z - z^*)$ is the imaginary part of z.

Remark: We see from (21) that a linear coupling operator L_1 produces, in $\mathcal{L}_1(x_1)$, only linear terms of the form $Mx_1\,dt$ with $M \in \mathbb{C}^{2n\times 2n}$, and constant noise vector fields because of the CCRs of x_1. Suppose now that L_1 is a quadratic form, i.e., $L_i = x_1^T\Gamma_1 x_1$, then the term $[L_1^\dagger, x_1]$ produces a bilinear term, however evaluating, for instance, $[L_1, x_1]L_1^\dagger$ generates a term of the form $M_1(x_1 \otimes x_1)$ with $M_1 \in \mathbb{C}^{2n\times(2n)^2}$. Even more, these terms cannot be embedded in a higher dimensional bilinear system since by doing so only produces polynomials of higher order of the oscillator system variables. This indicates that a QSDE describing a system of n harmonic oscillators cannot have terms of the form $B_i x_1 d\bar{W}_i$ when the coupling operator is a linear form.

For system (17), one has that

$$\begin{aligned} dx_2 = &-2\boldsymbol{i}\Theta^-(\Gamma_2^T)\Gamma_2^\dagger\,dt - 2\Theta^-(\alpha_2^T)x_2\,dt \\ &+ \Big(\Theta^-(\Gamma_2^T)\Theta^-(\Gamma_2^\dagger) + \Theta^-(\Gamma_2^\dagger)\Theta^-(\Gamma_2^T)\Big)x_2\,dt \\ &+ \boldsymbol{i}\Theta^-(\Gamma_2^\dagger - \Gamma_2^T)x\,d\bar{W}_1 - \Theta^-(\Gamma_2^T + \Gamma_2^\dagger)x_2\,d\bar{W}_2, \quad (22) \\ dy_2 = &\begin{pmatrix} \Gamma_2 + \Gamma_2^\# \\ i(\Gamma_2^\# - \Gamma_2) \end{pmatrix} x_2\,dt + d\bar{W}. \end{aligned}$$

Finally, (20) for the cascade of (17) driving (12) is

$$\begin{aligned} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = &\begin{pmatrix} 0 \\ -2\boldsymbol{i}\Theta^-(\Gamma_2^T)\Gamma_2^\dagger \end{pmatrix} dt \\ &+ \begin{pmatrix} \mathcal{R}_1 & -4\Theta\mathfrak{F}(\Gamma_1^T\Gamma_2^\#) \\ 0 & \mathcal{R}_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{i}\Theta^-(\Gamma_2^\dagger - \Gamma_2^T) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\bar{W}_1 \quad (23) \\ &- \begin{pmatrix} 0 & 0 \\ 0 & \Theta^-(\Gamma_2^T + \Gamma_2^\dagger) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\bar{W}_2 \\ &+ \begin{pmatrix} 2\boldsymbol{i}\Theta\Big(\Big(-\Gamma_1^\dagger + \Gamma_1^T\Big) \quad -\boldsymbol{i}\Big(\Gamma_1^\dagger + \Gamma_1^T\Big)\Big) \\ 0 \end{pmatrix} d\bar{W} \end{aligned}$$

with $\mathcal{R}_1 = 2\Theta\Big(R + \mathfrak{F}(\Gamma_1^\dagger\Gamma_1)\Big)$ and $\mathcal{R}_2 = -2\Theta^-(\alpha_2^T) + \Theta^-(\Gamma_2^T)\Theta^-(\Gamma_2^\dagger) + \Theta^-(\Gamma_2^\dagger)\Theta^-(\Gamma_2^T)$.

We observe that the QSDE (23) contains both additive and multiplicative noise terms, and its drift term is affine. Two question can now be asked. The first is under what conditions a general QSDE of such form (see equation (24) below) preserves the CCRs for x_1 and x_2 at the same time. This question is addressed in Section IV. Then, it will be desired to know under what conditions there exists $(S, L, \mathcal{H})$ as in (19) such that (20) can be written as in (3) (Section V).

IV. Preservation of CCRs

Consider an arbitrary n-dimensional bilinear QSDE interacting with a quadrature field. That is,

$$dx = A_0dt + Axdt + B_1xd\bar{W}_1 + B_2xd\bar{W}_2 + Bd\bar{W}, \quad (24a)$$

$$dy = Cxdt + d\bar{W}, \quad (24b)$$

where $A_0 \in \mathbb{R}^n$, $A, B_1, B_2 \in \mathbb{R}^{n\times n}$, $B \triangleq (\bar{B}_1 \ \ \bar{B}_2)$, $\bar{B}_1, \bar{B}_2 \in \mathbb{R}^n$, and $d\bar{W} = (d\bar{W}_1\ d\bar{W}_2)^T$.

In previous work ([11], [15]), the quantum noise appearing in the equations was either additive or multiplicative. This model differs from those in what it includes both additive and multiplicative noise, and the system models are such that their system variables can be partitioned into two mutually commuting sets each having different CCRs. Specifically, one set obeys the CCRs of harmonic oscillators, and the other follows the CCRs of a two-level system. That is,

$$\begin{aligned} [x, x^T] &= \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (x_1^T\ x_2^T)\right] \\ &= \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^-(x_2) \end{pmatrix}. \end{aligned} \quad (25)$$

Conversely, the imposition of these CCRs on an arbitrary x induces automatically a partition of x in a way that one set obeys harmonic oscillator CCRs, while the other obey the CCRs of $SU(2)$. Since this partition of x can always be obtained via a linear transformation, one can assume without loss of generality that x is always of the form $x^T = (x_1^T\ x_2^T)$.

Consider now the block partition of A_0, A, B_i and $\bar{B}_i$ as follows

$$A_0 = \begin{pmatrix} A_{01} \\ A_{02} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

$$B_i = \begin{pmatrix} B_{i11} & B_{i12} \\ B_{i21} & B_{i22} \end{pmatrix} \text{ and } \bar{B}_i = \begin{pmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \end{pmatrix}$$

for $i = 1, 2$. Recalling the fact that x_2 is self-adjoint, one can infer that

$$\bar{B}_{i2} = 0_{3\times 1}.$$

This agrees with the fact that a bilinear QSDE is driving a linear QSDE. In summary, the only source of additive noise is provided by the linear QSDE. Note that the bilinear QSDE system can only provide multiplicative noise to the composite system. Also, the equation for dx_1 can only have bilinear terms with respect to x_2. This means that

$$B_{i12} = 0_{2\times 2}.$$

Theorem 3: Let x be a vector of operators satisfying CCRs (25), a QSDE as in (24a) preserves such CCRs for all $t \geq 0$ if and only if the linear QSDE

$$dx_1 = A_{11}x_1\,dt + (\bar{B}_{11}\ \bar{B}_{21})\,d\bar{W}$$

and the bilinear QSDE

$$dx_2 = (A_{02} + A_{22}x_2)\,dt + B_{122}x_2\,d\bar{W}_1 + B_{222}x_2\,d\bar{W}_2$$

satisfy the conditions in Theorems 1 and 2, respectively, in addition to

$$\begin{aligned} (I_3 \otimes A_{12})F + (B_{122}^T \otimes \bar{B}_{21}) \\ - (B_{222}^T \otimes \bar{B}_{11}) = 0. \end{aligned} \quad (26)$$

Remark: The structure showed in (23) appears naturally from the preservation of mixed CCRs (see the proof of Theorem 3 in the appendix).

V. Cascade Physical Realizability

As mentioned in the introduction, physical realizability for linear and bilinear QSDEs has previously been treated independently of each other ([10], [11], [15]). However, a more natural setting for quantum systems is when linear and n-level systems are components of a larger system. The objective here is to give conditions for physical realizability for a bilinear QSDE driving a linear QSDE. The general notion of physical realizability is provided next. It basically ties QSDE's of arbitrary nature with an $(S, L, \mathcal{H})$ parametrization.

Definition 2: A QSDE is said to be *physically realizable* if there exist operators $\mathcal{H}$ and L such that the QSDE can be written as in (3) and (5).

In what follows a summary of the necessary and sufficient conditions for linear and bilinear QSDE's is given. Then the second main result of the paper is given. That is, necessary and sufficient conditions for physical realizability of the cascade of a bilinear QSDE followed by a linear QSDE.

A. Physical realizability of linear QSDEs

Definition 3: The system (12) is said to be physically realizable if there exist $\mathcal{H}_1$ and L_1 such that (12) can be written as in (3) and (5).

The explicit form of matrices A, B, C_1 and C_2 in (12) is given in terms of a Hamiltonian and coupling operator next, and can be identified from (21). The existence of an $(S_1, L_1, \mathcal{H}_1)$ parametrization of linear QSDEs with system variables as in (14) is given by the next theorem.

Theorem 4: (See [9], [15].) System (12) is physically realizable if and only if

$i.$ $A\Theta + \Theta A + BJB = 0,$

$ii.$ $B = \Theta C^T (J \otimes I_n),$

where $\mathcal{H}_1$ and Γ_1 are uniquely identified as

$$R = \frac{1}{4}\left(-\Theta A + A^T \Theta\right) \text{ and } \Gamma_1 = \frac{1}{2}\left(C_1 + \boldsymbol{i} C_2\right).$$

Note that (i) is identical to (13), however the latter is generated purely form algebraic considerations.

B. Physical realizability of bilinear QSDEs

Definition 4: System (17) is said to be physically realizable if there exist $\mathcal{H}$ and L such that (17a) can be written as in (3) and (5).

The explicit matrices A_0, A, B_1, B_2, C_1 and C_2 in terms of a Hamiltonian and coupling operator can be extracted from (22). The existence of an $(S_2, L_2, \mathcal{H}_2)$ parametrization of bilinear QSDEs with system variables as in (14) is given by the next theorem.

Theorem 5: (See [10], [11].) The system (17) with output equation (8) is physically realizable if and only if

$i.$ $A_0 = \frac{1}{2}(B_1 + \boldsymbol{i} B_2)(C_1 + \boldsymbol{i} C_2)^\dagger,$

$ii.$ $B_1 = \Theta^-(C_2^T),$

$iii.$ $B_2 = -\Theta^-(C_1^T),$

$iv.$ $A + A^T + B_1 B_1^T + B_2 B_2^T = 0.$

In which case, one can identify the matrix α_2 defining the system Hamiltonian and the coupling matrix Γ_2 as

$$\alpha_2 = \frac{1}{8}\mathrm{vec}(A - A^T)^T F, \text{ and } \Gamma_2 = \frac{1}{2}(C_1 + \boldsymbol{i} C_2).$$

Similar to the case of linear QSDEs, condition (iv) is identical to (18c), however (18c) is obtained form purely algebraic considerations.

C. Physical realizability of a class of cascade bilinear-linear QSDE's

The second main result of the paper is now presented. First, the definition of a physically realizable bilinear-linear cascade is given.

Definition 5: A QSDE is said to be a *physically realizable bilinear-linear cascade* if there exist operators $\mathcal{H}$ and L as in (19) such that QSDE (20) can be written as in (3) and (5).

The characterization of the physical realizability of a bilinear-linear cascade of QSDEs is given in the next theorem.

Theorem 6: The system (24) is physically realizable according to Definition 5 if and only if the following conditions hold

$i.$ The matrices A_0, A, B_1, B_2, B and C in (24) are of the following form

$$A_0 = \begin{pmatrix} 0 \\ A_{02} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$
$$B_i = \begin{pmatrix} 0 & 0 \\ 0 & B_{i22} \end{pmatrix}, \quad B = \begin{pmatrix} \bar{B}_{11} & \bar{B}_{21} \\ 0 & 0 \end{pmatrix},$$
$$\text{and } C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}.$$

$ii.$ System

$$dx_1 = A_{11} x_1 \, dt + (\bar{B}_{11} \ \bar{B}_{21}) \, d\bar{W},$$
$$dy = C_1 x_1 + d\bar{W}$$

is physically realizable in the sense of Definitions 3.

$iii.$ System

$$dx_2 = (A_{02} + A_{22} x_2)\, dt + B_{122} x_2 \, d\bar{W}_1 + B_{222} x_2 \, d\bar{W}_2,$$
$$dy = C_2 x_2 + d\bar{W}$$

is physically realizable in the sense of Definitions 4, where $C_2^T = (C_{21}^T \ C_{22}^T)$ is such that the following consistency condition holds:

$$A_{12} = \bar{B}_{11} C_{21} + \bar{B}_{21} C_{22}. \tag{27}$$

The following corollary is a consequence of the previous theorem.

Corollary 1: A bilinear-linear cascade physically realizable QSDE preserves (25).

VI. Conclusions and Future Research

Conditions for the preservation of mixed CCRs were developed. In particular, these conditions were obtained for bilinear systems having both additive and multiplicative quantum noise inputs. It was also shown that bilinear-linear QSDE cascades are physical realizable when the linear and bilinear subsystems are physically realizable and a consistency condition holds.

A future research direction is to consider an interactive Hamiltonian in the formalism (a hermitian operator $\mathcal{H}_I = x_1^T R_1 x_2$). This would allow our theory to capture some of the commonly used models in quantum optics. For example, an atom trapped in an optical cavity is described by the Jaynes-Cummings model, i.e. a model with a Hamiltonian of the form

$$\mathcal{H} = \frac{\hbar}{2}\omega_0\sigma_z + \frac{1}{2}\gamma a\sigma^+ + \frac{1}{2}\gamma^* a^\dagger\sigma^- + \hbar\omega_c a^\dagger a,$$

where ω_c and ω_0 are the frequencies of the cavity and atom, respectively, and γ is the interaction strength. In addition, the conditions provided in this manuscript will potentially allow the synthesis of coherent quantum observers for n-level systems in the Heisenberg picture.

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Appendix
Proofs of Results

Proof of Theorem 3: Using (2) and (7), it follows that $d[x, x^T]$ can be obtained by computing $d(xx^T)$ and $(d(xx^T))^T$. That is,

$$\begin{aligned}
&d(xx^T) = (dx)x^T + x(dx)^T + (dx)(dx)^T \\
&= (A_0x^T + xA_0^T)\,dt + (Axx^T + xx^TA^T)\,dt \\
&\quad + (B_1xx^T + xx^TB_1^T)\,d\bar{W}_1 + (B_2xx^T + xx^TB_2^T)\,d\bar{W}_2 \\
&\quad + B_1xx^TB_1^Td\bar{W}_1d\bar{W}_1 + B_1xx^TB_2^Td\bar{W}_1d\bar{W}_2 \\
&\quad + B_2xx^TB_1^Td\bar{W}_2d\bar{W}_1 + B_2xx^TB_2^Td\bar{W}_2d\bar{W}_2 \\
&\quad + (\bar{B}_1d\bar{W}_1 + \bar{B}_2d\bar{W}_2)(B_1xd\bar{W}_1 + B_2xd\bar{W}_2)^T \\
&\quad + (B_1xd\bar{W}_1 + B_2xd\bar{W}_2)(\bar{B}_1d\bar{W}_1 + \bar{B}_2d\bar{W}_2)^T \\
&\quad + (\bar{B}_1d\bar{W}_1 + \bar{B}_2d\bar{W}_2)(\bar{B}_1d\bar{W}_1 + \bar{B}_2d\bar{W}_2)^T \\
&= (A_0x^T + xA_0^T)\,dt + (Axx^T + xx^TA^T)\,dt \\
&\quad + (B_1xx^T + xx^TB_1^T)\,d\bar{W}_1 + (B_2xx^T + xx^TB_2^T)\,d\bar{W}_2 \\
&\quad + B_1xx^TB_1^Tdt + \boldsymbol{i}B_1xx^TB_2^Tdt \\
&\quad - \boldsymbol{i}B_2xx^TB_1^Tdt + B_2xx^TB_2^Tdt \\
&\quad + \bar{B}_1x^TB_1^T\,dt + \boldsymbol{i}\bar{B}_1x^TB_2^T\,dt \\
&\quad - \boldsymbol{i}\bar{B}_2x^TB_1^T\,dt + \bar{B}_2x^TB_2^T\,dt \\
&\quad + B_1x\bar{B}_1^T\,dt + \boldsymbol{i}B_1x\bar{B}_2^T\,dt \\
&\quad - \boldsymbol{i}B_2x\bar{B}_1^T\,dt + B_2x\bar{B}_2^T\,dt \\
&\quad + \bar{B}_1\bar{B}_1^T\,dt - \boldsymbol{i}\bar{B}_1\bar{B}_2^T\,dt \\
&\quad + \boldsymbol{i}\bar{B}_2\bar{B}_1^T\,dt + \bar{B}_2\bar{B}_2^T\,dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\big(d(xx^T)\big)^T \\
&= (A_0x^T + xA_0^T)\,dt + (A(xx^T)^T + (xx^T)^TA^T)\,dt \\
&\quad + (B_1(xx^T)^T + (xx^T)^TB_1^T)\,d\bar{W}_1 \\
&\quad + (B_2(xx^T)^T + (xx^T)^TB_2^T)\,d\bar{W}_2 \\
&\quad + B_1(xx^T)^TB_1^Tdt + \boldsymbol{i}B_1(xx^T)^TB_2^Tdt \\
&\quad - \boldsymbol{i}B_2(xx^T)^TB_1^Tdt + B_2(xx^T)^TB_2^Tdt
\end{aligned}$$

$$
\begin{aligned}
&+ B_1 x \bar{B}_1^T\, dt + \boldsymbol{i} B_2 x \bar{B}_1^T\, dt \\
&- \boldsymbol{i} B_1 x \bar{B}_2^T\, dt + B_2 x \bar{B}_2^T\, dt \\
&+ \bar{B}_1 x^T B_1^T\, dt + \boldsymbol{i} \bar{B}_2 x^T B_1^T\, dt \\
&- \boldsymbol{i} \bar{B}_1 x^T B_2^T\, dt + \bar{B}_2 x^T B_2^T\, dt \\
&+ \bar{B}_1 \bar{B}_1^T\, dt - \boldsymbol{i} \bar{B}_2 \bar{B}_1^T\, dt \\
&+ \boldsymbol{i} \bar{B}_1 \bar{B}_2^T\, dt + \bar{B}_2 \bar{B}_2^T\, dt.
\end{aligned}
$$

Hence, the commutator dynamics is

$$
\begin{aligned}
d\left[x, x^T\right] = {} & A[x, x^T] + [x, x^T]A^T \\
& + (B_1[x, x^T] + [x, x^T]B_1^T) d\bar{W}_1 \\
& + (B_2[x, x^T] + [x, x^T]B_2^T) d\bar{W}_2 \\
& + (B_1[x, x^T]B_1^T + B_2[x, x^T]B_2^T)\, dt \\
& + \boldsymbol{i}(B_1\{x, x^T\}B_2^T - B_2\{x, x^T\}B_1^T)\, dt \\
& + 2\boldsymbol{i} B_2 x \bar{B}_1^T\, dt + 2\boldsymbol{i} B_1 x \bar{B}_2^T\, dt \\
& + 2\boldsymbol{i} \bar{B}_1 x^T B_2^T\, dt + 2\boldsymbol{i} \bar{B}_2 x^T B_1^T\, dt \\
& + \boldsymbol{i} \bar{B}_1 \bar{B}_2^T\, dt - \boldsymbol{i} \bar{B}_2 \bar{B}_1^T\, dt.
\end{aligned}
$$

To preserve (25), (24a) has to satisfy

$$
d\left[x, x^T\right] = 2\boldsymbol{i} \begin{pmatrix} 0 & 0 \\ 0 & \Theta^-(dx_2) \end{pmatrix}, \tag{28}
$$

where

$$
\begin{aligned}
\Theta^-(dx_2) = {} & \Theta^-\left(A_{02}\right) dt + \Theta^-\left(A_{22} x_2\right) dt \\
& + \left(\Theta^-\left(B_{121} x_1\right) + \Theta^-\left(B_{122} x_2\right)\right) d\bar{W}_1 \\
& + \left(\Theta^-\left(B_{221} x_1\right) + \Theta^-\left(B_{222} x_2\right)\right) d\bar{W}_2.
\end{aligned}
$$

A_{01} does not play a role in the preservation of CCRs. Therefore, without loss of generality A_{01} is assumed to be zero. This goes in agreement with the fact that no term of this type is generated by quantum systems originating from harmonic oscillators of the class considered in this paper.

From [20, Proposition 27.3], one can also equate the integrands in (28) to zero. Recall that $x_2(0)$ is represented by the complete orthonormal set. This implies that any linear combination $\sum_{k=0}^{s} a_i x_i(0) \neq 0$ unless $a_i = 0$ for all i and $a_i \in \mathbb{C}$. In addition, no linear combination of Pauli matrices generates I_3. Therefore, any equation $Ax_2 = b$ ($A \in \mathbb{C}^{3\times 3}$ and $b \in \mathbb{C}^3$) implies $A = 0$ and $b = 0$. These facts are summarized in the following equations that have to be satisfied for the preservation of CCRs.

$$
\begin{aligned}
& B_{122} B_{222}^T - B_{222} B_{122}^T - \Theta^-(A_{02}) = 0, && \text{(29a)} \\
& B_{i21} \Theta = 0, && \text{(29b)} \\
& B_{i12} \Theta^-(x_2) = 0 && \text{(29c)} \\
& B_{i22} \Theta^-(x_2) + \Theta^-(x_2) B_{i22}^T - \Theta^-(B_{i22} x_2) = 0, && \text{(29d)} \\
& A_{11} \Theta + \Theta A_{11}^T + \boldsymbol{i}\left(\bar{B}_{11} \bar{B}_{21}^T - \bar{B}_{21} \bar{B}_{11}^T\right) = 0 && \text{(29e)} \\
& A_{12} \Theta^-(x_2) + \bar{B}_{21} x_2^T B_{122}^T - \bar{B}_{11} x_2^T B_{222}^T = 0 && \text{(29f)} \\
& A_{21} \Theta = 0 && \text{(29g)} \\
& A_{22} \Theta^-(x_2) + \Theta^-(x_2) A_{22}^T + B_{122} \Theta^-(x_2) B_{122}^T + B_{222} \Theta^-(x_2) B_{222}^T - \Theta^-(A_{22} x_2) = 0. && \text{(29h)}
\end{aligned}
$$

Relations (29a), (29d) and (29h) provide the preservation of CCRs of x_2 (Theorem 2). Similarly, (29e) assures the preservation of CCRs for x_1 (Theorem 1). Relations (29b), (29c) and (29g) impose a structure on the blocks in matrices A, B_1, B_2 and B. That is, one has that (29c) provides $B_{i12} = 0$ by the linear independence on the components of x_2. Since Θ only permutes the rows and columns of A_{12} and multiplies some of its components by -1 then $A_{12} = 0$. The same argument provides $B_{i21} = 0$. From Lemma 3 in [11], (29d) is always satisfied, and allows to write $B_{i22} = \Theta^-(b_i)$ with

$$
b_i = -\frac{1}{n} \begin{pmatrix} \mathrm{Tr}(F_1 B_{i22}) \\ \vdots \\ \mathrm{Tr}(F_s B_{i22}) \end{pmatrix}.
$$

Therefore, by fixing the CCRs of x, the matrices in (20) assume naturally the following structure

$$
A_0 = \begin{pmatrix} 0 \\ A_{02} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},
$$

$$
B_i = \begin{pmatrix} 0 & 0 \\ 0 & B_{i22} \end{pmatrix}, \text{ and } B = \begin{pmatrix} \bar{B}_{11} & \bar{B}_{21} \\ 0 & 0 \end{pmatrix}.
$$

To obtain (26), first recall $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$ for A, B and C of appropriate dimensions. Then, applying the stacking operator to (29f) the desired consistency condition (26) is obtained.

Conversely, since the steps used above to obtain (26) are reversible and the fact that the preservation of CCRs for x_1 and x_2 in Theorems 1 and 2 imply (29a), (29d), (29e) and (29h), then (28) holds. This finalizes the proof. ■

Proof of Theorem 6: If system (24) is bilinear-linear cascade physically realizable, then it can be written as in (23), and the systems formed by matrices $(A_{11}, (\bar{B}_{11}\, \bar{B}_{21}), C_1)$ and $(A_{02}, A_{22}, B_{122}, B_{222}, C_2)$ can be written as in (21) and (22), respectively. Therefore, $(I, L_1, \mathcal{H}_1)$ and $(I, L_2, \mathcal{H}_2)$ can be identified so that the parametrization $(S, L, \mathcal{H})$ as in (19) holds. It is only left to prove that A_{12} can be written as (27).

One has from Lemma 2 that

$$A_{12} = -4\Theta\mathfrak{F}(\Gamma_1^T\Gamma_2^\#) = 2\boldsymbol{i}\Theta\Gamma_1^T\Gamma_2^\# - 2\boldsymbol{i}\Theta\Gamma_1^\dagger\Gamma_2,$$

and that $2\boldsymbol{i}\Theta\Gamma_1^T = \bar{B}_{11} + \boldsymbol{i}\bar{B}_{21}$ and $-2\boldsymbol{i}\Theta\Gamma_1^\dagger = \bar{B}_{11} + \boldsymbol{i}\bar{B}_{21}$. Thus,

$$\begin{aligned} A_{12} &= (\bar{B}_{11} + \boldsymbol{i}\bar{B}_{21})\Gamma_2^\# + (\bar{B}_{11} + \boldsymbol{i}\bar{B}_{21})\Gamma_2 \\ &= \bar{B}_{11}(\Gamma_2 + \Gamma_2^\#) + \bar{B}_{21}\boldsymbol{i}(\Gamma_2^\# - \Gamma_2) \\ &= \bar{B}_{11}C_{21} + \bar{B}_{21}C_{22}. \end{aligned} \tag{30}$$

On the other hand, assuming (i)-(iii) hold, then from Theorems 4 and 5 the triples $(I, L_1, \mathcal{H}_1)$ and $(I, L_2, \mathcal{H}_2)$ are uniquely identified. In particular, $\Gamma_1 = \frac{1}{2}(C_{11} + \boldsymbol{i}C_{12})$ and $\Gamma_2 = \frac{1}{2}(C_{21} + \boldsymbol{i}C_{22})$. Finally, since (27) hold all the steps in (30) are reversible, then $A_{12} = -4\Theta\mathfrak{F}(\Gamma_1^T\Gamma_2^\#)$ as in (23). This completes the proof. ∎

Proof of Corollary 1: To prove this result using Theorem 6, only condition (26) of Theorem 3 need to be established. Given that the cascade is physically realizable, one has that $A_{12} = 2\boldsymbol{i}\Theta(\Gamma_1^T\Gamma_2^\# - \Gamma_1^\dagger\Gamma_2)$, $\bar{B}_{11} = \boldsymbol{i}\Theta(\Gamma_1^T - \Gamma_1^\dagger)$, $\bar{B}_{21} = \Theta(\Gamma_1^T + \Gamma_1^\dagger)$, $B_{122} = \boldsymbol{i}\Theta^-(\Gamma_2^\dagger - \Gamma_2^T)$ and $B_{222} = -\Theta^-(\Gamma_2^T + \Gamma_2^\dagger)$. Using Lemma 1, it then follows that

$$\begin{aligned} &\bar{B}_{21}x_2^T B_{122}^T \\ &= \boldsymbol{i}\Theta(\Gamma_1^\dagger + \Gamma_1^T)x_2^T\Theta^-(\Gamma_2^\dagger - \Gamma_2^T) \\ &= \boldsymbol{i}\Theta\left(\Gamma_1^T\Gamma_2 - \Gamma_1^T\Gamma_2^\# + \Gamma_1^\dagger\Gamma_2 - \Gamma_1^\dagger\Gamma_2^\#\right)\Theta^-(x_2), \end{aligned}$$

$$\begin{aligned} &\bar{B}_{11}x_2^T B_{222}^T \\ &= -\boldsymbol{i}\Theta(\Gamma_1^T - \Gamma_1^\dagger)x_2^T\Theta^-(\Gamma_2^T + \Gamma_2^\dagger) \\ &= \boldsymbol{i}\Theta\left(\Gamma_1^T\Gamma_2 + \Gamma_1^T\Gamma_2^\# - \Gamma_1^\dagger\Gamma_2 - \Gamma_1^\dagger\Gamma_2^\#\right)\Theta^-(x_2). \end{aligned}$$

Hence

$$A_{12}\Theta^-(x_2) + \bar{B}_{21}x_2^T B_{122}^T - \bar{B}_{11}x_2^T B_{222}^T = 0,$$

which is equivalent to (26) after applying the stacking operator and using the linear independence of the components of x_2. ∎