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Robustness Analysis of Nonlinear Systems with Feedback Linearizing Control

by

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ABSTRACT

FACULTY OF PHYSICAL SCIENCES AND ENGINEERING
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The feedback linearization approach is a control method which employs feedback to stabilize systems containing nonlinearities. In order to accomplish this, it assumes perfect knowledge of the system model to linearize the input-output relationship. In the absence of perfect system knowledge, modelling errors inevitably affect the performance of the feedback controller. This thesis introduces a design and analysis approach for robust feedback linearizing controllers for nonlinear systems. This approach takes into account these model errors and provides robustness margins to guarantee the stability of feedback linearized systems.

Based on robust stability theory, two important tools, namely the small gain theorem and the gap metric, are used to derive and validate robustness and performance margins for the feedback linearized systems. It is shown that the small gain theorem can provide unsatisfactory results, since the stability conditions found using this approach require the nonlinear plant to be stable. However, the gap metric approach is shown to yield general stability conditions which can be applied to both stable and unstable plants. These conditions show that the stability of the linearized systems depends on how exact the inversion of the plant nonlinearity is, within the nonlinear part of the controller.

Furthermore, this thesis introduces an improved robust feedback linearizing controller which can classify the system nonlinearity into stable and unstable components and preserve the stabilizing action of the inherently stabilizing nonlinearities in the plant, cancelling only the unstable nonlinear part of the plant. Using this controller, it is shown that system stability depends on the bound on the input nonlinear component of the plant and how exact the inversion of the unstable nonlinear of the plant is, within the nonlinear part of the controller.

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Declaration of Authorship

I, Abeer Khaldoon Al-Gburi declare that the thesis entitled

Robustness Analysis of Nonlinear Systems with Feedback Linearizing Control

and the work presented in the thesis are both my own and has been generated by me as the result of my own original research. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University;
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- Where I have consulted the published work of others, this is always clearly attributed;
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: A. Al-Gburi, M. French, and C. T. Freeman (2013)

Signed:

Date:

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To My Mother Mona

Nomenclature

$b_{P,C}$	Robust stability margin for the closed loop system P, C
B, B_u	Upper bound on (plant, unstable plant) nonlinear component of a system
ψ	Nonlinear component of a controller in a Hammerstein system
C_{Linear}	linear part of the controller
D, D_u	Upper bound on (input, unstable input) nonlinear component of a system
$\delta, \vec{\delta}$	Gap metric, directed gap metric
d_0, d_1, d_2, d_3	Disturbance signals added to the feedback signals
ε, ϵ	Lower bound on (input, unstable input) nonlinear component of a system
f^*	Plant nonlinear component of a transformed feedback linearizable system
f_u^*	Unstable plant nonlinear component of a transformed feedback linearizable system
f_s^*	Stable plant nonlinear component of a transformed feedback linearizable system
ϕ	Input nonlinear component of a Hammerstein system
g^*	Input nonlinear component of a transformed feedback linearizable system
g_u^*	Unstable input nonlinear component of a transformed feedback linearizable system
g_s^*	Stable input nonlinear component of a transformed feedback linearizable system
$\mathcal{G}_P, \mathcal{M}$	Graph of plant P
$\mathcal{G}_{P_i}, \mathcal{M}_i$	Graph of plant P_i
$\mathcal{G}_{P_i'}, \mathcal{M}_i$	Graph of perturbed plant P_i'
$\mathcal{G}_C, \mathcal{N}$	Graph of C
$\mathcal{H}_\infty$	Space of transfer functions of stable linear, time invariant, continuous time systems
$map(A, B)$	Relation between elements in A and B
M, N	Coprime factors of a system
P_1, P_2, P_3	Nominal plants in a network system
P_1', P_2', P_3'	Perturbed plants in a network system
P_{Linear}	Linear plant operator
$\tilde{P}$	Nominal plant operator
π, π'	Linear operators which replace the input nonlinearities in a

	linear configuration of the system
Π_x	Projection onto the subspace x
$\Pi_{P//C}$	Map from the disturbances w_0 to the plant signals w_1
$\Pi_{C//P}$	Map from the disturbances w_0 to the controller signals w_2
$\mathcal{RH}_\infty$	Space of rational $\mathcal{H}_\infty$ functions
$(T_t v)(\tau)$	The truncation operator of v to the interval $[0, \tau]$
$\mathcal{U}$	Space of L_p, $1 \leq p \leq \infty$ norm bounded input signals
v_1	Feedback signal to the nonlinear components of the plant
v_2	Feedback signal to the nonlinear components of the controller
$\tilde{v}_2$	Disturbed feedback signal to the nonlinear components of the controller
$\mathcal{V}$	Space of L_p, $1 \leq p \leq \infty$ norm bounded signals
$\mathcal{V}_e$	Space of L_p, $1 \leq p \leq \infty$ norm bounded signals on finite intervals
w_0	Disturbance signals $w_0 = (u_0, y_0)^\top$
w_1	Plant signals $w_1 = (u_1, y_1)^\top$
w_2	Controller signals $w_2 = (u_2, y_2)^\top$
$\mathcal{W}$	$\mathcal{U} \times \mathcal{Y}$
$\mathcal{W}_e$	$\mathcal{U}_e \times \mathcal{Y}_e$
$\mathcal{X}$	Space of L_p, $1 \leq p \leq \infty$ norm bounded signals
$\mathcal{Y}$	Space of L_p, $1 \leq p \leq \infty$ norm bounded output signals
z_1	Feedback signal to the nonlinear components of the plant
z_2	Feedback signal to the nonlinear components of the controller
$\tilde{z}_1$	Disturbed feedback signal to the nonlinear components of the plant
$\tilde{z}_2$	Disturbed feedback signal to the nonlinear components of the controller

Chapter 1

Introduction

Physical systems are nonlinear in nature, yet, most techniques and analysis available in the literature relate to linear systems. To approximate nonlinear systems with linear models a system can be linearized about an operating point and then analysis can be carried out on the resulting model. However, approaches based on linearization have limitations. Since it is an approximation method for system behavior near an operating point, it describes the nonlinear system behavior around that point and cannot be expected to capture its behavior far from the operating point. Another limitation is that linear systems are less rich in dynamics, so to describe nonlinear phenomena linearized models are inadequate. This has motivated the development of many control techniques introduced for nonlinear systems in the past three decades.

Feedback control has been used extensively to develop control techniques for nonlinear systems. Depending on the control problem different types of feedback are used. If all the state variables of the system are measured and can be controlled a state feedback control system may be used. Alternatively, if only some of the states of the system are measured and can be controlled an output feedback may be used.

Full state feedback techniques for continuous time nonlinear systems have been intensely discussed in the literature (e.g. in Isidori (1989), Sastry (1999) and Nijmeijer and van der Schaft (1990)). The feedback linearization approach Isidori (1989) is a well known state feedback technique where an exact linearization is performed to the system states via internal feedback. This method is based on linearizing the input-output relations of a nonlinear system but it assumes perfect knowledge of the system equations and it uses that knowledge to cancel the nonlinearity of the system. Since perfect system knowledge is not available in practice, the degree to which modelling error affects performance becomes an important issue. This problem and others will be discussed in the next section. It is the limitations associated with the feedback linearization approach in the presence of model uncertainties which prompt the work conducted in this thesis.

This thesis focuses on control design based on robust stability theory in which the feedback linearization is applied to control a nonlinear model. The conceptual idea of robust control can be briefly described as follows:

Consider the physical plant P_1 and its model P. The controller C is developed such that the feedback system $[P, C]$ shown in Figure 1.1 is stable. For a system with closed-loop $[P_1, C]$, a stability margin represents the amount that the plant P_1 can differ from P and $[P_1, C]$ still remain stable.

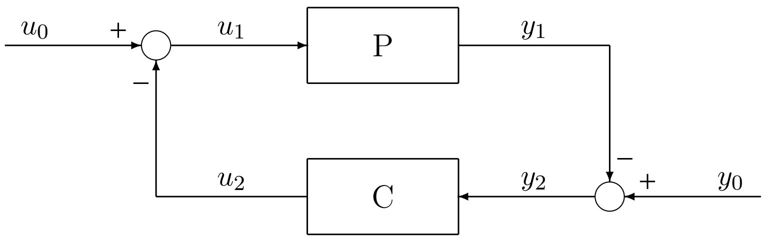

FIGURE 1.1: Feedback configuration [P,C]

There are many robustness tools which are used to analyze stability of nonlinear systems. One basic approach is the small gain theorem which was first introduced in Zames (1966). Despite its simplicity it forms a fundamental basis for many robustness results. A significant drawback to this approach is that it is not applicable to the case in which there is an unstable plant in the system (for more details see Chapters 3 and 4).

Another approach is the use of the gap metric, a tool which was first introduced directly in Zames and El-Sakkary (1980). This metric introduces a measure of the distance between two linear plants, and uses it to provide a quantitative description of a set of plants able to be stabilized by a robust controller. In particular a system $[P_1, C]$ has a stability margin if the gap between the two plants P and P_1 is less than the robust stability margin of $[P, C]$. Later on, the case where the gap metric is defined via bijective mappings between graphs was investigated in Georgiou and Smith (1997), and an input−output framework was developed for robustness analysis of nonlinear systems which generalized the linear gap metric introduced in Zames and El-Sakkary (1980).

In contrast to the small gain theorem which applies to the stable plant case only, the gap metric can be applied to more general cases including cases with unstable plants. It is closely related to the standard coprime factor uncertainty models which can include additive, multiplicative and inverse multiplicative uncertainty models.

In this thesis robustness of nonlinear systems with full state linearizing controllers will be investigated using the gap metric. Theoretical robustness and performance margins will be derived and validated for these systems. This work introduces the nonlinear gap metric and uses a 'network' result (Theorem (10)) Georgiou and Smith (1997) to undertake stability analysis for nonlinear systems.

The reminder of this chapter motivates the work performed in this thesis. First, the feedback linearization approach will be introduced, and its importance in solving many control problems will be addressed. This will lead to consideration of shortcomings and limitations related to robustness of systems designed using this approach which arise from the fact that feedback linearization uses an exact model of the plant to design a feedback controller. To solve this problem robustness analysis will be carried out using key tools within robust control (in particular, the small gain theorem and the gap metric).

1.1 Feedback Linearization

The feedback linearization approach, as comprehensively surveyed by Isidori (1989), efficiently deals with nonlinearities via feedback, assuming full state knowledge to linearize the input-output relationship. The main idea of this approach is to algebraically transform nonlinear system dynamics into a (fully or partly) linear counterpart, so that linear control techniques can be applied.

To introduce the idea of feedback linearization control, consider the single-input-single-output nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where f, g, and h are sufficiently smooth in a domain $D \subset \mathbb{R}^n$ where $\mathbb{R}$ is the set of real numbers. The mappings $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ are called vector fields on D. The output derivative ,$\dot{y}$, is given by:

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h(x) + L_g h(x)u$$

where

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is called the *Lie Derivative of* h with respect to f or along f. This is the familiar notion of the derivative of h along the trajectories of the system $\dot{x} = f(x)$. The new notation is convenient when we repeat the calculation of the derivative with respect to the same vector field or a new one. For example, the following notation is used:

$$\begin{aligned}
L_g L_f h(x) &= \frac{\partial L_f h}{\partial x} g(x), \\
L_f^2 h(x) &= L_f L_f h(x) = \frac{\partial L_f h}{\partial x} f(x), \\
L_f^k h(x) &= L_f L_f^{k-1} h(x) = \frac{\partial L_f^{k-1} h}{\partial x} f(x), \\
L_f^0 h(x) &= h.
\end{aligned}$$

If $L_g h(x) = 0$ then $\dot{y} = L_f h(x)$, independent of u. If we continue to calculate the second derivative of y, denoted by $y^{(2)}$, we obtain:

$$y^{(2)} = \frac{\partial L_f h}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)u.$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$, independent of u. Repeating this process, we see that if $h(x)$ satisfies:

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0, \tag{1.1}$$

then the input ,u, does not appear in the equations of $y, \dot{y}, \ldots, y^{(\rho-1)}$ and appears in the equation $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)u.$$

The forgoing equation shows clearly that the system is input-output linearizable, since the state feedback control:

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^\rho h(x) + v \right]$$

reduces the input-output map to:

$$y^{(\rho)} = v,$$

which is a chain of ρ integrators. In this case the integer ρ is called the relative degree of the system. This is the standard way to describe input−output feedback linearization, which was presented in, for example, Khalil (2002). For a nonlinear system, if $\rho = n$ (relative degree = state dimension) then the system is said to be full linearizable and a state feedback linearizing controller can be designed for this system. On the other hand, if $\rho < n$ the system is said to be partially linearizable and an input-output linearizing controller can be designed for this system.

When a nonlinear system is linearized using feedback linearization, the resulted system can be stabilized using additional state feedback $v = a^\top (y, y^{(1)}, \ldots, y^{(\rho)})^\top$ where $a^\top =$

$(a_1, \ldots, a_n)$ is such that the linear system matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & \ldots & & a_{n-1} & a_n \end{pmatrix}.$$

is stable. Exact linearization conditions can be satisfied for many physical systems.

In many practical control problems feedback linearization has been used successfully. These applications include, for example, the control of helicopters Koo and Sastry (1998), high performance aircraft Huang et al. (2009), biomedical devices Mohammed et al. (2007), stroke rehabilitation Freeman et al. (2012a) and extensively to the control of robot manipulators Sage et al. (1999).

However, feedback linearization has limitations. A major drawback is that it relies strongly on exact knowledge of nonlinearities and an exact model of the nonlinear process, which is generally not available. Another limitation is that not all nonlinear system classes are feedback linearizable, and the existence condition for the successive differentiations of the required outputs may not be satisfied. Also, since exact feedback linearization cancels the nonlinearity of the plant, it may destroy inherently stabilizing nonlinearities that can be used to stabilize the plant (an example is given in Section 5.3, Example 5.1). This problem was stated in, for example, Khalil (2002), Freeman and Kokotović (2008) and Sepulchre et al. (1997).

As noted in the previous section, this thesis is concerned with the robustness of nonlinear systems with full state linearizing controllers. To add robustness to state feedback linearization many approaches have been presented in the literature, including applications to systems with structured or unstructured uncertainties. Most of the work done is for systems with structured uncertainties, for example systems of the form $\dot{x} = f(x) + g(x)u + \kappa(x)$ where $\|\kappa(x)\| < M \quad \forall x$ and $M < \infty$. In the work presented in Spong and Vidyasagar (1987) a robust state feedback controller is designed to control a nonlinear robotic system. Assuming that the nonlinearities of the plant are bounded, the stability of this system was established using the small gain theorem.

While, in Spong et al. (1984) a state feedback controller was designed for a robotic manipulator with structured bounded uncertainties. However, this controller was designed based on the Lyapunov direct method and did not account for actuator saturation. To solve this problem, it was incorporated with an optimal decision strategy to realize a robust unsaturated controller. Meanwhile, Khalil (1994) uses a state feedback controller to drive the states of the system to a region of attraction and then depends on a servomechanism to recover the robustness and asymptotic tracking properties of this controller. Moreover, in Kravaris (1987) a robust nonlinear state feedback control design is proposed that is also based on input-output linearization. The robustness of the closed loop system in this paper is guaranteed using a Lyapunov based approach.

Another approach is to incorporate feedback linearization with adaptive control. In this approach an adaptive controller is used to add robustness to feedback linearized systems by helping to achieve asymptotic exact cancellation of the system nonlinearity in the presence of parametric uncertainty. This approach was developed through a number of publications, for example, Ortega and Spong (1989) and Sastry and Isidori (1989). However, in this approach a matching condition is required to be placed on the uncertainty of the system (the parameter uncertainty should appear at the same order of the differentiation as the control input). To overcome this problem, backstepping was introduced to adaptive nonlinear control. In Kanellakopoulos et al. (1991) the backstepping design scheme was illustrated. While in Freeman and Kokotović (2008) a study for robust backstepping controller designs was carried out. More results can be found in Marino and Tomei (1996) and Slotine and Hedrick (1993).

Although there has been little research to address robustness of these systems in the presence of unstructured uncertainties, a few works have dealt with the robustness of feedback linearization based controllers in the presence of input unstructured uncertainties (additive and multiplicative). In these works the small gain theorem is combined with backstepping to deal with unstructured uncertainty. This new approach is based on the input to state stability (ISS) concept introduced by Sontag (1995) and was presented in Jiang et al. (1994), Krstić et al. (1996), Praly and Wang (1996) and Jiang and Mareels (1997). However, these designs require the unmodelled dynamics to have bounded ISS-gain. Later on, this condition was replaced with a strict passivity condition on the class of the unmodelled dynamics in Janković et al. (1999) and Hamzi and Praly (2001). All small gain and strict passivity designs require the unstructured uncertainties to have relative degree zero. Finally, in Kokotović and Arcak (2001) the small gain and strict passivity conditions were relaxed by combining dynamic nonlinear normalizing design of Krstić et al. (1996) with the L_gV-backstepping scheme in Arcak et al. (1999).

Other works that address unstructured uncertainties include Taylor et al. (1989) where a robust state feedback linearization controller design is presented for nonlinear systems with parametric and multiplicative uncertainties. Then an adaptive parametric update law is introduced to the system to accommodate large parameters uncertainties. Robust stability for this design was established using LaSalle's theorem. In Chao (1995) an analysis was carried out for the stability robustness of a multiple input multiple output(MIMO) nonlinear system under feedback linearization which has a multiplicative unstructured uncertainty at the plant input. Meanwhile, Wang and Wen (2009) presented an approach to design robust backstepping controllers for MIMO systems with linear input unstructured uncertainty.

Despite the importance of the gap metric in robustness analysis, very few works have addressed the use of this tool to analyze the robustness of feedback linearization controllers. Xie (2004) used the gap metric framework to study the robustness of the backstepping design procedure for state feedback and output feedback designs and robust high-gain observer designs. Then it employed the robustness framework in Georgiou and Smith (1997) to design robust controllers to plant uncertainty for these approaches.

This motivates the work set out in this thesis, since most of the work done before focuses on input uncertainties and since there is a great need for robustness analysis for the controllers designed using the feedback linearization approach in the presence of output unstructured uncertainties (inverse muliplicative uncertainties) where more general cases can be included, but this analysis is absent from the literature.

1.2 Robust Control

In robust control a nominal model is defined and any perturbation to this model is considered as model uncertainty. The nominal and perturbed models can be represented as points in a ball and a robust controller will try to meet the performance objectives for any models inside this ball.

Given a plant model set Δ, if there exists a feedback controller C which stabilizes $P \in \Delta$ such that $[P, C]$ is stable then C is a robust stabilizing controller. A feedback structure is required in robust control to provide desirable performance of a system in the presence of uncertainty.

Two substantial tools in robust feedback stability analysis are next considered; the small gain theorem and the arguments related to gap metric.

The small gain theorem is a classical tool for input-output stability. It was introduced in Zames (1966) for closed loop stability. For nonlinear systems, it states that if the open loop gain is less than one, i.e. $\|P\|\|C\| < 1$, then the closed loop is stable. The small gain theorem is conservative, for example, in the case of an unstable plant $\|P\| = \infty$ the stability condition $\|P\|\|C\| < 1$ cannot be met but clearly some unstable Ps can be stabilized. The small gain theorem was formulated for nonlinear systems in Hill (1991) and was extended in the ISS framework by Jiang et al. (1994). However, these results still do not apply to unstable P.

Another important tool is the gap metric Zames and El-Sakkary (1980). This metric measures the size of coprime factor perturbations. This type of perturbation provides a good description of unstructured model uncertainties compared to other uncertainty models. Other models have restrictions: a stable and unstable model cannot be compared using an additive model, and parametric uncertainty does not permit changes in the model order. On the other hand the coprime factor uncertainty model describes unmodeled high and low frequency system dynamics well. This metric can be computed as a solution of an $\mathcal{H}_\infty$ optimization problem as was shown in Georgiou (1988). In Vidyasagar (1984) another related metric (but unfortunately not exactly computed), the graph metric, captures the robust perturbations of stable feedback systems. This metric can be found based on normalized coprime factorization. The gap and the graph metrics where studied in Vidyasagar (1984) and were shown to be equivalent. Finally, In Vinnicombe (1993) a new metric $\delta_v(P, P_1)$ was introduced. This metric is the tightest possible metric for the stability margin $b_{P,C}$.

Georgiou and Smith (1990) showed the equivalence between the robustness optimization problem in the gap metric and the robustness optimization for the normalized coprime factor perturbation. Moreover, in this work a robust stability condition was introduced for linear perturbed systems based on the gap metric. This condition is that any controller which can stabilize a nominal plant with stability margin $b_{P,C} > 0$ can stabilize plants within a distance measured by the gap metric.

The gap metric was generalized to a nonlinear setting in Georgiou and Smith (1997). In this paper the fundamental robustness theorem for nonlinear systems was presented. It states that, given the gap perturbations are smaller than the inverse of the norm of the parallel projection operator related to the feedback loop, then the feedback stability is preserved. Furthermore, in this paper it was illustrated that perturbations which are small in the gap are those which give small closed loop errors in a feedback loop. The basic tool used in Georgiou and Smith (1997) is to introduce a map from the nominal to the perturbed plant graph and measure the distance between these systems based on this map. Also, in Anderson and de Bruynet (1999), Vinnicombe (1999) a generalization of $\delta_v(P, P_1)$ was proposed for nonlinear systems.

In James et al. (2005) many expressions for the gap metric were presented and the connection between gap metrics and representations of the graph was studied.

In this thesis both small gain theorem and the gap metric will be used to study the robustness of nonlinear systems with full state linearizing controllers.

1.3 Chapter Organization

In Chapter 2 an introduction to the concepts and notation which are used in this thesis is presented. Starting with concepts related to signals and systems, like signal boundness, system stability and presentation of unstable systems using coprime factors. Following this, robust stability analysis for feedback control systems is introduced and a basic robust stability tool, namely the small gain theorem, is presented. Then a comprehensive description of uncertainty modelling is given, including additive, multiplicative, inverse multiplicative, coprime factor and the use of gap metric to measure the size of coprime factor perturbations. Finally, stability analysis using the gap metric is introduced. In this section, significant tools and results are provided, these tools will be used subsequently in the work done in this thesis.

Chapter 3 undertakes a robust stability analysis for a stable Hammerstein model using the small gain theorem and the gap metric approach. The gap metric analysis undertaken in this chapter uses Georgiou and Smith (1997)(Theorem 1) to study robust stability for this nonlinear system. This analysis is an introduction to the procedure carried out to study the robust stability for more complex nonlinear systems considered in the following chapters.

In Chapter 4 a stable affine nonlinear system with input nonlinearity is considered. Robustness analysis for this system is also carried out using the small gain theorem and the gap metric. However, a more complex configuration (rather than the two block structure used in Georgiou and Smith (1997)(Theorem 1)) is used. This configuration has a three plant structure P_1, P_2, P_3, and a stability condition is given for this system using the gap metric network result introduced in Georgiou and Smith (1997)(Theorem (10)). In addition, an illustrative example is introduced to compare the stability conditions found using the small gain theorem and the gap metric.

Chapter 5 undertakes a robustness analysis for an unstable affine nonlinear system using the gap metric. The gap analysis carried out in this chapter is more complicated than that of Chapter 4. Since the considered system is unstable, the stability assumption on the plant is dropped and coprime factors are used to represent the unstable plant in the system. Two cases of affine systems are considered here, the first case comprises an unstable affine nonlinear system with an unstable nonlinear part, the controller in this case carries out an inverting action to cancel all the nonlinear terms in the system. The second case considered is the affine nonlinear system with unstable linear and stable and unstable nonlinear components. In this case the system is assumed to have two nonlinear parts, an unstable nonlinear component also cancelled by control action, and a useful stable nonlinear component, whose control action is preserved in this approach. Stability conditions for these systems are given using the gap metric network result introduced in Georgiou and Smith (1997)(Theorem (10)).

In Chapter 6 a special class of unstable affine systems which have only a single nonlinear plant component is considered. This nonlinear component includes a stable part and an unstable part. The configuration used in the analysis undertaken in this chapter uses the linear stabilising component of the controller to stabilize the linear unstable part of the plant and produce a new stabilized plant. The gap analysis for this system also follows the procedure carried out in Chapter 4. Stability condition is given for this system using the gap metric network result introduced in Georgiou and Smith (1997)(Theorem (10)).

In Chapter 7 conclusions are drawn and future work is described in detail.

Chapter 2

Preliminaries

This chapter introduces the notation used in this thesis. Several concepts related to signals and systems are presented. Furthermore, the chapter illustrates the nonlinear robust stability framework and relates this framework to uncertainty modelling and gap metric analysis.

2.1 Signals and Norms

Signals can be defined as a pattern of variation of a physical quantity. A signal v can be considered as a map from time to a value, e.g. $v : t \to \mathbb{R}^n$, $t \in \mathbb{R}_+$ where $\mathbb{R}_+$ is the set of positive real numbers. A signal space can be defined as a vector space $\mathcal{V}$ which is a set of signals defined on a scalar field and satisfying a number of axioms.

In order to describe the performance of a control system we should be able to measure the size of certain signals in this system. For this purpose norms are used. Norms can be described as a yardstick with which we measure the size of vectors, or of real-valued function, or of vector-valued functions. A general definition of a norm can be stated as:

Definition 2.1. Let E be a vector space over either $\mathbb{R}$ the set of real numbers or $\mathbb{C}$ the set of complex numbers, a norm on E is a function $\|.\|$ with the following properties:
For all $a \in \mathbb{C}$ or $\mathbb{R}$ and all $x, y \in E$,

- $x = 0 \Leftrightarrow \|x\| = 0$: positivity,
- $\|ax\| = |a|\|x\|$: homogeneity,
- $\|x + y\| \leq \|x\| + \|y\|$: triangle inequality.

where E is closed under finite vector addition and scalar multiplication.

Examples on norms that are used to express physical properties of a signal defined in $\mathbb{R}$, $\mathbb{R}_+$ are:

$$\begin{aligned} \|x\|_p &= \left(\int_{\mathbb{R}} |x(t)|^p dt\right)^{\frac{1}{p}}, 1 \leq p < \infty \\ \|x\|_\infty &= ess \sup_{t\in\mathbb{R}} |x(t)|. \end{aligned}$$

By essential supremum we mean:

$$ess \sup_{t\in\mathbb{R}} |x(t)| = \inf\left(a \geq 0 : |x(t)| \leq a \quad \text{for almost every} \quad t\right)$$

that is, $|x(t)| \leq a$ except for a set of measure zero, and the ess sup is the smallest number which has that property. The corresponding normed spaces are called Lebesgue L_p and L_∞ spaces, respectively. The norms given above are defined for all functions for which these norms are finite so the signals they define are bounded, e.g. we say that:

$$f : \mathbb{R}_+ \to \mathbb{R} \quad \text{belongs to} \quad L_\infty \quad iff \quad ess \sup_{t\in R} |f(t)| < \infty.$$

Following the definitions given in Buchstaller (2010), we define a signal space $\mathcal{S}$ as the space of all measurable maps $\mathbb{T} \to \mathbb{R}^n$, where $n \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers and $\mathbb{T} = \mathbb{R}$, and define the corresponding signal space $\mathcal{V} \subset \mathcal{S}$ by:

$$\mathcal{V} := \{v \in \mathcal{S}, \forall t \in \mathbb{T}; \|v\| < \infty\}$$

where $\mathcal{V}$ is a normed vector space which includes norm bounded signals $\|v\| < \infty$.
To include signals that becomes unbounded in norm after infinite time, we extend the signal space $\mathcal{V}$ to $\mathcal{V}_e$. We can define the extended space $\mathcal{V} \subset \mathcal{V}_e \subset \mathcal{S}$ by:

$$\mathcal{V}_e := \{v \in \mathcal{S} | \forall t \in \mathbb{T} : T_t v \in \mathcal{V}\}.$$

where $(T_t v)(\tau)$ is called the truncation operator of v to the interval $[0, \tau]$:

$$(T_t v)(\tau) = \begin{cases} v(\tau), & 0 \leq \tau \leq t, \quad t \in \mathbb{T}, \\ 0, & \text{otherwise.} \end{cases}$$

A multidimensional extended spaces are defined as follows: if $\mathcal{L}_{\infty,e}$ is the extended $\mathcal{L}_\infty$ norm then $\mathcal{L}^n_{\infty,e} = \underbrace{\mathcal{L}_{\infty,e} \times ... \times \mathcal{L}_{\infty,e}}_{n}$ for any $n \in \mathbb{N}$,

Since the signals used in this thesis are all in $L_{\infty,e}$ space, the following Lemma is useful.

Lemma 2.2. *Let $a(\cdot)$ and $b(\cdot)$ be two functions that belongs to $L_{\infty,e}$ space, then:*

$$\|a(\cdot)b(\cdot)\| \leq \|a(\cdot)\|\|b(\cdot)\|.$$

Proof. Since $a(\cdot)$ and $b(\cdot)$ belong to $L_{\infty,e}$ space, then:

$$\begin{aligned}\|a(\cdot)b(\cdot)\|_{\infty,e} &= \sup_{t>0}|a(t)b(t)|,\\ &\leq \sup_{t>0}|a(t)|\sup_{t>0}|b(t)|,\\ &= \|a(\cdot)\|_{\infty,e}\|b(\cdot)\|_{\infty,e}.\end{aligned}$$

□

2.2 Frequency Domain: Operators and Stability

Control systems are often thought of as being operators. An operator is defined as the mapping from an input vector space to an output vector space. For example an operator G that acts on signal $u \in \mathcal{U}$ and produces an output signal $y \in \mathcal{Y}$ is defined as $G : \mathcal{U} \to \mathcal{Y}$. Operators can exhibit many properties such as linearity, continuity and boundedness.

Norms can be used to measure the "gain" of operators. For example the maximum gain for G is given by the induced operator norm

$$\|G\| = \sup_{u\neq 0, u\in\mathcal{U}} \frac{\|Gu\|}{\|u\|}$$

G is said to be bounded (or gain stable if $\|G\| < \infty$).

Another important property of operators is causality, a causal operator is:

Definition 2.3. An operator $G : \mathcal{U} \to \mathcal{Y}$ is called causal if, and only if,

$$\forall x, y \in \mathcal{U} \quad \forall \tau \in dom(x) \cap dom(y) \quad [T_\tau x = T_\tau y \Rightarrow T_\tau(Gx) = T_\tau(Gy)]. \tag{2.1}$$

This property ensures that an output of an operator depends only on past and current inputs but not future inputs. Using causal operators we can define the following:

Definition 2.4. A causal operator $G : \mathcal{U} \to \mathcal{Y}$ is called gain stable if $G(\mathcal{U}) \subset \mathcal{Y}$, G(0)=0 and

$$\|G\| := \sup\left\{\frac{\|T_\tau Gx\|_\tau}{\|T_\tau x\|_\tau} \quad : \quad x \in \mathcal{U}, \tau > 0, T_\tau x \neq 0\right\} < \infty, \tag{2.2}$$

where $\|T_\tau Gx\|_\tau$, $\|T_\tau x\|_\tau$ are the truncated norms of Gx, x, respectively to the interval $[0, \tau]$.

To study the stability of causal operators, it is necessary to consider the relations between time and frequency domain spaces (Based on the definitions given in Vinnicombe (2001)). For example, the L_2 space in the frequency domain is related to the l_2 space in the time domain, given that a signal $\hat{v} \in L_2$ is the Laplace transform of a signal $v \in l_2$. In either space, the norm of v is equivalent to the norm of $\hat{v}$ (by Parseval's theorem) and can be used to measure the energy of these signals.

Defining $\mathcal{H}_2$ as the space which contains all bounded signals in the frequency domain. A stable system is a system that maps an input signal in $\mathcal{H}_2$ to an output signal which is also in $\mathcal{H}_2$. For a linear, time invariant, continuous time, stable system we consider $\mathcal{H}_\infty$ space which is a Hardy space. Given an operator $P \in \mathcal{H}_\infty$, this has norm

$$\|P\|_\infty = \sup_{u\in\mathcal{H}_2, u\neq 0} \frac{\|Pu\|_2}{\|u\|_2}.$$

This norm measures the maximum energy gain of the system P. A stable P system has a finite $\|P\|_\infty$. Another expression for $\mathcal{H}_\infty$ is

$$\|P\|_\infty = \sup_{s:\Re(s)>0} \bar{\sigma}(P(s)),$$

where $(s : \Re(s) > 0)$ is the space of functions of the complex variable s that are analytic for all s in the open RHP and $\bar{\sigma}$ denotes the maximum singular value, see for example Vinnicombe (2001).

For an unstable P we have $\|P\|_\infty = \infty$ and coprime factors are required to represent P.

2.3 Coprime Factor Representation of an Operator

Suppose that $P \in \mathcal{R}$ is unstable, where $\mathcal{R}$ denotes the space of all real rational transfer functions, then P can be represented as a quotient of two $\mathcal{RH}_\infty$ stable functions M, N. Given that there exist $X, Y \in \mathcal{RH}_\infty$ such that $NX + MY = I$ a right coprime factorization of a plant P is defined when $P = NM^{-1}$, where the zeros of M are the closed RHP poles of P.

One approach to find the coprime factors M, N for an unstable plant P is using a state-space approach. Consider the system:

$$\begin{aligned} P &: u \mapsto y, \\ \dot{x} &= Ax + Bu, && (2.3) \\ y &= Cx + Du, && (2.4) \end{aligned}$$

where A, B, C, D are the state, input and output matrices.
A matrix L is chosen such that $A + BL$ is stable, and an external signal $v = u - Lx$ is introduced. Then from (2.3) and (2.4), P can be written as:

$$\begin{aligned} P &: u \mapsto y, \\ \dot{x} &= (A + BL)x + Bv, && (2.5) \\ u &= Lx + v, && (2.6) \\ y &= (C + DL)x + Dv, && (2.7) \end{aligned}$$

which can be given as two stable operators M and N as follows:

$$\begin{aligned} M &: v \mapsto u, \\ \dot{x} &= (A + BL)x + Bv, \\ u &= Lx + v, \end{aligned}$$

and

$$\begin{aligned} N &: v \mapsto y, \\ \dot{x} &= (A + BL)x + Bv, \\ y &= (C + DL)x + Dv. \end{aligned}$$

Clearly, M and N are stable because $(A + BL)$ is stable.

2.4 Feedback System Configuration

For the closed loop system shown in Figure 2.1, the plant P is given as:

$$P : \mathcal{U}_e \to \mathcal{Y}_e \tag{2.8}$$

and the controller C is given as:

$$C : \mathcal{Y}_e \to \mathcal{U}_e \tag{2.9}$$

and we assume $P(0) = 0, C(0) = 0$.

The closed loop system [P,C] is defined in the following set of equations:

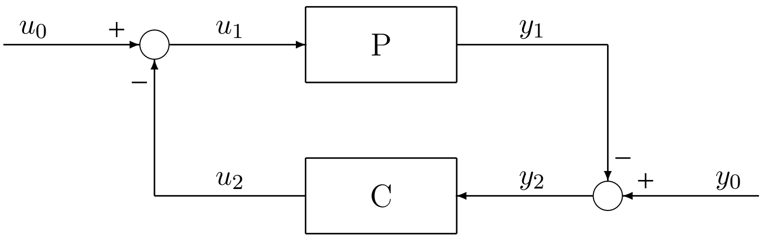

FIGURE 2.1: Closed loop [P,C]

$$y_1 = Pu_1 \tag{2.10}$$
$$u_2 = Cy_2 \tag{2.11}$$
$$u_0 = u_1 + u_2 \tag{2.12}$$
$$y_0 = y_1 + y_2 \tag{2.13}$$

For $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and $\mathcal{W}_e = \mathcal{U}_e \times \mathcal{Y}_e$, where $\mathcal{U}$ and $\mathcal{Y}$ are appropriate input and output signal spaces and $\mathcal{U}_e$ and $\mathcal{Y}_e$ are the extended input and output signal spaces, we have $w_0 = (u_0, y_0)^\top \in \mathcal{W}$ represents the input and output disturbance acting on the plant P, and $w_1 = (u_1, y_1)^\top \in \mathcal{W}_e$ represents the plant input and output, also $w_2 = (u_2, y_2)^\top \in \mathcal{W}_e$ represents the controller input and output. We can define the graph of a system as the set of all possible input-output pairs which are compatible with the description of the system.
Formally the graph of P is defined as:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} : u \in \mathcal{U}, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

The graph of C is defined as:

$$\mathcal{G}_C := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} : Cy \in \mathcal{U}, y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

To study stability, robustness and performance of such closed loop systems [P,C], we would first like to assume a priori that the system is well-posed.

Definition 2.5. A closed loop system [P,C] given by (2.10)-(2.13) is said to be well-posed if for all $w_0 \in \mathcal{W}$ there exists a unique solution $(w_1, w_2) \in \mathcal{W}_e \times \mathcal{W}_e$.

The fundamental nonlinear robust stability framework for this system was developed in Georgiou and Smith (1997), and the following notation and results follow from this work. The closed-loop operator $H_{P,C}$ is defined as the mapping from the external to internal signals as:

$$H_{P,C} \quad : \quad \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e : \quad w_0 \to (w_1, w_2)$$

$H_{P,C}$ can be decomposed into the operator $\Pi_{P//C}$ (which is the map from the disturbances $w_0 \in \mathcal{W}$ to the plant signals $w_1 \in \mathcal{W}_e$) and the operator $\Pi_{C//P}$ (which is the map from the disturbances $w_0 \in \mathcal{W}$ to the controller signals $w_2 \in \mathcal{W}_e$), i.e.

$$\begin{aligned}
\Pi_{P//C} \quad &: \quad \mathcal{W} \to \mathcal{W}_e : \quad w_0 \to w_1, \\
\Pi_{C//P} \quad &: \quad \mathcal{W} \to \mathcal{W}_e : \quad w_0 \to w_2
\end{aligned}$$

so

$$H_{P,C} = (\Pi_{C//P}, \Pi_{P//C}).$$

The stability of a system can be defined by the boundedness of the induced norm of $\Pi_{P//C}$

Lemma 2.6. *Let the closed loop system [P, C] shown in Figure 2.1 and given by (2.10)-(2.13), be well-posed. [P, C] is said to be gain stable if there exists a $M > 0$ such that:*

$$\sup_{w_0 \in \mathcal{W}, w_0 \neq 0} \frac{\|\Pi_{P//C} w_0\|}{\|w_0\|} = \|\Pi_{P//C}\| \leq M < \infty.$$

Proof. Since for all $w_0 \in \mathcal{W}$ we have

$$\Pi_{P//C} w_0 + \Pi_{C//P} w_0 = w_1 + w_2 = w_0,$$

then

$$\Pi_{P//C} + \Pi_{C//P} = I$$

This means that the gain stability of $\Pi_{P//C}$ also ensures gain stability for $\Pi_{C//P}$ and $H_{P,C}$, this can be proved as follows:

$$\begin{aligned}
\|\Pi_{P//C}\| < \infty \quad &\Rightarrow \quad \|\Pi_{P//C} w_0\| \leq M\|w_0\| \quad \forall w_0 \in W \\
&\Rightarrow \quad \|\Pi_{C//P} w_0\| = \|(I - \Pi_{P//C}) w_0\| \\
&\leq \quad \|w_0\| + \|\Pi_{P//C} w_0\| \\
&\leq \quad (M+1)\|w_0\| \quad \forall w_0 \in W \\
&\Rightarrow \quad \|\Pi_{C//P}\| \leq M + 1 < \infty.
\end{aligned}$$

□

A BIBO stable system is defined as:

Definition 2.7. The closed loop system $[P, C]$, is a bounded input bounded output BIBO stable system if the induced norm of the closed loop operator given by:

$$\Pi_{P//C} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}$$

is gain stable (that is $\|\Pi_{P//C}\| < \infty$).

2.5 Stability Analysis using the Small Gain Theorem

The small gain theorem, first introduced in Zames (1966), is the basis for many stability results. Consider the feedback configuration of a nonlinear system consisting of a nonlinear plant P with nonlinear feedback controller C shown in Figure 2.2.

The small gain theorem has been stated in several publications, for example, as follows:

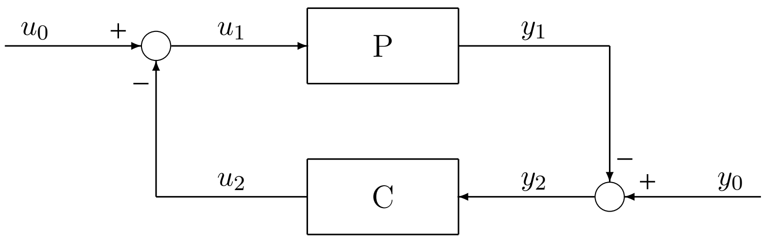

FIGURE 2.2: Feedback control system

Theorem 2.8. *For the system shown in Figure 2.2 with the inputs u_0, y_0 and the outputs y_1, u_2. Assume that the systems $P : L_{pe} \to L_{pe}, C : L_{pe} \to L_{pe}$ are both causal and finite gain stable, that is, there exist $\gamma_1, \gamma_2, \beta_1, \beta_2$ such that:*

$$\| y_1 \| = \| P(u_1) \|_p \leq \gamma_1 \| u_1 \| + \beta \qquad \forall u_1 \in L_{pe}, \tag{2.14}$$

$$\| u_2 \| = \| C(y_2) \|_p \leq \gamma_2 \| y_2 \| + \beta \qquad \forall y_2 \in L_{pe}. \tag{2.15}$$

Further, assume that the loop is well posed in the sense that for given $u_0 \in L_{pe}, y_0 \in L_{pe}$ there are unique $u_1, u_2 \in L_{pe}; y_1, y_2 \in L_{pe}$, then the closed loop system is also finite gain stable from u_0, y_0 to y_1, u_2 if

$$\gamma_1 \gamma_2 < 1.$$

Proof. See Sastry (1999). □

This theorem will be used in Chapters 3 and 4 to find stability conditions for a Hammerstein structure and an affine system with input nonlinearity, respectively.

2.6 Modelling the Uncertainty

Choosing an appropriate nominal model of a system when designing a robust controller determines how well this controller will work on the real system. The mismatch between the real physical plant P_1 and the nominal model P of the plant is referred to as uncertainty. As mentioned in Chapter 1, uncertainty can be structured or unstructured. Structured uncertainty corresponds to the inaccuracies in the terms actually included in the model (unknown plant parameters), e.g. $P_1 = \frac{1}{s+\theta}, \theta_{min} \leq \theta \leq \theta_{max}$. This type of uncertainty modelling has a very restrictive form since using it uncertainties cannot be expressed outside a defined structure.

The other type of uncertainty is unstructured uncertainty which may correspond to unmodelled dynamics or underestimation of the system order. Next, the unstructured uncertainty models will be described following the standard presentation given in Vinnicombe (2001) .

2.6.1 Unstructured Uncertainty Models

Unstructured uncertainty modelling can take many forms such as additive, multiplicative, inverse or coprime factor models, these models are described in this section.

- **Additive Uncertainty**

  In the additive model the uncertainty is assumed to be represented by an additive perturbation. The additive uncertainty set for a plant P is typically given as

$$\{P + \Delta, \|W_1 \Delta W_2\| < 1, \Delta, W_1, W_2 \in \mathcal{RH}_\infty\}.$$

  The weights W_1 and W_2 are chosen sufficiently large at frequencies where the plant P response is well known, driving Δ to be small, and are chosen to be small at frequencies where the plant P response is unknown, driving Δ to be large, (Vinnicombe, 2001). Figure 2.3 shows a block diagram for the additive uncertainty representation.

  However, this uncertainty model has disadvantages. For example, in the presence of additive uncertainty, to ensure that $\|C(1-PC)^{-1}\|$ is small, $\|C\|$ must be small. Moreover, the number of RHP poles should not be changed. This is impractical in feedback control, since it is preferred to have large loop gain in the system.

- **Multiplicative and Inverse Multiplicative Uncertainty**

  Similarly to additive uncertainty a multiplicative uncertainty model can be described using the multiplicative uncertainty set

$$\{(I + \Delta)P, \|W_1 \Delta W_2\| < 1, \Delta, W_1, W_2 \in \mathcal{RH}_\infty\}.$$

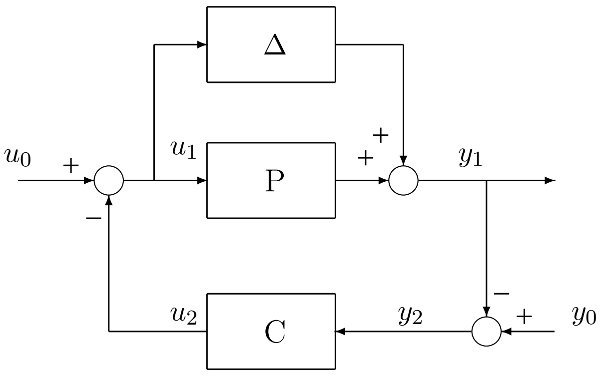

FIGURE 2.3: Additive uncertainty

Multiplicative uncertainty model are often regarded as arising from high frequency unmodelled dynamics. Figure 2.4 shows a block diagram for the multiplicative uncertainty representation.

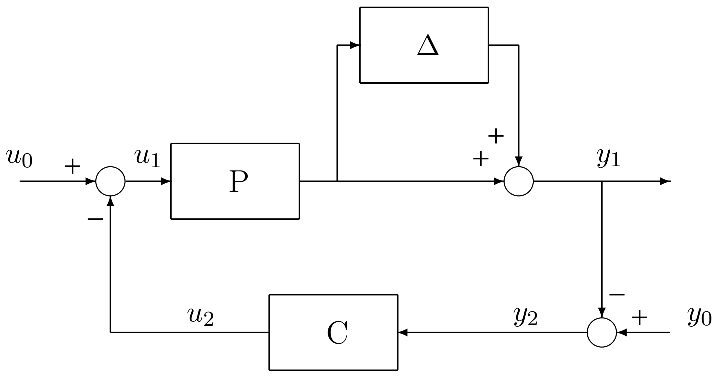

FIGURE 2.4: Multiplicative uncertainty

In contrast, for low frequency unmodelled dynamics the inverse multiplicative uncertainty model is used (Vinnicombe, 2001). If $[P, C]$ is stable and satisfies $\|W_2^{-1}(I - PC)^{-1}W_1^{-1}\| < 1$, then the inverse multiplicative uncertainty set

$$\left\{(I + \Delta)^{-1}P, \|W_1\Delta W_2\| < 1, \Delta, W_1, W_2 \in \mathcal{RH}_\infty\right\}.$$

Figure 2.5 shows a block diagram for the inverse multiplicative uncertainty representation.

In some frequency range, inverse multiplicative uncertainty can be dealt with using large feedback gains. Since in this case robust stability can be guaranteed by insuring that the sensitivity function is small (Vinnicombe, 2001). On the other hand, in another frequency range, multiplicative uncertainty can be dealt with using small loop gain. Since in this case robust stability can be guaranteed by ensuring that the complementary sensitivity function is small.

These two models can be mixed to include different frequency range uncertainties in the system. However, using such a model has a number of drawbacks, e.g., the

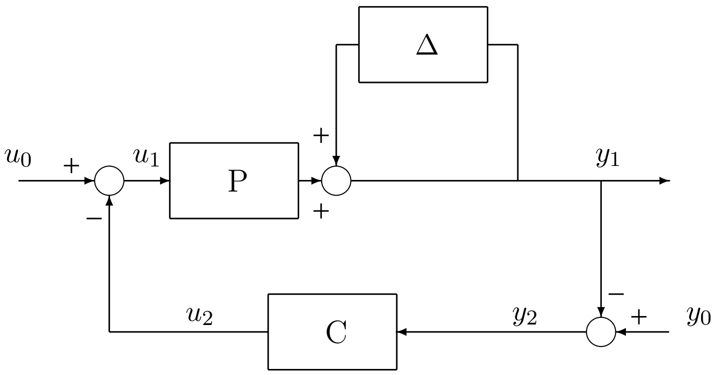

FIGURE 2.5: Inverse multiplicative uncertainty

specifications imposed by the multiplicative uncertainty and inverse multiplicative uncertainty in the system could be contradictory, and the resulting mathematical structure could be complicated.

The next model can combine different types of uncertainty (additive, multiplicative and inverse multiplicative uncertainty models).

- **Coprime Factor Uncertainty**

Coprime factor uncertainty models combine features from all other uncertainty models. Given that $M, N \in \mathcal{RH}_\infty$ are the right coprime factors of a plant P defined when $P = NM^{-1}$ and there exist $X, Y \in \mathcal{RH}_\infty$ such that $XM + YN = I$. Moreover, if $M^*M + N^*N = I$ (where M^*, N^* are the conjugates of M, N, respectively) then the pair M, N are considered normalized. Then the coprime factor uncertainty set is of the form

$$\left\{ (N + \Delta_N)(M + \Delta_M)^{-1} : \left\| \begin{matrix} \Delta_N \\ \Delta_M \end{matrix} \right\| < \frac{1}{\gamma} \right\}.$$

where $\gamma > 1$.

Figure 2.6 shows a block diagram for the coprime factor uncertainty representation.

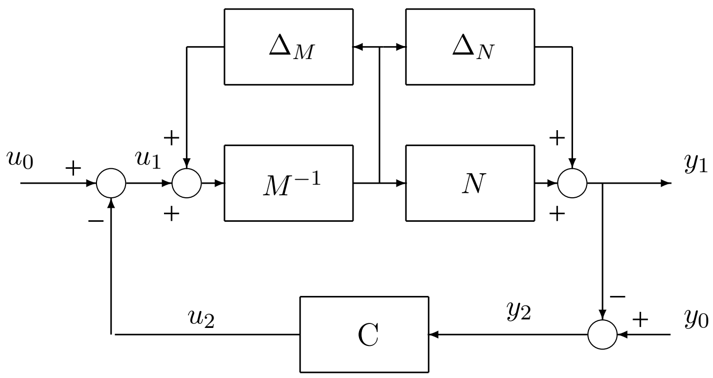

FIGURE 2.6: Coprime factor uncertainty

As discussed in Chapter 1, the linear gap metric presented in Zames and El-Sakkary

(1980) measures the size of coprime factor perturbations.

$$\vec{\delta}(P, P_1) = \inf_{(\Delta_N \Delta_M) \in \mathcal{RH}_\infty} \left\{ \left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\| : \left(\begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right) \in \mathcal{RH}_\infty, P_1 = (N + \Delta_N)(M + \Delta_M)^{-1} \right\}.$$

Based on normalized coprime factorization, this type of perturbation provides a good description of unstructured model uncertainties compared to other uncertainty models (Vinnicombe, 2001).

2.6.2 Considering Nonlinear Components as Uncertainties in a Nonlinear System

As mentioned in Chapter 1, this thesis studies the robustness for nonlinear systems control design. This study considers a nominal model, which will include in most cases the linear component of the actual nonlinear system, and a perturbed plant which will include the nominal linear model along with the nonlinear component as the uncertainty presented in the real plant.

As an example, consider the SISO nonlinear system given by the Hammerstein model shown in Figure 2.7 (the Hammerstein model will be considered in Chapter 3).

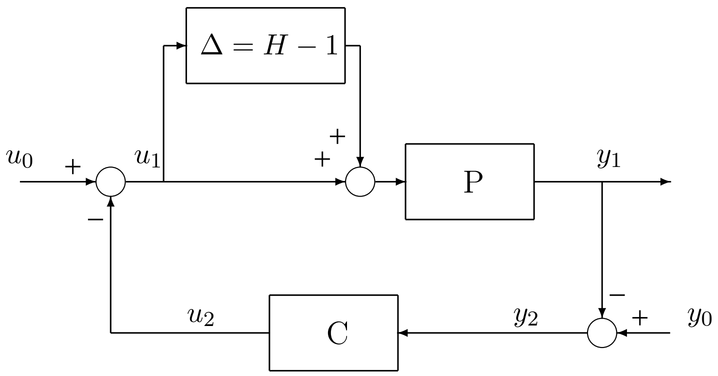

FIGURE 2.7: Multiplicative uncertainty

Note that the considered system is an example of a perturbed system with multiplicative uncertainty, however, the uncertainty model in this case lies at the input part of the plant (while the multiplicative uncertainty model lies at the output of the plant). Given that the nominal linear plant is P and let $h(u)$ be a bounded memoryless continuous increasing nonlinear function, where $\|h\| < D, 0 < D < \infty$, we can define an operator $H : u \mapsto y$ such that $H(u)(t) = h(u(t))$ then the uncertainty $\Delta = H - 1$ and the uncertainty model set is given by:

$$\{P(1 + (H - 1)) : H \in \mathcal{RH}_\infty, \|H\| < D\}.$$

2.7 Robust Stability Analysis Using the Gap Metric

In the presence of uncertainty, stabilization of feedback systems is established using robust stability margins. According to Lemma 2.6 a closed loop system $[P, C]$ is gain stable if the induced norm $\|\Pi_{P//C}\| < \infty$. A robust stability margin for the system $[P, C]$ is defined as the inverse of the gain $\|\Pi_{P//C}\|$ which is given as

$$b_{P,C} = \begin{cases} \frac{1}{\|\Pi_{P//C}\|} & \text{for } \|\Pi_{P//C}\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $b_{P,C}$ has the same definition for the system $[P, C]$ whether the operators P, C are LTI systems or nonlinear systems. Given this margin, a stability condition for a perturbed linear system P_1 is found using the gap metric as follows

Theorem 2.9. *Let $P, P_1, C \in \mathcal{R}$ and let the closed loop system $[P, C]$ shown in Figure 2.1 and given by (2.10)-(2.13), be well-posed and gain stable. Then the well-posed closed loop system $[P_1, C]$ is gain stable if:*

$$\vec{\delta}(P, P_1) < b_{P,C}.$$

Proof. See Georgiou and Smith (1990). □

To undertake stability analysis for nonlinear feedback control systems, the gap metric approach was generalized to be suitable to study the robustness of nonlinear systems in Georgiou and Smith (1997).

This metric is defined as

Definition 2.10. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{W}$ where $\mathcal{W}$ is a signal space then the gap metric can be given as:

$$\begin{aligned} \vec{\delta}(\mathcal{X}, \mathcal{Y}) &= \begin{cases} \inf\{\|(\Phi - I)\mid_{\mathcal{X}}\| : \Phi \text{ is a causal, surjective map from } \mathcal{X} \text{ to } \mathcal{Y} \text{ with} \\ \quad \Phi(0) = 0\}, \\ \infty \text{ if no such operator } \Phi \text{ exists,} \end{cases} \\ \delta(\mathcal{X}, \mathcal{Y}) &= \max\left\{\vec{\delta}(\mathcal{X}, \mathcal{Y}), \vec{\delta}(\mathcal{Y}, \mathcal{X})\right\}. \end{aligned}$$

Using the gap metric, to have a stable system Georgiou and Smith (1997) (Theorem 1) states that:

Theorem 2.11. *Consider the system shown in Figure 2.1 and denote* $\mathcal{M} := \mathcal{G}_P$ *,* $\mathcal{N} := \mathcal{G}_C$ *, and let* $\mathbf{H}_{P,C}$ *be stable. If a system* P_1*, with* $\mathcal{M}_1 := \mathcal{G}_{P_1}$ *is such that*

$$\vec{\delta}(\mathcal{M}, \mathcal{M}_1) < b_{P,C}. \tag{2.16}$$

then $\mathbf{H}_{P_1,C}$ *is stable and*

$$\|\Pi_{P_1//C}\| \leq \|\Pi_{P//C}\| \frac{1 + \vec{\delta}(\mathcal{M}, \mathcal{M}_1)}{1 - \|\Pi_{P//C}\| \vec{\delta}(\mathcal{M}, \mathcal{M}_1)}.$$

Proof. See Georgiou and Smith (1997). □

This theorem deals with robustness of global stability of nonlinear feedback systems. In Chapter 3, Theorem 2.11 will be used to study the robust stabilization of Hammerstein systems in the sense that the induced norm of the input-to-error mapping is finite and remains finite for suitable perturbations of the nominal plant P. The original condition imposed on the map Φ in the definition of $\vec{\delta}(\mathcal{X}, \mathcal{Y})$ was that this map should be bijective, however, this condition was relaxed in Georgiou and Smith (1997) (Section 3D) to the requirement that Φ can be subjective for the robust stability theorem to remain valid.

The gap analysis carried out in this thesis will consider many cases of nonlinear systems, where stable, unstable and more complex networked systems than the one shown in Figure 2.1 are involved. The following subsections provide critical background results, based on the gap metric, which will be used subsequently in this thesis.

2.7.1 Stability Analysis for a Network System Using the Gap Metric

Consider an interconnection of three systems as shown in Figure 2.8. Let $\mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$ be the input, state and output signal spaces, respectively, where we define a signal space to be an extended space, e.g., $\mathcal{L}_{\infty,e}$ with $u_i \in \mathcal{U}$ $(i = 0, 1, 2, 3)$ and denote $\mathcal{W} := \mathcal{U} \times \mathcal{X} \times \mathcal{Y}$.

The closed-loop operator H_{P_1,P_2,P_3} is defined as the mapping from external to internal signals, given by

$$H_{P_1,P_2,P_3} :$$
$$\begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix} \rightarrow \left(\begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ y_3 \end{pmatrix} \right)$$

We denote by $\Pi_i (i = 1, 2, 3)$ the natural projection from $\mathcal{W}$ onto $\mathcal{U}, \mathcal{X}$ and $\mathcal{Y}$ respectively.

The graphs for P_i in $\mathcal{W}$ are written as

$$\mathcal{G}_{P_1} = \left\{ \begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix} : \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P_1 u_1, \|u_1\| < \infty, \left\| \begin{matrix} x_1 \\ y_1 \end{matrix} \right\| < \infty \right\},$$

$$\mathcal{G}_{P_2} = \left\{ \begin{pmatrix} u_2 \\ x_2 \\ y_2 \end{pmatrix} : u_2 = 0, y_2 = P_2 x_2, \|x_2\| < \infty, \|y_2\| < \infty \right\},$$

$$\mathcal{G}_{P_3} = \left\{ \begin{pmatrix} u_3 \\ x_3 \\ y_3 \end{pmatrix} : u_3 = P_3 y_3, x_3 = 0, \|u_3\| < \infty, \|y_3\| < \infty \right\}.$$

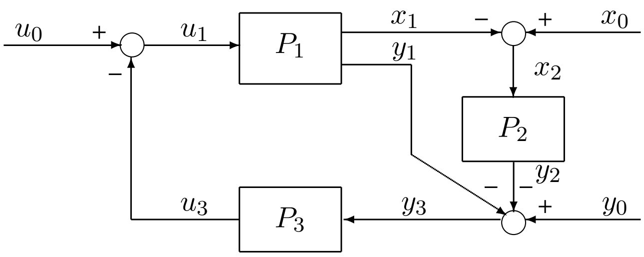

FIGURE 2.8: Feedback interconnection of three subsystems $P_i, (i = 1, 2, 3)$

Writing $\mathcal{M}_i := \mathcal{G}_{P_i}, (i = 1, 2, 3)$ define the summation operator as:

$$\Sigma_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3} : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \to \mathcal{W},$$

so that:

$$H_{P_1,P_2,P_3} = \Sigma^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}.$$

Let $\Pi_{(i)}$ denote the mapping

$$\Pi_{(i)} := \Pi_i \Sigma^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}, \qquad (i = 1, 2, 3).$$

Definition 2.12. The closed-loop $[P_1, P_2, P_3]$ is called gain stable if the operator $\Pi_{(i)}$, where $i = 1, 2, 3$, has a finite induced norm, i.e.

$$\|\Pi_{(i)}\| = \sup_{\|w\| \neq 0} \frac{\|\Pi_{(i)} w\|}{\|w\|} < \infty. \tag{2.17}$$

Now consider perturbed systems P_1', P_2', P_3', acting on the appropriate spaces, with graphs $\mathcal{M}'_i := \mathcal{G}_{P_i'}$. Accordingly, define

$$\Pi'_{(i)} := \Pi_i \Sigma^{-1}_{\mathcal{M}'_1,\mathcal{M}'_2,\mathcal{M}'_3}, \qquad (i = 1, 2, 3).$$

According to Georgiou and Smith (1997) the gap metric between the nominal plant P_i

and any perturbed plant P'_i is defined as follows:

$$\vec{\delta}(\mathcal{M}_i, \mathcal{M'}_i) = \begin{cases} \inf\{\|(\Phi - I)\mid_{\mathcal{M}_i}\| : \Phi \text{ is a causal,} \\ \quad \text{surjective map from } \quad \mathcal{M}_i \\ \quad \text{to} \quad \mathcal{M}'_i \quad \text{with} \quad \Phi(0) = 0\}, \\ \infty \text{ if no such operator } \Phi \quad \text{exists,} \end{cases}$$

$$\delta(\mathcal{M}_i, \mathcal{M'}_i) = \max\left\{\vec{\delta}(\mathcal{M}_i, \mathcal{M'}_i), \vec{\delta}(\mathcal{M'}_i, \mathcal{M}_i)\right\}.$$

The robust stability theorem, Georgiou and Smith (1997) (Theorem 10) states that:

Theorem 2.13. *Let H_{P_1,P_2,P_3} be gain stable. If*

$$\alpha := \sum_{i=1}^{3} \vec{\delta}(\mathcal{M}_i, \mathcal{M'}_i)\|\Pi_{(i)}\| < 1 \tag{2.18}$$

then $\Sigma^{-1}_{\mathcal{M'}_1,\mathcal{M'}_2,\mathcal{M'}_3}$ is gain stable and

$$\|\Pi'_{(i)}\| \leq \|\Pi_{(i)}\| \frac{1 + \vec{\delta}(\mathcal{M}_i, \mathcal{M'}_i)}{1 - \alpha}.$$

Proof. The proof of this theorem can be found in Georgiou and Smith (1997). □

This provides a condition on the gap $\vec{\delta}(\mathcal{M}_i, \mathcal{M'}_i)$ for plant P_i, and hence defines a plant set able to be stabilized by the controller. In Chapters 4, 5 and 6, Theorem 2.13 will be used to study robust stability for the nonlinear systems considered in these chapters.

2.7.2 Finding the Gap Bound for Nonlinear Stable Systems

Consider the system shown in Figure 2.1, the graphs for stable P and P_1 are given as follows:

$$\mathcal{G}_P := \left\{\begin{pmatrix} u \\ Pu \end{pmatrix} : \|u\| < \infty, \|Pu\| < \infty\right\},$$

$$\mathcal{G}_{P_1} := \left\{\begin{pmatrix} u \\ P_1 u \end{pmatrix} : \|u\| < \infty, \|P_1 u\| < \infty\right\}.$$

A map Φ between the graphs of P and P_1 can be defined as:

$$\begin{aligned} \Phi &: \mathcal{G}_P \to \mathcal{G}_{P_1} \\ \Phi \begin{pmatrix} u \\ Pu \end{pmatrix} &= \begin{pmatrix} u \\ P_1 u \end{pmatrix}. \end{aligned} \tag{2.19}$$

This map is surjective by the following proposition:

Proposition 2.14. *Consider the system shown in Figure 2.1. Suppose P, P_1 are stable, then the map Φ given in (2.19) is surjective.*

Proof. The map Φ is surjective if the following condition is satisfied:

$$\forall y \in \mathcal{G}_{P_1} \quad \exists x \in \mathcal{G}_P \quad s.t. \quad \Phi(x) = y.$$

Let us choose an element $y \in \mathcal{G}_{P_1}$,where:

$$y = \begin{pmatrix} u \\ P_1 u \end{pmatrix},$$

for some $\|u\| < \infty$. As P_1 is stable, then $\|P_1 u\| < \infty$.
Define:

$$x = \begin{pmatrix} u \\ Pu \end{pmatrix}.$$

Since $\|u\| < \infty$ and as P is a stable plant, then $\|Pu\| < \infty$ and $x \in \mathcal{G}_P$.
Finally using the mapping:

$$\Phi(x) = \Phi \begin{pmatrix} u \\ Pu \end{pmatrix} = \begin{pmatrix} u \\ P_1 u \end{pmatrix} = y.$$

□

Using the identity:

$$\begin{aligned} (\Phi - I)x &= (\Phi - I) \begin{pmatrix} u \\ Pu \end{pmatrix}, \\ &= \Phi \begin{pmatrix} u \\ Pu \end{pmatrix} - I \begin{pmatrix} u \\ Pu \end{pmatrix}, \\ &= \begin{pmatrix} u \\ P_1 u \end{pmatrix} - \begin{pmatrix} u \\ Pu \end{pmatrix}, \\ &= \begin{pmatrix} 0 \\ (P_1 - P)u \end{pmatrix}. \end{aligned}$$

A bound on the gap distance is:

$$\begin{aligned}
\vec{\delta}(\mathcal{M},\mathcal{M}_1) &\leq \|(\Phi - I)\mid_{\mathcal{G}_P}\|, \\
&= \sup_{x\in\mathcal{G}_P\setminus\{0\}} \frac{\|(\Phi-I)x\|}{\|x\|}, \\
&\leq \sup_{\begin{pmatrix} u \\ Pu \end{pmatrix}\in\mathcal{G}_P\setminus\{0\}} \frac{\left\|(\Phi-I)\begin{pmatrix} u \\ Pu \end{pmatrix}\right\|}{\left\|\begin{pmatrix} u \\ Pu \end{pmatrix}\right\|}, \\
&= \sup_{\begin{pmatrix} u \\ Pu \end{pmatrix}\in\mathcal{G}_P\setminus\{0\}} \frac{\left\|\begin{pmatrix} 0 \\ (P_1-P)u \end{pmatrix}\right\|}{\left\|\begin{pmatrix} u \\ Pu \end{pmatrix}\right\|},
\end{aligned}$$

Since $\left\|\begin{matrix} u \\ Pu \end{matrix}\right\| \geq \|u\|$ and since $\begin{pmatrix} u \\ Pu \end{pmatrix} \in \mathcal{G}_P\setminus\{0\}$ implies $\begin{pmatrix} u \\ Pu \end{pmatrix} \neq \{0\}$ then $\|u\| \neq 0$ and $\|Pu\| \neq 0$, it follows that:

$$\vec{\delta}(\mathcal{M},\mathcal{M}_1) \leq \sup_{\|u\|\neq 0} \frac{\|(P_1-P)u\|}{\|u\|}. \tag{2.20}$$

The approach described in this subsection will be used in Chapters 3 and 4 to establish a bound on the gap metric $\vec{\delta}(\mathcal{M},\mathcal{M}_1) < \infty$ for the stable nonlinear systems considered in these chapters.

2.7.3 Finding the Gap Bound for Unstable Linear Systems

This thesis studies the robust stability control of unstable nonlinear systems, where coprime factors are used to represent the unstable part of the plant and to find the gap metric (in Chapters 5 and 6). An introduction to this approach is presented in this subsection.

Consider the system shown in Figure 2.1. To represent unstable linear plant P we use coprime factorization which is a significant tool in the study of robustness of stability for linear feedback systems. In this case P is written as the ratio of coprime functions,

$$P = \frac{N}{M}$$

and the graph of P is defined as

$$\mathcal{G}_P = \left\{\begin{pmatrix} M \\ N \end{pmatrix} v : v \in \mathcal{U}\right\}.$$

also for the perturbed system the plant P_1 is defined as:

$$P_1 = \frac{N_1}{M_1};$$

with its graph

$$\mathcal{G}_{P_1} = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} v : v \in \mathcal{U} \right\}.$$

A possible map Φ between $\mathcal{G}_P$ and $\mathcal{G}_{P_1}$ is given as

$$\begin{aligned} \Phi \quad &: \quad \mathcal{G}_P \rightarrow \mathcal{G}_{P_1}, \\ \Phi \begin{pmatrix} M \\ N \end{pmatrix} v \quad &= \quad \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} v. \end{aligned}$$

Using Φ a bound on the gap between P and P_1 is:

$$\vec{\delta}(P, P_1) = \sup_{\begin{pmatrix} M \\ N \end{pmatrix} v \neq 0} \frac{\left\| \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} v - \begin{pmatrix} M \\ N \end{pmatrix} v \right\|}{\left\| \begin{pmatrix} M \\ N \end{pmatrix} v \right\|}$$

In the above equation the operators M, N appear in the denominator of the expression used to find the gap metric. Using the relation:

$$XM + YN = I \Rightarrow (XY) \begin{pmatrix} M \\ N \end{pmatrix} = I$$

then

$$\begin{aligned} \Phi \begin{pmatrix} M \\ N \end{pmatrix} v \quad &= \quad \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} v, \\ \Phi \begin{pmatrix} M \\ N \end{pmatrix} v \quad &= \quad \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} (XY) \begin{pmatrix} M \\ N \end{pmatrix} v, \end{aligned}$$

and if there is a signal $w \in \mathcal{G}_P$ such that:

$$\begin{aligned} w &= \begin{pmatrix} M \\ N \end{pmatrix} v \\ (X \quad Y)w &= (X \quad Y) \begin{pmatrix} M \\ N \end{pmatrix} v \\ &= Iv = v, \end{aligned}$$

then

$$\begin{aligned} (I - \Phi) \mid_{\mathcal{G}_P} w &= (I - \Phi) \begin{pmatrix} M \\ N \end{pmatrix} v, \\ &= \begin{pmatrix} M \\ N \end{pmatrix} v - \Phi \begin{pmatrix} M \\ N \end{pmatrix} v, \\ &= \begin{pmatrix} M \\ N \end{pmatrix} v - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} v, \\ &= \begin{pmatrix} M - M_1 \\ N - N_1 \end{pmatrix} v, \\ &= \begin{pmatrix} M - M_1 \\ N - N_1 \end{pmatrix} (X \quad Y)w. \end{aligned}$$

Taking the norms for both sides, gives:

$$\begin{aligned} \|(I - \Phi) \mid_{\mathcal{G}_P} w\| &= \sup_{0 \neq w \in \mathcal{G}_P} \frac{\|(\Phi - I)w\|}{\|w\|}, \\ &= \sup_{0 \neq w \in \mathcal{G}_P} \frac{\left\| \begin{pmatrix} M - M_1 \\ N - N_1 \end{pmatrix} (X \quad Y)w \right\|}{\|w\|} \\ &= \left\| \begin{pmatrix} M - M_1 \\ N - N_1 \end{pmatrix} (X \quad Y) \right\|, \end{aligned}$$

and a bound on the gap is:

$$\vec{\delta}(P, P_1) \leq \left\| \begin{pmatrix} M - M_1 \\ N - N_1 \end{pmatrix} \right\| \|X \quad Y\| \tag{2.21}$$

Chapter 3

Stability Analysis for Hammerstein Systems Using the Small Gain Theorem and the Gap Metric

3.1 Introduction

In this chapter a simple nonlinear system structure is considered, the Hammerstein model, and a procedure is presented to study the robustness of feedback linearization controllers designed for this model using the small gain theorem and the gap metric. The analysis performed in this chapter will help to explain the approach undertaken to study the robustness of feedback linearizing controllers designed for more complex nonlinear system structures, which will be considered in the following chapters.

First, the Hammerstein model is presented, then the nonlinear feedback control for this system is introduced. Then, the small gain theorem is used to find a simple stability condition for this feedback control system. Finally, the main stability result for the Hammerstein model is given using the gap metric analysis.

3.2 Hammerstein Models

The Hammerstein model is a type of nonlinear structure which is composed of a memoryless nonlinear function and a linear dynamic subsystem, as shown in Figure 3.1. The identification of Hammerstein models was first suggested by (Narendra and Gallman, 1966). Along with other types, such as Wiener, Wiener-Hammerstein and others (Haber and Unbehauen, 1990), the Hammerstein model has proven to be successful in describing the nonlinear dynamics of many chemical, biological and electrical processes (Sung, 2002). In chemical processes, this model can account for nonlinear effects encountered where the nonlinear behaviour of many distillation columns, pH neutralization processes, heat exchangers as well as furnaces and reactors can be effectively modelled by a nonlinear static element followed by a linear dynamic element (Fruzzetti et al., 1997).

In particular, the Hammerstein structure has been particularly useful for describing biological systems that involve neural encoding, like the stretch reflex. An important application for this is stroke rehabilitation using robotic systems and functional electrical stimulation, where the Hammerstein structure is used to model the nonlinear isometric recruitment curve and the linear activation dynamics of an electrically stimulated muscle (Le et al., 2010). For this application, the Hammerstein model has been found to accurately capture the force or torque generated by electrically stimulated muscle, facilitating increasingly accurate model-based control approaches to be applied (Freeman et al., 2012b).

To describe the Hammerstein model, consider the system shown in Figure 3.1. The input, $u(t)$, is transformed by a memoryless, i.e.(static), invertible nonlinearity ϕ producing the intermediate signal, $v(t)$. This, in turn, is operated on by a dynamic linear system P, producing $y(t)$, the system output.

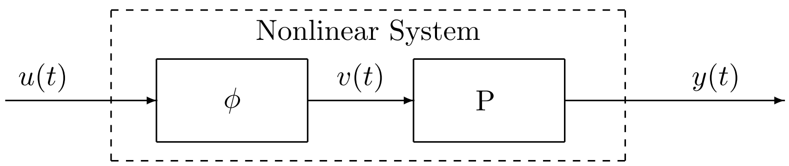

FIGURE 3.1: Hammerstein Model

We can define the memoryless property of a system as follows:

Definition 3.1. In a memoryless (static) nonlinear system, the output of the system at time t depends only on the instantaneous input values at time t and not on any of the past values of its input. Formally:

$$\begin{aligned} \Phi : L_{pe} \to L_{pe} \quad &\text{is memoryless if} \quad \exists \phi : \mathbb{R} \to \mathbb{R} \quad \text{s.t.} \\ \Phi(u)(t) = \phi(u(t)) \qquad &\forall u \in L_{pe}, 1 \leq p < \infty. \end{aligned}$$

The input-output relations for the Hammerstein model can be given as:

$$\begin{aligned} v(t) &= \phi(u(t)) \\ y(t) &= Pv(t) \\ &= P\phi(u(t)). \end{aligned}$$

In the literature, a common method to control the Hammerstein system is to use a model of the inverse of the static nonlinearity in order to cancel it (Samuelsson et al., 2005), (Hwang and Hsu, 1995). Another possibility is to use a gain scheduling technique (Åstrm and Wittenmark, 1995) or approximate the inverse implicitly using high-gain feedback (Goodwin et al., 2000). Another approach which is based on the dissipativity theory is described in Haddad and Chellaboina (2001). This approach assumes that the linear part of the system is passive and the nonlinear controller is exponential passive, while the class of the input nonlinearities is general. In this paper, the global closed loop stability is achieved by modifying the input to the nonlinear compensator to counteract the effects of the input nonlinearity.

In the following sections, robustness analysis will be performed for a feedback linearizing controller designed for a Hammerstein model. This analysis will be implemented using the small gain theorem and the gap metric.

3.3 Nonlinear Feedback Control of a Hammerstein Model

In this section the nonlinear feedback control of a SISO Hammerstein model system is investigated. In this system we consider the nonlinear plant to consist of P which is the linear component and ϕ which is the memoryless invertible nonlinear component. The nonlinear feedback controller consists of the linear component C and $-\phi^{-1}$ which is the inverse of the memoryless nonlinearity ϕ. The term $-\phi^{-1}$ forms the nonlinear component of the controller. This configuration is shown in Figure 3.2.

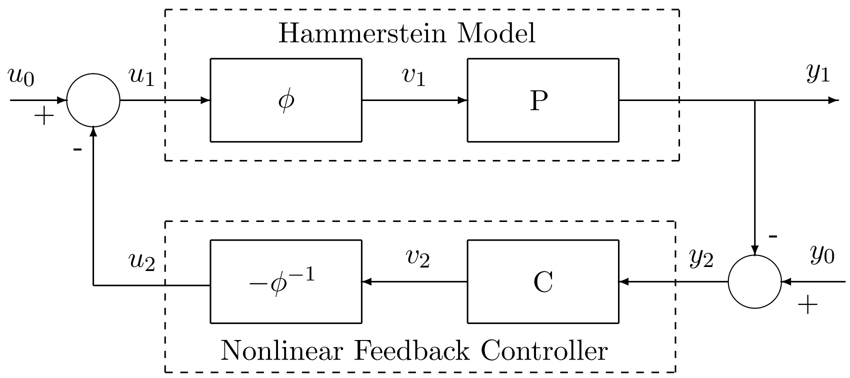

FIGURE 3.2: Nonlinear feedback control system for a Hammerstein model

In this configuration, if we set the input disturbance u_0 to zero, as shown in Figure 3.3(a), then we have an exact cancelation for the nonlinearity in the system since $v_1 = \phi(\phi^{-1}(v_2)) = v_2$, and classical LTI feedback control design can be used to control the plant P, as shown in Figure 3.3(b), as the closed loop system is equivalent to the LTI interconnection of P and C.

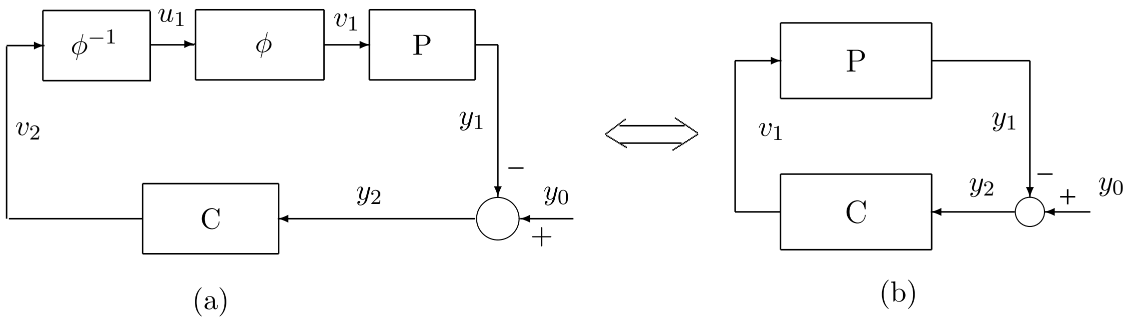

FIGURE 3.3: Reduction to the feedback configuration [P,C]

However, $u_0 \neq 0$ and hence $v_1 = \phi(u_1) = \phi(u_0 - u_2) = \phi(u_0 + \phi^{-1}(v_2))$, then to analyze the system in Figure 3.2 further consideration is required. Let us consider the more general Hammerstein system:

$$\begin{aligned} M_h \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ & : u_1 \mapsto y_1, \\ & y_1 = P\phi(u_1), \end{aligned} \tag{3.1}$$

with the nonlinear component specified as:

$$\begin{aligned} \phi \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ & : u_1 \mapsto v_1, \\ & v_1 = f(u_1), \end{aligned} \tag{3.2}$$

while P can be suitably represented by:

$$\begin{aligned} P \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ & : v_1 \mapsto y_1, \\ & y_1 = Pv_1, \end{aligned} \tag{3.3}$$

A nonlinear feedback linearizing controller N_h is defined to be:

$$\begin{aligned} N_h \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ & : y_2 \mapsto u_2, \\ & u_2 = \psi(Cy_2). \end{aligned} \tag{3.4}$$

This controller consists of two components, the nonlinear component ψ which tries to cancel the nonlinear component ϕ of the plant, and the linear component C which stabilizes the linear component P of the plant. These components can be described as:

$$\begin{aligned} C \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ &: y_2 \mapsto v_2, \\ & v_2 = Cy_2, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \psi \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ &: v_2 \mapsto u_2, \\ & u_2 = -f^{-1}(v_2). \end{aligned} \tag{3.6}$$

Note that $\psi = -\phi^{-1}$ and as $u_0 \neq 0$ then $[M_h, N_h]$ is not equivalent to the linear system $[P, C]$. The configuration in Figure 3.4 shows this system and in the following section robustness analysis will be carried out for this configuration.

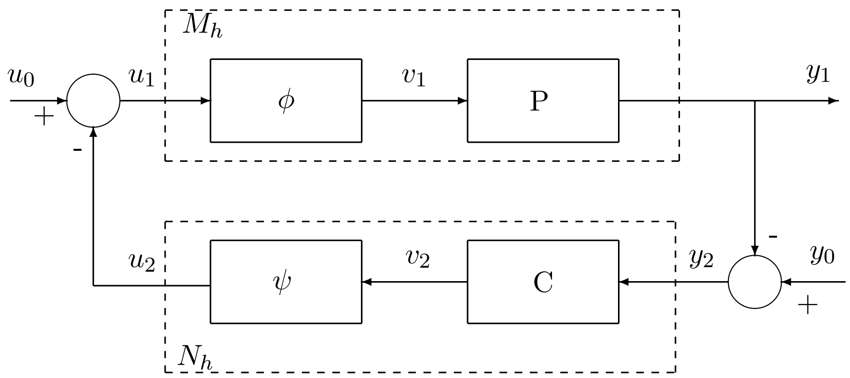

FIGURE 3.4: Hammerstein feedback control system configuration.

The closed loop equations for the system shown in Figure 3.4 are:

$$u_0 = u_1 + u_2, \tag{3.7}$$
$$y_0 = y_1 + y_2, \tag{3.8}$$
$$v_1 = f(u_1), \tag{3.9}$$
$$v_2 = Cy_2, \tag{3.10}$$
$$y_1 = Pf(u_1), \tag{3.11}$$
$$u_2 = -f^{-1}(Cy_2). \tag{3.12}$$

3.4 Stability Analysis for a Hammerstein Model Control System Using the Small Gain Theorem

As mentioned in Chapter 1 the small gain theorem is a classical tool for input-output stability. This theorem provides a straightforward stability condition for nonlinear systems, i.e. $\|P\|\|C\| < 1$. However, the theorem does not apply to unstable P.

In this analysis the following assumption on the form of $f(u_1)$ is required:

Assumption 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a memoryless continuous nonlinear function, satisfying the following condition:

$$\begin{aligned} &\exists \varepsilon > 0, \quad \exists D < \infty \text{ and} \\ &\varepsilon x \leq f(x) \leq Dx \quad \forall x. \end{aligned}$$

Hence

$$\|f(x)\| \leq D\|x\| \quad \forall x \in \mathbb{R}, \tag{3.13}$$

$$\left\|f^{-1}(x)\right\| \leq \frac{1}{\varepsilon}\|x\| \quad \forall x \in \mathbb{R}. \tag{3.14}$$

In this section, a stability condition based the small gain theorem 2.8 for the nonlinear closed loop system $[M_h, N_h]$ shown in Figure 3.4 is given in the following theorem:

Theorem 3.3. *Consider the nonlinear closed loop system $[M_h, N_h]$ given by Figure 3.4 and equations (3.7-3.12). Suppose P is stable and f satisfies Assumption 3.2. Then the closed loop system $[M_h, N_h]$ is a bounded input bounded output stable system if the following condition holds:*

$$\frac{D}{\varepsilon}\|P\|\|C\| < 1.$$

Proof. First we consider y_1 and u_2 in the system shown in Figure 3.4 and given by equations (3.11) and (3.12), respectively, as follows:

$$\begin{aligned} y_1 &= Pv_1 = Pf(u_1) \\ \|y_1\| &\leq \|P\|\|f(u_1)\|, \end{aligned}$$

Using(3.13) we have:

$$\|y_1\| \leq D\|P\|\|u_1\|, \tag{3.15}$$

for u_2:

$$\begin{aligned} u_2 &= -f^{-1}(Cy_2), \\ \|u_2\| &\leq \|f^{-1}(Cy_2)\|, \end{aligned}$$

using (3.14) we have

$$\|u_2\| \quad \leq \quad \frac{1}{\varepsilon}\|C\|\|y_2)\|, \tag{3.16}$$

Now consider equations (2.14) and (2.15) in Theorem 2.8, then:

$$\begin{aligned} \| y_1 \| &= \| P(u_1) \| \leq \gamma_1 \| u_1 \| + \beta, \\ \| u_2 \| &= \| C(y_2) \| \leq \gamma_2 \| y_2 \| + \beta. \end{aligned}$$

Let $\beta_1, \beta_2 = 0$, so that:

$$\| y_1 \| = \| P(u_1) \| \leq \gamma_1 \| u_1 \|, \tag{3.17}$$

$$\| u_2 \| = \| C(y_2) \| \leq \gamma_2 \| y_2 \| . \tag{3.18}$$

Comparing the expressions in (3.17), (3.18) with those found for $\| y_1 \|$, $\| u_2 \|$ given by (3.15) and (3.16), respectively, we have:

$$\gamma_1 = D\|P\|.$$

and

$$\gamma_2 = \frac{1}{\varepsilon}\|C\|.$$

from the above two expressions for γ_1 and γ_2, the closed loop system $[M_h, N_h]$ is a bounded input bounded output stable system if

$$\gamma_1\gamma_2 < 1 \Rightarrow \frac{D}{\varepsilon}\|P\|\|C\| < 1,$$

as required. □

This stability condition is simple, it states that the stability of closed loop system is ensured if the product of the system components gains is less than one. However, it shows that the closed loop system $[M_h, N_h]$ is stable if the linear part P of the plant is stable.

3.5 Stability Analysis for a Hammerstein Model Control System Using the Gap Metric

In this section we will consider the application of the gap metric to study the stability of the nonlinear feedback control system shown in Figure 3.4.

The main result is stated as follows:

Theorem 3.4. *Consider the nonlinear closed loop system $[M_h, N_h]$ given by Figure 3.4 and (3.7)-(3.12). Suppose P is stable and f satisfies Assumption 3.2. Then $[M_h, N_h]$ has a robust stability margin.*

The proof requires results that are developed in this section, and appears subsequently. The route taken is as follows:

Although it is possible to apply the robust stability result Theorem 2.11 directly to the system shown in Figure 3.4, the presence of nonlinear elements in multiple blocks leads to significant conservatism. Hence a new system representation is required where the nonlinear component of the plant and the nonlinear component of the controller are both included together with the linear component of the plant in block P_1', and only the linear component of the controller is included in the block C_1'. This configuration is shown in Figure 3.5. This new configuration of the perturbed (actual) system allows us

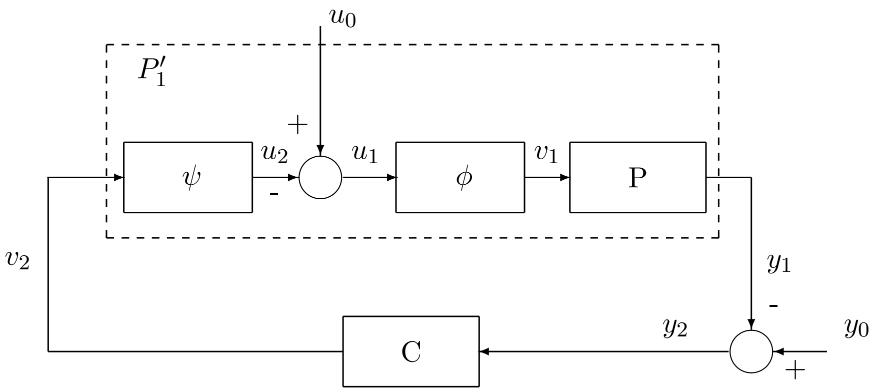

FIGURE 3.5: Second Hammerstein model feedback control system configuration

to consider the nonlinear components of the plant and controller in calculating a bound for the gap metric and minimizing this value will mean that we can make this nonlinear system approximate to its linear version.

The linear configuration for this system is shown in Figure 3.6, where the unperturbed linear system is taken to be the system components P_1, C_1 with the nonlinearities ϕ, ψ set to $\phi = 1, \psi = -1$.

To present the nonlinear and linear configurations of this system shown in Figures 3.5 and 3.6, respectively, in the form of Figure 2.1, we add $x_0 = 0$ as an external input to the systems shown in Figures 3.5 and 3.6, as shown in Figures 3.7 and 3.8, respectively.

The following augmented signals will be considered to define input and output signals in the new forms of these systems. Hence, let the external input u_0 be changed to $u_0' = \begin{pmatrix} u_0 \\ x_0 \end{pmatrix}$ since $x_0 = 0$ then $u_0' = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$, let $u_2' = \begin{pmatrix} 0 \\ -v_2 \end{pmatrix}$ and let $u_1' = u_0' - u_2' =$

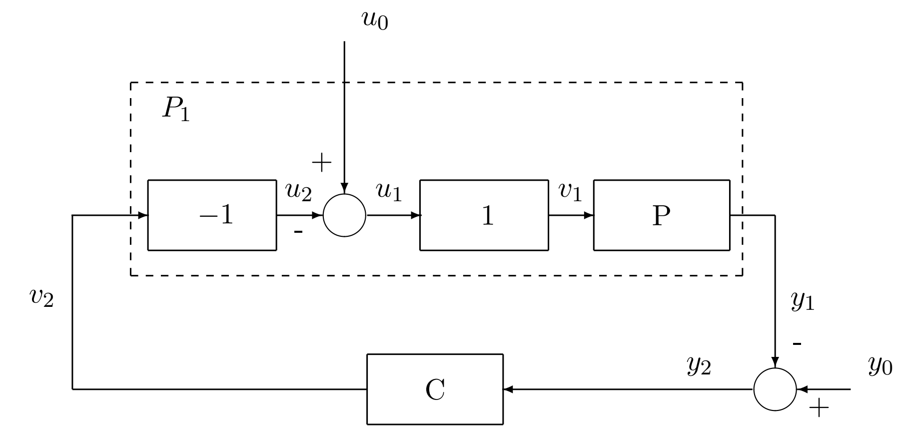

FIGURE 3.6: Hammerstein model feedback control system, linear configuration

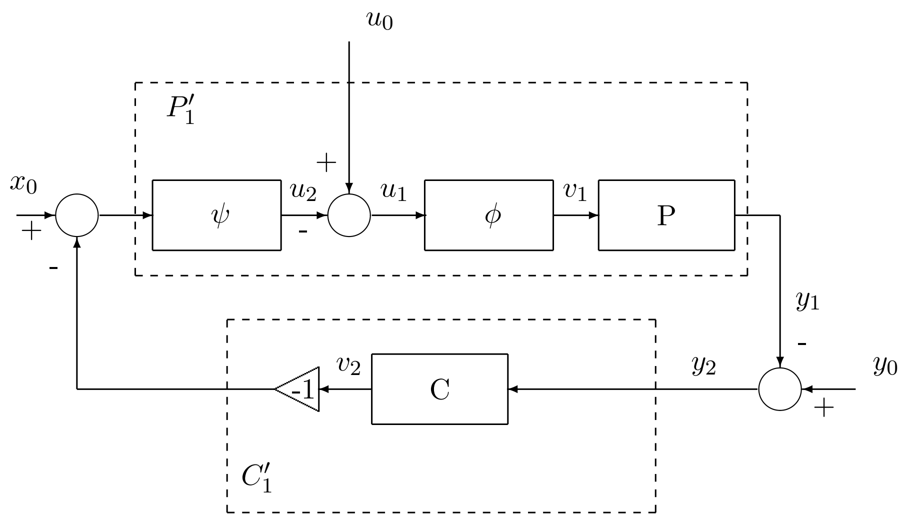

FIGURE 3.7: Second Hammerstein model feedback control system configuration with extra input x_0

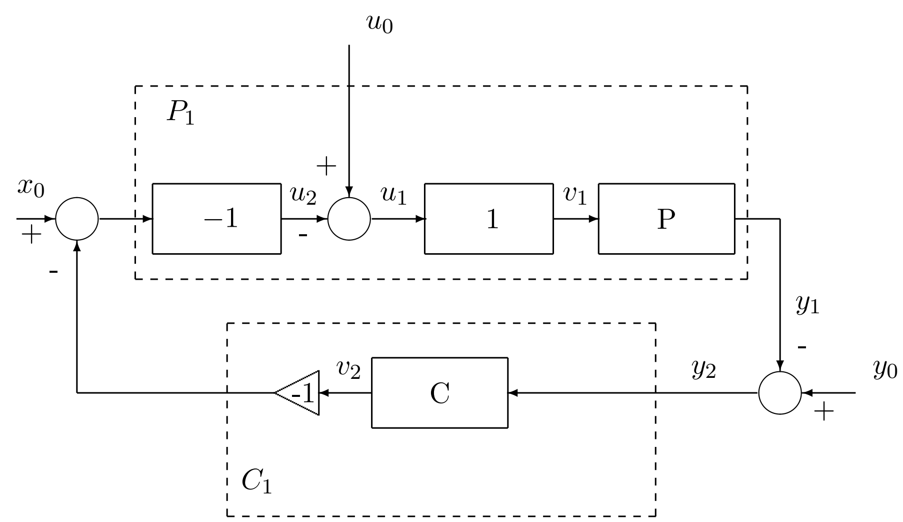

FIGURE 3.8: Hammerstein model feedback control system, linear configuration with extra input x_0

$\begin{pmatrix} u_0 \\ v_2 \end{pmatrix}$. Also let $y_0' = y_0$, $y_1' = y_1$ and $y_2' = y_2$. The augmented system is shown in Figure 3.9.

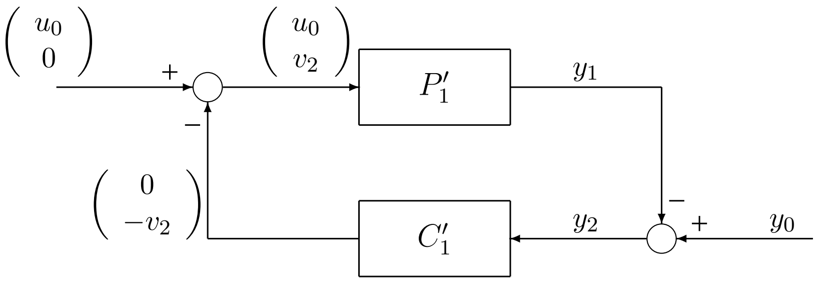

FIGURE 3.9: Nonlinear feedback system configuration

The system shown in Figure 3.9 corresponds exactly to the system form shown in Figure 3.10, which in turn has identical structure to that of Figure 2.1.

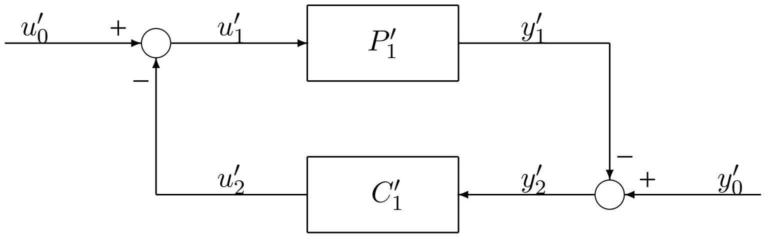

FIGURE 3.10: Nonlinear control system configuration

The linear configuration of this system is shown in Figure 3.11,

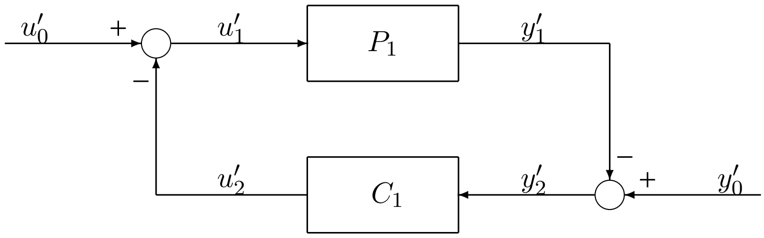

FIGURE 3.11: linear control system configuration

The gap metric measures the difference between the linear nominal plant $P_1 : \left(u_0, v_2 \right) \mapsto y_1', y_1' = P(u_0 + v_2)$ and the nonlinear perturbed plant $P_1' : \left(u_0, v_2 \right) \mapsto y_1', y_1' = P\phi(u_0 - \psi(v_2))$. The unperturbed and perturbed plants P_1 and P_1' are shown in Figure 3.12.

Before providing a complete description of the operators P_1', C_1', P_1, C_1 shown in Figures 3.10, 3.11 we briefly state the motivation for the proceeding manipulations.

A stability condition on the gap between P_1 and P_1' is provided using Theorem 2.11 as:

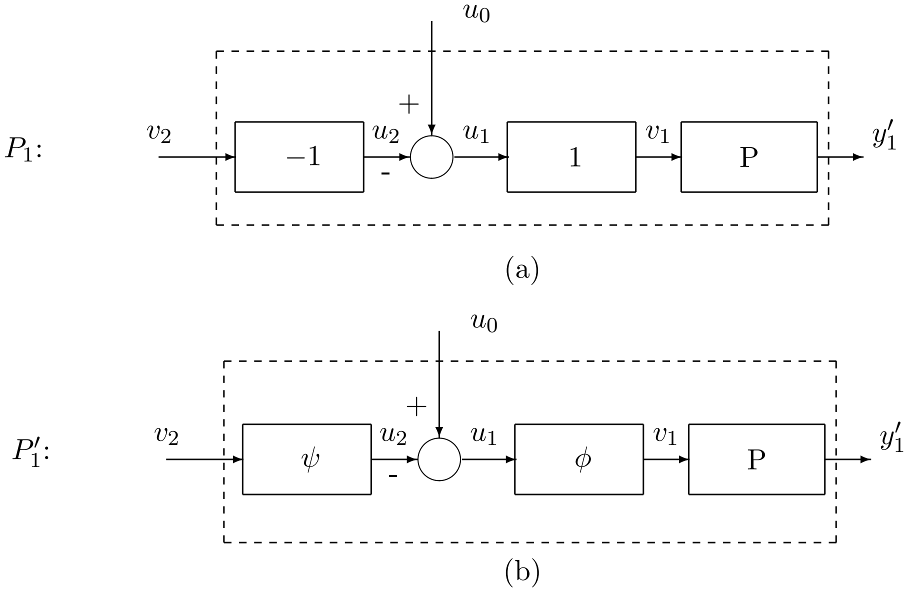

FIGURE 3.12: Hammerstein plant mapping:(a) purely linear, (b) with nonlinear part.

$$\vec{\delta}(P_1, P_1') < \|\Pi_{P_1//C_1}\|^{-1}, \tag{3.19}$$

where C_1 is the linear controller which stabilizes P_1 and P_1'. This stability condition can be related to the original system configuration shown in Figure 3.4. It will be shown later in the proof of Theorem 3.4 that the stability margin for the system shown in Figure 3.11 is greater than or equal to the stability margin for the system shown in Figure 3.6 which in turn equals the stability margin for the original system shown in Figure 3.4 with $\phi = 1, \psi = -1$. Now a description for the closed loop operators P_1', C_1' and P_1, C_1 shown in Figures 3.10, 3.11, respectively, is given. The perturbed system shown in Figure 3.10 can be described as

$$\begin{aligned} P_1' \quad &: \quad \mathcal{L}^2_{\infty,e} \to \mathcal{L}_{\infty,e} \\ &: \quad u_1' \mapsto y_1', \\ y_1' \quad &= \quad P\phi(u_0 - \psi(v_2)), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} C_1' \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}^2_{\infty,e} \\ &: \quad y_2' \mapsto u_2', \\ &\quad u_2' = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \\ &\quad v_2 = -Cy_2'. \end{aligned} \tag{3.21}$$

While the unperturbed system shown in Figure 3.11 can be described as

$$\begin{aligned} P_1 &: \mathcal{L}^2_{\infty,e} \to \mathcal{L}_{\infty,e} \\ &: u'_1 \mapsto y'_1, \\ y'_1 &= P(u_0 + v_2), \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} C_1 &: \mathcal{L}_{\infty,e} \to \mathcal{L}^2_{\infty,e} \\ &: y'_2 \mapsto u'_2, \\ &\quad u'_2 = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \\ &\quad v_2 = -Cy'_2. \end{aligned}$$

The closed loop equations for the system shown in Figure 3.10 are

$$u'_0 = u'_1 + u'_2, \tag{3.23}$$

$$y'_0 = y'_1 + y'_2. \tag{3.24}$$

$$v_1 = f(u'_1), \tag{3.25}$$

$$v_2 = -Cy'_2, \tag{3.26}$$

$$y'_1 = Pf(u'_1), \tag{3.27}$$

$$u'_2 = (0, -Cy'_2). \tag{3.28}$$

To apply Theorem 2.11 to this system, we must satisfy inequality (3.19). In the following two subsections the two sides of this inequality will be evaluated, namely the linear gain $\|\Pi_{P_1//C_1}\|$ and the gap value $\vec{\delta}(P_1, P'_1)$.

3.5.1 Finding $\|\Pi_{P_1//C_1}\|$ for a Hammerstein Model Control System

Let us consider the RHS of inequality (3.19) first. The parallel projection $\Pi_{P_1//C_1}$ is the mapping from the external signals (u'_0, y'_0) to the internal signals (u'_1, y'_1) in the configuration shown in Figure 3.11. The inverse of its gain, $\|\Pi_{P_1//C_1}\|$, is the stability margin for the plant P_1. The value of $\|\Pi_{P_1//C_1}\|$ is found as

$$\begin{aligned}
\begin{pmatrix} u_1' \\ y_1' \end{pmatrix} &= \Pi_{P_1//C_1}\begin{pmatrix} u_0' \\ y_0' \end{pmatrix}, \\
\|\Pi_{P_1//C_1}\| &= \sup_{\|u_0',y_0'\|\neq 0}\frac{\|u_1',y_1'\|}{\|u_0',y_0'\|}. && (3.29)\\
&= \sup_{\left\|\begin{pmatrix} u_0 \\ 0 \end{pmatrix},y_0\right\|\neq 0}\frac{\left\|\begin{pmatrix} u_0 \\ -v_2 \end{pmatrix},y_1\right\|}{\left\|\begin{pmatrix} u_0 \\ 0 \end{pmatrix},y_0\right\|}, \\
&= \sup_{\|u_0',y_0'\|\neq 0}\frac{\left\|\begin{pmatrix} u_0 \\ -v_2 \end{pmatrix},P\begin{pmatrix} u_0 \\ -v_2 \end{pmatrix}\right\|}{\|u_0',y_0'\|},
\end{aligned}$$

$$(3.30)$$

$$\|\Pi_{P_1//C_1}\| \leq \sup_{\|u_0',y_0'\|\neq 0}\frac{\left\|\begin{pmatrix} I \\ P \end{pmatrix}\begin{pmatrix} u_0 \\ v_2 \end{pmatrix}\right\|}{\|u_0',y_0'\|}. \tag{3.31}$$

An expression for (u_0, v_2) in terms of (u_0', y_0') is found to be:

$$\begin{aligned}
\begin{pmatrix} u_0 \\ v_2 \end{pmatrix} &= \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + Cy_2 \\
&= \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + C(y_0 - y_1) \\
&= \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + (Cy_0) - (CP\begin{pmatrix} u_0 \\ v_2 \end{pmatrix}) \\
\begin{pmatrix} u_0 \\ v_2 \end{pmatrix}(I+CP) &= \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + Cy_0.
\end{aligned}$$

If the sensitivity function $(I + CP)^{-1}$ exists, then:

$$\begin{pmatrix} u_0 \\ v_2 \end{pmatrix} = (I+CP)^{-1}\left(\begin{pmatrix} u_0 \\ 0 \end{pmatrix} + Cy_0\right),$$

and since $u_0' = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$ and $y_0' = y_0$ then:

$$\begin{pmatrix} u_0 \\ v_2 \end{pmatrix} = (I+CP)^{-1}(I \quad 0 \quad C)\begin{pmatrix} u_0' \\ y_0' \end{pmatrix}.$$

Substituting this value in (3.31); it follows that:

$$\|\Pi_{P_1//C_1}\| \leq \sup_{\|u_0',y_0'\|\neq 0} \frac{\left\|\begin{pmatrix} I \\ P \end{pmatrix}(I+CP)^{-1}(I \quad 0 \quad C)\right\| \left\|\begin{matrix} u_0' \\ y_0' \end{matrix}\right\|}{\|u_0', y_0'\|},$$

$$\leq \left\| \begin{matrix} (I+CP)^{-1} & 0 & C(I+CP)^{-1} \\ P(I+CP)^{-1} & 0 & PC(I+CP)^{-1} \end{matrix} \right\|$$

From the above inequality it can be noted that the components of $\|\Pi_{P_1//C_1}\|$ are the closed loop transfer functions of the linear system $[P_1, C_1]$, indicating that $\|\Pi_{P_1//C_1}\|$ is finite. Let:

$$Q_h = \begin{pmatrix} (I+CP)^{-1} & 0 & C(I+CP)^{-1} \\ P(I+CP)^{-1} & 0 & PC(I+CP)^{-1} \end{pmatrix}$$

Hence from (3.19) the gap between perturbed and unperturbed plants must satisfy:

$$\vec{\delta}(P_1, P_1') < \|Q_h\|^{-1} \tag{3.32}$$

3.5.2 Finding the Gap Metric for a Hammerstein Model Control System

In this subsection the LHS of the condition in (3.19) is considered. As mentioned previously the gap metric $\vec{\delta}(P_1, P_1')$ is a measure of the distance between the models P_1 and P_1', given that there exists a surjective map between the graphs of these plants.

To express $\vec{\delta}(P_1, P_1')$ in terms of the individual plant parameters contained within P_1 and P_1', we need to define the graphs for these systems, these graphs are defined to be:

$$\mathcal{G}_{P_1'} \quad : \quad = \left\{ \begin{pmatrix} u_0 \\ v_2 \\ y_1' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ v_2 \\ y_1' \end{matrix} \right\| < \infty, y_1' = P(\phi(u_0 - \psi(v_2))) \right\}, \tag{3.33}$$

$$\mathcal{G}_{P_1} \quad : \quad = \left\{ \begin{pmatrix} u_0 \\ v_2 \\ y_1' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ v_2 \\ y_1' \end{matrix} \right\| < \infty, y_1' = P(u_0 + v_2) \right\}. \tag{3.34}$$

In order to find a bound on the gap metric a surjective map Φ is required between $\mathcal{G}_{P_1}$ and $\mathcal{G}_{P_1'}$. To define this map, the following two lemmas are used. First consider the nonlinear component of the plant P_1' shown in Figure 3.12 (b), for this component the following lemma is used.

Lemma 3.5. *Let f satisfy Assumption 3.2, and consider the following equation:*

$$\begin{aligned} v_1 &= \phi(u_0 - \psi(v_2)), & (3.35)\\ &= f(u_0 - f^{-1}(v_2)). & (3.36) \end{aligned}$$

Then

$$\|v_2\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

Proof. Let $\|v_2\| < \infty, \|u_0\| < \infty$, and using Assumption 3.2 equation (3.14) since f^{-1} is a bounded function and since $\|v_2\| < \infty$ then $\|f^{-1}(v_2)\| < \infty$. Since $\|u_0\| < \infty$, it follows that:

$$\|u_0 - f^{-1}(v_2)\| \leq \|u_0\| + \|f^{-1}(v_2)\| < \infty.$$

Since f is a bounded function, then

$$\|v_1\| = \|f(u_0 - f^{-1}(v_2))\| < \infty,$$

as required.

□

In this analysis Φ is defined to be the map between stable P_1 and P_1'. These two plants are stable if the plant P is stable, as proved in the following lemma.

Lemma 3.6. *Suppose P is BIBO stable and let f satisfy Assumption 3.2. Then P_1 and P_1' given by Figure 3.12 and (3.22) and (3.20), respectively, are stable.*

Proof. First we prove that if P is stable, then P_1 is stable. In order to do that we must prove that if $\|u_1'\| < \infty$ then $\|P_1 u_1'\| < \infty$. So, let $\|u_1'\| < \infty$. While $u_1' = \begin{pmatrix} u_0 \\ v_2 \end{pmatrix}$, so $\|u_0\|, \|v_2\| < \infty$. Then by definition:

$$\begin{aligned} \|y_1'\| &= \|P_1 u_1'\|,\\ &= \|P(u_0 + v_2)\|,\\ &\leq \|P\|(\|u_0\| + \|v_2\|),\\ &< \infty. \end{aligned}$$

Hence P_1 is stable.

Similarly to prove that if P is stable then P_1' is stable it is required to prove that if

$\|u_1'\| < \infty$ then $\|P_1'u_1'\| < \infty$. Let $\|u_1'\| < \infty$, and since $u_1' = \begin{pmatrix} u_0 \\ v_2 \end{pmatrix}$, then $\|u_0\|, \|v_2\| < \infty$.
by definition:

$$\begin{aligned} y_1' &= P_1'u_1' = P(\phi(u_0 - \psi(v_2))), \\ \|y_1'\| &= \|P(\phi(u_0 - \psi(v_2)))\|, \end{aligned}$$

Using Lemma 3.5, since $\|u_0\|, \|\tilde{v}_2\| < \infty$:

$$\|v_1\| = \|\phi(u_0 - \psi(v_2))\| < \infty.$$

Since P is stable, it follows that:

$$\|y_1'\| = \|P\|\|\phi(u_0 - \psi(v_2))\| < \infty,$$

and hence P_1' is stable, as required. $\square$

Since P_1 and P_1' are stable, the graphs for P_1 and P_1' can be written as given in the following proposition:

Proposition 3.7. *Let P be stable and let f satisfy Assumption 3.2, for the systems P_1 and P_1' given by Figure 3.12 and (3.22) and (3.20), respectively, the graphs $\mathcal{G}_{P_1}$ and $\mathcal{G}_{P_1'}$ are given by:*

$$\mathcal{G}_{P_1} = \left\{ \begin{pmatrix} \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \\ P_1 \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \end{pmatrix} : \left\| \begin{matrix} u_0 \\ v_2 \end{matrix} \right\| < \infty \right\}, \tag{3.37}$$

$$\mathcal{G}_{P_1'} = \left\{ \begin{pmatrix} \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \\ P_1' \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \end{pmatrix} : \left\| \begin{matrix} u_0 \\ v_2 \end{matrix} \right\| < \infty \right\}. \tag{3.38}$$

Proof. To show that if P is stable and f satisfies Assumption 3.2 then $\mathcal{G}_{P_1'}$ given in (3.33) can be written in the form given in (3.38), and denote the set given in (3.38) as $\mathcal{A}$.
Let $\begin{pmatrix} (u_0, v_2) & P_1'(u_0, v_2) \end{pmatrix}^\top \in \mathcal{A}$, i.e $\|(u_0, v_2)\| < \infty$, P is stable and f satisfies Assumption 3.2. Hence using lemma 3.6, P_1' is stable. Since $\|(u_0, v_2)\| < \infty$ and P_1' is stable then $\|y_1'\| = \|P_1'(u_0, v_2)\| < \infty$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_1'}$.

Next we prove that $\mathcal{G}_{P_1'} \subset \mathcal{A}$. Let $\begin{pmatrix} u_0 & v_2 & y_1' \end{pmatrix}^\top \in \mathcal{G}_{P_1'}$, i.e $\|(u_0, v_2, y_1')\| < \infty$ and $y_1' = P(\phi(u_0 - \psi(v_2)))$. Then $\|(u_0, v_2)\| < \infty$ and $y_1' = P(\phi(u_0 - \psi(v_2))) = P_1'(u_0, v_2)$.

This leads to $\mathcal{G}_{P_1'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_1'} = \mathcal{A}$.

To show that $\mathcal{G}_{P_1}$ given in (3.37) is equivalent to that given in (3.34), set $\phi(x) = 1, \psi(x) = -1$ for all x. In this case $\mathcal{G}_{P_1}$ follows as a special case, as required. $\square$

The map Φ between $\mathcal{G}_{P_1}$ and $\mathcal{G}_{P_1'}$ is defined using the following proposition:

Proposition 3.8. *Let P be stable and let f satisfy Assumption 3.2. Let P_1 and P_1' be given by Figure 3.12 and (3.22) and (3.20), respectively, then there exists a map $\Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_1'}$ given by:*

$$\Phi\begin{pmatrix} \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \\ P_1\begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \\ P_1'\begin{pmatrix} u_0 \\ v_2 \end{pmatrix} \end{pmatrix}, \tag{3.39}$$

Furthermore this map is surjective.

Proof. First we need to prove that if

$$x = \begin{pmatrix} (u_0'', v_2'') & P_1(u_0'', v_2'') \end{pmatrix}^\top \in \mathcal{G}_{P_1},$$

then $\Phi(x) \in \mathcal{G}_{P_1'}$. Since $x \in \mathcal{G}_{P_1}$ then $\|(u_0'', v_2'')\|, \|y_1''\| = \|P_1(u_0'', v_2'')\| < \infty$. Let $y = \begin{pmatrix} (u_0, v_2) & P_1'(u_0, v_2) \end{pmatrix}^\top = \Phi(x)$. We need to show that $y_1' = P_1'(u_0, v_2)$, $\|(u_0, v_2)\| < \infty$. It follows from (3.39) that $(u_0, v_2) = (u_0'', v_2'')$ and $y_1' = P_1'(u_0'', v_2'')$, then $\|(u_0, v_2)\| < \infty$.

Using Proposition 3.7 since P is stable, f satisfies Assumption 3.2 and $\|(u_0, v_2)\| < \infty$, then $\|y_1'\| = \|P_1'(u_0, v_2)\| < \infty$. Hence:

$$\begin{aligned} y &= \begin{pmatrix} (u_0, v_2) & P_1'(u_0, v_2) \end{pmatrix}^\top \\ &= \begin{pmatrix} (u_0, v_2) & y_1' \end{pmatrix}^\top \in \mathcal{G}_{P_1'}. \end{aligned}$$

as required.

Next, to prove that Φ is surjective, let $u = (u_0, v_2)$ and using Proposition 2.14 since P_1 and P_1' are stable and since $\|(u_0, v_2)\| < \infty$ then the map given in (3.39) is surjective.

$\square$

Using the previous results, a bound on the gap between P_1 and P_1' is given using the following theorem

Theorem 3.9. *Let P be stable and let f satisfy Assumption 3.2. Let P_1 and P_1' be given by Figure 3.12 and (3.22) and (3.20), respectively. Then a bound on the gap between P_1*

and P_1' is

$$\vec{\delta}(P_1, P_1') \leq \|P\| F_\delta, \tag{3.40}$$

where

$$F_\delta = \sup_{\left\| \begin{smallmatrix} u_0 \\ v_2 \end{smallmatrix} \right\| \neq 0} \frac{\| \phi(u_0 - \psi(v_2)) - (u_0 + v_2) \|}{\|u_0, v_2\|}.$$

Proof. Using Proposition 3.8, since P is stable and f satisfies Assumption 3.2. Since P_1 and P_1' are given by Figure 3.12 and (3.22), (3.20). Then there exists a surjective map $\Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_1'}$ given by (3.39). Then the gap between P_1 and P_1' is given by:

$$\begin{aligned} \vec{\delta}(P_1, P_1') &\leq \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\}} \frac{\|(\Phi - I)x\|}{\|x\|}, \\ &\leq \sup_{\left\| \begin{smallmatrix} u_0 \\ v_2 \end{smallmatrix} \right\| \neq 0} \frac{\|P\| \, \| \phi(u_0 - \psi(v_2)) - (u_0 + v_2) \|}{\|u_0, v_2\|}. \end{aligned}$$

Let:

$$F_\delta = \sup_{\left\| \begin{smallmatrix} u_0 \\ v_2 \end{smallmatrix} \right\| \neq 0} \frac{\| \phi(u_0 - \psi(v_2)) - (u_0 + v_2) \|}{\|u_0, v_2\|},$$

then

$$\vec{\delta}(P_1, P_1') \leq \|P\| F_\delta.$$

as required.

□

Recall from Chapter 2 that a robust stability margin for the system $[P_1', C_1']$ exists if there is a finite gap metric between P_1 and P_1', and this gap is less than $\|\Pi_{P_1//C_1}\|$. Then a robust stability result for the rearranged block diagram shown in Figure 3.10 is stated as follows:

Proposition 3.10. *Consider the nonlinear closed loop system $[P_1', C_1']$ given by Figure 3.10 and (3.23)-(3.28). Suppose P is stable and f satisfies Assumption 3.2. Then $[P_1', C_1']$ has a robust stability margin.*

Proof. Since P is stable and f satisfies Assumptions 3.2, then by Lemmas 3.5, 3.6, and using Proposition 3.7 for the systems P_1 and P_1' given by Figure 3.12 and equations (3.22) and (3.20), respectively, the graphs $\mathcal{G}_{P_1}$ and $\mathcal{G}_{P_1'}$ can be given by (3.37) and (3.38), respectively. Using Proposition 3.8, there exists a map $\Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_1'}$ given by

(3.39). This leads to the presence of a finite gap value between the linear and nonlinear configurations of this system given by the inequality (3.40). Then the closed loop system $[P_1', C_1']$ given by Figure 3.10 and (3.23)- (3.28) have a robust stability margin.

$\square$

The main result Theorem 3.4 easily follows from Theorem 3.11.

Theorem 3.11. *Consider the nonlinear closed loop system $[M_h, N_h]$ given by Figure 3.4 and (3.7)-(3.12). Suppose P is stable and f satisfies Assumption 3.2. Then $[M_h, N_h]$ has a robust stability margin $b_{P,C}$ that satisfies the inequality*

$$b_{P,C} \geq \|Q_h\|^{-1}. \tag{3.41}$$

Proof. Let $\|\Pi_{P_1//C_1}\|^{-1} = \|Q_h\|^{-1}$ be a stability margin for the system $[P_1', C_1']$ shown in Figure 3.10, let $\|\Pi'_{P_1//C_1}\|^{-1}$ be a stability margin for the system $[P_1', C_1']$ shown in Figure 3.5, finally let $\|\Pi_{P//C}\|^{-1}$ be a stability margin $b_{P,C}$ for the system $[M_h, N_h]$ shown in Figure 3.4, then

$$\begin{aligned}
\left\|\Pi_{P_1//C_1}\begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix}\right\| &= \sup_{\|u_0,x_0,y_0\|\neq 0} \frac{\left\|\Pi_{P_1//C_1}\begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix}\right\|}{\|u_0, x_0, y_0\|}, \\
&\geq \sup_{\|u_0,0,y_0\|\neq 0} \frac{\left\|\Pi'_{P_1//C_1}\begin{pmatrix} u_0 \\ 0 \\ y_0 \end{pmatrix}\right\|}{\|u_0, 0, y_0\|}, \\
&= \sup_{\|u_0,y_0\|\neq 0} \frac{\left\|\Pi_{P//C}\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}\right\|}{\|u_0, y_0\|}, \\
&= \|\Pi_{P//C}\|.
\end{aligned}$$

This leads us to

$$b_{P,C} = \frac{1}{\Pi_{P//C}} = \frac{1}{\Pi'_{P_1//C_1}} \geq \frac{1}{\Pi_{P_1//C_1}} = \|Q_h\|^{-1}.$$

Moreover the existence of a stability margin for the system shown in Figure 3.10 guarantees the existence of a stability margin for the system $[M_h, N_h]$ shown in Figure 3.4. Also, since P is stable and f satisfies Assumption 3.2, by Proposition 3.10, the nonlinear closed loop system $[P_1', C_1']$ given by Figure 3.10 and (3.23)-(3.28), has a robustness stability margin. This leads to that the system $[M_h, N_h]$ given by Figure 3.4 and (3.7)-(3.12) also has a robust stability margin.

$\square$

Based on Theorems 3.9 and 3.11 we can give the following corollary:

Corollary 3.12. *Consider the nonlinear closed loop system $[M_h, N_h]$ given by Figure 3.4 and (3.7)-(3.12). Suppose P is stable and f satisfies Assumption 3.2. Then this system is stable if*

$$\|P\|F_\delta < \|Q_h\|^{-1}.$$

Proof. Using Theorem 3.9 inequality (3.40):

$$\vec{\delta}(P_1, P_1') \leq \|P\|F_\delta,$$

and using Theorem 3.11 inequality (3.41), it follows that if

$$\|P\|F_\delta < \|Q_h\|^{-1},$$

we have:

$$\vec{\delta}(P_1, P_1') \leq \|P\|F_\delta < \|Q_h\|^{-1} \leq b_{P,C}.$$

Then $\vec{\delta}(P_1, P_1') < b_{P,C}$ and the conditions hold from Theorem 2.11, hence stability.

as required. $\square$

3.6 Summary

Chapter 3 introduced a procedure to study the robustness of a feedback linearization controller designed for a Hammerstein model nonlinear system using the small gain theorem and the gap metric. First, the small gain theorem was employed to find a stability condition for this feedback control system. The stability condition found using small gain theorem is conservative, since the closed loop system is stable if the product of the system components is less than one and holds if the linear component P of the plant is stable.

To perform the gap analysis, a new configuration of the perturbed (actual) system which allowed for the consideration of the nonlinear components of the plant and controller in calculating a bound for the gap metric was investigated. In this configuration minimizing the gap metric made the nonlinear system approximates its linear version. The stability condition found using this approach depends on the linear gain of the system and the bound on the gap between the linear and the nonlinear configurations of the system. The result found using the gap metric is also conservative, since the bounds on the gap scale with the uncertainty in the plant, hence having large nonlinearities in the plant will make this stability condition harder to be met, on the other hand the linear component P of the plant is required to be stable.

In later chapters affine systems will be considered. The gap metric will be used to find more general results where the stability assumption on P is dropped and Proposition 3.7 does not hold. In this case the system characterization will be more complex and analysis involving coprime factors will be required.

Chapter 4

Stability Analysis for Affine Systems with Input Nonlinearity Using the Small Gain Theorem and the Gap Metric

4.1 Introduction

This chapter undertakes robustness analysis for a stable affine nonlinear system with input nonlinearity using the small gain theorem and the gap metric. First a stability condition is provided for this system using the small gain theorem. Then the gap metric analysis is performed to find stability conditions for this system. In this chapter the gap analysis will follow the same procedure carried out in Chapter 3, however, a MIMO affine nonlinear system will be considered which is significantly more complicated than the SISO Hammerstein model considered in Chapter 3. To apply the gap analysis for this new system, a more complex configuration (rather than the two block structure used in Chapter 3) is needed. Hence, a three plant configuration P_1, P_2, P_3 will be used and a stability condition will be given for this system using the gap metric network result introduced in (Theorem (10)) Georgiou and Smith (1997). An illustrative example is then used to compare the stability conditions found using the small gain theorem and the gap metric. This example will investigate the validity of each result in providing stability for a nonlinear system.

4.2 Affine Systems with Input Nonlinearity

Consider an affine nonlinear system described as

$$
\begin{aligned}
P_1 \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^n \\
&: \quad u \mapsto y, \\
\dot{x} \quad &= \quad f(x) + g(x)u, && (4.1) \\
y \quad &= \quad h(x), && (4.2)
\end{aligned}
$$

where $x = (x_1, x_2, \cdots, x_n)$, f, g, and h are smooth in $\mathbb{R}^n$. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are called vector fields on $\mathbb{R}^n$. The derivative $\dot{y}$ is given by:

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h(x) + L_g h(x)u$$

where

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

The new notation is convenient when we repeat the calculation of the derivative with respect to the same vector field or a new one. For example, the following notation is used:

$$
\begin{aligned}
L_g L_f h(x) &= \frac{\partial L_f h}{\partial x} g(x), \\
L_f^2 h(x) &= L_f L_f h(x) = \frac{\partial L_f h}{\partial x} f(x), \\
L_f^k h(x) &= L_f L_f^{k-1} h(x) = \frac{\partial L_f^{k-1} h}{\partial x} f(x), \\
L_f^0 h(x) &= h.
\end{aligned}
$$

If $L_g h(x) = 0$ then $\dot{y} = L_f h(x)$ is independent of u. If we continue to calculate the second derivative of y, denoted by $y^{(2)}$, we obtain:

$$y^{(2)} = \frac{\partial L_f h}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)u.$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$ is independent of u. Repeating this process, we see that if f, g, h satisfy:

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0,$$

then u does not appear in the equations of $y, \dot{y}, \ldots, y^{(\rho-1)}$ and appears in the equation $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)u.$$

The system is input-output linearizable since the state feedback control:

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^{\rho} h(x) + v \right]$$

reduces the input-output map to:

$$y^{(\rho)} = v,$$

which is a chain of ρ integrators Khalil (2002). In this case the integer ρ is the relative degree of the system, defined as follows:

Definition 4.1. The nonlinear system (4.1)-(4.2) is said to have relative degree ρ, $1 \leq \rho \leq n$, if

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0. \tag{4.3}$$

for all $x \in R^n$.

The case considered in this analysis is when the relative degree of P_1 equals the order of this system i.e. $\rho = n$. A change of variables $x^* = T(x)$ is needed to transform the state equation from $x - coordinates$ to $x^* - coordinates$. The map T used must be a diffeomorphism where

Definition 4.2. A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a diffeomorphism if it is smooth, and if its inverse T^{-1} exists and is smooth.

The form of T considered is defined as follows

Definition 4.3. Suppose the nonlinear system (4.1)-(4.2) where f, g, and h are sufficiently smooth in $\mathbb{R}^n$, is full relative degree (i.e relative degree = state dimension). Then the map $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix}, \tag{4.4}$$

is a diffeomorphism and transforms the system into the normal form ($x^* = T(x)$), where

$$\begin{aligned} P_1 \quad : \quad & \mathcal{L}_{\infty,e}^{n+1} \to \mathcal{L}_{\infty,e}^{n} : (u_1, z_1) \mapsto (y_1), \\ & \dot{x}^* = Ax^* + B(f^*(z_1) + g^*(z_1)u_1), & (4.5) \\ & y_1 = (y_{11}, \ldots, y_{1n}) = x^*, & (4.6) \\ & z_1 = (z_{11}, \ldots, z_{1n}) = x^*, & (4.7) \end{aligned}$$

In this system the feedback input y_1 is considered as an input, z_1, to the nonlinear parts of the plant $g^*(z_1)$ and $f^*(z_1)$, and $A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & \dots & & a_{n-1} & a_n \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$,

$$\begin{aligned} f^*(x^*) &= L_f^n h(T^{-1}(x^*)), \\ g^*(x^*) &= L_g L_f^{n-1} h(T^{-1}(x^*)). \end{aligned}$$

Such a system is said to be feedback linearizable.

For simplicity, in this chapter we consider the transformed plant to only have a nonlinear input component. This can be done by setting $f^*(z_1) = 0$ in (4.5), so that we have the system

$$\begin{aligned} P_1' \quad &: \quad \mathcal{L}_{\infty,e}^{n+1} \to \mathcal{L}_{\infty,e}^{n} : (u_1, z_1) \mapsto (y_1), \\ &\dot{x}^* = Ax^* + Bg^*(z_1)u_1, && (4.8) \\ &y_1 = (y_{11}, \dots, y_{1n}) = x^*, && (4.9) \\ &z_1 = (z_{11}, \dots, z_{1n}) = x^*, \end{aligned}$$

The feedback control connection for this system is shown in Figure 4.1.

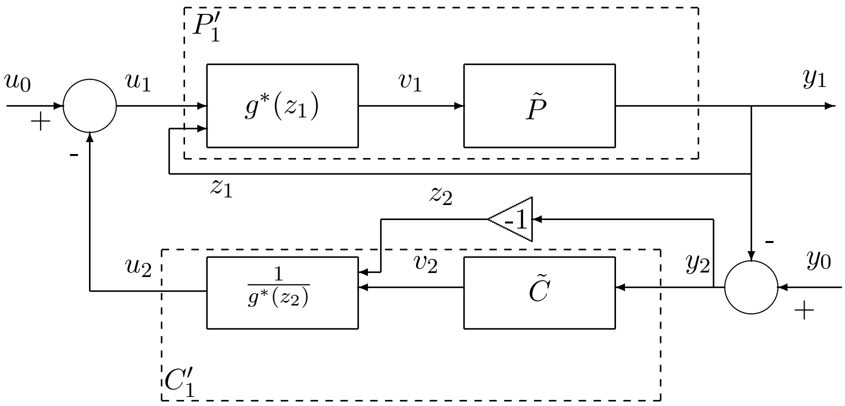

FIGURE 4.1: Nonlinear Feedback System

Here the linear part of the plant is

$$\begin{aligned} \tilde{P} \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^{n} : v_1 \mapsto y_1, \\ &\dot{x}^* = Ax^* + Bv_1, && (4.10) \\ &y_1 = x^*, && (4.11) \end{aligned}$$

given that $v_1 = g^*(z_1)u_1$ and $\tilde{P}$ is assumed to be stable (this is done because the plant required for the small gain theorem analysis must be stable), then we can define P_1' as

$$\begin{aligned} P_1' \quad : \quad & \mathcal{L}_{\infty,e}^{n+1} \to \mathcal{L}_{\infty,e}^{n} : (u_1, z_1) \mapsto (y_1), \\ & y_1 = \tilde{P} g^*(z_1) u_1 \\ & z_1 = (z_{11}, \ldots, z_{1n}) = y_1. \end{aligned} \tag{4.12}$$

Motivated by the form of (4.8), the feedback linearizing controller for this system is given as:

$$\begin{aligned} C_1' \quad : \quad & \mathcal{L}_{\infty,e}^{2n} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\ & u_2 = \frac{1}{g^*(z_2)} \tilde{C} y_2, \\ & z_2 = (z_{21}, \ldots, z_{2n}) = -y_2, \end{aligned} \tag{4.13}$$

where $y_2 = (y_{21}, \ldots, y_{2n})$, the signal $-y_2$ is considered as an input z_2 to the nonlinear part $\frac{1}{g^*(z_2)}$ and $\tilde{C}$ is the linear component of the controller C_1',

$$\tilde{C} \quad : \quad \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_2 \tag{4.14}$$

$$v_2 = c^\top y_2. \tag{4.15}$$

and

$$c = (c_1, \ldots, c_n)^\top.$$

For the system shown in Figure 4.1 the closed loop equations can be written as:

$$u_0 = u_1 + u_2, \tag{4.16}$$

$$y_0 = y_1 + y_2, \tag{4.17}$$

$$v_1 = g^*(z_1) u_1, \tag{4.18}$$

$$v_2 = \tilde{C} y_2, \tag{4.19}$$

$$y_1 = \tilde{P} v_1, \tag{4.20}$$

$$u_2 = \frac{1}{g^*(z_2)} v_2. \tag{4.21}$$

We note that the system shown in Figure 4.1 is not equivalent to the closed loop system $[P, C]$ shown in Figure 2.1. This is due to the presence of the two signals z_1, z_2 which feed the signals $y_1, -y_2$ to the blocks P_1', C_1', respectively. Hence, we cannot apply Theorem 2.11 to find stability conditions for this system.

However, the linear configuration which will be needed to find stability conditions for this system corresponds to that shown in Figure 2.1, since replacing $g^*(z_1)$ and $\frac{1}{g^*(z_2)}$ by the linear operators $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, respectively, will result in the system $[\tilde{P}, \tilde{C}]$ shown in Figure 4.2.

The blocks π, π' cancel the effect of the signals z_1, z_2 on the closed loop system $[P_1, C_1]$.

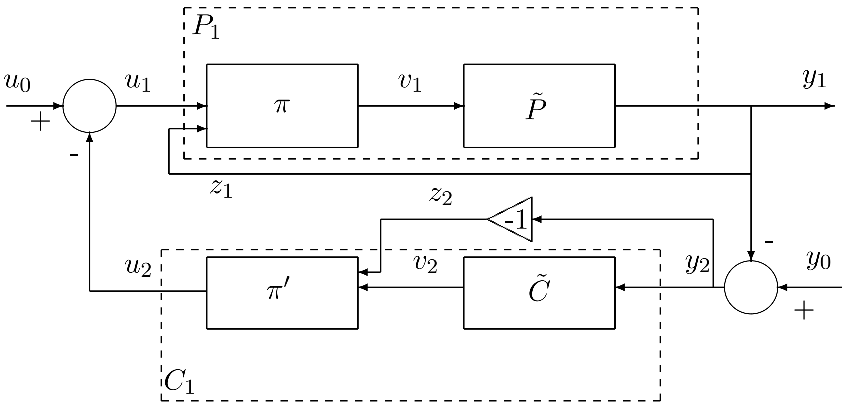

FIGURE 4.2: Linear configuration of the system in Figure 4.1

Hence, the signals z_1, z_2 can be removed, with the mappings π, π' unchanged as shown in Figure 4.3.

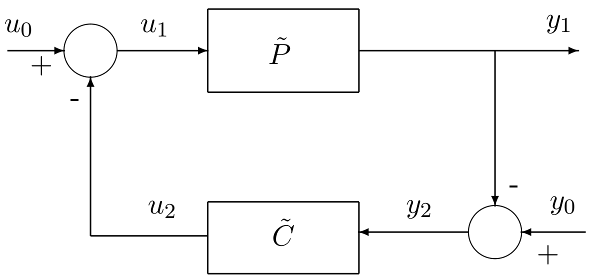

FIGURE 4.3: Linear configuration of the system in Figure 4.1 with z_1, z_2 removed and the mappings π, π' applied to the system

Note that $\Pi_{\tilde{P}//\tilde{C}}$ is the closed loop operator for the system shown in Figure 4.3.

Robust stability analysis is carried out for the system shown in Figure 4.1 using two approaches, the small gain theorem and the gap metric. Then a simple example is used to compare the stability conditions found using these approaches.

4.3 Stability Analysis for Stable Affine Systems with Input Nonlinearity Using the Small Gain Theorem

This section will carry out stability analysis for the closed loop system shown in Figure 4.1 using the small gain theorem stated in Theorem 2.8. In this analysis, Definition 2.7 will be used to define the stability for this system. In this analysis the following assumption on the form of $g^*(x)$ is required:

Assumption 4.4. Assume $g^* : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinear function, satisfying the following condition:

$$\exists \varepsilon > 0, \quad \exists D < \infty \text{ and } \quad \varepsilon \leq \|g^*(x)\| \leq D \quad \forall x \in \mathbb{R},$$

Hence

$$\|g^*(x)\| \leq D \quad \forall x \in \mathbb{R}, \tag{4.22}$$
$$\left\|\frac{1}{g^*(x)}\right\| \leq \frac{1}{\varepsilon} \quad \forall x \in \mathbb{R}. \tag{4.23}$$

Then a stability condition based on Theorem 2.8 for the nonlinear closed loop system shown in Figure 4.1 is given in the following theorem:

Theorem 4.5. *Consider the nonlinear closed loop system given by Figure 4.1 and (4.16)-(4.21). Suppose P is stable and $g^*(x)$ satisfies Assumption 4.4. Then this closed loop system is a bounded input, bounded output stable system if the following condition holds:*

$$\frac{D}{\varepsilon}\|\tilde{P}\|\|\tilde{C}\| < 1.$$

Proof. First we consider y_1 and u_2 in the system shown in Figure 4.1 and given by (4.20) and (4.21), respectively, as follows:

$$\begin{aligned} y_1 &= \tilde{P}v_1 \\ &= \tilde{P}g^*(z_1)u_1 \\ \|y_1\| &\leq \|\tilde{P}\|\|g^*(z_1)\|\|u_1\|, \end{aligned}$$

Using(4.22) we have:

$$\|y_1\| \leq D\|\tilde{P}\|\|u_1\|, \tag{4.24}$$

for u_2:

$$\begin{aligned} u_2 &= \frac{1}{g^*(z_2)}v_2. \\ &= \frac{1}{g^*(z_2)}(\tilde{C}y_2) \\ \|u_2\| &\leq \left\|\frac{1}{g^*(z_2)}\right\|\|\tilde{C}\|\|y_2\|, \end{aligned}$$

using (4.23) we have

$$\|u_2\| \leq \frac{1}{\varepsilon}\|\tilde{C}\|\|y_2)\|, \tag{4.25}$$

Now consider (2.14) and (2.15) in Theorem 2.8, then:

$$\begin{aligned} \| y_1 \| &= \| P(u_1) \| \leq \gamma_1 \| u_1 \| + \beta, \\ \| u_2 \| &= \| C(y_2) \| \leq \gamma_2 \| y_2 \| + \beta. \end{aligned}$$

Let $\beta_1, \beta_2 = 0$, so that:

$$\| y_1 \| = \| P(u_1) \| \leq \gamma_1 \| u_1 \|, \tag{4.26}$$
$$\| u_2 \| = \| C(y_2) \| \leq \gamma_2 \| y_2 \| . \tag{4.27}$$

Comparing the expressions in (4.26), (4.27) with those found for $\| y_1 \|$, $\| u_2 \|$ given by (4.24) and (4.25), respectively, we have:

$$\gamma_1 = D\|\tilde{P}\|.$$

and

$$\gamma_2 = \frac{1}{\varepsilon}\|\tilde{C}\|.$$

From the above expressions for γ_1 and γ_2, it follows that the closed loop system shown in Figure 4.1 is a bounded input bounded output stable system if

$$\gamma_1\gamma_2 < 1 \Rightarrow \frac{D}{\varepsilon}\|\tilde{P}\|\|\tilde{C}\| < 1,$$

as required.

□

Comparing the stability condition in Theorem 4.5 to that given in Theorem 3.3 for Hammerstein systems, this new condition also states that the stability of closed loop system shown in Figure 4.1 is ensured if the product of the system components gains is less than one and it requires the linear part $\tilde{P}$ of the plant P_1' to be stable.

4.4 Stability Analysis for Stable Affine Systems with Input Nonlinearity Using the Gap Metric

In this section the nonlinear system shown in Figure 4.1 is considered, and the gap metric framework is applied to study the stability of this system using the following theorem:

Theorem 4.6. *Consider the nonlinear closed loop system shown in Figure 4.1 and given by (4.16)-(4.21). Suppose $\tilde{P}$ is stable and g^* satisfies Assumption 4.4. Then this system has a robust stability margin.*

The analysis carried out in this section will follow the approach of Chapter 3, and analogous to that for Theorem 3.4 , this proof also requires results that are first developed in this section.

To find a stability condition for the nonlinear system shown in Figure 4.1, the route taken is as follows:

As discussed in Chapter 3, the presence of nonlinear elements in multiple blocks leads to significant conservatism. Hence, a new representation for the system shown in Figure 4.1 is required. For this system, the stability condition in Theorem 2.11 can not be applied (as was done in Chapter 3 for the Hammerstein model), since we cannot put this system in a configuration that corresponds exactly to the form of the system shown in Figure 2.1 (this is due to the extra feedback signal z_1 in the system shown in Figure 4.1 which feeds back the signal y_1 to the block $g^*(z_1)$). However, this system can be configured in a form that allows Theorem 2.13 to be applied. In the new configuration it is also required that the nonlinear components of the plant and controller be included in one block to calculate a bound on the gap metric, and minimizing this value means that we can make the nonlinear system correspond more closely to its linear counterpart.

The new configuration is shown in Figure 4.4. In this configuration the nonlinear component of the plant P_1 and the nonlinear component of the controller C_1 are considered to be both included in the block P_3' and an external input x_0 is added to the system. Also the feedback input $x_0 - y_1$ is considered as an input, z_1, to the nonlinear part of the plant $g^*(z_1)$, and the feedback input $-y_2$ is considered as an input z_2 to the nonlinear part $\frac{1}{g^*(z_2)}$.

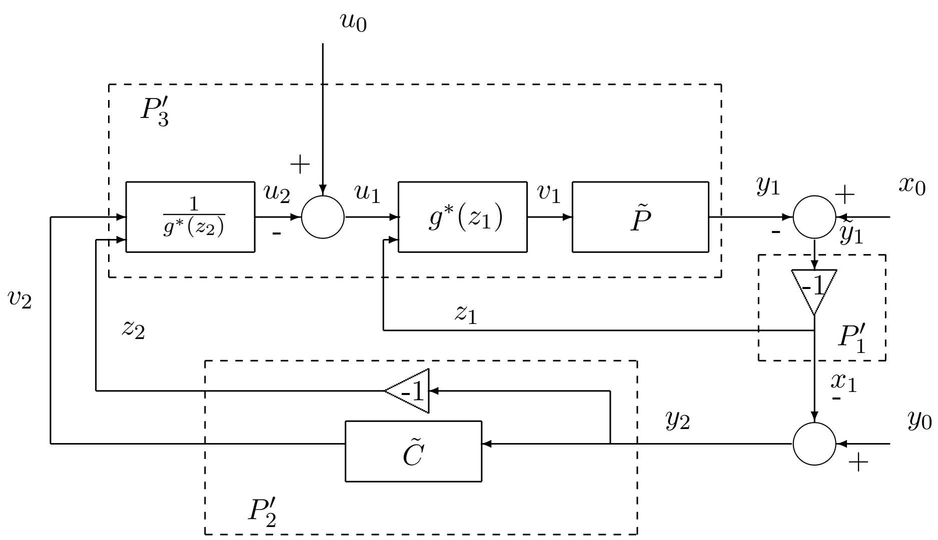

FIGURE 4.4: Nonlinear system with input nonlinearity configuration

To do stability analysis for this new configuration, we consider the triple system configuration shown in Figure 4.5. For this system a 'network' result which was introduced (as Theorem (10)) in Georgiou and Smith (1997) can be applied to find stability condtions. This theorem was stated in Subsection 2.7.1 (Theorem 2.13).

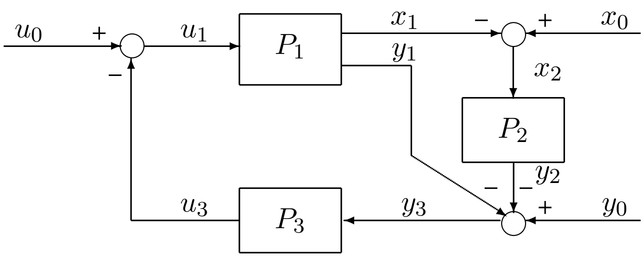

FIGURE 4.5: Feedback interconnection of three subsystems $P_i, (i = 1, 2, 3)$ shown previously in Figure 2.8

The linear configuration for the system shown in Figure 4.4 is taken to be the system shown in Figure 4.6. The components P_1, P_2, P_3 are taken to be with the nonlinearities $g^*(z_1)$ and $\frac{1}{g^*(z_2)}$ replaced by the linear operators $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, respectively.

To apply Theoreom 2.13 we must put the nonlinear and linear configurations of

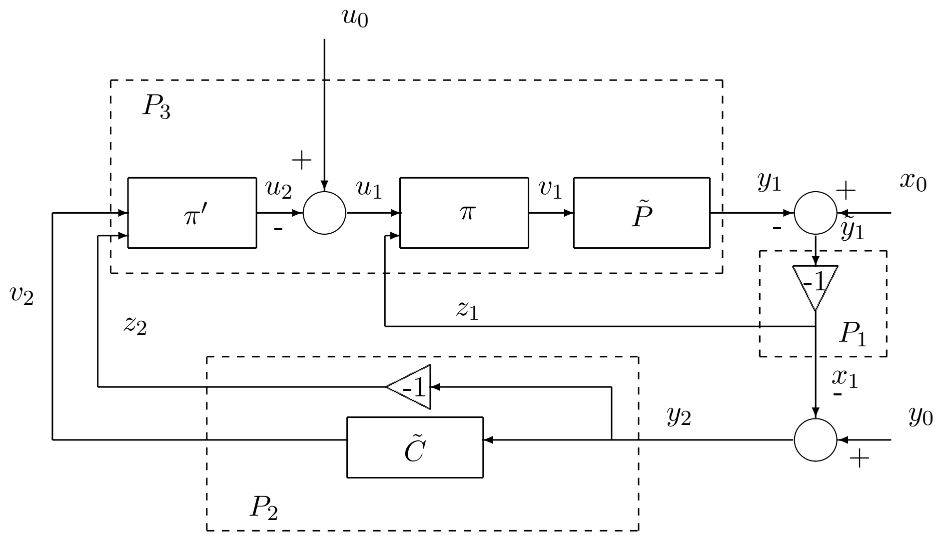

FIGURE 4.6: Linear control system with linear plant configuration

the systems shown in Figures 4.4 and 4.6 in a form comparable to that given in Figure 4.5. In order to do this we consider three signal spaces $\mathcal{U} = \mathcal{L}^n_{\infty,e}$, $\mathcal{X} = \mathcal{L}^n_{\infty,e}$ and $\mathcal{Y} = \mathcal{L}^n_{\infty,e}$, together with the following augmented signals, let $\hat{v}_2 = -v_2$ and let $u'_1 = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top$ and let $u'_2 = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top$ also let the external input u_0 be changed to $u'_0 = \begin{pmatrix} u_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top$, let $u'_3 = u'_0 - u'_2 - u'_1 = \begin{pmatrix} u_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top - \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top - \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top = \begin{pmatrix} u_0 & d_1 - \hat{v}_2 & d_2 - z_1 & d_3 - z_2 \end{pmatrix}^\top$, let $\tilde{v}_2 = d_1 - \hat{v}_2$, $\tilde{z}_1 = d_2 - z_1$, $\tilde{z}_2 = d_3 - z_2$ then $u'_3 = \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$. Finally, let $x'_0 = y_0$, $y'_0 = x_0$, $y'_3 = y_1$, $x'_1 = x_1$, $x'_2 = y_2$ and let $y'_1 = y'_0 - y'_3 = x_0 - y_1$. The change made to the nonlinear configuration of this system is shown in Figure 4.7.

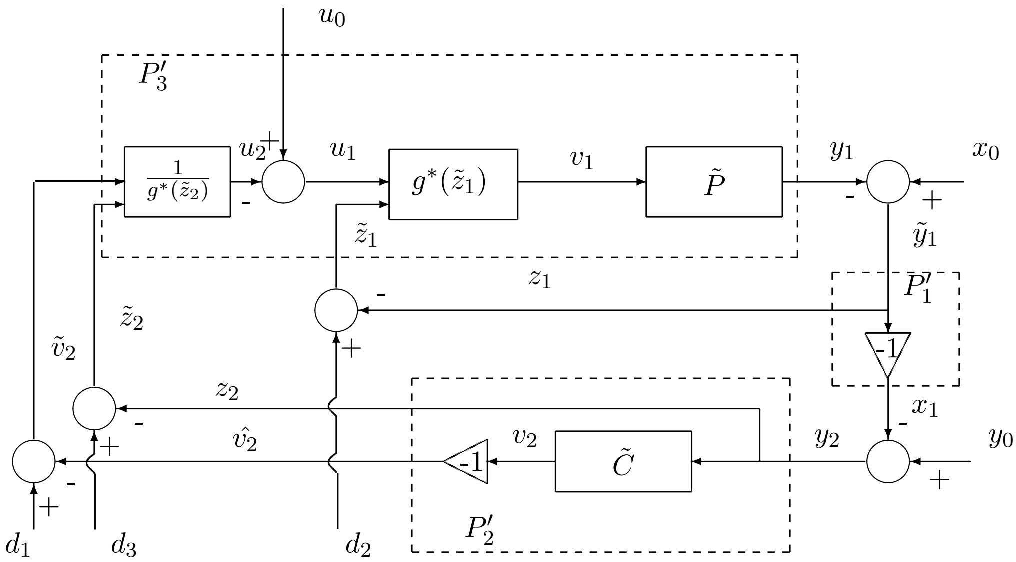

FIGURE 4.7: Nonlinear system with input nonlinearity and injected disturbance

While the change made to the linear configuration of this system is shown in Figure 4.8.

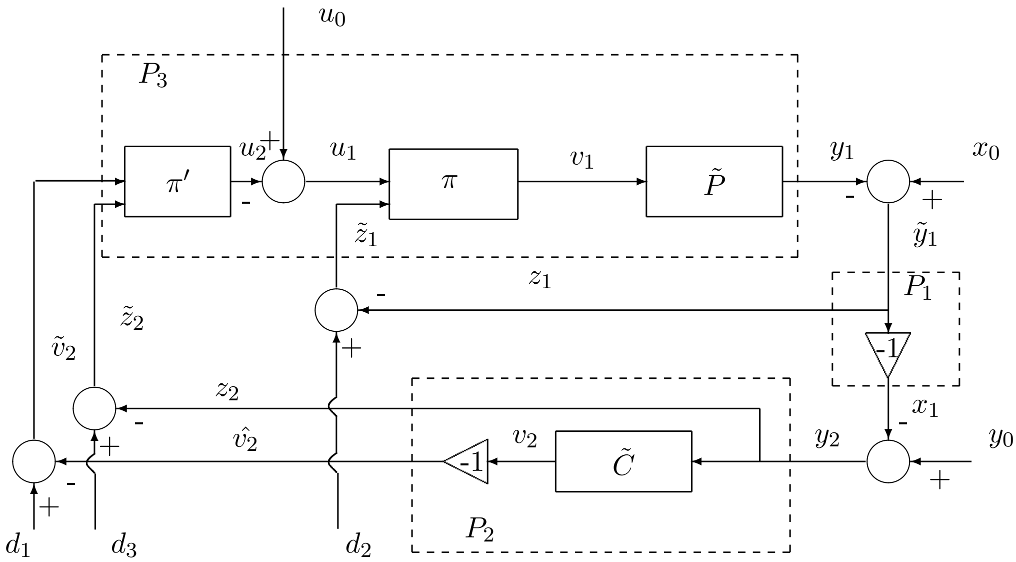

FIGURE 4.8: Linear system with input nonlinearity and injected disturbance

Note from the two systems shown in Figure 4.7 and Figure 4.8 that $P_1 = P_1'$ and $P_2 = P_2'$.

These configurations correspond to those of Figures 4.4, 4.6, respectively, except for the presence of d_1, d_2 and d_3. Figures 4.7, 4.8 correspond exactly to the forms shown in Figures 4.9, 4.10, respectively, which in turn have identical structure to that of Figure 4.5.

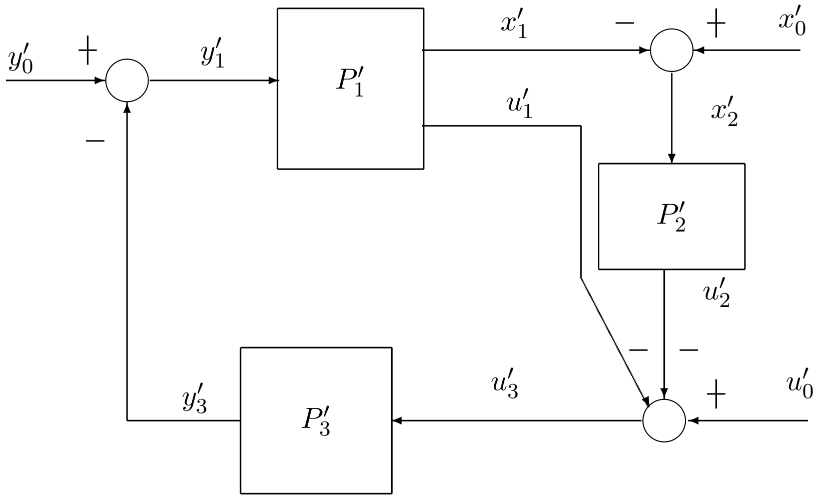

FIGURE 4.9: Nonlinear control system with input nonlinearity, second configuration

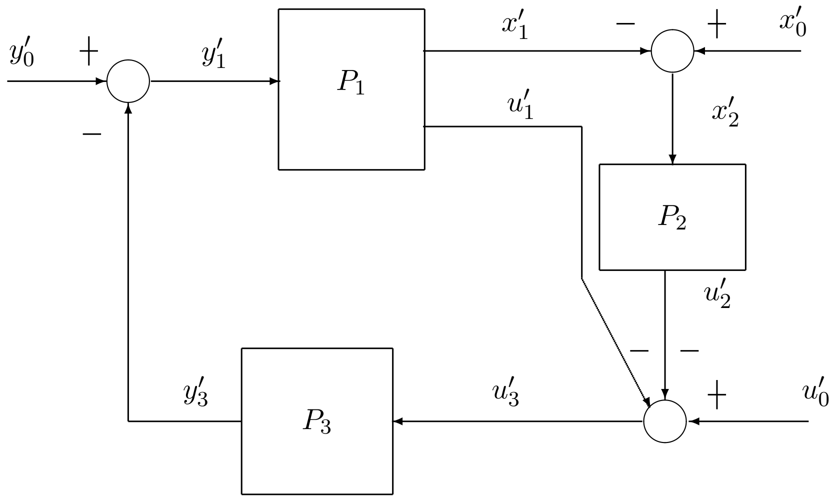

FIGURE 4.10: Linear control system with linear input, second configuration

Since $P_1 = P_1'$ and $P_2 = P_2'$, then

$$\vec{\delta}(P_1, P_1') = 0, \vec{\delta}(P_2, P_2') = 0.$$

Using Theorem 2.13, the robust stability condition is given as:

$$\sum_{i=1}^{3} \vec{\delta}(P_i, P_i') < \|\Pi_{(i)}\|^{-1},$$

For our system this condition becomes:

$$\vec{\delta}(P_3, P_3') < \|\Pi_{(3)}\|^{-1}. \tag{4.28}$$

Then the gap metric measures the difference between the linear nominal plant P_3 :

$u_3' \mapsto y_3', y_3' = \tilde{P}\pi(u_0 - \pi'(\tilde{v}_2, \tilde{z}_2), \tilde{z}_1) = \tilde{P}(u_0 - \tilde{v}_2)$ and the nonlinear perturbed plant $P_3' : u_3' \mapsto y_3', y_3' = \tilde{P}g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)$. The plants P_3 and P_3' are shown in Figure 4.11.

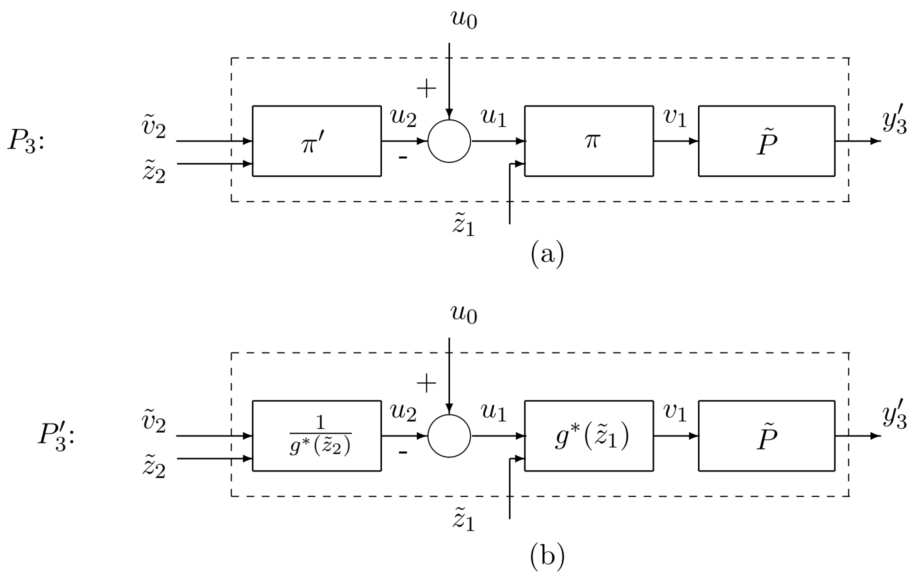

FIGURE 4.11: Second plant mapping:(a) purely linear, (b) with nonlinear components $g(z_1)$ and $g(z_2)$.

Analogous to the analysis in Section 3.5, before providing a complete description of the operators P_1', P_2' and P_3' and P_1, P_2 and P_3 shown in Figures 4.9 and 4.10, respectively, we briefly state the motivation for the proceeding manipulations.

The stability condition (4.28) can be related to the original system configuration shown in Figure 4.1 as follows: It will be shown later in the proof of Theorem 4.6 that the stability margin for the system shown in Figure 4.7 is less than or equal to the stability margin corresponding to the system shown in Figure 4.4 which in turn is less than or equal to the stability margin corresponding to the original system shown in Figure 4.1. This is because for each pair the latter is a special case of the former.

A description for the closed loop operators P_1', P_2' and P_3' shown in Figure 4.9 is given as

$$\begin{aligned} P_1' \quad : \quad & \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{2n} : y_1' \mapsto (x_1', u_1') \\ & x_1' = -y_1', u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, \\ & z_1 = y_1', \end{aligned}$$

where $y_1' = \tilde{y}_1$, and

$$\begin{aligned} P_2' \;:\; & \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{n+1}_{\infty,e} : x_2' \mapsto u_2', \\ & u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, \\ & z_2 = x_2', \\ & \hat{v}_2 = -\tilde{C}x_2', \end{aligned}$$

the configuration for the block P_3' is given as:

$$\begin{aligned} P_3' \;&:\; \mathcal{L}^{2n+2}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u_3' \mapsto y_3', \\ u_3' \;&=\; \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top \\ y_3' \;&=\; \tilde{P}g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2). \end{aligned} \tag{4.29}$$

While the closed loop operators P_1, P_2, P_3 for the linear configuration shown in Figure 4.10 are given as

$$\begin{aligned} P_1 \;:\; & \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{2n}_{\infty,e} : y_1' \mapsto (x_1', u_1') \\ & x_1' = -y_1', u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, \\ & z_1 = y_1', \end{aligned}$$

and

$$\begin{aligned} P_2 \;:\; & \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{n+1}_{\infty,e} : x_2' \mapsto u_2' \\ & u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, z_2 = x_2', \\ & \hat{v}_2 = -\tilde{C}x_2', \end{aligned}$$

also,

$$\begin{aligned} P_3 \;&:\; \mathcal{L}^{2n+2}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u_3' \mapsto y_3', \\ u_3' \;&=\; \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top, \\ y_3' \;&=\; \tilde{P}\pi(u_0 - \pi'(\tilde{v}_2, \tilde{z}_2), \tilde{z}_1), \\ &=\; \tilde{P}(u_0 - \tilde{v}_2). \end{aligned} \tag{4.30}$$

To apply Theorem 2.13 to this system, we must satisfy inequality (4.28). In the following two subsections the two sides of this inequality will be evaluated, namely the linear gain $\|\Pi_{(3)}\|$ and the gap value $\vec{\delta}(P_3, P_3')$.

4.4.1 Finding $\|\Pi_{(3)}\|$ for a Nonlinear System with Input Nonlinearity

Starting with the RHS of the inequality (4.28). The parallel projection $\Pi_{(3)}$ is the mapping from the external signals (u_0', x_0', y_0') to the internal signals $(u_3', 0, y_3')$ in the configuration shown in Figure 4.10. To find the linear gain $\|\Pi_{(3)}\|$ consider the relation:

$$\begin{aligned}\begin{pmatrix} u_3' \\ 0 \\ y_3' \end{pmatrix} &= \Pi_{(3)} \begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}, \\ \|\Pi_{(3)}\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\|u_3', 0, y_3'\|}{\|u_0', x_0', y_0'\|}.\end{aligned}$$

Then:

$$\begin{aligned}\|\Pi_{(3)}\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\|u_3', y_3'\|}{\|u_0', x_0', y_0'\|}, \\ &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}. \qquad (4.31)\end{aligned}$$

Now to find expressions for u_3' and P_3u_3' in terms of u_0', x_0', y_0', we start with P_3u_3' as follows:

$$P_3 u_3' = \tilde{P}(u_0 - \tilde{v}_2). \qquad (4.32)$$

Then $\tilde{v}_2$ is found as:

$$\begin{aligned}\tilde{v}_2 &= d_1 - \hat{v}_2, \\ &= d_1 + v_2, \\ &= d_1 + \tilde{C}x_2', \\ &= d_1 + \tilde{C}(x_0' - x_1'), \\ &= d_1 + \tilde{C}(x_0' + y_1'), \\ &= d_1 + \tilde{C}(x_0' + y_0' - y_3'), \\ &= d_1 + \tilde{C}(x_0' + y_0' - \tilde{P}(u_0 - \tilde{v}_2)), \\ (I - \tilde{C}\tilde{P})\tilde{v}_2 &= d_1 + \tilde{C}(x_0' + y_0' - \tilde{P}u_0), \\ \tilde{v}_2 &= (I - \tilde{C}\tilde{P})^{-1}(d_1 + \tilde{C}(x_0' + y_0' - \tilde{P}u_0)), \qquad (4.33) \\ &= (I - \tilde{C}\tilde{P})^{-1}(-\tilde{C}\tilde{P} \quad I \quad \tilde{C} \quad \tilde{C}) \begin{pmatrix} u_0 \\ d_1 \\ x_0' \\ y_0' \end{pmatrix} \qquad (4.34)\end{aligned}$$

Using (4.32) we have:

$$\begin{aligned}
P_3 u_3' &= \tilde{P}\left(u_0 - (I - \tilde{C}\tilde{P})^{-1}(-\tilde{C}\tilde{P} \quad I \quad \tilde{C} \quad \tilde{C}) \begin{pmatrix} u_0 \\ d_1 \\ x_0' \\ y_0' \end{pmatrix} \right) \\
&= \tilde{P}\left(u_0 + ((I - \tilde{C}\tilde{P})^{-1}\tilde{C}\tilde{P}u_0 - (I - \tilde{C}\tilde{P})^{-1}d_1 + (I - \tilde{C}\tilde{P})^{-1}\tilde{C}x_0' - (I - \tilde{C}\tilde{P})^{-1}\tilde{C}y_0') \right), \\
&= \tilde{P}\left(I + (I - \tilde{C}\tilde{P})^{-1}\tilde{C}\tilde{P} \quad -(I - \tilde{C}\tilde{P})^{-1} \quad -(I - \tilde{C}\tilde{P})^{-1}\tilde{C} \quad -(I - \tilde{C}\tilde{P})^{-1}\tilde{C} \right) \begin{pmatrix} u_0 \\ d_1 \\ x_0' \\ y_0' \end{pmatrix}.
\end{aligned}$$

Since $I + (I - \tilde{C}\tilde{P})^{-1}\tilde{C}\tilde{P} = (I - \tilde{C}\tilde{P})^{-1}$ and let:

$$c = \left(\tilde{P}((I - \tilde{C}\tilde{P})^{-1} \quad -(I - \tilde{C}\tilde{P})^{-1} \quad 0 \quad 0 \quad -(I - \tilde{C}\tilde{P})^{-1}\tilde{C} \quad -(I - \tilde{C}\tilde{P})^{-1}\tilde{C}) \right), \tag{4.35}$$

then:

$$P_3 u_3' = c \begin{pmatrix} u_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{pmatrix}^\top. \tag{4.36}$$

Next we find u_3' as:

$$u_3' = \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ d_2 - z_1 \\ d_3 - z_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ d_2 - (y_0' - y_3') \\ d_3 - (x_0' - (y_0' - y_3') \end{pmatrix},$$

using (4.34) and (4.36) and letting:
$\Lambda =$

$$\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
-(I - \tilde{C}\tilde{P})^{-1}\tilde{C}\tilde{P} & (I - \tilde{C}\tilde{P})^{-1} & 0 & 0 & (I - \tilde{C}\tilde{P})^{-1}\tilde{C} & (I - \tilde{C}\tilde{P})^{-1}\tilde{C} \\
\tilde{P}(I - \tilde{C}\tilde{P})^{-1} & -\tilde{P}(I - \tilde{C}\tilde{P})^{-1} & I & 0 & -\tilde{P}(I - \tilde{C}\tilde{P})^{-1}\tilde{C} & -(I + \tilde{P}(I - \tilde{C}\tilde{P})^{-1}\tilde{C}) \\
\tilde{P}(I - \tilde{C}\tilde{P})^{-1} & -\tilde{P}(I - \tilde{C}\tilde{P})^{-1} & 0 & I & -(I - \tilde{P}(I - \tilde{C}\tilde{P})^{-1}\tilde{C}) & -(I + \tilde{P}(I - \tilde{C}\tilde{P})^{-1}\tilde{C})
\end{pmatrix}$$

we have:

$$u_3' = \Lambda \begin{pmatrix} u_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{pmatrix}^\top$$

using (4.31) and defining $Q = \begin{pmatrix} \Lambda \\ c \end{pmatrix}$ we have:

$$\begin{aligned} \|\Pi_{(3)}\| &= \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}, \\ &\leq \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}, \end{aligned}$$

since $\begin{pmatrix} u_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top = u_0'$, then:

$$\begin{aligned} \|\Pi_{(3)}\| &\leq \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0' & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}, \\ &= \|Q\|. \end{aligned}$$

The components of $\|\Pi_{(3)}\|$ are the closed loop transfer functions of the linear system $[\tilde{P}, \tilde{C}]$, confirming that $\|\Pi_{(3)}\|$ is finite. Hence from (4.28) the gap between P_3 and P_3' must satisfy:

$$\vec{\delta}(P_3, P_3') < \frac{1}{\|Q\|}. \tag{4.37}$$

4.4.2 Finding the Gap Metric for a Nonlinear System with Input Nonlinearity

In this subsection the LHS, $\vec{\delta}(P_3, P_3')$, of the inequality (4.28) is considered. To find $\vec{\delta}(P_3, P_3')$ an analogous analysis to that of Subsection 3.5.2 is carried out. First, the graphs for P_3 and P_3' are defined to be:

$$\mathcal{G}_{\tilde{P}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : y = \tilde{P}u, \|u\| < \infty, \|y\| < \infty \right\}, \tag{4.38}$$

$$\mathcal{G}_{P_3} := \left\{ \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{matrix} \right\| < \infty, y_3' = \tilde{P}(u_0 - \tilde{v}_2) \right\}, \tag{4.39}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{matrix} \right\| < \infty, y_3' = \tilde{P}g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2) \right\}. \tag{4.40}$$

To find a bound on the gap between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$, a surjective map Φ is required between these graphs. The following two lemmas are used to define the map Φ. First, consider the nonlinear part of the plant P_3' shown in Figure 4.11b, for this component the following lemma is used.

Lemma 4.7. *Let g^* satisfy Assumption 4.4, and consider the following equation:*

$$v_1 = g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2). \tag{4.41}$$

Then:

$$\|\tilde{v}_2\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

and

$$\|v_1\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Proof. We will first prove that:

$$\|\tilde{v}_2\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty.$$

Let $\|\tilde{v}_2\| < \infty, \|u_0\| < \infty$, and using Assumption 4.4 since $\frac{1}{g^*}$ is a bounded function, $\|\frac{1}{g^*(\tilde{z}_2)}\| < \infty$. Using Lemma 2.2 and since $\|\tilde{v}_2\| < \infty$,

$$\|\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2\| \leq \|\frac{1}{g^*(\tilde{z}_2)}\|\|\tilde{v}_2\| < \infty.$$

Since $\|u_0\| < \infty$, it follows that:

$$\|u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2\| \leq \|u_0\| + \|\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2\| < \infty.$$

Since g^* is a bounded function, then $\|g^*(\tilde{z}_1)\| < \infty$, and hence by Lemma 2.2,

$$\|v_1\| = \|g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)\| \leq \|g^*(\tilde{z}_1)\|\|(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)\| < \infty.$$

as required.
Next we will prove that:

$$\|v_1\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Let $\|v_1\| < \infty, \|u_0\| < \infty$, where $\tilde{v}_2$ can be obtained from equation (4.41):

$$\tilde{v}_2 = g^*(\tilde{z}_2)(u_0 - \frac{1}{g^*(\tilde{z}_1)}v_1).$$

Using Assumption 4.4 then $\frac{1}{g^*}$ is a bounded function, then $\|\frac{1}{g^*(\tilde{z}_1)}\| < \infty$, also using

Lemma 2.2 and since $\|v_1\| < \infty$ then:

$$\|\frac{1}{g^*(\tilde{z}_1)}v_1\| \leq \|\frac{1}{g^*(\tilde{z}_1)}\|\|v_1\| < \infty.$$

Since $\|u_0\| < \infty$, it follows that

$$\|u_0 - \frac{1}{g^*(\tilde{z}_1)}v_1\| \leq \|u_0\| + \|\frac{1}{g^*(\tilde{z}_1)}v_1\| < \infty.$$

Since g^* is a bounded function, then $\|g^*(\tilde{z}_2)\| < \infty$, and

$$\|\tilde{v}_2\| = \|g^*(\tilde{z}_2)(u_0 - \frac{1}{g^*(\tilde{z}_1)}v_1)\| \leq \|g^*(\tilde{z}_2)\|\|(u_0 - \frac{1}{g^*(\tilde{z}_1)}v_1)\| < \infty.$$

as required.

□

In this analysis Φ is defined to be the map between stable P_3 and P_3'. The plants P_3 and P_3' are stable if the plant $\tilde{P}$ is stable, as proved in the following lemma.

Lemma 4.8. *Let $\tilde{P}$ be stable and let g^* satisfy Assumption 4.4. Then P_3 and P_3' given by Figure 4.11 and (4.30) and (4.29), respectively, are stable.*

Proof. First we prove that if $\tilde{P}$ is stable then P_3 is stable. In order to do that we must prove that if $\|u_3'\| < \infty$ then $\|P_3 u_3'\| < \infty$. Also let $\|u_3'\| < \infty$ and hence since $u_3' = \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$, this implies $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$. Then by definition:

$$\begin{aligned} \|y_3'\| &= \|P_3 u_3'\|, \\ &= \|\tilde{P}(u_0 - \tilde{v}_2)\|, \\ &\leq \|\tilde{P}\|(\|u_0\| + \|\tilde{v}_2\|), \\ &< \infty. \end{aligned}$$

Hence P_3 is stable.

Similarly to prove that if $\tilde{P}$ is stable then P_3' is stable we must prove that if $\|u_3'\| < \infty$ then $\|P_3' u_3'\| < \infty$. So, let $\|u_3'\| < \infty$. Since $u_3' = \begin{pmatrix} u_0' \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$, it follows that

$\|u_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$. By definition:

$$\begin{aligned} y_3' &= P_3' u_3' = \tilde{P} g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2), \\ \|y_3'\| &= \|\tilde{P} g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)\|, \end{aligned}$$

Using Lemma 4.7, since $\|u_0\|, \|\tilde{v}_2\| < \infty$ then:

$$\|v_1\| = \|g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)\| < \infty.$$

Since $\tilde{P}$ is stable, it follows that:

$$\|y_3'\| = \|\tilde{P}\|\|g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)\| < \infty.$$

Therefore P_3' is stable as required. □

Since P_3 and P_3' are stable, the graphs for P_3 and P_3' can be written in the form given in the following proposition:

Proposition 4.9. *Let $\tilde{P}$ be stable and let g^* satisfy Assumption 4.4, for the systems P_3 and P_3' given by Figure 4.11 and (4.30) and (4.29), respectively. Then the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ satisfy:*

$$\mathcal{G}_{P_3} := \left\{ \left(\begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3 \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{matrix} \right\| < \infty \right\}, \tag{4.42}$$

$$\mathcal{G}_{P_3'} := \left\{ \left(\begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3' \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{matrix} \right\| < \infty \right\}. \tag{4.43}$$

Proof. To show that if $\tilde{P}$ is stable and g^* satisfies Assumption 4.4 then $\mathcal{G}_{P_3'}$ given in (4.40) can be written as that given in (4.43), and denote the set given in (4.43) as $\mathcal{A}$. Let $\left(\begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3' \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) \in \mathcal{A}$, i.e $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$, $\tilde{P}$ is stable and g^* satisfies Assumption 4.4. Hence using Lemma 4.8, P_3' is stable. Since $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ and P_3' is stable then $\|y_3'\| = \|P_3'(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_3'}$.

Next we prove that $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Let $\begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} \in \mathcal{G}_{P_1'}$, i.e $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2, y_3')\| < \infty$ and $y_3' = \tilde{P}g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)$. Then $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ and $y_3' = \tilde{P}g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2) = P_3'(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$.

This leads to $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_3'} = \mathcal{A}$.

To show that $\mathcal{G}_{P_3}$ given by (4.42) is equivalent to that given by (4.39), set $g^* = \pi$, $\frac{1}{g^*} = \pi'$. In this case $\mathcal{G}_{P_3}$ follows as a special case, as required. □

The map Φ between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ is defined using the following proposition:

Proposition 4.10. *Let $\tilde{P}$ be stable and let $g*$ satisfy Assumption 4.4. Let P_3 and P_3' given by Figure 4.11 and (4.30) and (4.29), respectively. Then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by:*

$$\Phi \begin{pmatrix} \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ P_3 \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ P_3' \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \end{pmatrix}, \tag{4.44}$$

Furthermore this map is surjective.

Proof. First we prove that if

$$x = \begin{pmatrix} (u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2') & P_3(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2') \end{pmatrix}^\top \in \mathcal{G}_{P_3},$$

then $\Phi(x) \in \mathcal{G}_{P_3'}$. Since $x \in \mathcal{G}_{P_3}$ then $\|(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')\|, \|y_3''\| = \|P_3(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')\| < \infty$. Let $y = \begin{pmatrix} (u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & P_3'(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) \end{pmatrix}^\top = \Phi(x)$. We need to show that $y_3' = P_3'(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$ and $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$. It follows from (4.44) that $(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) = (u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')$ and $y_3' = P_3'(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')$, then $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$.

Using Proposition 4.9 since $\tilde{P}$ is stable and $g*$ satisfies Assumption 4.4 and $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| <$

∞, then $\|y_3'\| = \|P_3'(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$, and hence:

$$\begin{aligned} y &= \begin{pmatrix} (u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & P_3'((u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) \end{pmatrix}^\top \\ &= \begin{pmatrix} (u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}. \end{aligned}$$

as required.
Next, to prove that Φ is surjective, let $u = (u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$ and using Proposition 2.14 since P_3 and P_3' are stable and since $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ then the map given in (4.44) is surjective, as required.

□

Using the previous results, a bound on the gap between P_3 and P_3' appearing in the inequality (2.20) can be obtained. This is done using (4.28) as follows

Theorem 4.11. *Let $\tilde{P}$ be stable and let g^* satisfy Assumption (4.4). Let P_3 and P_3' be given by Figure 4.11 and (4.30) and (4.29), respectively. Then a bound on the gap between P_3 and P_3' is*

$$\vec{\delta}(P_3, P_3') \leq \|\tilde{P}\|\|M_g, N_g\|. \tag{4.45}$$

where $\|g^*(\tilde{z}_1) - 1\| \leq M_g$ *and* $\left\|1 - g^*(\tilde{z}_1)\frac{1}{g^*(\tilde{z}_2)}\right\| \leq N_g$,

Proof. Using Proposition 4.10, since $\tilde{P}$ is stable and g^* satisfies Assumption 4.4. Then there exists a surjective map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (4.44) and the gap between P_3 and P_3' is given as:

$$\begin{aligned} \vec{\delta}(P_3, P_3') &\leq \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\}} \frac{\|(\Phi - I)x\|}{\|x\|}, \\ &\leq \sup_{\left\|\begin{smallmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}(g^*(\tilde{z}_1)(u_0 - \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2)) - \tilde{P}(u_0 - \tilde{v}_2)\|}{\|u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2\|} \\ &\leq \sup_{\left\|\begin{smallmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\| \left(\|g^*(\tilde{z}_1) - 1\|\|u_0\| + \|1 - g^*(\tilde{z}_1)\frac{1}{g^*(\tilde{z}_2)}\|\|\tilde{v}_2\|\right)}{\|u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2\|}. \end{aligned} \tag{4.46}$$

Using Assumption 4.4 since $g^*, \frac{1}{g^*}$ are bounded functions we have:

$$\|g^*(\tilde{z}_1) - 1\| \leq M_g \leq \infty$$
$$\left\|1 - g^*(\tilde{z}_1)\frac{1}{g^*(\tilde{z}_2)}\right\| \leq N_g \leq \infty,$$

using the above inequality in (4.46), gives:

$$\begin{aligned}
\vec{\delta}(P_3, P_3') &\leq \sup_{\left\|\begin{smallmatrix} u_0 \\ \tilde{v}_2 \end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\|(M_g\|u_0\| + N_g\|\tilde{v}_2\|)}{\|u_0, \tilde{v}_2\|}, \\
&= \sup_{\left\|\begin{smallmatrix} u_0 \\ \tilde{v}_2 \end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\|\|M_g, N_g\|.\|u_0, \tilde{v}_2\|}{\|u_0, \tilde{v}_2\|}, \\
&\leq \|\tilde{P}\|\|M_g, N_g\|. \qquad (4.47)
\end{aligned}$$

as required.

□

Hence according to the following proposition robust stability is preserved for the system shown in Figure 4.7.

Proposition 4.12. *Consider the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 4.7. Suppose $\tilde{P}$ is stable and g^* satisfies Assumption 4.4. Then $[P_1', P_2', P_3']$ has a robust stability margin.*

Proof. Let $\tilde{P}$ be stable and let g^* satisfy Assumption 4.4, then by Lemmas 4.7, 4.8, and using Proposition 4.9 for the systems P_3 and P_3' given by Figure 4.11 and (4.30) and (4.29), respectively, the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ can be given by (4.42) and (4.43), respectively. Using Proposition 4.10, then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (4.44). This leads to the presence of a finite gap value between the linear and nonlinear configurations of this system given by (4.47). Then the system $[P_1', P_2', P_3']$ given by Figure 4.7 and (4.16)-(4.21) has a robust stability margin.

□

The main result Theorem 4.6 easily follows from Theorem 4.13 which we establish next.

Theorem 4.13. *Consider the nonlinear closed loop system shown in Figure 4.1 and given by (4.16)-(4.21). Suppose $\tilde{P}$ is stable and g^* satisfies Assumption 4.4. Then this system has a robust stability margin b_{P_1,C_1} which satisfies the inequality*

$$b_{P_1,C_1} \geq \|Q\|^{-1}. \qquad (4.48)$$

Proof. Let $\frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 4.7, let $\frac{1}{\|\Pi'_{(3)}\|}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 4.4, finally let $b_{P_1,C_1} = \frac{1}{\|\Pi_{\tilde{P}//\tilde{C}}\|}$ be a stability margin for the system shown in Figure 4.1. Then

$$\begin{aligned}
\|Q\| = \left\|\Pi_{(3)}\right\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}\right\|}{\|u_0', x_0', y_0'\|}, \\
&= \sup_{\|u_0,d_1,d_2,d_3,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0 & d_1 & d_2 & d_3 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, d_1, d_2, d_3, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,0,0,0,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_{(3)}\begin{pmatrix} u_0 & 0 & 0 & 0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, 0, 0, 0, y_0, x_0\|} = \|\Pi'_{(3)}\|, \\
\|\Pi'_{(3)}\| &= \sup_{\|u_0,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_3\begin{pmatrix} u_0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,y_0,0\|\neq 0} \frac{\left\|\Pi_{\tilde{P}//\tilde{C}}\begin{pmatrix} u_0 & y_0 & 0 \end{pmatrix}^\top\right\|}{\|u_0, y_0, 0\|}, \\
&= \|\Pi_{\tilde{P}//\tilde{C}}\|.
\end{aligned}$$

This leads to

$$b_{P_1,C_1} = \frac{1}{\|\Pi_{\tilde{P}//\tilde{C}}\|} \geq \frac{1}{\|\Pi'_{(3)}\|} \geq \frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}.$$

Then the existence of a stability margin for the system shown in Figure 4.9 guarantees the existence of a stability margin for the system $[P_1, C_1]$ shown in Figure 4.1. Also, let $\tilde{P}$ be stable and let g^* satisfy Assumption 4.4, then by Proposition 4.12, the nonlinear closed loop system $[P_1', P_2', P_3']$ given by Figure 4.9 and (4.16)-(4.21), has a robust stability margin. This leads to the conclusion that the system $[P_1, C_1]$ given by Figure 4.1 and (4.16)-(4.21) also has a robust stability margin.

□

Based on Theorems 4.11 and 4.13 we can write the following corollary:

Corollary 4.14. *Consider the nonlinear closed loop system shown in Figure 4.1 and given by (4.16)-(4.21). Suppose that $g^*(z)$ satisfies Assumption 4.4. Then this system is stable if*

$$\|\tilde{P}\|\|M_g, N_g\| < \|Q\|^{-1}. \tag{4.49}$$

Proof. Using Theorem 4.11 inequality (4.45), since:

$$\vec{\delta}(P_3, P_3') \leq \|\tilde{P}\| \|M_g, N_g\|,$$

and using Theorem 4.13 inequality (4.48), since:

$$b_{P_1,C_1} \geq \|Q\|^{-1},$$

It follows that if

$$\|\tilde{P}\| \|M_g, N_g\| < \|Q\|^{-1},$$

then:

$$\vec{\delta}(P_3, P_3') \leq \|\tilde{P}\| \|M_g, N_g\| < \|Q\|^{-1} \leq b_{P_1,C_1}.$$

Hence $\vec{\delta}(P_3, P_3') < b_{P_1,C_1}$ and the conditions hold from Theorem 2.11, hence stability.

□

Next we will compare the validity of the two results given in Theorem 4.5 and Corollary 4.14 for a nonlinear control system in the following example.

4.5 Example

In this example, we wish to find robust stability conditions for the following feedback system using the small gain theorem (Theorem 4.5) and gap metric based analysis (Corollary 4.14). Then we compare these conditions by examining them for this system.

$$\begin{aligned} P_1 &: \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \\ &: u_1 \mapsto y_1, \\ \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - 2x_2 + (1 + \delta \sin(z_1))u_1, \\ y_1 &= x_1, \\ z_1 &= y_1, \end{aligned}$$

where $0 < \delta < 1$. Comparing this system with the one given in (4.8) allows us to set $g^*(x) = (1 + \delta \sin(z_1))$, Now $0 < 1 - \delta \leq \|g^*(x)\| \leq 1 + \delta$ and it is hence clear that Assumption 4.4 is satisfied with $\varepsilon = 1 - \delta$ and $D = 1 + \delta$. A feedback linearizing controller for this system is given as:

$$\begin{aligned} C \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\ & u_2 = 0.4 * \frac{1}{(1+\delta\sin(z_2))} y_2, \\ & z_2 = -y_2. \end{aligned}$$

First, the small gain theorem is applied to the system and using Theorem 4.5 the small gain stability condition is given as

$$\frac{1+\delta}{1-\delta}\|\tilde{P}\|\|\tilde{C}\| < 1.$$

In this example we have $\|\tilde{P}\| = 1$ and $\|\tilde{C}\| = 0.4$, then the stability condition is reduced to

$$0.4 * \frac{1+\delta}{1-\delta} < 1. \tag{4.50}$$

Next, we examine the stability condition found using the gap metric analysis. Using Corollary 4.14 inequality (4.49) we have

$$\|\tilde{P}\|\|M_g, N_g\| < \frac{1}{\|Q\|},$$

let

$$\begin{aligned} M_g &= \max_{z_1} \|1+\delta\sin(z_1) - 1\| = \delta, \\ N_g &= \max_{z_1,z_2} \left\| 1 - (1+\delta\sin(z_1))\frac{1}{1-\delta\sin(z_2)} \right\|, \\ &= \left\| 1 - (1+\delta)\frac{1}{1-\delta} \right\| = \left\| \frac{-2\delta}{(1-\delta)} \right\| \end{aligned}$$

In this example we have $\|\tilde{P}\| = 1$ and $\|Q\| = 1.0079$, then the stability condition reduces to

$$\left\| \delta, \frac{2\delta}{(1-\delta)} \right\| * 1.0079 < 1.$$

where $\left\| \delta, \frac{2\delta}{(1-\delta)} \right\|$ can be found as

$$\begin{aligned}\left\|\delta, \frac{2\delta}{(1-\delta)}\right\| &= \sqrt{\delta^2 + \frac{4\delta^2}{(1-\delta)^2}} \\ &= \sqrt{\frac{\delta^2(1-\delta)^2 + 4\delta^2}{(1-\delta)^2}} \\ &= \sqrt{\frac{\delta^2(\delta^2 - 2\delta + 5)}{(1-\delta)^2}}\end{aligned}$$

then in this case our stability condition will be

$$\sqrt{\frac{\delta^2(\delta^2 - 2\delta + 5)}{(1-\delta)^2}} * 1.0079 < 1. \tag{4.51}$$

The two conditions (4.50) and (4.51) have been plotted in Figure 4.12, where $f_1(\delta) = 0.4 * \frac{1+\delta}{1-\delta}$ and $f_2(\delta) = \sqrt{\frac{\delta^2(\delta^2-2\delta+5)}{(1-\delta)^2}} * 1.0079$:

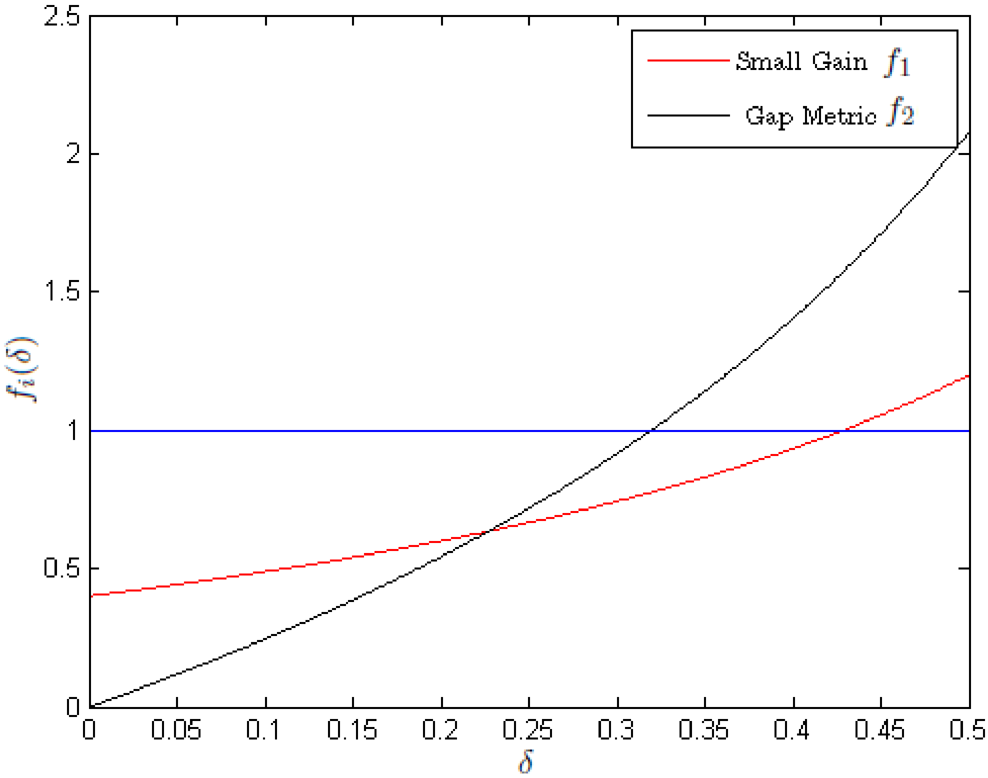

Figure 4.12: plot of the stability conditions

Let $f_i(\delta) < \sigma < 1, i = \{1, 2\}$, we note that for small σ more plant uncertainty can be tolerated without effecting the system stability. This plot shows that for

- $0 < \delta < 0.227$ the gap metric gives a better stability condition (smaller σ) than the small gain theorem.
- $\delta = 0.227$ both conditions gives the same result.
- $0.227 < \delta < 0.32$ the small gain theorem gives a better stability condition than the gap metric.
- $0.32 < \delta < 0.43$ the stability condition in (4.50) is met while the stability condition in (4.51) is not (the small gain theorem gives a better stability condition than the gap metric).
- $0.43 < \delta < 1$ both conditions are violated.

4.6 Summary

This chapter had introduced two results (Theorem 4.5 and Corollary 4.14) which enable the study of the stability of an affine nonlinear control system with input nonlinearity using the small gain theorem and the gap metric analysis. Theorem 4.5 applies the small gain theorem to find a stability condition for the nonlinear system. This stability condition is similar to that given for the Hammerstein system, and states that closed loop system is stable if the product of the system components is less than one and holds if the linear component P of the plant is stable.

The gap analysis had followed the same procedure carried out in Chapter 3, but for a stable affine nonlinear system which is more complicated than the Hammerstein model considered in Chapter 3. This necessitated the use of a triple plant configuration, in order to employ the gap metric network result introduced in (Theorem (10)) Georgiou and Smith (1997). Using this approach the stability condition found depends on the linear gain of the system and the bounds on the input nonlinearity and how exact the inversion is of the nonlinear part of the plant within the nonlinear part of the controller. To investigate the validity of each result, an example was introduced to compute the two stability conditions found in Theorems 4.5 and 5.9 to a nonlinear control system. This example showed that for small gap value, the gap metric gives a better stability condition than the small gain theorem.

In Chapter 5 an unstable affine nonlinear system will be considered. The analysis undertaken for this system will follow the same procedure carried out in this chapter. However, the stability assumption on $\tilde{P}$ will be dropped and Proposition 4.9 will no longer be assumed to hold. Hence, stability analysis will be more complex and the use of coprime factors will be required.

Chapter 5

Robustness Analysis for Unstable Affine Systems Using the Gap Metric

5.1 Introduction

In this chapter the robustness analysis for an unstable affine nonlinear system using the gap metric is considered. The analysis undertaken will follow the same procedure carried out in Chapter 4, however, since the system considered is unstable, the small gain theorem can no longer be used to find stability conditions for this system. The gap analysis carried out in this chapter is more complicated than that of Chapter 4 as the stability assumption on the linear part of the plant is dropped and Proposition 4.9 is no longer assumed to hold. To address this, coprime factors will be required to represent unstable plants in this chapter.

The gap metric will be used to study the stability of two cases of affine systems. The first case comprises an unstable affine nonlinear system with an unstable nonlinear component, for example, consider the system $\dot{x} = x^2 + u, y = x$ where x^2 is an unstable nonlinear component which we will try to cancel by control action. In this case the nominal plant will include only the unstable linear component of the plant. The second system is an affine nonlinear system with both stable and unstable nonlinear components, for example, the system $\dot{x} = x^2 - x^3 + u, y = x$, here we have two nonlinear parts in this system, x^2 which is an unstable nonlinear component also cancelled by control action, and $-x^3$, is a useful stable nonlinear part of the plant, which we will preserve its stabilizing rule on the plant. Thus, the nominal plant in this case will include the linear part of the plant and the stable nonlinear part of the plant. The motive for considering the second case is to use the inherently stabilizing nonlinearities to stabilize the plant instead of trying to cancel them by the control action, as is done in the feedback linearization approach (this will be explained fully in Section 5.3, Example 5.1).

5.2 Robustness Analysis of a Nonlinear System with an Unstable Nonlinear Part using the Gap Metric

This section will consider an unstable affine nonlinear system with an unstable nonlinear part and carry out a stability analysis for this system using the gap metric.

5.2.1 Affine Nonlinear Systems With Unstable Nonlinear Part

Consider the normal form of the nonlinear system described previously in Section 4.2, (4.5)-(4.6), this form is given by:

$$P \quad : \quad \mathcal{L}^{n+1}_{\infty,e} \to \mathcal{L}^{n}_{\infty,e} : (u_1, z_1) \mapsto (y_1),$$

$$\dot{x}^* = Ax^* + B(f^*(z_1) + g^*(z_1)u_1), \tag{5.1}$$

$$y_1 = (y_{11}, \ldots, y_{1n}) = x^*, \tag{5.2}$$

$$z_1 = (z_{11}, \ldots, z_{1n}), \tag{5.3}$$

In this system the feedback input y_1 is considered as an input, z_1, to the nonlinear parts of the plant $g^*(z_1)$ and $f^*(z_1)$, A is unstable and is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & \ldots & & a_{n-1} & a_n \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{aligned} f^*(x^*) &= L_f^n h(T^{-1}(x^*)), \\ g^*(x^*) &= L_g L_f^{n-1} h(T^{-1}(x^*)). \end{aligned}$$

We select the form $u_1 = l(x^*, v)$ to stabilize the system in (5.1)-(5.3) and linearize the terms in (5.1). We next choose a vector

$$c = (c_1, \ldots, c_n)^\top,$$

such that $A_c = A - Bc^\top$ is stable. A real function $l(x^*, v) = a(x^*) + b(x^*)v$ is then chosen such that this function will cancel the nonlinear terms $f^*(x^*)$ and $g^*(x^*)$ while stabilizing the linear part of the plant. To do this, it follows that $Ax^* + B(f^*(x^*) + g^*(x^*)l(x^*, v)) = A_c x^* + Bv$ for some input v. This leads to the conclusion that $l(x^*, v)$ must take the form:

$$l(x^*, v) = \frac{-c^\top x^* - f^*(x^*) + v}{g^*(x^*)}.$$

Hence, a feedback linearizing controller which generates the term $l(x^*, v)$ is given as:

$$\begin{aligned} C \quad : \quad & \mathcal{L}^{2n}_{\infty,e} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\ & u_2 = -l(y_2, z_2) = \frac{1}{g^*(z_2)}(C_s y_2 + f^*(z_2) - \tilde{C} y_2), \\ & z_2 = (z_{21}, \ldots, z_{2n}) = -y_2, \end{aligned}$$

where $l(x^*, v) = l(y_2, z_2)$, $y_2 = (y_{21}, \ldots, y_{2n})$, C_s is the linear stabilizing part of the controller C,

$$\begin{aligned} C_s \quad : \quad & \mathcal{L}^{n}_{\infty,e} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_s \\ & v_s = c^\top y_2. \end{aligned}$$

The term $\tilde{C}$ is required to generate v using a feedback action, and is given by

$$\begin{aligned} \tilde{C} \quad : \quad & \mathcal{L}^{n}_{\infty,e} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v \\ & v = -\tilde{c}^\top y_2, \end{aligned}$$

where

$$\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)^\top,$$

The feedback control connection for this system is shown in Figure 5.1.

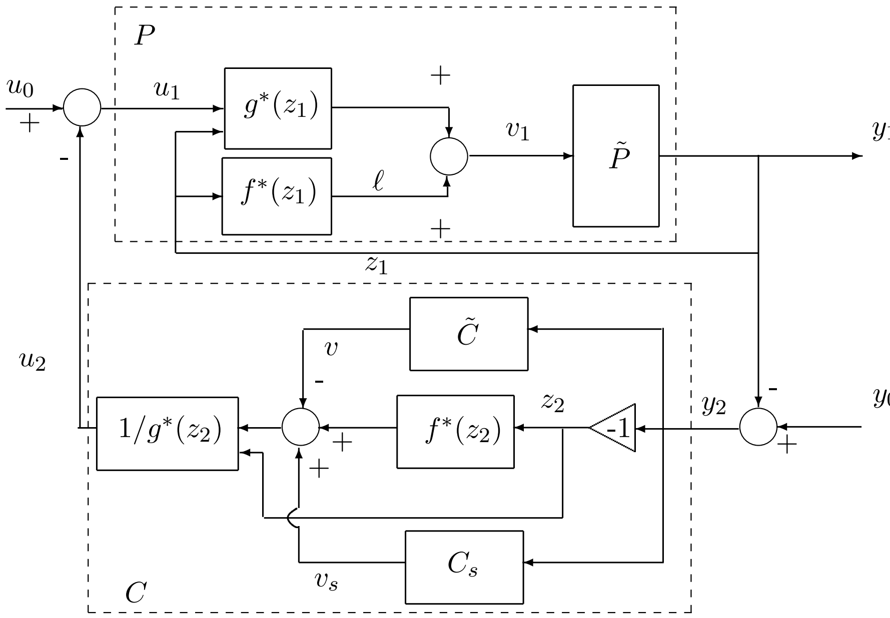

FIGURE 5.1: Nonlinear control system with input/state plant nonlinearity

Here $\tilde{P}$, is the linear potentially unstable component of the plant P. This component can be written using linear coprime factorization as $\tilde{P} = NM^{-1}$ where N and M satisfy $NX + MY = I$ for some X and Y, where N, M, X and Y are stable. $\tilde{P}$ is given by:

$$\begin{aligned}\tilde{P} \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : v_1 \mapsto y_1, \\ & \dot{x}^* = Ax^* + Bv_1, y_1 = x^*,\end{aligned}$$

where $v_1 = f^*(z_1) + g^*(z_1)u_1$. It follows that M can be written as the linear operator from v_n to v_1 which give rise to

$$\begin{aligned}M \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : v_n \mapsto v_1, \\ \dot{x}^* \quad &= \quad A_c x^* + Bv_n, \\ v_1 \quad &= \quad Lx^* + v_n, \\ z_1 \quad &= \quad (z_{11}, \dots, z_{1n}) = x^*,\end{aligned}$$

and N is the linear operator from v_n to y_1:

$$\begin{aligned}N \quad &: \quad \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : v_n \mapsto y_1, \\ \dot{x}^* \quad &= \quad A_c x^* + Bv_n, \\ y_1 \quad &= \quad x^*,\end{aligned}$$

where v_n is an external signal, and $\tilde{P}$ is stabilized by a linear controller of the form $v_1 = v_n + Lx^*$, where $Lx^* = -c^\top x^*$ (which is the linear part of our controller).

Then we can write P as

$$\begin{aligned}P \quad : \quad & \mathcal{L}^{n+1}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : (u_1, z_1) \mapsto y_1, \\ & y_1 = \tilde{P}(f^*(z_1) + g^*(z_1)u_1) \\ & z_1 = (z_{11}, \dots, z_{1n}) = y_1.\end{aligned} \tag{5.4}$$

In addition, let $v_2 = v_s - v$ so that $v_2 = C_s y_2 - \tilde{C} y_2 = (C_s - \tilde{C}) y_2$, and let $C_{Linear} = C_s - \tilde{C}$ then $v_2 = C_{Linear} y_2$, then we can write

$$\begin{aligned}C \quad : \quad & \mathcal{L}^{2n}_{\infty,e} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\ & u_2 = -l(y_2, z_2) = \frac{1}{g^*(z_2)}(C_{Linear} y_2 + f^*(z_2)), \\ & z_2 = (z_{21}, \dots, z_{2n}) = -y_2,\end{aligned}$$

This new feedback control connection for the system is shown in Figure 5.2.

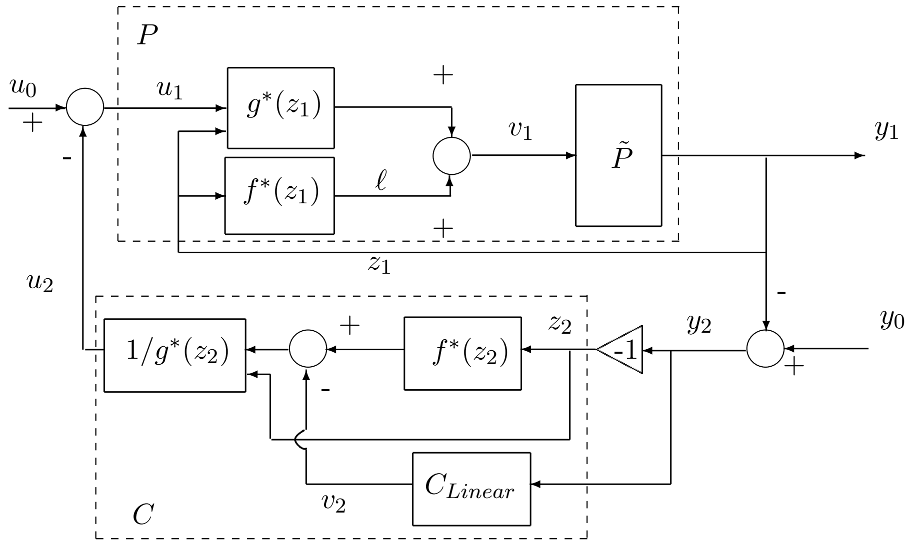

FIGURE 5.2: Nonlinear control system with input/state plant nonlinearity

For the system shown in Figure 5.2 the closed loop equations can be written as:

$$u_0 = u_1 + u_2, \tag{5.5}$$

$$y_0 = y_1 + y_2, \tag{5.6}$$

$$v_1 = f^*(z_1) + g^*(z_1)u_1, \tag{5.7}$$

$$v_2 = C_{Linear}y_2, \tag{5.8}$$

$$y_1 = \tilde{P}v_1, \tag{5.9}$$

$$u_2 = 1/g^*(z_2)(f^*(z_2) + v_2). \tag{5.10}$$

We note that the system shown in Figure 5.2 is not equivalent to the closed loop system $[P, C]$ shown in Figure 2.1. This is due to the presence of the two signals z_1, z_2 which feed the signals $y_1, -y_2$ to the blocks P, C, respectively. Hence, we cannot apply Theorem 2.11 to find stability conditions for this system.

Following the same approach undertaken in Chapter 4, we need to consider the linear operator $\Pi_{\tilde{P}//C_{Linear}}$ to find stability conditions for the system shown in Figure 5.2. This operator corresponds to the closed loop system $[\tilde{P}, C_{Linear}]$ which is the linear configuration of the system shown in Figure 5.2 and can be found by replacing $g^*(z_1)$ and $\frac{1}{g^*(z_2)}$ by the linear operators $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, respectively, and setting $f^*(z_1) = f^*(z_2) = 0$ in this system, as shown in Figure 5.3.

The blocks π, π' cancel the effect of the signals z_1, z_2 on the closed loop system $[P_1, C_1]$. Hence, the signals z_1, z_2 can be removed, with the mappings π, π' unchanged as shown in Figure 5.4.

This linear configuration corresponds to the system shown in Figure 2.1.

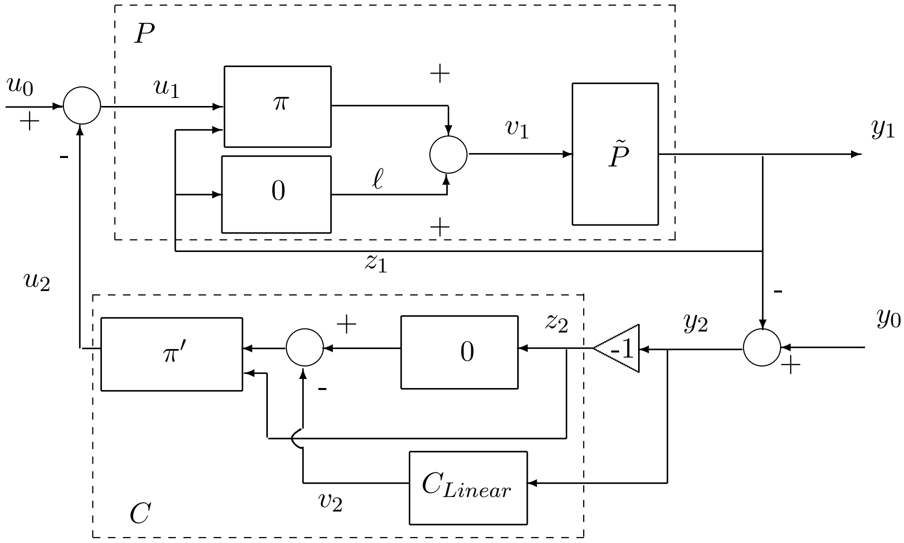

FIGURE 5.3: Linear configuration of the system in Figure 5.2

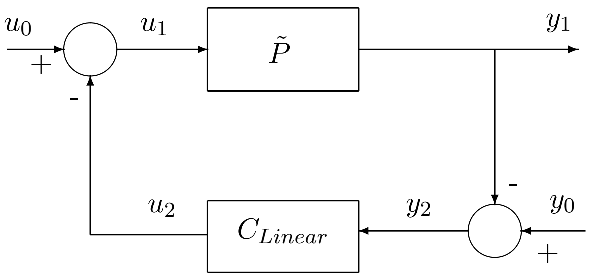

FIGURE 5.4: Linear configuration of the system in Figure 5.2 with z_1, z_2 removed and the mappings π, π' applied to the system

5.2.2 Gap Metric for a Nonlinear System with an Unstable Nonlinear Part

This section undertakes robustness stability analysis for the affine nonlinear system shown in Figure 5.2. The analysis carried out in this section will follow the approach presented in Chapter 4. However, as was mentioned previously in Section 5.1, the stability assumption on $\tilde{P}$ is dropped and Proposition 4.9 will no longer be assumed to hold. The following assumptions on the forms of $f^*(z)$ and $g^*(z)$ are required in this analysis:

Assumption 5.1. Let $f^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the following condition:

$$\exists \mathbf{B} < \infty \text{ and } \quad |f^*(x)| \leq \mathbf{B} \quad \forall x \in \mathbb{R}^n. \tag{5.11}$$

Assumption 5.2. Let $g^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the following condition:

$$\exists \varepsilon > 0, \quad \exists \mathbf{D} < \infty \text{ and } \quad \varepsilon \leq |g^*(x)| \leq \mathbf{D} \quad \forall x \in \mathbb{R}^n. \tag{5.12}$$

The gap metric framework is applied to the system shown in Figure 5.2 using the following theorem:

Theorem 5.3. *Consider the nonlinear closed loop system shown in Figure 5.2 and given by (5.5)-(5.10). Let f^*, g^* satisfy Assumptions 5.1, 5.2. Then this system has a robust stability margin.*

Analogous to the proof of Theorem 4.6, this proof also requires results that are developed later in this section. This analysis will also consider the triple system configuration shown in Figure 4.5 and apply the 'network' result in (Theorem 2.13) to find a stability condition for the nonlinear system shown in Figure 5.2.

The route taken is as follows: Since the presence of nonlinear elements in multiple blocks in the system shown in Figure 5.2 leads to significant conservatism, and to apply Theorem 2.13 to this system, the new system configuration shown in Figure 5.5 is used. In this configuration the nonlinear component of the plant P and the nonlinear component of the controller C are considered to be both included in the block P_3' and an external input x_0 is added to the system. Also the feedback input $x_0 - y_1$ is considered as an input, z_1, to the nonlinear components of the plant $f^*(z_1), g^*(z_1)$, and the feedback input $-y_2$ is considered as an input z_2 to the nonlinear components $f^*(z_2), \frac{1}{g^*(z_2)}$.

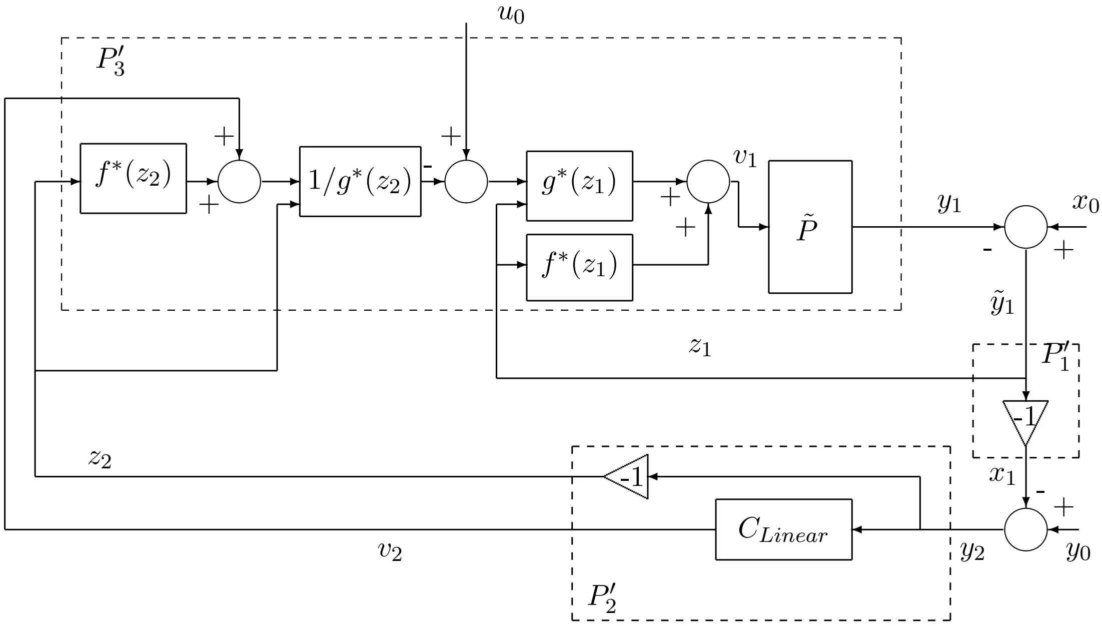

FIGURE 5.5: Nonlinear system with unstable nonlinearity configuration

The linear configuration for this system is taken to comprise the system components P_1, P_2, P_3 with the nonlinearities $g^*(z_1)$ and $\frac{1}{g^*(z_2)}$ replaced by the linear operators $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, respectively. In addition, $f^*(z_1) = f^*(z_2) = 0$. This configuration is shown in Figure 5.6.

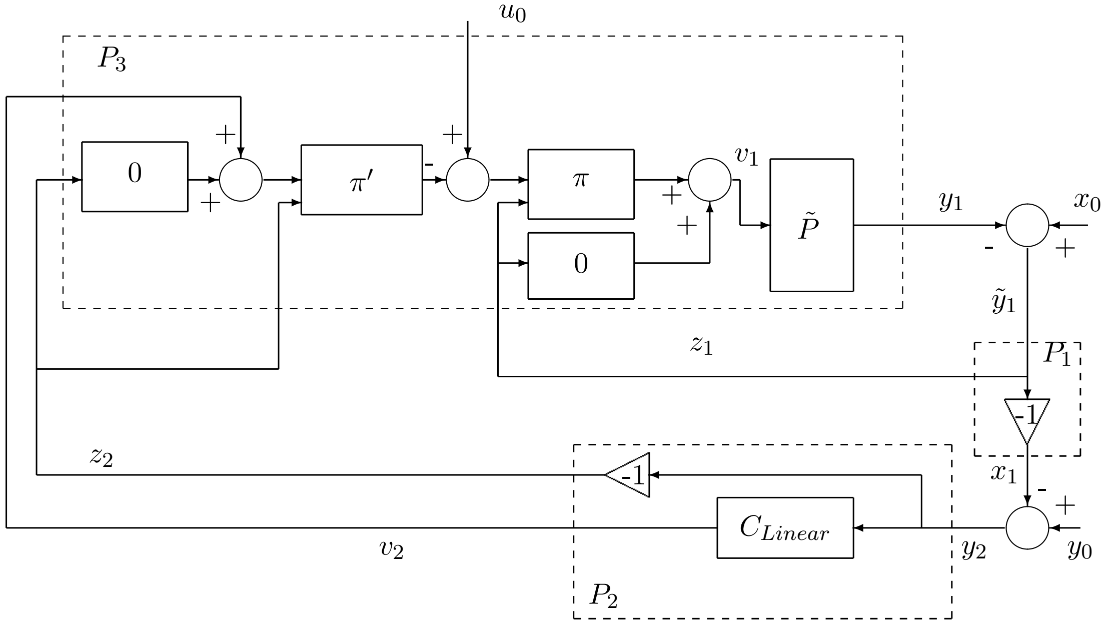

FIGURE 5.6: Linear configuration of an affine system

To apply Theoreom 2.13 we must rearrange the nonlinear and linear configurations of the systems shown in Figures 5.5 and 5.6 in a form comparable to that given in Figure 4.5. In order to do this we consider three signal spaces $\mathcal{U} = \mathcal{L}^n_{\infty,e}, \mathcal{X} = \mathcal{L}^n_{\infty,e}$ and $\mathcal{Y} = \mathcal{L}^n_{\infty,e}$, together with the following augmented signals; let $\hat{v}_2 = -v_2$ and let $u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top$ and let $u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top$, also let the external input u_0 be changed to $u_0' = \begin{pmatrix} u_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top$, where $d_2 = (d_{21}, \ldots, d_{2n})$ and $d_3 = (d_{31}, \ldots, d_{3n})$. Let $u_3' = u_0' - u_2' - u_1' = \begin{pmatrix} u_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top - \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top - \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top = \begin{pmatrix} u_0 & d_1 - \hat{v}_2 & d_2 - z_1 & d_3 - z_2 \end{pmatrix}^\top$, let $\tilde{v}_2 = d_1 - \hat{v}_2$, $\tilde{z}_1 = d_2 - z_1$, $\tilde{z}_2 = d_3 - z_2$ then $u_3' = \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$. Finally, let $x_0' = y_0$, $y_0' = x_0$, $y_3' = y_1$, $x_1' = x_1$, $x_2' = y_2$ and $y_1' = y_0' - y_3' = x_0 - y_1$. The resulting system is shown in Figure 5.7.

The corresponding linear configuration of this system is shown in Figure 5.8.

From the two systems shown in Figure 5.7 and Figure 5.8 it follows that $P_1 = P_1'$ and $P_2 = P_2'$.

These configurations correspond to those of Figures 5.5 and 5.6, respectively, except for the presence of d_1, d_2 and d_3. Furthermore, Figures 5.7 and 5.8 correspond exactly to the forms shown in Figures 5.9 and 5.10, respectively, which in turn have identical structure to that of Figure 4.5. Hence, our stability condition will be applied to the systems of Figures 5.9 and 5.10.

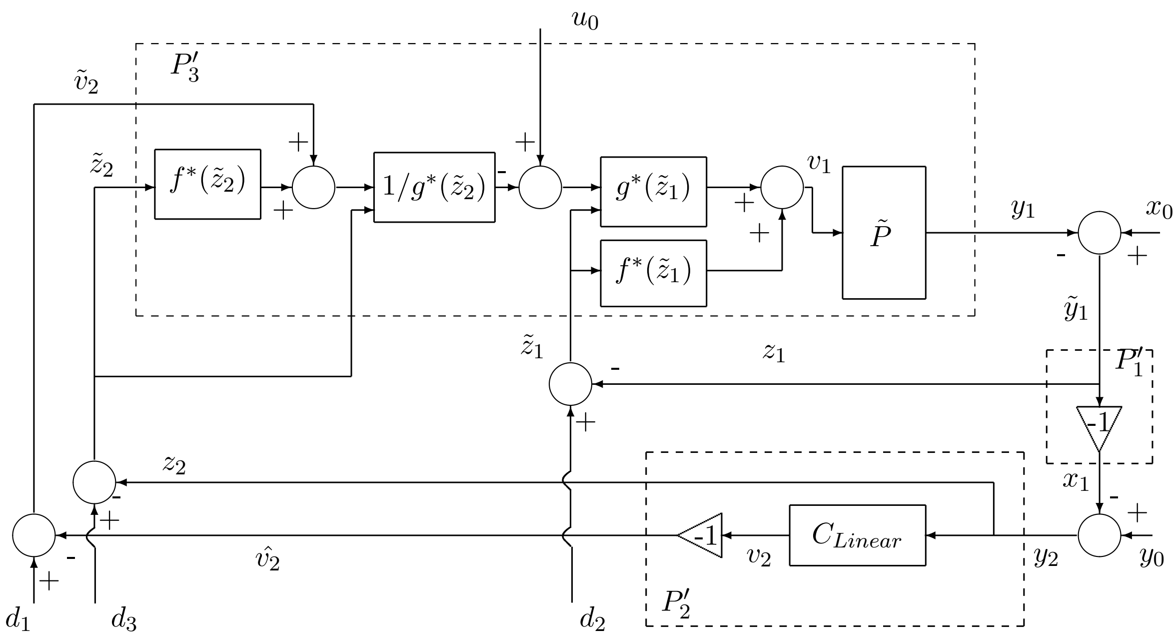

FIGURE 5.7: Augmented affine nonlinear system

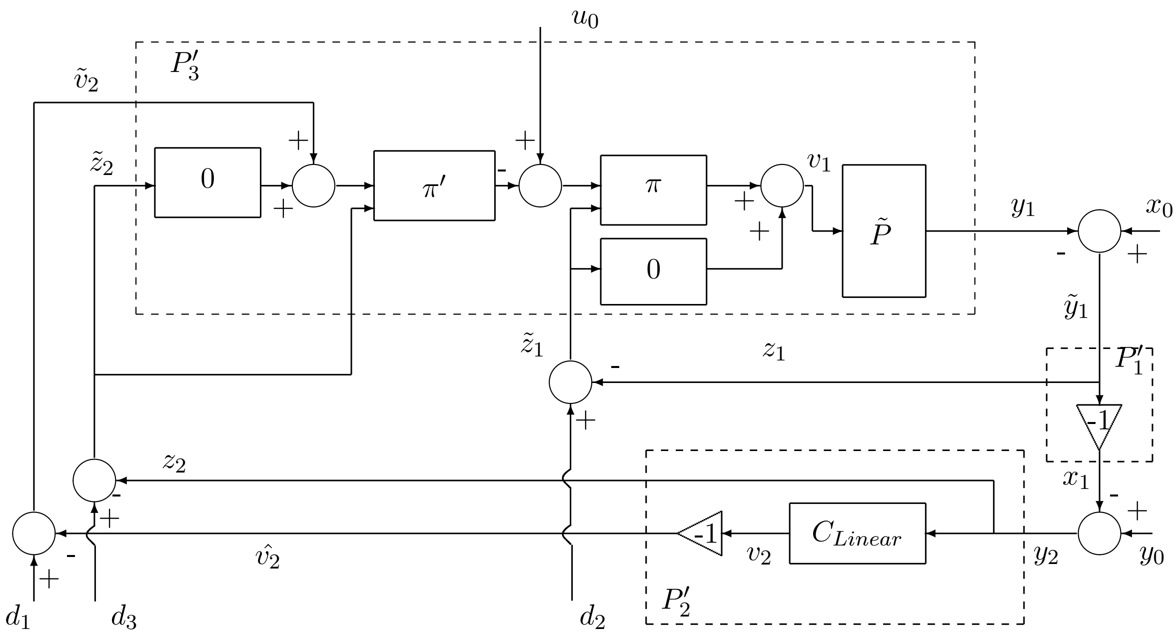

FIGURE 5.8: Linear configuration of affine nonlinear system

Since $P_1 = P_1'$ and $P_2 = P_2'$, then

$$\vec{\delta}(P_1, P_1') = 0, \vec{\delta}(P_2, P_2') = 0.$$

Using Theorem 2.13, the robust stability condition is given as:

$$\sum_{i=1}^{3} \vec{\delta}(P_i, P_i') < \|\Pi_{(i)}\|^{-1},$$

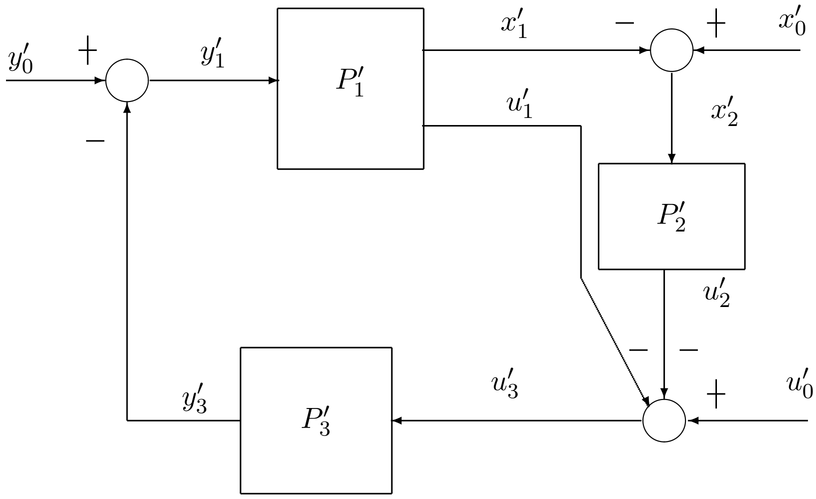

FIGURE 5.9: Nonlinear configuration of affine system with unstable nonlinearity

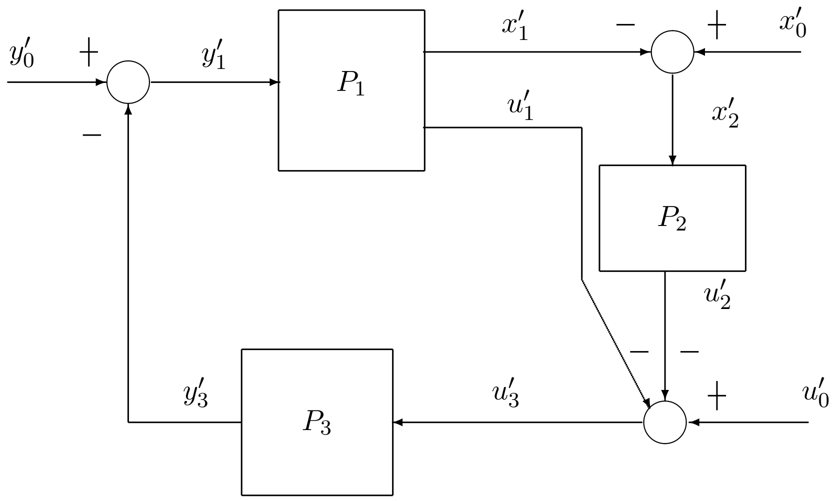

FIGURE 5.10: Linear configuration of affine system with unstable nonlinearity

For our system this condition is:

$$\vec{\delta}(P_3, P_3') < \|\Pi_{(3)}\|^{-1}. \tag{5.13}$$

Then the gap metric measures the difference between the linear nominal plant $P_3 : u_3' \mapsto y_3', y_3' = \tilde{P}\pi(u_0 - \pi'(\tilde{v}_2, \tilde{z}_2), \tilde{z}_1) = \tilde{P}(u_0 - \tilde{v}_2)$ and the nonlinear perturbed plant $P_3' : u_3' \mapsto y_3', y_3' = \tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)((f^*(\tilde{z}_2) + \tilde{v}_2))))$. The plants P_3 and P_3' are shown in Figure 5.11.

Before providing a complete description of the operators P_1', P_2' and P_3' and P_1, P_2 and P_3 shown in Figures 5.9 and 5.10, respectively, we briefly state the motivation for the proceeding manipulations (as was done in Chapter 4).

The stability condition (5.13) can be related to the original system configuration shown

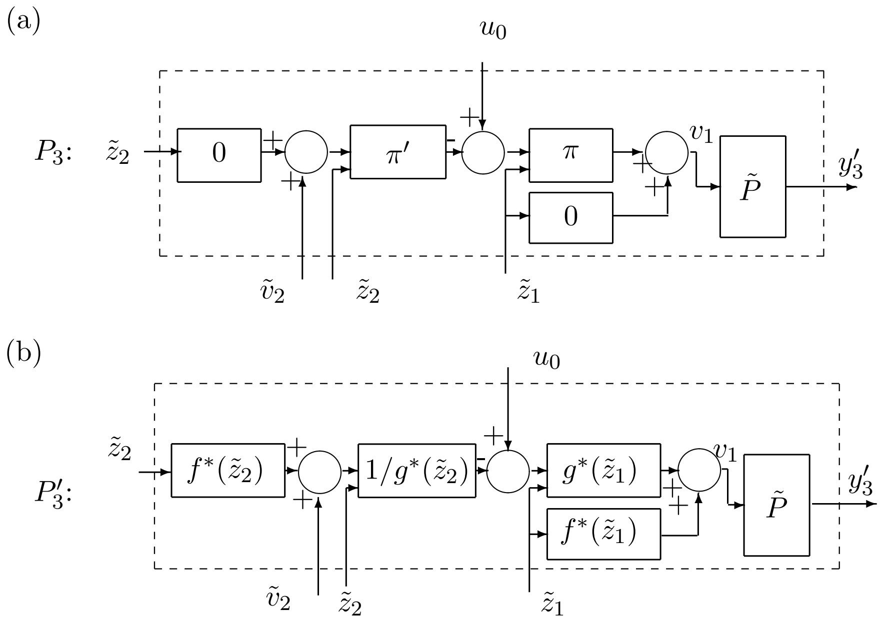

FIGURE 5.11: Nonlinear plant mapping:(a) unperturbed, (b) perturbed

in Figure 5.2 as follows: It will be shown later in the proof of Theorem 5.3 that the stability margin for the system shown in Figure 5.7 is less than or equal to the stability margin corresponding to the system shown in Figure 5.5 which in turn is less than or equal to the stability margin corresponding to the original system shown in Figure 5.2. This is because for each pair the latter is a special case of the former.

The closed loop operators P_1', P_2' and P_3' shown in Figure 5.9 are given by

$$\begin{aligned} P_1' \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{2n}_{\infty,e} : y_1' \mapsto (x_1', u_1'), x_1' = -y_1', \\ &\quad u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1', \end{aligned}$$

where $y_1' = \tilde{y}_1$, and :

$$\begin{aligned} P_2' \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{n+1}_{\infty,e} : x_2' \mapsto u_2', u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, \\ &\quad z_2 = x_2', \hat{v}_2 = -C_{linear} x_2', \end{aligned}$$

and the block P_3' is given by:

$$\begin{aligned} P_3' \quad &: \quad \mathcal{L}^{2n+2}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u_3' \mapsto y_3', \qquad (5.14) \\ y_3' \quad &= \quad P_3' u_3', \\ &= \quad \tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)((f^*(\tilde{z}_2) + \tilde{v}_2)))). \qquad (5.15) \end{aligned}$$

The linear configuration shown in Figure 5.10 comprises subsystems:

$$\begin{aligned} P_1 \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{2n}_{\infty,e} : y'_1 \mapsto (x'_1, u'_1) \\ & \quad x'_1 = -y'_1, u'_1 = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y'_1, \end{aligned}$$

$$\begin{aligned} P_2 \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{n+1}_{\infty,e} : \quad x'_2 \mapsto u'_2 \\ & u'_2 = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, z_2 = x'_2, \hat{v}_2 = -C_{linear} x'_2, \end{aligned}$$

and

$$\begin{aligned} P_3 \quad &: \quad \mathcal{L}^{2n+2}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u'_3 \mapsto y'_3, \\ y'_3 \quad &= \quad P_3 u'_3 = \tilde{P}(u_0 - \tilde{v}_2). \end{aligned} \tag{5.16}$$

From the above definitions given for P_1, P_2, P_3, we conclude that the linear configuration shown in Figure 5.10 is equivalent to the configuration shown in Figure 4.10. This leads to the conclusion that the linear gain $\|\Pi_{(3)}\|$ calculated for these systems is the same.

Similar to the approach taken in Chapter 4, to apply Theorem 2.13 to this system, we must satisfy inequality (5.13). In the following two subsections, the two sides of this inequality will be evaluated, namely the linear gain $\|\Pi_{(3)}\|$ and the gap value $\vec{\delta}(P_3, P'_3)$.

5.2.3 Finding $\|\Pi_{(3)}\|$ for an Affine Nonlinear System with Unstable Nonlinearity

We start with the RHS of inequality (5.13). Since the linear gain $\|\Pi_{(3)}\|$ calculated for the system shown in Figure 5.10 is the same as the linear gain $\|\Pi_{(3)}\|$ for the system shown in Figure 4.10, the procedure in Subsection 4.4.1 can be used to calculate this value. In Subsection 4.4.1 it was found that:

$$\|\Pi_3\| \quad \leq \quad \sup_{\|u'_0, x'_0, y'_0\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u'_0 & x'_0 & y'_0 \end{matrix} \right\|^\top}{\|u'_0, x'_0, y'_0\|} = \|Q\|.$$

where $Q = \begin{pmatrix} \Lambda \\ c \end{pmatrix}$, with Λ, c matrices of dimension 4×6 and 1×6 respectively, their terms comprising closed loop functions of system $[\tilde{P}, C_{linear}]$. Hence from (5.13) and since $\|\Pi_3\| \leq \|Q\|$ then for the considered system, the stability condition is:

$$\vec{\delta}(P_3, P'_3) < \frac{1}{\|Q\|}. \tag{5.17}$$

5.2.4 Finding the Gap Metric for a Nonlinear System with Unstable Nonlinearity

In this subsection the LHS, $\vec{\delta}(P_3, P_3')$, of the inequality (5.13) is considered. To find $\vec{\delta}(P_3, P_3')$ an analogous approach to that developed in Subsection 4.4.2 is used. However, the procedure followed in this subsection is more complicated. First, the graphs for $\tilde{P}$, P_3 and P_3' are defined to be:

$$\mathcal{G}_{\tilde{P}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : y = \tilde{P}u, \|u\| < \infty, \|y\| < \infty \right\}, \tag{5.18}$$

$$\mathcal{G}_{P_3} := \left\{ \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{matrix} \right\| < \infty, y_3' = \tilde{P}(u_0 - \tilde{v}_2) \right\}, \tag{5.19}$$

$$\mathcal{G}_{P_3'} :=$$
$$\left\{ \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{matrix} \right\| < \infty, y_3' = \tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2))) \right\}. \tag{5.20}$$

In order to find a bound on the gap between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$, a surjective map Φ is required between these graphs. The following lemma is used to define this map. First, consider the nonlinear part of the plant P_3' shown in Figure 5.11b. For this component the following lemma is used.

Lemma 5.4. *Let f^* satisfy Assumption 5.1 and let g^* satisfy Assumption 5.2, and consider the following equation:*

$$v_1 = f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)). \tag{5.21}$$

then:

$$\|\tilde{v}_2\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

and

$$\|v_1\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Proof. We will first prove that:

$$\|\tilde{v}_2\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty.$$

Since f^* is a bounded function via Assumption 5.1, then $\|f^*(\tilde{z}_1)\| < \infty$ and $\|f^*(\tilde{z}_2)\| < \infty$. Since g^* is a bounded function via Assumption 5.2, then $\|g^*(\tilde{z}_1)\| < \infty$ and $\|1/g^*(\tilde{z}_2)\| < \infty$ and since $\|\tilde{v}_2\| < \infty$, $\|u_0\| < \infty$, then:

$$\begin{aligned}
\|v_1\| \quad &= \|f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2))\|, \\
&\leq \|f^*(\tilde{z}_1)\| + \|g^*(\tilde{z}_1)\|\|u_0\| + \|g^*(\tilde{z}_1)\|\|1/g^*(\tilde{z}_2)\| + \|f^*(\tilde{z}_2)\| + \\
&\|g^*(\tilde{z}_1)\|\|1/g^*(\tilde{z}_2)\|\|\tilde{v}_2\|, \\
&< \infty.
\end{aligned}$$

as required. Next we will prove that:

$$\|v_1\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

By (5.21):

$$\tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2).$$

Also since f^* and g^* are both bounded functions, and since $\|v_1\| < \infty$, $\|u_0\| < \infty$, then:

$$\begin{aligned}
\|\tilde{v}_2\| \quad &= \quad \| - g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2)\|, \\
&\leq \quad \|g^*(\tilde{z}_2)\|\|1/g^*(\tilde{z}_1)\|\|v_1\| + \|g^*(\tilde{z}_2)\|\|1/g^*(\tilde{z}_1)\| + \|f^*(\tilde{z}_1)\| + \\
&\quad\;\; \|g^*(\tilde{z}_2)\|\|u_0\| + \|f^*(\tilde{z}_2)\|, \\
&< \quad \infty.
\end{aligned}$$

as required. □

The graphs for P_3 and P_3' can be written using coprime factorization functions as shown in the following proposition:

Proposition 5.5. *Let $\tilde{P}$ be unstable, let f^* satisfy Assumption 5.1 and let g^* satisfy Assumption 5.2, for the systems P_3 and P_3' given by Figure 5.11 and (5.15) and (5.16), respectively. Then the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ satisfy:*

$$\mathcal{G}_{P_3} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = u_0 - v_1, v_n, \tilde{z}_1, \tilde{z}_2, u_0 \in \mathcal{U} \end{array} \right\}, \tag{5.22}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - \\ f^*(\tilde{z}_2), v_n, \tilde{z}_1, \tilde{z}_2, , u_0 \in \mathcal{U} \end{array} \right\}. \tag{5.23}$$

where M, N form a right coprime factorization of $\tilde{P}$ i.e. $\tilde{P} = NM^{-1}$.

Proof. To show that $\mathcal{G}_{P_3'}$ given in (5.23) is equivalent to that given in (5.20), denote the set given in (5.23) as $\mathcal{A}$.

First we prove that $\mathcal{A} \subset \mathcal{G}_{P_3'}$. Let $\begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{A}$, i.e $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2)$ where $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, u_0 \in \mathcal{U}$. Since $u_0 \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}$ and $\tilde{z}_1 \in \mathcal{U}$ we have $\|u_0\| < \infty, \|\tilde{z}_1\| < \infty$ and $\|\tilde{z}_2\| < \infty$, respectively. Since $v_1 = Mv_n$, $v_n \in \mathcal{U}$ and M is a bounded operator it follows that $\|v_1\| < \infty$. In the same way, since $y_3' = Nv_n$, $v_n \in \mathcal{U}$ and N is a bounded operator it follows that $\|y_3'\| < \infty$. Since $\|v_1\| < \infty$ and $\|u_0\| < \infty$, and $v_1 = f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) - \tilde{v}_2))$, it follows from Lemma 5.4 (second statement), that $\|\tilde{v}_2\| < \infty$. Also given that $y_3' = Nv_n = NM^{-1}v_1 = \tilde{P}v_1$, it follows that $y_3' = \tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)))$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_3'}$.

Next we prove that $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Let $\begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}$. Then we have $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$ and $y_3' = \tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)))$. We need to show that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2)$ and $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, u_0 \in \mathcal{U}$. Here $u_0 \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}$ follow from the definition of $\mathcal{G}_{P_3'}$ and since $\|y_3'\| < \infty$ then $\|\tilde{P}(f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) - \tilde{v}_2)))\| < \infty$, also given that $\|u_0\|, \|\tilde{v}_2\| < \infty$ and defining $v_1 = f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2))$, by Lemma 5.4 first statement, it follows that $\|v_1\| < \infty$, this leads to $\begin{pmatrix} v_1 & \tilde{P}v_1 \end{pmatrix}^\top = \begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{\tilde{P}}$. Now any element in $\mathcal{G}_{\tilde{P}}$ can be written in the form $\begin{pmatrix} M & N \end{pmatrix}^\top v_n$ for some $v_n \in \mathcal{U}$. So let:

$$\begin{pmatrix} M & N \end{pmatrix}^\top v_n = \begin{pmatrix} v_1 & \tilde{P}v_1 \end{pmatrix}^\top = \begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top,$$

which leads to $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_3'} = \mathcal{A}$.

Similarly, to show that $\mathcal{G}_{P_3}$ given in (5.19) is equivalent to that given in (5.22), set $g^*(z_1) = \pi, \frac{1}{g^*(z_1)} = \pi'$ and $f^*(z_1) = f^*(z_2) = 0$. In this case $\mathcal{G}_{P_3}$ follows as a special case, as required. □

The map Φ between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ is defined using the following proposition:

Proposition 5.6. *Let $\tilde{P}$ be unstable, let f^* satisfy Assumption 5.1 and let g^* satisfy Assumption 5.2, for the systems P_3 and P_3' given by Figure 5.11 and (5.15) and (5.16),* *respectively. Then there exists a map* $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ *given by:*

$$\Phi \begin{pmatrix} u_0 \\ (u_0 - Mv_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix} = \begin{pmatrix} u_0 \\ -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(Mv_n - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix}. \tag{5.24}$$

Furthermore this map is surjective.

Proof. First we need to prove that if

$$x = \begin{pmatrix} u_0'' & (u_0'' - Mv_n') & \tilde{z}_1' & \tilde{z}_2' & Nv_n' \end{pmatrix}^\top \in \mathcal{G}_{P_3},$$

then $\Phi(x) \in \mathcal{G}_{P_3'}$. Since $x \in \mathcal{G}_{P_3}$ then $\|u_0''\|, \|\tilde{v}_2'\| = \|u_0'' - v_1'\| = \|(u_0'' - Mv_n')\|, \|\tilde{z}_1'\|, \|\tilde{z}_2'\|, \|y_3''\| < \infty$, $\begin{pmatrix} v_1' & y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n', \tilde{v}_2' = u_0'' - v_1'$.
Let $y = \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top = \Phi(x)$. We need to show that $\tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(Mv_n - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2)$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$, $v_n, \tilde{z}_1, \tilde{z}_2, u_0 \in \mathcal{U}$. It follows from (5.24) that $u_0 = u_0''$, $\tilde{z}_1 = \tilde{z}_1'$,$\tilde{z}_2 = \tilde{z}_2'$, $y_3' = y_3''$, $\tilde{v}_2 = -g^*(\tilde{z}_2')(1/g^*(\tilde{z}_1')(Mv_n' - f^*(\tilde{z}_1')) - u_0'') - f^*(\tilde{z}_2')$, then $\|u_0\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$.

Since $v_1' = u_0'' - \tilde{v}_2'$, then by Proposition 5.5 (5.22) there exist $v' \in \mathcal{U}$ such that $\begin{pmatrix} v_1' & y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n'$. It follows that $y_3'' = Nv_n' = NM^{-1}v_1'$. Now let $v_1 = v_1'$, and note that $y_3' = y_3'' = NM^{-1}v_1' = NM^{-1}v_1$, then there exists $v_n = v_n'$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = Mv_n$, $v \in \mathcal{U}$ and M is a bounded operator it follows that $\|v_1\| < \infty$. Using Lemma 5.4 (second statement) as $\tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(Mv_n - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2) = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2)$ and since $\|u_0\|, \|v_1\| < \infty$ then $\|\tilde{v}_2\| < \infty$. Then:

$$\begin{aligned} \tilde{v}_2 &= -g^*(\tilde{z}_2')(1/g^*(\tilde{z}_1')(v_1' - f^*(\tilde{z}_1')) - u_0'') - f^*(\tilde{z}_2'), \\ &= -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2), \end{aligned}$$

and hence:

$$y = \begin{pmatrix} u_0 \\ -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(Mv_n - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix}$$

$$= \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}.$$

as required.

Next, to prove that Φ is surjective, so that if $y \in \mathcal{G}_{P_3'}$ then there exists $x \in \mathcal{G}_{P_3}$ such that $\Phi(x) = y$, choose an element $y = \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}$ where $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)$ $(v_1 - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2), v_n \in \mathcal{U}$. Letting

$$x = \begin{pmatrix} u_0 \\ u_0 - f^*(\tilde{z}_1) - g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix},$$

we need to show that $x \in \mathcal{G}_{P_3}$ i.e. $v_n, \tilde{z}_1, \tilde{z}_2, \tilde{v}_2' = u_0 - f^*(\tilde{z}_1) - g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)) \in \mathcal{U}$, $\tilde{v}_2' = u_0 - v_1$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = f^*(\tilde{z}_1) + g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2))$ then $\tilde{v}_2' = u_0 - v_1$, Since $\|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$ then $\tilde{z}_1, \tilde{z}_2 \in \mathcal{U}$. Using Assumption 5.1, since $g^*, \frac{1}{g^*}$ is a bounded function, then $\|g^*(\tilde{z}_1)\|, \|\frac{1}{g^*(\tilde{z}_2)}\| < \infty$, also using Assumption 5.2, since f^* is a bounded function, then $\|f^*(\tilde{z}_1)\|, \|f^*(\tilde{z}_2)\| < \infty$ and since $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$, it follows that:

$$\|\tilde{v}_2'\| \leq \|u_0\| + \|f^*(\tilde{z}_1)\| + \|g^*(\tilde{z}_1)\|(\|u_0 + \|1/g^*(\tilde{z}_2)\|(\|f^*(\tilde{z}_2)\| + \|\tilde{v}_2\|)) < \infty.$$

Then by Proposition 5.5 (5.23) there exists $v_n \in \mathcal{U}$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Hence $x \in \mathcal{G}_{P_3}$ is such that:

$$\Phi \begin{pmatrix} u_0 \\ u_0 - f^*(\tilde{z}_1) - g^*(\tilde{z}_1)(u_0 - 1/g^*(\tilde{z}_2)(f^*(\tilde{z}_2) + \tilde{v}_2)) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix}. \tag{5.25}$$

as required. □

Using the previous results, a bound on the gap between P_3 and P_3' is given using the following theorem

Theorem 5.7. *Let $\tilde{P}$ be unstable, let f^* satisfy Assumption 5.1 and let g^* satisfy Assumption 5.2, for the systems P_3 and P_3' given by Figure 5.11 and (5.15) and (5.16),* *respectively. Then a bound on the gap between P_3 and P_3' is*

$$\vec{\delta}(P_3, P_3') \leq \max \left| \mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B} \right|. \tag{5.26}$$

Proof. Since $\tilde{P}$ is unstable, f^* satisfies Assumption 5.1 and g^* satisfies Assumption 5.2, then using Proposition 5.6 there exists a surjective map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by equation 5.24. The gap between P_3 and P_3' is given as:

$$\begin{aligned}
\vec{\delta}(P_3, P_3') \quad &\leq \sup_{x \in \mathcal{G}_{P_3} \setminus \{0\}} \frac{\|(\Phi - I)x\|}{\|x\|}, \\
&\leq \sup_{\| \ u_0 \quad u_0 - Mv_n \quad \tilde{z}_1 \quad \tilde{z}_2 \ \| \neq 0} \\
&\frac{\| - g^*(\tilde{z}_2)(1/g^*(\tilde{z}_1)(Mv_n - f^*(\tilde{z}_1)) - u_0) - f^*(\tilde{z}_2) - (u_0 - Mv_n)\|}{\|u_0, u_0 - Mv_n, \tilde{z}_1, \tilde{z}_2\|}, \\
&\leq \sup_{\| \ u_0 \quad u_0 - Mv_n \quad \tilde{z}_1 \quad \tilde{z}_2 \ \| \neq 0} \\
&\frac{\left\| \left(g^*(\tilde{z}_2) - 1\right) u_0 + \left(1 - g^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_1)}\right) Mv_n + \left(g^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_1)} f^*(\tilde{z}_1) - f^*(\tilde{z}_2)\right) \right\|}{\|u_0, u_0 - Mv_n, \tilde{z}_1, \tilde{z}_2\|},
\end{aligned}$$

Replacing $u_0 - Mv_n$, which is bounded only for $\|u_0\| < \infty, \|M\| < \infty$, with $s \in \mathcal{U}$ produces:

$$\begin{aligned}
&\vec{\delta}(P_3, P_3') \leq \sup_{\| \ u_0 \quad s \quad \tilde{z}_1 \quad \tilde{z}_2 \ \| \neq 0} \\
&\frac{\left\| \left(g^*(\tilde{z}_2) - 1\right) u_0 + \left(1 - g^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_1)}\right) s + \left(g^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_1)} f^*(\tilde{z}_1) - f^*(\tilde{z}_2)\right) \right\|}{\|u_0, s, \tilde{z}_1, \tilde{z}_2\|}
\end{aligned} \tag{5.27}$$

Using Assumptions 5.1 and 5.2 we have

$$\|f^*(\tilde{z}_1)\|, \|f^*(\tilde{z}_2)\| \leq \mathbf{B}, \left\| \frac{1}{g^*(\tilde{z}_1)} \right\| \leq \frac{1}{\varepsilon}, \|g^*(\tilde{z}_2)\| \leq \mathbf{D},$$

and using the above inequality in (5.27), a bound for the gap is

$$\vec{\delta}(P_3, P_3') \leq \max \left| \mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B} \right|. \tag{5.28}$$

as required.

□

This theorem states that a bound on $\vec{\delta}(P_3, P_3')$ depends on the upper bound on the nonlinear input part of the controller and how exact the inversion of the plant nonlinearity is, within the nonlinear part of the controller.

Then according to the following proposition, robust stability is preserved for the system shown in Figure 5.9.

Proposition 5.8. *Consider the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 5.7. Let $f^*(z)$ and $g^*(z)$ satisfy Assumptions 5.1, 5.2, respectively. Then $[P_1', P_2', P_3']$ has a robust stability margin.*

Proof. Let $f^*(z)$ and $g^*(z)$ satisfy Assumptions 5.1 and 5.2, respectively, then by Lemma 5.4, and Proposition 5.5 for the systems P_3 and P_3' given by Figure 5.11 and equations (5.15) and (5.16), respectively, the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ are given by (5.22) and (5.23), respectively. Using Proposition 5.6, then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (5.24). This leads to the presence of a finite gap value between the linear and nonlinear configurations of this system given by (5.26). Then the system $[P_1', P_2', P_3']$ shown in Figure 5.7 has a robust stability margin.

□

The main result of Theorem 5.3 follows directly from Theorem 5.9 which we establish next.

Theorem 5.9. *Consider the nonlinear closed loop system shown in Figure 5.2 and given by (5.5)-(5.10). Suppose $f^*(z)$ and $g^*(z)$ satisfy Assumptions 5.1 and 5.2, respectively. Then this system has a robust stability margin b_{P_1,C_1} which satisfies the inequality*

$$b_{P_1,C_1} \geq \|Q\|^{-1}. \tag{5.29}$$

Proof. Let $\frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 5.7, let $\frac{1}{\|\Pi_{(3)}'\|}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 5.5, finally let $b_{P_1,C_1} = \frac{1}{\|\Pi_{\tilde{P}//C_{Linear}}\|}$ be a stability margin for the system shown in Figure

5.2. Then

$$\begin{aligned}
\|Q\| = \|\Pi_{(3)}\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}\right\|}{\|u_0',x_0',y_0'\|}, \\
&= \sup_{\|u_0,d_1,d_2,d_3,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0 & d_1 & d_2 & d_3 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,d_1,d_2,d_3,y_0,x_0\|}, \\
&\geq \sup_{\|u_0,0,0,0,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_{(3)}\begin{pmatrix} u_0 & 0 & 0 & 0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,0,0,0,y_0,x_0\|} = \|\Pi'_{(3)}\|, \\
\|\Pi'_{(3)}\| &= \sup_{\|u_0,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_3\begin{pmatrix} u_0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,y_0,x_0\|}, \\
&\geq \sup_{\|u_0,y_0,0\|\neq 0} \frac{\left\|\Pi_{\tilde{P}//C_{Linear}}\begin{pmatrix} u_0 & y_0 & 0 \end{pmatrix}^\top\right\|}{\|u_0,y_0,0\|}, \\
&= \|\Pi_{\tilde{P}//C_{Linear}}\|.
\end{aligned}$$

This leads us to

$$b_{P_1,C_1} = \frac{1}{\|\Pi_{\tilde{P}//C_{Linear}}\|} \geq \frac{1}{\|\Pi'_{(3)}\|} \geq \frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}.$$

Therefore the existence of a stability margin for the system shown in Figure 5.7 guarantees the existence of a stability margin for the system $[P_1, C_1]$ shown in Figure 5.2. Also, since $f^*(z)$ and $g^*(z)$ satisfy Assumptions 5.1 and 5.2, respectively, then by Proposition 5.8, the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 5.7, has a robust stability margin. This leads to the conclusion that the system $[P_1, C_1]$ given by Figure 5.2 and (5.5)-(5.10) also has a robust stability margin. As required.

□

Based on Theorems 5.7 and 5.9 we can write the following corollary:

Corollary 5.10. *Consider the nonlinear closed loop system shown in Figure 5.2 and given by (5.5)-(5.10). Suppose $f^*(z)$ and $g^*(z)$ satisfy Assumptions 5.1 and 5.2, respectively. Then this system is stable if*

$$\left|\mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B}\right| < \|Q\|^{-1}.$$

Proof. Using Theorem 5.7 inequality (5.26), since:

$$\vec{\delta}(P_3, P_3') \leq \left|\mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B}\right|,$$

and using Theorem 5.9 inequality (5.29), since:

$$b_{P_1,C_1} \geq \|Q\|^{-1},$$

It follows that if

$$\left|\mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B}\right| < \|Q\|^{-1}.$$

we have:

$$\vec{\delta}(P_3, P_3') \leq \left|\mathbf{D} - 1, 1 - \frac{\mathbf{D}}{\varepsilon}, \frac{\mathbf{D}}{\varepsilon}\mathbf{B} - \mathbf{B}\right| < \|Q\|^{-1} \leq b_{P_1,C_1},$$

then $\vec{\delta}(P_3, P_3') < b_{P_1,C_1}$ and the conditions hold from Theorem 2.11, hence stability.

as required. □

5.3 Robustness Analysis for Nonlinear Systems with Stable and Unstable Nonlinear Parts Using the Gap Metric

This section will consider an affine nonlinear system which has a stable and an unstable nonlinear components and will carry out a stability analysis for this system. The reason for considering this case (as was mentioned in Section 5.1) is that exact feedback linearization usually cancels all the nonlinearity of the plant, so it may destroy inherently stabilizing nonlinearities that can be used to stabilize the plant. A motivating example (from Freeman and Kokotović (2008)) which illustrates this point is given next.

Example 5.1. *Consider the system:*

$$\begin{aligned} \dot{x} &= -x^3 + u + \omega x \\ y &= x, \end{aligned}$$

where u is an unconstrained control input, ω is a disturbance which takes values in the interval $[-1, 1]$. A robustly stabilizing feedback controller for this system is

$$u = x^3 - 2x.$$

This control law is a result of feedback linearization, however, it is a bad choice since it wastefully cancels a beneficial nonlinearity x^3. Furthermore, considering the other *uncertainties in the system the term x^3 in this control law adds a positive feedback which increases the risk of instability in the control system.*

The need for a control law that can classify the nonlinearity in the system to stable and unstable components so that the feedback controller cancels only the unstable nonlinear component of the plant motivated Freeman and Kokotovic (Freeman and Kokotović (2008)) to introduce an "Inverse Optimal" design in which they replaced feedback linearization by robust backstepping, and achieved a form of worst case optimality, however, using the backstepping in this approach restricts the design. This motivates the work done in this section.

5.3.1 Nonlinear Systems with Stable and Unstable Nonlinear Parts

In this section we again consider the nonlinear system described previously in Section 4.2, equations (4.5)-(4.6), this form is given by:

$$\begin{aligned} P_1 \quad : \quad & \mathcal{L}^{n+1}_{\infty,e} \to \mathcal{L}^{n}_{\infty,e} : (u_1, z_1) \mapsto (y_1), \\ & \dot{x}^* = Ax^* + B(f^*(z_1) + g^*(z_1)u_1), \\ & y_1 = (y_{11}, \ldots, y_{1n}) = x^*, \\ & z_1 = (z_{11}, \ldots, z_{1n}) = x^*, \end{aligned}$$

where the feedback input y_1 is considered as an input, z_1, to the nonlinear parts of the plant $g^*(z_1)$ and $f^*(z_1)$, $A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & \ldots & & a_{n-1} & a_n \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$,

$f^*(x^*)$ and $g^*(x^*)$ represent the nonlinear part of the transformed system, $f^*(x^*) = L_f^n h(T^{-1}(x^*))$, $g^*(x^*) = L_g L_f^{n-1} h(T^{-1}(x^*))$. In this section $f^*(x^*), g^*(x^*)$ both have a stable and an unstable components within them. So, we divide $f^*(x^*)$ and $g^*(x^*)$ into stable and unstable components, and cancel the latter.

To achieve this last objective, a controller of the form $u_1 = l(x^*, v)$ is selected such that it stabilizes the system $\dot{x}^* = Ax^* + B(f^*(x^*) + g^*(x^*)u_1), y_1 = x^*$ and cancels only the unstable components of $f^*(x^*), g^*(x^*)$.

As will be seen this controller action requires a vector:

$$c = (c_1, \ldots, c_n)^\top,$$

to be chosen such that $A_c = A - Bc^\top$ is stable. We introduce the stable components of $f^*(x^*)$ and $g^*(x^*)$, which we will call f_s^* and g_s^*, according to the following definition:

Definition 5.11. Given that $A_c = A - Bc^\top$ is stable, a function $g_s^* : R^n \to R$ and a function $f_s^* : R^n \to R$ are called stable if the mapping $u \mapsto y$, with

$$\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)u), y = x^*$$

is stable.

The real function $l(x^*, v) = a(x^*) + b(x^*)v$ is chosen such that this function will cancel only the unstable nonlinear terms of $f^*(x^*)$ and $g^*(x^*)$ while stabilizing the linear part of the plant. To do this, it follows that $Ax^* + B(f^*(x^*) + g^*(x^*)l(x^*, v)) = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v)$ for some input v. This leads to the conclusion that $l(x^*, v)$ has the form:

$$l(x^*, v) = \frac{-c^\top x^* - f^*(x^*) + f_s^*(x^*) + g_s^*(x^*)v}{g^*(x^*)}. \tag{5.30}$$

Let

$$f_u^*(x^*) = \frac{(f^*(x^*) - f_s^*(x^*))}{g_s^*(x^*)}, \tag{5.31}$$

and

$$g_u^*(x^*) = \frac{g^*(x^*)}{g_s^*(x^*)}, \tag{5.32}$$

then $l(x^*, v) = \frac{-c^\top x^*}{g^*(x^*)} - \frac{1}{g_u^*(x^*)} f_u^*(x^*) + \frac{1}{g_u^*(x^*)} v$. Hence, a feedback linearizing controller which generates the term $l(x^*, v)$ is given as:

$$\begin{aligned} C_1 \quad : \quad & \mathcal{L}_{\infty,e}^{2n} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\ & u_2 = -l(y_2, z_2) = \frac{1}{g^*(z_2)} C_{Linear} y_2 + \frac{1}{g_u^*(z_2)} f_u^*(z_2) + \frac{1}{g_u^*(z_2)} \tilde{C} y_2, \\ & z_2 = (z_{21}, \ldots, z_{2n}) = -y_2, \end{aligned}$$

where $l(x^*, v) = l(y_2, z_2)$, $y_2 = (y_{21}, \ldots, y_{2n})$, C_{Linear} is the linear stabilizing part of the controller C_1,

$$\begin{aligned} C_{Linear} \quad : \quad & \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_2 \\ & v_2 = c^\top y_2. \end{aligned}$$

and $\tilde{C}$ is the linear part which implements the feedback loop which is needed to generate v, and is given by

$$\begin{aligned} \tilde{C} \quad : \quad & \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v \\ & v = -\tilde{c}^\top y_2, \end{aligned}$$

where

$$\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)^\top,$$

The feedback control connection for this system is shown in Figure 5.12.

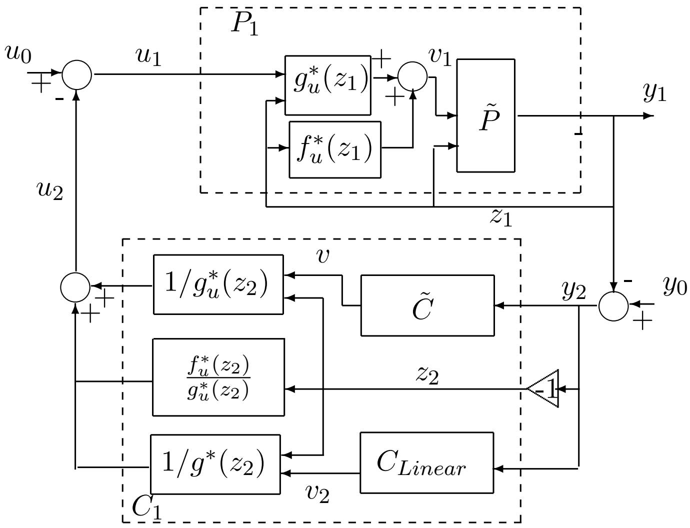

FIGURE 5.12: Nonlinear control system with stable/unstable plant nonlinearity

Compared to the case in Section 5.2 this system has a new $\tilde{P}$, which includes the stable nonlinear part as well as the linear potentially unstable part of the plant P_1. This is given by:

$$\begin{aligned} \tilde{P} \quad : \quad & \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : v_1 \mapsto y_1, \\ & \dot{x}^* = Ax^* + B(f_s^*(x^*) + g_s^*(x^*)v_1), && (5.33) \\ & y_1 = x^*, && (5.34) \end{aligned}$$

It follows that $\tilde{P}$ can be written using nonlinear coprime factorization as $\tilde{P} = NM^{-1}$ where N and M satisfy $L(M,N)^\top = I$, and L is a causal stable mapping $L : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ and here

$$\begin{aligned} M \quad &: \quad v_n \mapsto v_1, \\ \dot{x}^* \quad &= \quad A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), && (5.35) \\ v_1 \quad &= \quad -\frac{1}{g_s^*(x^*)} C_{Linear} x^* + v_n, , && (5.36) \end{aligned}$$

and

$$\begin{aligned} N &: v_n \mapsto y_1, \\ \dot{x}^* &= A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), \qquad (5.37) \\ y_1 &= x^*, \qquad (5.38) \end{aligned}$$

then $\tilde{P}$ is stabilized by a nonlinear controller of the form $v_1 = v_n + L_1 x^*$, where $L_1 x^* = -\frac{1}{g_s^*(x^*)} C_{Linear} x^*$.

We can write P_1 as

$$\begin{aligned} P_1 &: \mathcal{L}_{\infty,e}^{n+1} \to \mathcal{L}_{\infty,e}^{n} : (u_1, z_1) \mapsto (y_1), \\ & y_1 = \tilde{P}(f_u^*(z_1) + g_u^*(z_1)u_1) \qquad (5.39) \\ & z_1 = (z_{11}, \ldots, z_{1n}) = y_1. \end{aligned}$$

For the system shown in Figure 5.12 the closed loop equations can be written as:

$$\begin{aligned} u_0 &= u_1 + u_2, \qquad (5.40) \\ y_0 &= y_1 + y_2, \qquad (5.41) \\ v_1 &= f_u^*(z_1) + g_u^*(z_1)u_1, \qquad (5.42) \\ v_2 &= C_{Linear} y_2, \qquad (5.43) \\ y_1 &= \tilde{P} v_1, \qquad (5.44) \\ u_2 &= \frac{1}{g^*(z_2)} C_{Linear} y_2 + \frac{1}{g_u^*(z_2)} f_u^*(z_2) + \frac{1}{g_u^*(z_2)} \tilde{C} y_2. \qquad (5.45) \end{aligned}$$

We also note that the system shown in Figure 5.12 is not equivalent to the closed loop system $[P_1, C_1]$. This is due to the presence of the two signals z_1, z_2 which feed the signals $y_1, -y_2$ to the blocks P_1, C_1, respectively. If $z_1, z_2 \neq 0$, then the system in Figure 5.12 does not correspond exactly to the system shown in Figure 2.1. Hence, we cannot apply Theorem 2.11 to find stability conditions for this system.

However, the linear configuration of this system, which will be needed to find stability conditions for this system, does correspond to the system shown in Figure 2.1, since replacing $g_u^*(z_1), g_s^*(z_1)$ by the linear operator $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and replacing $\frac{1}{g_u^*(z_2)}, \frac{1}{g^*(z_2)}$ by the linear operator $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, and setting $f_u^*(z_1) = f_s^*(z_1) = f_u^*(z_2) = 0$ will result in the linear system $[P_{Linear}, C_{Linear} + \tilde{C}]$ shown in Figure 5.13.

This system is equivalent to the linear system shown in Figure 5.14.

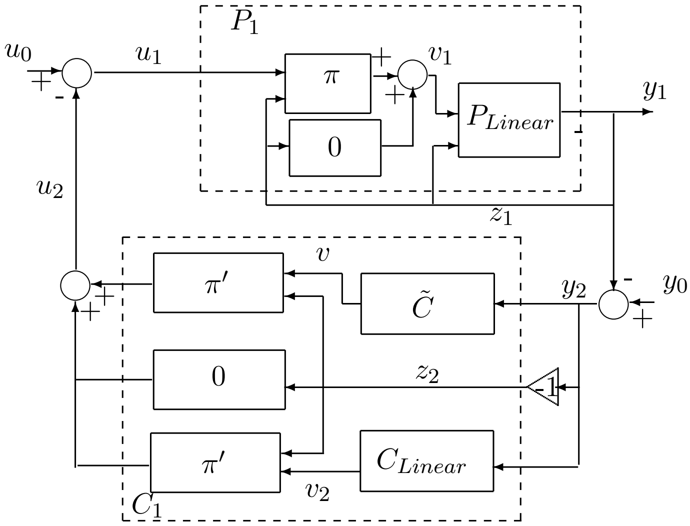

FIGURE 5.13: Linear configuration of a nonlinear system with stable/unstable plant nonlinearity

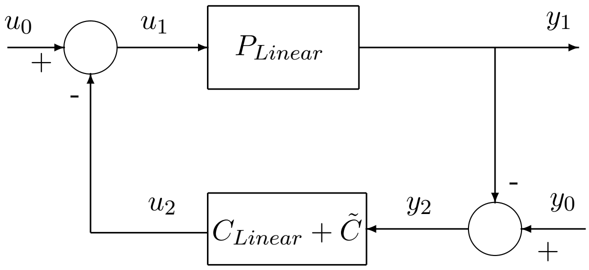

FIGURE 5.14: Equivalent linear configuration of a nonlinear system with stable/unstable plant nonlinearity

Here P_{Linear} is given by

$$P_{Linear} \;:\; \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u_1 \mapsto y_1,$$

$$\dot{x}^* = Ax^* + Bu_1, \tag{5.46}$$

$$y_1 = x^*, \tag{5.47}$$

The closed loop system $[P_{Linear}, C_{Linear}+\tilde{C}]$ has a closed loop operator $\Pi_{P_{linear}//(C_{Linear}+\tilde{C})}$, which will be used to find stability conditions for the system shown in Figure 5.12.

5.3.2 Gap Metric for Nonlinear Systems with Stable and Unstable Nonlinear Part

This section undertakes robustness stability analysis for the affine nonlinear system shown in Figure 5.12. The analysis carried out in this section will follow the approach given in Chapter 4. However, in this analysis $\tilde{P}$ is a potentially unstable nonlinear plant with a stable nonlinear part.

The following assumptions on the forms of g_u^* and f_u^* are required in subsequent analysis:

Assumption 5.12. Let $g_u^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the following condition:

$$\exists \epsilon > 0, \quad \exists \mathbf{D}_u < \infty \text{ and } \quad \epsilon \leq |g_u^*(x)| \leq \mathbf{D}_u \quad \forall x \in \mathbb{R}^n.$$

Assumption 5.13. Let $f_u^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the following condition:

$$\exists \mathbf{B}_u < \infty \text{ and } \quad |f_u^*(x)| \leq \mathbf{B}_u \quad \forall x \in \mathbb{R}^n.$$

The gap metric framework is applied to the system shown in Figure 5.12 to result in the following theorem:

Theorem 5.14. *Consider the nonlinear closed loop system shown in Figure 5.12 and given by (5.40)-(5.45). Let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13 and let g^* satisfy Assumption 5.2. Then this system has a robust stability margin.*

Parallel to the proof of Theorem 4.6, this proof also requires results that are developed subsequently in this section. This analysis will also consider the triple system configuration shown in Figure 4.5 and apply the 'network' result in (Theorem 2.13) to find a stability condition for the nonlinear system shown in Figure 5.12.

The route taken is as follows: Since the presence of nonlinear elements in multiple blocks in the system shown in Figure 5.12 leads to significant conservatism, and to apply Theorem 2.13 to this system, a new system configuration shown in Figure 5.15 is used. In this configuration the unstable nonlinear component of the plant P_1 and the nonlinear component of the controller C_1 are considered to be included along with the nominal plant $\tilde{P}$ in the block P_3' and an external input x_0 is added to the system. Also the feedback input $x_0 - y_1$ is considered as an input, z_1, to the nonlinear components of the plant $f_u^*(z_1), g_u^*(z_1)$, $f_s^*(z_1)$ and $g_s^*(z_1)$, and the feedback input $-y_2$ is considered as an input z_2 to the nonlinear components $\frac{1}{g_u^*(z_2)} f_u^*(z_2)$, $\frac{1}{g^*(z_2)}$ and $\frac{1}{g_u^*(z_2)}$.

The nominal system configuration is taken to comprise the system components P_1, P_2, P_3 with the nonlinearity $g_u^*(z_1)$ being replaced by the linear operator $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and the nonlinearity $\frac{1}{g^*(z_2)}$ being replaced by $\frac{1}{g_s^*(z_2)}$, also $\frac{1}{g_u^*(z_2)}$ being replaced by the linear operator $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, and setting $f_u^*(z_1) = f_u^*(z_2) = 0$. This configuration is shown in Figure 5.16.

To apply Theorem 2.13 we must put the real and the nominal nonlinear systems shown in Figures 5.15 and 5.16 in a form comparable to that given in Figure 4.5. In order to do this we consider three signal spaces $\mathcal{U} = \mathcal{L}^n_{\infty,e}, \mathcal{X} = \mathcal{L}^n_{\infty,e}$ and $\mathcal{Y} = \mathcal{L}^n_{\infty,e}$, together with the following augmented signals; let $\hat{v}_2 = -v_2$ and let $u_1' = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top$

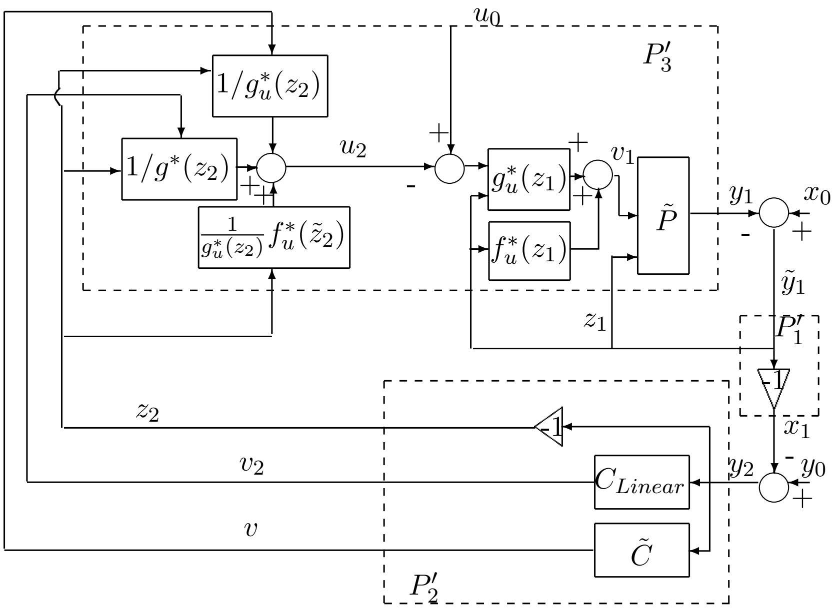

FIGURE 5.15: Nonlinear system with stable and unstable nonlinearity configuration

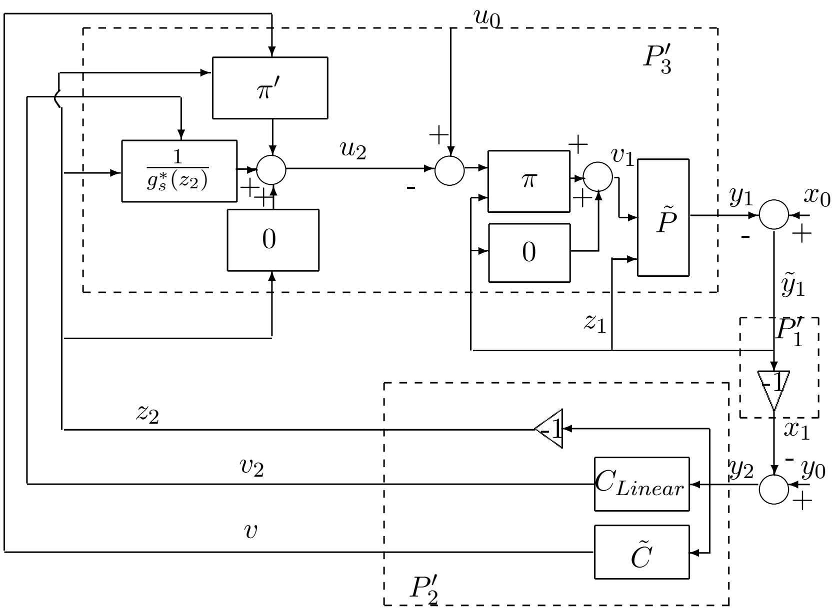

FIGURE 5.16: Nominal configuration of an affine system with only stable nonlinear components

and let $u_2' = \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top$ also let the external input u_0 be changed to $u_0' = \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top$, where $d_2 = (d_{21}, \ldots, d_{2n})$ and $d_3 = (d_{31}, \ldots, d_{3n})$, also let $u_3' = u_0' - u_2' - u_1' = \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top - \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top - \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top = \begin{pmatrix} u_0 & d_0 + v & d_1 - \hat{v}_2 & d_2 - z_1 & d_3 - z_2 \end{pmatrix}^\top$, let $\tilde{v} = d_0 + v$, $\tilde{v}_2 = d_1 - \hat{v}_2$, $\tilde{z}_1 = d_2 - z_1$, $\tilde{z}_2 = d_3 - z_2$ then $u_3' = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$. Also let $x_0' = y_0$, $y_0' = x_0$, $y_3' = y_1$, $x_1' = x_1$, $x_2' = y_2$ and finally $y_1' = y_0' - y_3' = x_0 - y_1$. The resulting system is shown in Figure 5.17.

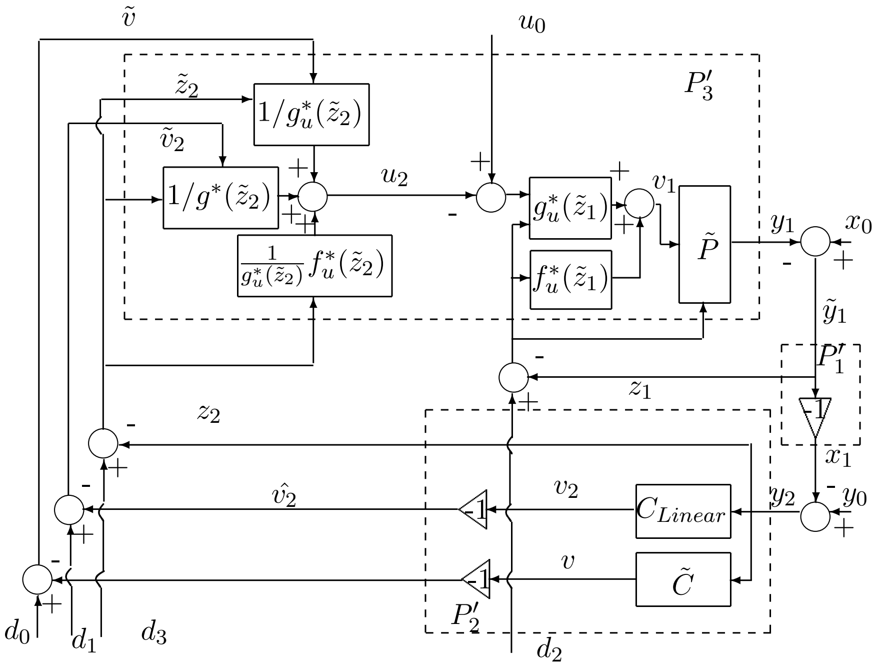

FIGURE 5.17: Augmented nonlinear system with stable and unstable nonlinear part

The corresponding nominal system is shown in Figure 5.18.

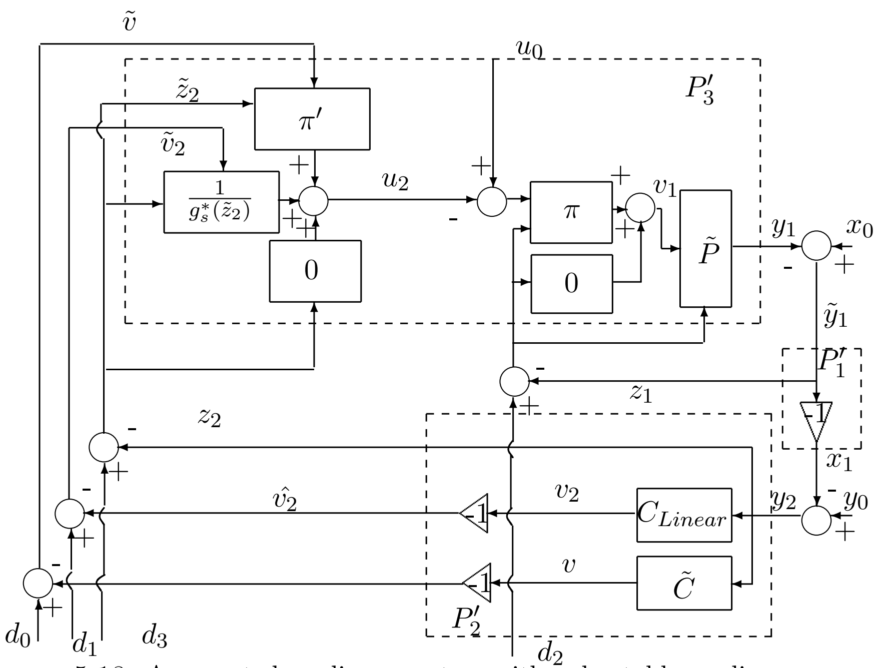

FIGURE 5.18: Augmented nonlinear system with only stable nonlinear component

Note from the two systems shown in Figure 5.17 and Figure 5.18 that $P_1 = P_1'$ and $P_2 = P_2'$.

These configurations correspond to those of Figures 5.15 and 5.16, respectively, except for the presence of d_0, d_1, d_2 and d_3. Figures 5.17 and 5.18 correspond exactly to the forms shown in Figures 5.19 and 5.20, respectively, which in turn have identical structure to that of Figure 4.5. Hence, the stability condition will be applied to the systems of Figures 5.19, 5.20.

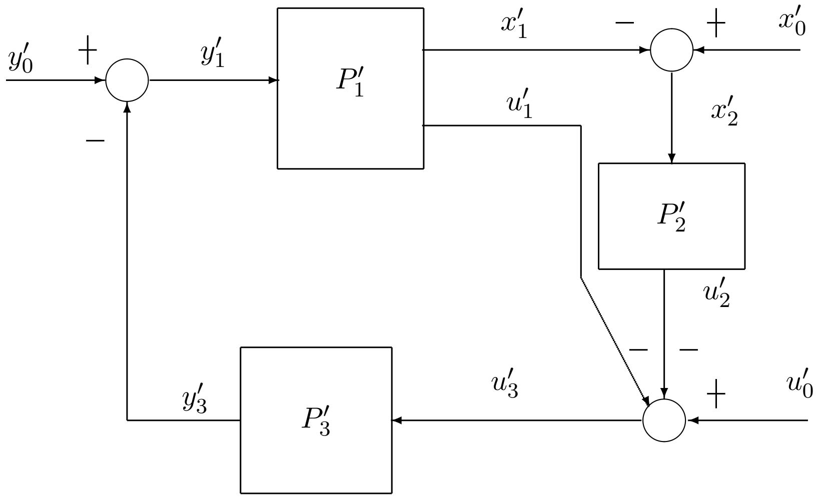

FIGURE 5.19: Nonlinear configuration of affine system with unstable nonlinearity

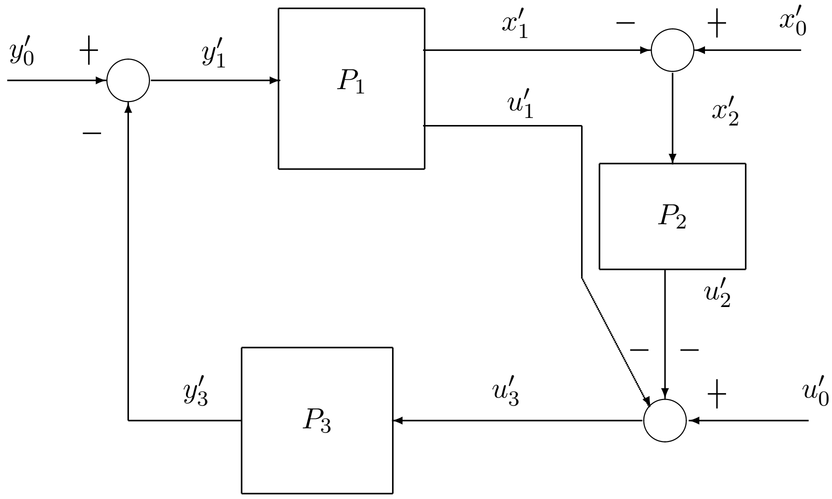

FIGURE 5.20: Nominal configuration of affine system with only stable nonlinearity

Since $P_1 = P_1'$ and $P_2 = P_2'$, then

$$\vec{\delta}(P_1, P_1') = 0, \vec{\delta}(P_2, P_2') = 0.$$

Using Theorem 2.13, the robust stability condition is given as:

$$\sum_{i=1}^{3} \vec{\delta}(P_i, P_i') < \|\Pi_{(i)}\|^{-1},$$

For our system this condition becomes:

$$\vec{\delta}(P_3, P_3') < \|\Pi_{(3)}\|^{-1}. \tag{5.48}$$

Then the gap metric measures the difference between the nominal plant $P_3 : u_3' \mapsto y_3', y_3' = \tilde{P}\pi(u_0 - (\frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 + \pi'(\tilde{v}, \tilde{z}_2)), \tilde{z}_1) = \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v})$ and the perturbed plant $P_3' : u_3' \mapsto y_3', y_3' = \tilde{P}\left(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)\left(u_0 - \left(\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)\right)\right)$. The plants P_3 and P_3' are shown in Figure 5.21,

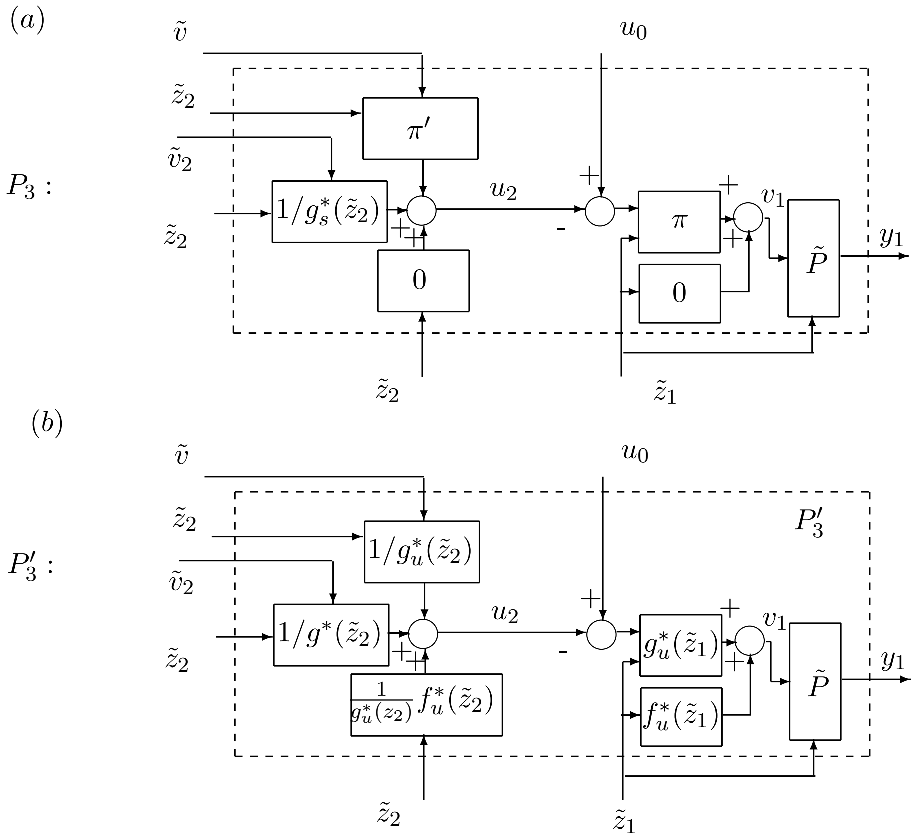

FIGURE 5.21: Nonlinear plant mapping:(a) unperturbed, (b) perturbed

Before providing a complete description of the operators P_1', P_2' and P_3' and P_1, P_2 and P_3 shown in Figures 5.19 and 5.20, respectively, we also briefly state the motivation for the proceeding manipulations (as was done in Chapter 4).

The stability condition (5.48) can be related to the original system configuration shown in Figure 5.12 as follows: It will be shown later in the proof of Theorem 5.14 that the stability margin for the system shown in Figure 5.17 is less than or equal to the stability margin corresponding to the system shown in Figure 5.15 which in turn is less than or equal to the stability margin corresponding to the original system shown in Figure 5.12. This is because for each pair the latter is a special case of the former.

The closed loop operators P_1', P_2' and P_3' shown in Figure 5.19 are given by

$$\begin{aligned} P_1' &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{2n} : y_1' \mapsto (x_1', u_1'), x_1' = -y_1', \\ & u_1' = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1', \end{aligned}$$

where $y_1' = \tilde{y}_1$, and :

$$\begin{aligned} P_2' &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{n+2} : x_2' \mapsto u_2', u_2' = \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, \\ & z_2 = x_2', \hat{v}_2 = -C_{Linear}x_2', v = \tilde{C}x_2', \end{aligned}$$

and the block P_3' is given as:

$$\begin{aligned} P_3' &: \mathcal{L}_{\infty,e}^{3n+3} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\ y_3' &= P_3'u_3', \\ &= \tilde{P}v_1, \\ &= \tilde{P}\left(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)\left(u_0 - \left(\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)\right)\right) \end{aligned} \tag{5.49}$$

The configuration shown in Figure 5.20 comprises of the subsystems:

$$\begin{aligned} P_1 &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{2n} : y_1' \mapsto (x_1', u_1') \\ & x_1' = -y_1', u_1' = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1', \end{aligned}$$

$$\begin{aligned} P_2 &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{n+2} : \quad x_2' \mapsto u_2' \\ & u_2' = \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, z_2 = x_2', \hat{v}_2 = -C_{linear}x_2', v = \tilde{C}x_2', \end{aligned}$$

and

$$\begin{aligned} P_3 &: \mathcal{L}_{\infty,e}^{3n+3} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\ y_3' &= P_3u_3', \\ &= \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v}), \end{aligned} \tag{5.50}$$

In a similar manner to the approach taken in Chapter 4, to apply Theorem 2.13 to this system, we must satisfy inequality (5.48). In the following two subsections, the two sides of this inequality will be evaluated, namely the linear gain $\|\Pi_{(3)}\|$ and the gap value $\vec{\delta}(P_3, P_3')$.

5.3.3 Finding $\|\Pi_{(3)}\|$ for an Affine Nonlinear System with Stable and Unstable Nonlinearity

Starting with the RHS of inequality (5.48), the parallel projection $\Pi_{(3)}$ is the mapping from the external signals (u_0', x_0', y_0') to the internal signals $(u_3', 0, y_3')$ in the configuration shown in Figure 5.20 with $\frac{1}{g_s^*(\tilde{z}_2)}$ replaced by π and $\tilde{P}$ replaced by P_{Linear}.

Note that in this linear configuration of the system, shown in details in Figure 5.18, the signal $\tilde{v}_2$ is dependent on $\tilde{v}$ then for this linear configuration of the system let $\tilde{v}_c = \tilde{v} + \tilde{v}_2$ we can write u_3' as $u_3' = \begin{pmatrix} u_0 & \tilde{v}_c & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$, also let $P_L = P_{Linear}$ and $C_L = C_{Linear}$, then to find the linear gain $\|\Pi_{(3)}\|$ consider the relation:

$$\begin{aligned} \begin{pmatrix} u_3' \\ 0 \\ y_3' \end{pmatrix} &= \Pi_{(3)} \begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}, \\ \|\Pi_{(3)}\| &= \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|u_3', 0, y_3'\|}{\|u_0', x_0', y_0'\|}. \end{aligned}$$

Then:

$$\begin{aligned} \|\Pi_{(3)}\| &= \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|u_3', y_3'\|}{\|u_0', x_0', y_0'\|}, \\ &= \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}. \end{aligned} \tag{5.51}$$

To find expressions for u_3' and $P_3 u_3'$ in terms of u_0', x_0' and y_0', we start with $P_3 u_3'$ as follows:

$$P_3 u_3' = P_L(u_0 - \tilde{v}_2 - \tilde{v}), \tag{5.52}$$

since the signals $\tilde{v}_2$ and $\tilde{v}$ in the linear configuration of the system shown in Figure 5.20 are both dependant on the y_2 signal and can be summed to produce $\tilde{v}_c = \tilde{v} + \tilde{v}_2$. Now we can write y_3' as

$$P_3 u_3' = P_L(u_0 - \tilde{v}_c). \tag{5.53}$$

We can find an expression for $\tilde{v}_c$ in terms of u'_0, x'_0, y'_0 as follows:

$$
\begin{aligned}
\tilde{v}_c \quad & = \tilde{v} + \tilde{v}_2, \\
& = d_0 + d_1 + v + v_2, \\
& = d_0 + d_1 + \tilde{C}x'_2 + C_L x'_2, \\
& = d_0 + d_1 + \tilde{C}(x'_0 - x'_1) + C_L(x'_0 - x'_1), \\
& = d_0 + d_1 + \tilde{C}(x'_0 - y'_1) + C_L(x'_0 - y'_1), \\
& = d_0 + d_1 + \tilde{C}(x'_0 + y'_0 - y'_3) + C_L(x'_0 + y'_0 - y'_3), \\
\tilde{v} + \tilde{v}_2 \quad & = d_0 + d_1 + \tilde{C}(x'_0 + y'_0 - P_L(u_0 - \tilde{v}_2 - \tilde{v})) + \\
& C_L(x'_0 + y'_0 - P_L(u_0 - \tilde{v}_2 - \tilde{v})), \\
& (I - C_L P_L - \tilde{C}P_L)\tilde{v} + (I - C_L P_L - \tilde{C}P_L)\tilde{v}_2 = d_0 + d_1 + \tilde{C}(x'_0 + y'_0 - P_L u_0) + \\
& C_L(x'_0 + y'_0 - P_L u_0), \\
& \tilde{v} + \tilde{v}_2 = (I - C_L P_L - \tilde{C}P_L)^{-1}(d_0 + d_1 + \tilde{C}(x'_0 + y'_0 - P_L u_0) + C_L(x'_0 + y'_0 - P_L u_0)), \\
& \tilde{v}_c = (I - C_L P_L - \tilde{C}P_L)^{-1}(-(C_L P_L + \tilde{C}P_L) \quad I \quad I \quad (C_L + \tilde{C}) \quad (C_L + \tilde{C})) * \\
& \begin{pmatrix} u_0 & d_0 & d_1 & x'_0 & y'_0 \end{pmatrix}^\top
\end{aligned}
\tag{5.54}
$$

Let $(I - C_L P_L - \tilde{C}P_L) = G$. Using (5.53), we have:

$$
\begin{aligned}
P_3 u'_3 \quad & = P_L u_0 - P_L \Big(G^{-1}(-(C_L P_L + \tilde{C}P_L) \quad I \quad I \quad (C_L + \tilde{C}) \quad (C_L + \tilde{C})) \Big) * \\
& \begin{pmatrix} u_0 & d_0 & d_1 & x'_0 & y'_0 \end{pmatrix}^\top \\
& = P_L(u_0 + (G^{-1}(C_L P_L + \tilde{C}P_L)u_0 - G^{-1}d_0 - G^{-1}d_1 - G^{-1}(C_L + \tilde{C})x'_0 - \\
& G^{-1}(C_L + \tilde{C})y'_0)), \\
& = P_L \begin{pmatrix} I + G^{-1}(C_L P_L + \tilde{C}P_L) & -G^{-1} & -G^{-1} & -G^{-1}(C_L + \tilde{C}) & -G^{-1}(C_L + \tilde{C}) \end{pmatrix} * \\
& \begin{pmatrix} u_0 & d_0 & d_1 & x'_0 & y'_0 \end{pmatrix}^\top .
\end{aligned}
$$

Since $I + (I - C_L P_L - \tilde{C}P_L)^{-1}(C_L P_L + \tilde{C}P_L) = (I - C_L P_L - \tilde{C}P_L)^{-1} = G^{-1}$ and let:

$$
c = \Big(P_L(G^{-1} \quad - G^{-1} \quad - G^{-1} \quad 0 \quad 0 \quad - G^{-1}(C_L + \tilde{C}) \quad - G^{-1}(C_L + \tilde{C})) \Big), \tag{5.55}
$$

then:

$$
P_3 u'_3 \quad = \quad c \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & x'_0 & y'_0 \end{pmatrix}^\top . \tag{5.56}
$$

Next we find u_3' as follows:

$$u_3' = \begin{pmatrix} u_0 \\ \tilde{v}_c \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v}_c \\ d_2 - z_1 \\ d_3 - z_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v}_c \\ d_2 - (y_0' - y_3') \\ d_3 - (x_0' - (y_0' - y_3') \end{pmatrix},$$

using (5.54) and (5.56) and on letting:
$\Lambda =$

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ -G^{-1}(C_L P_L + \tilde{C} P_L) & G^{-1} & G^{-1} & 0 & 0 & G^{-1}(C_L + \tilde{C}) & G^{-1}(C_L + \tilde{C}) \\ P_L G^{-1} & -P_L G^{-1} & -P_L G^{-1} & I & 0 & -P_L G^{-1}(C_L + \tilde{C}) & -(I + P_L G^{-1}(C_L + \tilde{C})) \\ P_L G^{-1} & -P_L G^{-1} & -P_L G^{-1} & 0 & I & -(I - P_L G^{-1}(C_L + \tilde{C})) & -(I + P_L(I - G^{-1}(C_L + \tilde{C}))) \end{pmatrix}$$

we have:

$$u_3' = \Lambda \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{pmatrix}^\top$$

using (4.31) and defining $Q = \begin{pmatrix} \Lambda \\ c \end{pmatrix}$ we have:

$$\begin{aligned} \|\Pi_{(3)}\| &= \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}, \\ &\leq \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0 & d_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}. \end{aligned}$$

Since $\begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top = u_0'$, then:

$$\begin{aligned} \|\Pi_{(3)}\| &\leq \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0' & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}, \\ &= \|Q\|. \end{aligned}$$

The components of $\|\Pi_{(3)}\|$ are the closed loop transfer functions of the linear system $[P_L, C_L + \tilde{C}]$, confirming that $\|\Pi_{(3)}\|$ is finite. Hence from (5.48) the gap between P_3 and P_3' must satisfy:

$$\vec{\delta}(P_3, P_3') < \frac{1}{\|Q\|}. \tag{5.57}$$

5.3.4 Finding the Gap Metric for a Nonlinear System with Stable and Unstable Nonlinearity

In this subsection the LHS, $\vec{\delta}(P_3, P_3')$, of the inequality (5.48) is considered. To find $\vec{\delta}(P_3, P_3')$ an analogous approach to that in Subsection 4.4.2 is developed. However, in this analysis $\tilde{P}$ is a potentially unstable nonlinear plant. First, the graphs for P_3 and P_3' are defined to be:

$$\mathcal{G}_{\tilde{P}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : y = \tilde{P}u, \|u\| < \infty, \|y\| < \infty \right\}, \tag{5.58}$$

$$\mathcal{G}_{P_3} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : y_3' = \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v}), \\ \left\| \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \right\| < \infty \end{array} \right\}. \tag{5.59}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - \\ (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))), \left\| \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \right\| < \infty \end{array} \right\} \tag{5.60}$$

To find a bound on the gap between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$, a surjective map Φ is required between these graphs. The following lemma is used to define Φ. First, consider the nonlinear part of the plant P_3' shown in Figure 5.21b. For this component the following lemma is used.

Lemma 5.15. *Let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2 and consider the nonlinear part of the plant P_3' shown in Figure 5.21b, where:*

$$v_1 = (f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))). \tag{5.61}$$

then:

$$\|\tilde{v}_2\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

and

$$\|v_1\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Proof. We will first prove that:

$$\|\tilde{v}_2\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty.$$

Using Assumption 5.13 since f_u^* is a bounded function, we have $\|f_u^*(\tilde{z}_1)\|, \|f_u^*(\tilde{z}_2)\| < \infty$, using Assumption 5.12 since g_u^* is a bounded function, we have $\|g_u^*(\tilde{z}_1)\|, \|1/g_u^*(\tilde{z}_2)\| < \infty$, and $\|\frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2)\| < \infty$, and using Assumption 5.2 since g^* is a bounded function, we have $\|1/g^*(\tilde{z}_2)\| < \infty$ and since $\|\tilde{v}_2\| < \infty$, $\|\tilde{v}\| < \infty$, $\|u_0\| < \infty$, then:

$$\begin{aligned}
\|v_1\| \quad &= \left\| f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})) \right\|, \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|g_u^*(\tilde{z}_1)\| \left(\|u_0\| + \left\| \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 \right\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) \right\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right\| \right) \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|g_u^*(\tilde{z}_1)\| \|u_0\| + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g^*(\tilde{z}_2)} \right\| \|\tilde{v}_2\| + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \\
&\|f_u^*(\tilde{z}_2)\| + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \|\tilde{v}\| \\
&< \infty.
\end{aligned}$$

as required. Next we will prove that:

$$\|v_1\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

By (5.61):

$$\tilde{v}_2 = -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right).$$

Since g_u^*, f_u^*, and g^* are all bounded functions, and since $\|v_1\| < \infty$, $\|\tilde{v}\| < \infty$, $\|u_0\| < \infty$, then:

$$\begin{aligned}
\|\tilde{v}_2\| \quad &= \quad \left\| -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right) \right\|, \\
&\leq \quad \|g^*(\tilde{z}_2)\| (\| \frac{1}{g_u^*(\tilde{z}_1)} \| (\|v_1\| + \|f_u^*(\tilde{z}_1)\|) + \|u_0\| + \| \frac{1}{g_u^*(\tilde{z}_2)} \| \|f_u^*(\tilde{z}_2)\| + \\
&\quad\quad \| \frac{1}{g_u^*(\tilde{z}_2)} \| \|\tilde{v}\|) \\
&< \quad \infty.
\end{aligned}$$

as required.

□

The graphs for P_3 and P_3' can be written using coprime factorization functions as shown in the following proposition:

Proposition 5.16. *Let $\tilde{P}$ be the unstable system given by (5.33)-(5.34), let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2, for the systems P_3 and P_3' given by Figure 5.21 and (5.50) and (5.49), respectively. Then the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ satisfy:*

$$\mathcal{G}_{P_3} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1), v_n, \tilde{z}_1, \tilde{z}_2, u_0, \tilde{v} \in \mathcal{U} \end{array} \right\}, \tag{5.62}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right), \\ v_n, \tilde{z}_1, \tilde{z}_2, u_0, \tilde{v} \in \mathcal{U} \end{array} \right\}. \quad (5.63)$$

where M, N given in equations (5.35-5.36) and (5.37-5.38), respectively, form a right coprime factorization of $\tilde{P}$ i.e. $\tilde{P} = NM^{-1}$.

Proof. To show that $\mathcal{G}_{P_3'}$ given in (5.63) is equivalent to that given in (5.60), denote the set given in (5.63) as $\mathcal{A}$.

First we prove that $\mathcal{A} \subset \mathcal{G}_{P_3'}$. Let $\begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{A}$, i.e $\begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)$ where $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}$. Since $u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}$ and $\tilde{z}_2 \in \mathcal{U}$ we have $\|u_0\| < \infty, \|\tilde{v}\| < \infty, \|\tilde{z}_1\| < \infty$ and $\|\tilde{z}_2\| < \infty$, respectively. Since $v_1 = Mv_n$, since M is as given in (5.35)-(5.36) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), v_1 = -\frac{1}{g_s^*(x^*)} C_{Linear} x^* + v_n$, since f_s^* and g_s^* are as defined in Definition 5.11 and A_c is stable, then M is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|v_1\| < \infty$. In the same way, since $y_3' = Nv_n$, since N is as given in (5.37)-(refnpe) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), y_1 = x^*$, since f_s^* and g_s^* are as defined in Definition 5.11 and A_c is stable, then N is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|y_3'\| < \infty$. Since $\|v_1\| < \infty$, $\|u_0\| < \infty$ and $\|\tilde{v}\| < \infty$, and $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))$, it follows from Lemma 5.15 (second statement), that $\|\tilde{v}_2\| < \infty$. Also given that $y_3' = Nv_n = NM^{-1}v_1 = \tilde{P}v_1$, it follows that $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)$
$(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_3'}$.

Next we prove that $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Let $\begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}$. Then we have $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$ and $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$.

We need to show that $\begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$ and $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, u_0 \in \mathcal{U}$. Here $u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}$ follow from the definition of $\mathcal{G}_{P_3'}$ and since $\|y_3'\| < \infty$ then $\|\tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))\| < \infty$, also given that $\|u_0\|, \|\tilde{v}\|, \|\tilde{v}_2\| < \infty$ and defining $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$, by Lemma 5.15 first statement, it follows that $\|v_1\| < \infty$, this leads to $\begin{pmatrix} v_1 \\ \tilde{P}v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} \in \mathcal{G}_{\tilde{P}}$.

Now any element in $\mathcal{G}_{\tilde{P}}$ can be written in the form $\begin{pmatrix} M \\ N \end{pmatrix} v_n$ for some $v_n \in \mathcal{U}$. So let:

$$\begin{pmatrix} M \\ N \end{pmatrix} v_n = \begin{pmatrix} v_1 \\ \tilde{P} v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ y_3' \end{pmatrix}.$$

This leads to $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_3'} = \mathcal{A}$. To show that $\mathcal{G}_{P_3}$ given in (5.59) is equivalent to that given in (5.62), set $g^*(\tilde{z}_2) = g_s^*(\tilde{z}_2)$, $g_u^*(\tilde{z}_1) = \pi$, $\frac{1}{g_u^*(\tilde{z}_2)} = \pi'$ and $f_u^*(\tilde{z}_1) = f_u^*(\tilde{z}_2) = 0$. In this case $\mathcal{G}_{P_3}$ follows as a special case, as required. $\square$

The map Φ between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ is defined using the following proposition:

Proposition 5.17. *Let $\tilde{P}$ be the unstable system given by (5.33)-(5.34), let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2, for the systems P_3 and P_3' given by Figure 5.21 and (5.50) and (5.49), respectively. Then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by:*

$$\Phi \begin{pmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ N v \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v} \\ -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(M v_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ N v \end{pmatrix}. \tag{5.64}$$

Furthermore this map is surjective.

Proof. First we need to prove that if

$$x = \begin{pmatrix} u_0'' & \tilde{v}' & g_s^*(\tilde{z}_2')(u_0'' - M v_n') & \tilde{z}_1' & \tilde{z}_2' & N v_n' \end{pmatrix}^\top \in \mathcal{G}_{P_3},$$

then $\Phi(x) \in \mathcal{G}_{P_3'}$. Since $x \in \mathcal{G}_{P_3}$ then $\|u_0''\|, \|\tilde{v}'\| \|\tilde{v}_2'\| = \|g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - v_1')\| = \|g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - M v_n')\|, \|\tilde{z}_1'\|, \|\tilde{z}_2'\|, \|y_3''\| < \infty$, $\begin{pmatrix} v_1' \\ y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n', \tilde{v}_2' = g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - v_1')$.
Let $y = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top = \Phi(x)$. We need to show that
$\tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(M v_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))\right)$,

$\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$, $v_n, \tilde{z}_1, \tilde{z}_2, u_0 \in \mathcal{U}$. It follows from (5.64) that $u_0 = u_0''$, $\tilde{v} = \tilde{v}'$, $\tilde{z}_1 = \tilde{z}_1'$, $\tilde{z}_2 = \tilde{z}_2'$, $y_3' = y_3''$,
$\tilde{v}_2 = -g^*(\tilde{z}_2')\left(\frac{1}{g_u^*(\tilde{z}_1')}(M v_n' - f_u^*(\tilde{z}_1')) - (u_0'' - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2') - \frac{1}{g_u^*(\tilde{z}_2')\tilde{v}'})\right)$,
then $\|u_0\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$. Since $v_1' = u_0'' - \tilde{v}' - \frac{1}{g_s^*(\tilde{z}_2')}\tilde{v}_2'$, then by Proposition 5.16 (5.62) there exist $v_n' \in \mathcal{U}$ such that $\begin{pmatrix} v_1' & y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n'$. It follows that

$y_3'' = Nv_n' = NM^{-1}v_1'$. Now let $v_1 = v_1'$, and note that $y_3' = y_3'' = NM^{-1}v_1' = NM^{-1}v_1$, then there exists $v_n = v_n'$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = Mv_n$, since M is as given in (5.35)-(5.36) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), v_1 = -\frac{1}{g_s^*(x^*)}C_{Linear}x^* + v_n$, since f_s^* and g_s^* are as defined in Definition 6.1 and A_c is stable, then M is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|v_1\| < \infty$.
Using Lemma 5.15 (second statement) as $\tilde{v}_2 = -g^*(\tilde{z}_2)(\frac{1}{g_u^*(\tilde{z}_1)}(Mv_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})) = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right)$ and since $\|u_0\|, \|\tilde{v}\|, \|v_1\| < \infty$ then $\|\tilde{v}_2\| < \infty$. Then:

$$\begin{aligned}
\tilde{v}_2 &= -g^*(\tilde{z}_2')\left(\frac{1}{g_u^*(\tilde{z}_1')}(v_1' - f_u^*(\tilde{z}_1')) - (u_0'' - \frac{1}{g_u^*(\tilde{z}_2')}f_u^*(\tilde{z}_2') - \frac{1}{g_u^*(\tilde{z}_2')}\tilde{v}')\right), \\
&= -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right),
\end{aligned}$$

and hence:

$$\begin{aligned}
y &= \begin{pmatrix} u_0 \\ \tilde{v} \\ -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(Mv_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix} \\
&= \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}.
\end{aligned}$$

as required.
Next, to prove that Φ is surjective, so that if $y \in \mathcal{G}_{P_3'}$ then there exists $x \in \mathcal{G}_{P_3}$ such that $\Phi(x) = y$, let us choose an element:

$$y = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}.$$

where $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$, $\tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right), v_n \in \mathcal{U}$. Let

$$x = \begin{pmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)} + 1\right)u_0 - g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix},$$

we need to show that $x \in \mathcal{G}_{P_3}$ i.e. $v_n, \tilde{v}, \tilde{z}_1, \tilde{z}_2, \tilde{v}_2' = g_s^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)} + 1\right)u_0 -$

$g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right) \in \mathcal{U}$, $\tilde{v}_2' = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1)$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))$ then $\tilde{v}_2' = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1)$, Since $\|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$ then $\tilde{z}_1, \tilde{z}_2 \in \mathcal{U}$. Using Assumption 5.2, since $g^*, \frac{1}{g^*}$ is a bounded function, using Assumption 5.12, since $g_u^*, \frac{1}{g_u^*}$ is a bounded function, also using Assumption 5.13, since f_u^* is a bounded function, since $g_s^*(\tilde{z}_2)$ is a bounded function then $\|f_u^*(\tilde{z}_1)\| < \infty$ and since $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$, it follows that:

$$\begin{aligned}
\|\tilde{v}_2'\| &\leq \left\| g_s^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)} + 1\right)u_0 \right\| + \\
&\quad \left\| g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right) \right\|, \\
&\leq \|g_s^*(\tilde{z}_2)\| \left\| \frac{1}{g^*(\tilde{z}_2)} \right\| \|\tilde{v}_2\| + \|g_s^*(\tilde{z}_2)\| \left(\left\| \frac{1}{g_u^*(\tilde{z}_1)} \right\| + 1 \right) \|u_0\| + \\
&\quad \|g_s^*(\tilde{z}_2)\| \left(\left\| \frac{1}{g_u^*(\tilde{z}_1)} \right\| \|f_u^*(\tilde{z}_1)\| + \|\frac{1}{g_u^*(\tilde{z}_2)}\| \|f_u^*(\tilde{z}_2)\| + \|\frac{1}{g_u^*(\tilde{z}_2)}\| \|\tilde{v}\| \right) \| < \infty.
\end{aligned}$$

Then by Proposition 5.16 (5.63) there exists $v_n \in \mathcal{U}$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$.

Hence $x \in \mathcal{G}_{P_3}$ is such that:

$$\Phi \begin{pmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)} + 1\right)u_0 - g_s^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix}$$
$$= \begin{pmatrix} u_0 \\ \tilde{v} \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix}. \tag{5.65}$$

as required. □

Using the previous results, a bound on the gap between P_3 and P_3' appearing in the inequality (5.48) is given using the following theorem:

Theorem 5.18. *Let $\tilde{P}$ be the unstable system given by (5.33)-(5.34), let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2. Let P_3 and P_3' given by Figure 5.21 and equations (5.50) and (5.49), respectively. Then a* *bound on the gap between* P_3 *and* P_3' *is*

$$\vec{\delta}(P_3, P_3') \quad \leq \quad F_{\delta 2}, \tag{5.66}$$

where

$$F_{\delta 2} = \sup_{\left\| \begin{smallmatrix} u_0 \\ s \in \mathcal{U} \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix} \right\| \neq 0} \frac{\left\| (g^*(\tilde{z}_2) - g_s^*(\tilde{z}_2)) u_0 + \left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2) \frac{1}{g_u^*(\tilde{z}_1)} \right) s + g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)} f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) \right) \right\|}{\| u_0, s, \tilde{z}_1, \tilde{z}_2 \|},$$

Proof. Using Proposition 5.17, since $\tilde{P}$ is an unstable system, since g_u^* satisfies Assumption 5.12, f_u^* satisfies Assumption 5.13, and since g^* satisfies Assumption 5.2. Then there exists a surjective map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (5.64). Then the gap between P_3 and P_3' is given as:

$$\begin{aligned}
\vec{\delta}(P_3, P_3') &\leq \sup_{x \in \mathcal{G}_{P_3} \setminus \{0\}} \frac{\|(\Phi - I)x\|}{\|x\|}, \\
&\leq \sup_{\left\| \begin{smallmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix} \right\| \neq 0} \frac{\left\| \begin{matrix} -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)} (M v_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)} \tilde{v}) \right) - \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \end{matrix} \right\|}{\| u_0, g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n), \tilde{z}_1, \tilde{z}_2 \|}, \\
&\leq \sup_{\left\| \begin{smallmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix} \right\| \neq 0} \frac{\left\| \begin{matrix} (g^*(\tilde{z}_2) - g_s^*(\tilde{z}_2))\, u_0 + \left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2) \frac{1}{g_u^*(\tilde{z}_1)} \right) M v_n + \\ g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)} f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) \right) + \left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2) \frac{1}{g_u^*(\tilde{z}_2)} \right) \tilde{v} \end{matrix} \right\|}{\| u_0, g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n), \tilde{z}_1, \tilde{z}_2 \|},
\end{aligned}$$

Given that $\left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2) \frac{1}{g_u^*(\tilde{z}_2)} \right) = 0$ and replacing $g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n)$ which is bounded for $\|g_s^*\tilde{z}_2)\|, \|u_0\|, \|\tilde{v}\| < \infty$ and $\|M\| < \infty$, with $s \in \mathcal{U}$ produces:

$$\vec{\delta}(P_3, P_3') \leq \sup_{\left\|\begin{smallmatrix} u_0 \\ s \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right\| \neq 0} \frac{\left\|(g^*(\tilde{z}_2) - g_s^*(\tilde{z}_2))u_0 + \left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2)\frac{1}{g_u^*(\tilde{z}_1)}\right)s + g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2)\right)\right\|}{\|u_0, s, \tilde{z}_1, \tilde{z}_2\|} \tag{5.67}$$

let

$$F_{\delta 2} = \sup_{\left\|\begin{smallmatrix} u_0 \\ s \\ \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right\| \neq 0} \frac{\left\|(g^*(\tilde{z}_2) - g_s^*(\tilde{z}_2))u_0 + \left(g_s^*(\tilde{z}_2) - g^*(\tilde{z}_2)\frac{1}{g_u^*(\tilde{z}_1)}\right)s + g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}f_u^*(\tilde{z}_1) - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2)\right)\right\|}{\|u_0, s, \tilde{z}_1, \tilde{z}_2\|},$$

then a bound on the gap is

$$\vec{\delta}(P_3, P_3') \leq F_{\delta 2}. \tag{5.68}$$

as required.

□

Theorem 5.18 states that a bound on the gap value depends on the difference between the input nonlinear component and the stable part of the input nonlinear component of the controller and on how exact the inversion of the unstable part of the plant nonlinearity is, within the nonlinear part of the controller.

Then according to the following proposition robust stability is preserved for the system shown in Figure 5.17.

Proposition 5.19. *Consider the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 5.17. Let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2. Then $[P_1', P_2', P_3']$ has a robust stability margin.*

Proof. Since g_u^* satisfies Assumption 5.12, f_u^* satisfies Assumption 5.13, and g^* satisfies Assumption 5.2, then by Lemma 5.15, and Proposition 5.16 for the systems P_3 and P_3' given by Figure 5.21 and (5.50) and (5.49), respectively, the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ can be given by (5.62) and (5.63), respectively. Using Proposition 5.17, then there exists a map

$\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (5.64). This leads to the presence of a finite gap value between the perturbed and the unperturbed configurations of this system given by the inequality (5.67). Then the system $[P_1', P_2', P_3']$ shown in Figure 5.17 has a robust stability margin. □

The main result Theorem 5.14 easily follows from Theorem 5.20 which we establish next.

Theorem 5.20. *Consider the nonlinear closed loop system shown in Figure 5.12 and given by (5.40)-(5.45). Let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2. Then this system has a robust stability margin b_{P_1,C_1} which satisfies the inequality*

$$b_{P_1,C_1} \geq \|Q\|^{-1}. \tag{5.69}$$

Proof. Let $\frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 5.17, let $\frac{1}{\|\Pi_{(3)}'\|}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 5.15, finally let $b_{P_1,C_1} = \frac{1}{\|\Pi_{P_{Linear}//C_{Linear}+\tilde{C}}\|}$ be a stability margin for the system shown in Figure 5.12. Then

$$\begin{aligned}
\|Q\| = \left\|\Pi_{(3)}\right\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}\right\|}{\|u_0', x_0', y_0'\|}, \\
&= \sup_{\|u_0,d_0,d_1,d_2,d_3,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, d_0, d_1, d_2, d_3, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,0,0,0,0,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}'\begin{pmatrix} u_0 & 0 & 0 & 0 & 0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, 0, 0, 0, 0, y_0, x_0\|} = \|\Pi_{(3)}'\|, \\
\|\Pi_{(3)}'\| &= \sup_{\|u_0,y_0,x_0\|\neq 0} \frac{\left\|\Pi_3'\begin{pmatrix} u_0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,y_0,0\|\neq 0} \frac{\left\|\Pi_{P_{Linear}//C_{Linear}+\tilde{C}}\begin{pmatrix} u_0 & y_0 & 0 \end{pmatrix}^\top\right\|}{\|u_0, y_0, 0\|}, \\
&= \|\Pi_{P_{Linear}//C_{Linear}+\tilde{C}}\|.
\end{aligned}$$

This leads us to

$$b_{P_1,C_1} = \frac{1}{\Pi_{P_{Linear}//C_{Linear}+\tilde{C}}} \geq \frac{1}{\Pi_{(3)}'} \geq \frac{1}{\Pi_{(3)}} = \|Q\|^{-1}.$$

Therefore the existence of a stability margin for the system shown in Figure 5.17 guarantees the existence of a stability margin for the system $[P_1, C_1]$ shown in Figure 5.12. Also, since g_u^* satisfies Assumption 5.12, f_u^* satisfies Assumption 5.13, and since g^* satisfies Assumption 5.2, then by Proposition 5.19, the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 5.17, has a robust stability margin. This leads to the conclusion that the system $[P_1, C_1]$ given by Figure 5.12 and (5.40)-(5.45) also has a robust stability margin. as required.

□

Based on Theorems 5.18 and 5.20 we can write the following corollary:

Corollary 5.21. *Consider the nonlinear closed loop system shown in Figure 5.12 and given by (5.40)-(5.45). Let g_u^* satisfy Assumption 5.12, let f_u^* satisfy Assumption 5.13, and let g^* satisfy Assumption 5.2. Then this system is stable if*

$$F_{\delta 2} < \|Q\|^{-1}.$$

Proof. Using Theorem 5.18 inequality (5.66), since:

$$\vec{\delta}(P_3, P_3') \leq F_{\delta 2},$$

and using Theorem 5.20 inequality (5.69), since:

$$b_{P_1,C_1} \geq \|Q\|^{-1},$$

It follows that if

$$F_{\delta 2} < \|Q\|^{-1}.$$

we have:

$$\vec{\delta}(P_3, P_3') \leq F_{\delta 2} < \|Q\|^{-1} \leq b_{P_1,C_1},$$

so $\vec{\delta}(P_3, P_3') < b_{P_1,C_1}$ and the conditions hold from Theorem 2.11, hence stability is assured.

□

5.4 Summary

This chapter has considered two important cases of unstable affine systems. The first case is the affine system with unstable linear and unstable nonlinear parts, where we consider all the nonlinear terms in the plant to be unstable. The stability condition found for this system showed that the gap value depends on the nonlinear input part of the controller and how exact the inversion of the plant nonlinearity is, within the nonlinear part of the controller (equation (5.26)). This result shows that in this case the controller realizes an inverting action to cancel all the nonlinear terms in the system, including inherently stabilizing nonlinearities which can be used to stabilize the plant, hence adding large positive feedback to cancel useful nonlinearities in the plant. This increases the risk of instability in the control system.

An improved approach was developed in the second case where an affine nonlinear system with an unstable linear and a stable and an unstable nonlinear components was considered. In this case the system was assumed to have two nonlinear components; an unstable nonlinear component also cancelled by control action, and a useful stable nonlinear component, which we wish to preserve within the stabilizing action. The stability condition found for this system showed that the gap value depends on the difference between the input nonlinear component and the stable part of the input nonlinear component of the controller and on how exact the inversion of the unstable part of the plant nonlinearity is, within the nonlinear part of the controller (equation (5.66)). This result showed that the new approach prevented adding large positive feedback to cancel useful nonlinearities in the plant (as was done in the previous case). On the other hand, following this approach does not place restrictions on the control design of the system.

Chapter 6

Robustness Analysis for Nonlinear Systems with Stable and Unstable Plant Nonlinearities Using the Gap Metric

6.1 Introduction

This chapter considers the robustness analysis of a special class of unstable affine systems which have only a single nonlinear component, $f^*(x^*)$. The linear part of the plant is assumed to be unstable and the nonlinear part, $f^*(x^*)$ includes a stable component, $f_s^*(x^*)$, and an unstable component, $f_u^*(x^*)$. This system is a special case of the system considered in Section 5.3, however, in this chapter the analysis undertaken will not follow the procedure carried out in Section 5.3. Instead the linear stabilising component of the controller will be used to stabilize the linear unstable part of the plant, aligning the choice of the stabilizing linear component of the feedback with the implicit stabilizing feedback used in the construction of the underlying coprime factors. Then, the resulting stable linear plant is incorporated with the stable nonlinear component of the plant, $f_s^*(x^*)$, to construct the nominal plant which will be considered in the robustness analysis carried out in this chapter. The gap analysis for this system will follow the procedure carried out in Chapter 4.

6.2 Robustness Analysis of a Nonlinear System with Plant Nonlinearity Using the Gap Metric

6.2.1 Affine Nonlinear Systems with Stable and Unstable Plant Nonlinearity

Consider the normal form of the nonlinear system described previously in Section 4.2, given by (4.5)-(4.6), this form is:

$$\begin{aligned} P \quad : \quad & \mathcal{L}^{n+1}_{\infty,e} \to \mathcal{L}^{n}_{\infty,e} : (u_1, z_1) \mapsto (y_1), \\ & \dot{x}^* = Ax^* + B(f^*(z_1) + g^*(z_1)u_1), \qquad (6.1) \\ & y_1 = (y_{11}, \dots, y_{1n}) = x^*, \qquad (6.2) \\ & z_1 = (z_{11}, \dots, z_{1n}) = x^*, \qquad (6.3) \end{aligned}$$

In this system the feedback input y_1 is considered as an input, z_1, to the nonlinear parts of the plant $g^*(z_1)$ and $f^*(z_1)$, $A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & \dots & & a_{n-1} & a_n \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, $f^*(z_1)$ and $g^*(z_1)$ represent the nonlinear part of the transformed system, $f^*(x^*) = L_f^n h(T^{-1}(x^*))$, $g^*(x^*) = L_g L_f^{n-1} h(T^{-1}(x^*))$.

In this chapter we consider the transformed plant to have only a single nonlinear plant component, $f^*(z_1)$, which is composed of a stable and an unstable component. This corresponds to replacing $g^*(z_1)$ by the linear operator $\pi : (u_1, z_1) \mapsto v_i, v_i = u_1$, so that the system becomes

$$\begin{aligned} P \quad : \quad & \mathcal{L}^{n+1}_{\infty,e} \to \mathcal{L}^{n}_{\infty,e} : (u_1, z_1) \mapsto (y_1), \\ & \dot{x}^* = Ax^* + B(f^*(z_1) + u_1), \qquad (6.4) \\ & y_1 = (y_{11}, \dots, y_{1n}) = x^*, \qquad (6.5) \\ & z_1 = (z_{11}, \dots, z_{1n}) = x^*, \end{aligned}$$

The approach is then to select a controller of the form $u_1 = a(x^*) + b(x^*)v$ to stabilize the system $\dot{x}^* = Ax^* + B(f^*(x^*) + u_1), y_1 = x^*$ and cancel only the unstable component of $f^*(x^*)$. As will be seen this controller action requires a vector:

$$c = (c_1, \dots, c_n)^\top$$

to be chosen such that $A_c = A - Bc^\top$ is Hurwitz. Analogous to the work done in Section

5.3, we introduce the stable component of $f^*(x^*)$, which we will call f_s^*, according to the following definition:

Definition 6.1. Given that $A_c = A - Bc^\top$ is Hurwitz, a function $f_s^* : \mathbb{R}^n \to \mathbb{R}$ is called stable if the mapping $u \mapsto y$, with

$$\dot{x}^* = A_c x^* + B(f_s^*(x^*) + u), y = x^*$$

is stable.

Then to find an admissible form for $f_u^*(x^*)$, which is the unstable component of $f^*(x^*)$, we consider

$$\begin{aligned} P &: (u_1, z_1) \mapsto y_1, \\ \dot{x}^* &= Ax^* + B(f^*(z_1) + u_1), \\ &= Ax^* + Bf_s^*(z_1) - Bf_s^*(z_1) + B(f^*(z_1) + u_1), \\ &= Ax^* + Bf_s^*(z_1) + B(f^*(z_1) - f_s^*(z_1) + u_1), \\ & \quad y_1 = x^*, \end{aligned}$$

let $f_u^*(x^*) = (f^*(x^*) - f_s^*(x^*))$ then P can be written as

$$P : (u_1, z_1) \mapsto y_1, \tag{6.6}$$

$$\dot{x}^* = Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + u_1), \tag{6.7}$$

$$y_1 = x^*. \tag{6.8}$$

Since P is unstable, it can be represented using nonlinear coprime factorization as $P = NM^{-1}$ where N and M are two stable operators satisfying $L(M, N)^\top = I$, where L is a causal stable mapping $L : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$. The operator M is given by

$$M : (v, z_1) \mapsto (u_1, z_1),$$

$$\dot{x}^* = Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + l(x^*) + v), \tag{6.9}$$

$$u_1 = l(x^*) + v, \tag{6.10}$$

$$z_1 = (z_{11}, \ldots, z_{1n}) = x^*,$$

and N is given by

$$
\begin{aligned}
N \quad &: \quad (v, z_1) \mapsto y_1, \\
\dot{x}^* \quad &= \quad Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + l(x^*) + v), \qquad (6.11) \\
y_1 \quad &= \quad x^*. \qquad (6.12)
\end{aligned}
$$

Hence $u_1 = a(x^*) + b(x^*)v = l(x^*) + v$. The real function $l(x^*)$ is chosen such that this function will cancel only the unstable nonlinear term $f_u^*(x^*)$ while stabilizing the linear part of the plant so that M, N are stable. To do this, consider $Ax^* + B(f^*(x^*) + l(x^*) + v) = A_c x^* + B(f_s^*(x^*) + v)$ for some input v. This leads to the conclusion that $l(x^*)$ can be given as:

$$
l(x^*) = -c^\top x^* - (f^*(x^*) - f_s^*(x^*)). \qquad (6.13)
$$

Since

$$
f_u^*(x^*) = (f^*(x^*) - f_s^*(x^*)), \qquad (6.14)
$$

then $l(x^*) = -c^\top x^* - f_u^*(x^*)$.

Hence, a feedback linearizing controller which generates the term $l(x^*)$ and adds a feedback loop $\tilde{C}y_2$ to generate v is given as:

$$
\begin{aligned}
C \quad : \quad & \mathcal{L}_{\infty,e}^{2n} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto u_2 \\
& u_2 = -l(y_2, z_2) + \tilde{C}y_2 = f_u^*(z_2) - C_{linear} y_2 - \tilde{C}y_2, \\
& z_2 = (z_{21}, \ldots, z_{2n}) = -y_2,
\end{aligned}
$$

where $l(x^*) = l(y_2, z_2)$, $y_2 = (y_{21}, \ldots, y_{2n})$, C_{linear} is the linear stabilizing part of the controller C,

$$
\begin{aligned}
C_{linear} \quad : \quad & \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_s \\
& v_s = c^\top y_2.
\end{aligned}
$$

and $\tilde{C}$ is the linear part which adds a feedback loop to generate v, is given as

$$
\begin{aligned}
\tilde{C} \quad : \quad & \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_2 \\
& v_2 = -\tilde{c}^\top y_2,
\end{aligned}
$$

where $v_2 = v$ and

$$
\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)^\top,
$$

In the presence of disturbances, $l(x^*)$ stabilizes the linear part of the plant P. This is because the linear part of this function, $c^\top x^*$, is employed as the stabilizing feedback to the underlying coprime factors of this system to produce a new plant with a stable linear part. To show that, consider (6.7) and proceed as follows:

$$
\begin{aligned}
\dot{x}^* &= Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + u_1), \\
&= Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + u_0 - u_2) \\
&= Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) + u_0 - f_u^*(z_2) + C_{linear}y_2 + \tilde{C}y_2), \\
&= Ax^* + Bf_s^*(z_1) + B(f_u^*(z_1) - f_u^*(z_2) + C_{linear}(y_0 - y_1) + \tilde{C}y_2 + u_0), \\
&= (A - BC_{linear})x^* + Bf_s^*(z_1) + B(f_u^*(z_1) - f_u^*(z_2) + C_{linear}y_0 + \tilde{C}y_2 + u_0), \\
&= A_c x^* + Bf_s^*(z_1) + B(f_u^*(z_1) - f_u^*(z_2) + C_{linear}y_0 + \tilde{C}y_2 + u_0),
\end{aligned}
$$

Note the link to the coprime factor construction given in (6.9)-(6.12), where the above the equation is the stabilized version of the dynamic equation (6.9). Then given that

$$
\begin{aligned}
\tilde{P} \;:\; & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^n : (v_1, z_1) \mapsto y_1, \\
& \dot{x}^* = A_c x^* + B(f_s^*(z_1) + v_1), \\
& z_1 = (z_{11}, \ldots, z_{1n}) = y_1, \\
& y_1 = x^*,
\end{aligned}
$$

where $v_1 = (u_0 + C_{linear}y_0) + f_u^*(z_1) - f_u^*(z_2) + \tilde{C}y_2$ and let $\tilde{u}_1 = \tilde{u}_0 - \tilde{u}_2$ where $\tilde{u}_0 = u_0 + C_{linear}y_0$ and $\tilde{u}_2 = f_u^*(z_2) - \tilde{C}y_2$. Hence P is given as:

$$
\begin{aligned}
P_1 \;:\; & \mathcal{L}_{\infty,e}^{n+1} \to \mathcal{L}_{\infty,e}^n : (\tilde{u}_1, z_1) \mapsto (y_1), \\
& y_1 = \tilde{P}v_1, && (6.15) \\
& = \tilde{P}(f_u^*(z_1) + \tilde{u}_1) && (6.16) \\
& z_1 = (z_{11}, \ldots, z_{1n}) = y_1.
\end{aligned}
$$

For this new plant, a feedback controller is given as

$$
\begin{aligned}
C_1 \;:\; & \mathcal{L}_{\infty,e}^{2n} \to \mathcal{L}_{\infty,e} : (y_2, z_2) \mapsto \tilde{u}_2 \\
& \tilde{u}_2 = f_u^*(z_2) - \tilde{C}y_2, \\
& z_2 = (z_{21}, \ldots, z_{2n}) = -y_2,
\end{aligned}
$$

The feedback control connection for this system is shown in Figure 6.1. Note that in this configuration, C_{Linear} operator is not shown in the block diagram of the system,

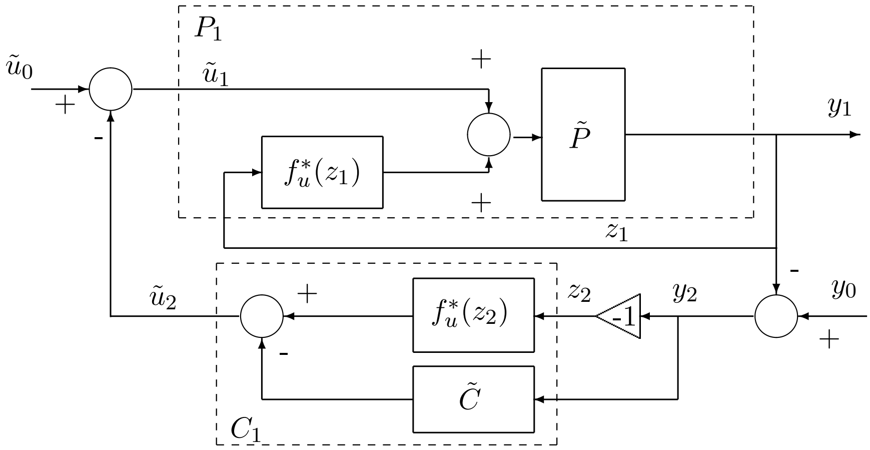

FIGURE 6.1: Nonlinear control system with stable/unstable plant nonlinearity

this is because C_{Linear} has been included in the stabilized nominal plant $\tilde{P}$. For the system shown in Figure 6.1 the closed loop equations can be written as:

$$\tilde{u}_0 = \tilde{u}_1 + \tilde{u}_2, \quad (6.17)$$

$$y_0 = y_1 + y_2, \quad (6.18)$$

$$v_1 = f_u^*(z_1) + \tilde{u}_1, \quad (6.19)$$

$$v_2 = \tilde{C}y_2, \quad (6.20)$$

$$y_1 = \tilde{P}v_1, \quad (6.21)$$

$$\tilde{u}_2 = f_u^*(z_2) - \tilde{C}y_2. \quad (6.22)$$

Next, to derive conditions for robust stability for the system shown in Figure 6.1, we note that this system is not equivalent to the closed loop system $[P_1, C_1]$. This is due to the presence of the two signals z_1, z_2 which feed the signals $y_1, -y_2$ to the blocks P_1, C_1, respectively. If $z_1, z_2 \neq 0$, then the system in Figure 6.1 does not correspond exactly to the system shown in Figure 2.1. Hence, we cannot apply Theorem 2.11 to find stability conditions for this system.

However, the linear configuration of this system, which will be needed to find stability conditions for this system, does correspond to the system shown in Figure 2.1, since setting $f_u^*(z_1) = f_u^*(z_2) = f_s^*(z_1) = 0$ will result in the linear system $[P_{Linear}, \tilde{C}]$ shown in Figure 6.2.

This system is equivalent to the linear system shown in Figure 6.3, where P_{Linear} is given by

$$P_{Linear} \; : \; \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : \tilde{u}_1 \mapsto y_1,$$

$$\dot{x}^* = A_c x^* + B\tilde{u}_1, \quad (6.23)$$

$$y_1 = x^*, \quad (6.24)$$

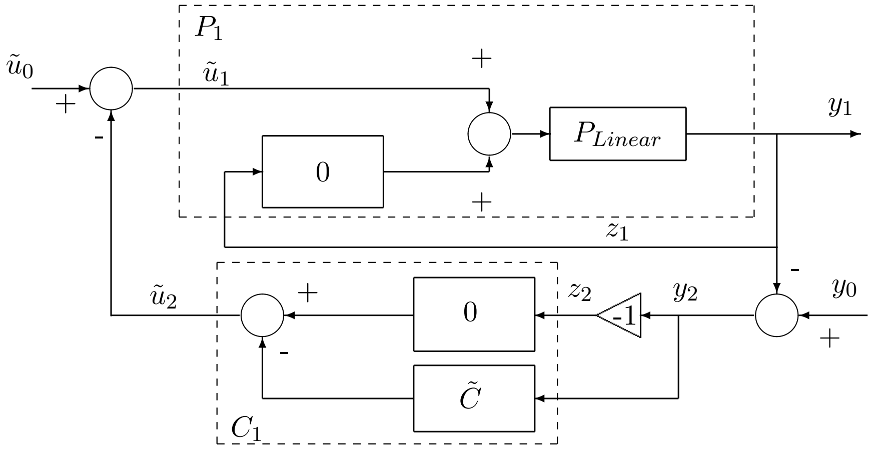

FIGURE 6.2: Linear configuration of a nonlinear system with stable/unstable plant nonlinearity

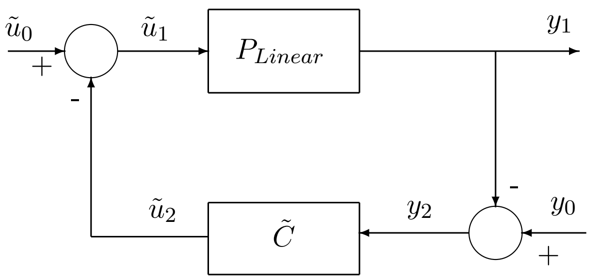

FIGURE 6.3: Equivalent linear configuration of a nonlinear system with stable/unstable plant nonlinearity

The closed loop system $[P_{Linear}, \tilde{C}]$ has a closed loop operator $\Pi_{P_{linear}//\tilde{C}}$, which will be used to find stability conditions for the system shown in Figure 6.1.

6.3 Gap Metric for Nonlinear Systems with Stable and Unstable Plant Nonlinearities

This section undertakes robustness stability analysis for the nonlinear system shown in Figure 6.1. The analysis carried out in this section will follow the approach presented in Chapter 4 since the system considered in this analysis is stable, however, $\tilde{P}$ is now a stable *nonlinear* plant with stable linear and nonlinear plant components.

The following assumptions on the form of f_u^* is required in subsequent analysis:

Assumption 6.2. Let $f_u^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the following conditions:

$$\exists \mathbf{Q}_u < \infty \text{ and } \quad |f_u^*(x)| \leq \mathbf{Q}_u \quad \forall x \in \mathbb{R}^n. \tag{6.25}$$

The gap metric framework is applied to the system shown in Figure 6.1 to result in the following theorem:

Theorem 6.3. *Consider the nonlinear closed loop system shown in Figure 6.1 and given by (6.17)-(6.22). Let f_u^* satisfy Assumption 6.2, then this system has a robust stability margin.*

Parallel to the proof of Theorem 4.6, this proof also requires results that are developed subsequently in this section. This analysis will also consider the triple system configuration shown in Figure 4.5 and apply the 'network' result in (Theorem 2.13) to find a stability condition for the nonlinear system shown in Figure 6.1.

The route taken is as follows: Since the presence of nonlinear elements in multiple blocks in the system shown in Figure 6.1 leads to significant conservatism, and to apply Theorem 2.13 to this system, a new system configuration shown in Figure 6.4 is used. In this configuration the unstable nonlinear part of the plant P_1 and the nonlinear part of the controller C_1 along with the plant $\tilde{P}$ are considered to be included in the block P_3' and an external input x_0 is added to the system. Also the feedback input $x_0 - y_1$ is considered as an input, z_1, to the nonlinear components of the plant $f_u^*(z_1)$,$f_s^*(z_1)$, and the feedback input $-y_2$ is considered as an input z_2 to the nonlinear component $f_u^*(z_2)$.

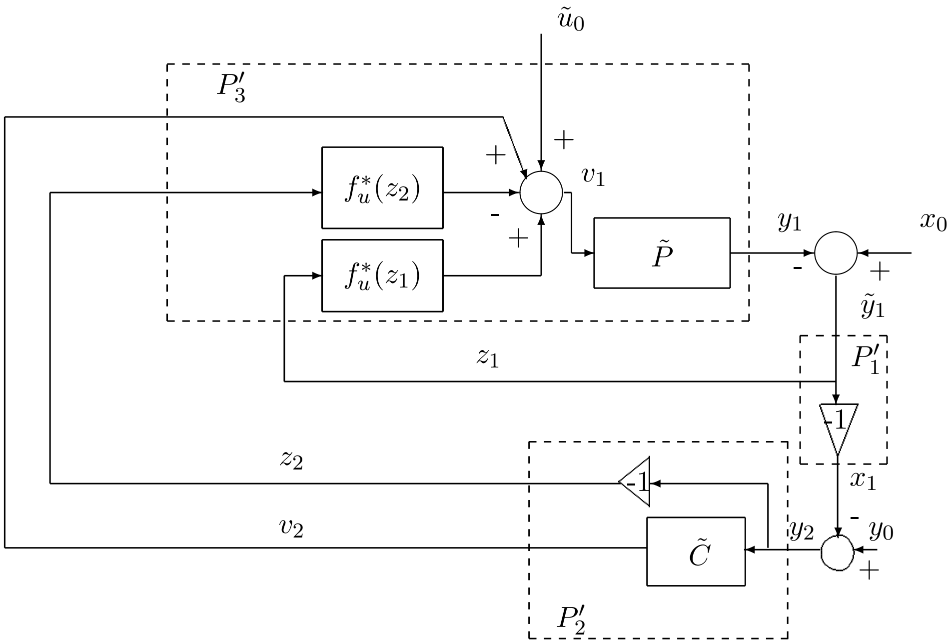

FIGURE 6.4: Nonlinear system with stable and unstable plant nonlinearity configuration

The nominal system configuration is taken to comprise the system components P_1, P_2 and P_3 with setting $f_u^*(z_1) = f_u^*(z_2) = 0$. This configuration is shown in Figure 6.5.

To apply Theorem 2.13 we must put the real and the nominal nonlinear systems shown in Figures 6.4 and 6.5 in a form comparable to that given in Figure 4.5. In order to

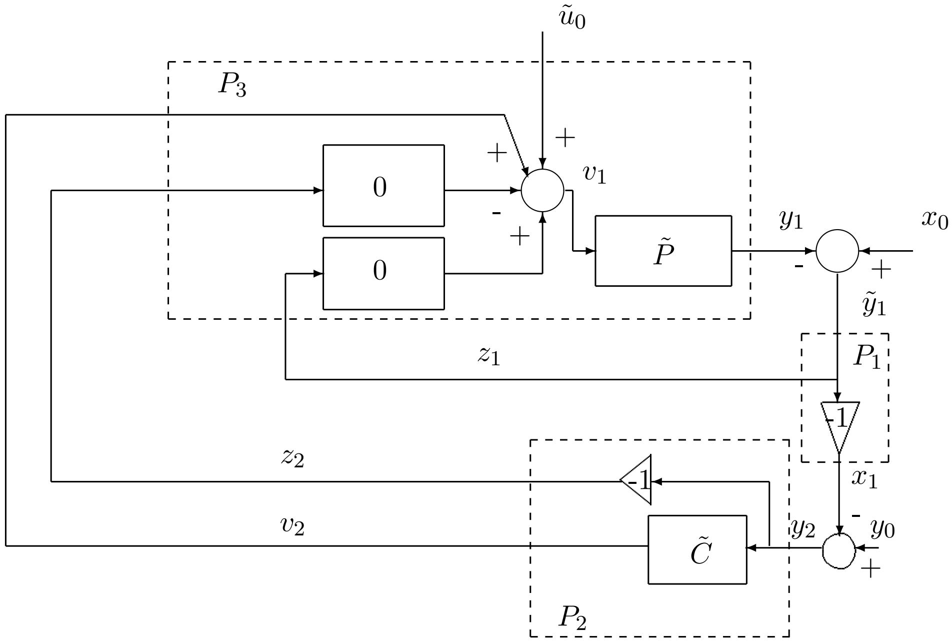

FIGURE 6.5: Linear configuration of an affine system with stable and unstable plant nonlinear part

do this we consider three signal spaces $\mathcal{U} = \mathcal{L}^n_{\infty,e}, \mathcal{X} = \mathcal{L}^n_{\infty,e}$ and $\mathcal{Y} = \mathcal{L}^n_{\infty,e}$, together with the following augmented signals; let $\hat{v}_2 = -v_2$ and let $u'_1 = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top$ and let $u'_2 = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top$ also let the external input u_0 be changed to $u'_0 = \begin{pmatrix} \tilde{u}_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top$, where $d_2 = (d_{21}, \ldots, d_{2n})$ and $d_3 = (d_{31}, \ldots, d_{3n})$, also let $u'_3 = u'_0 - u'_2 - u'_1 = \begin{pmatrix} \tilde{u}_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top - \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top - \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top = \begin{pmatrix} \tilde{u}_0 & d_1 - \hat{v}_2 & d_2 - z_1 & d_3 - z_2 \end{pmatrix}^\top$, let $\tilde{v}_2 = d_1 - \hat{v}_2$, $\tilde{z}_1 = d_2 - z_1$, $\tilde{z}_2 = d_3 - z_2$ then $u'_3 = \begin{pmatrix} \tilde{u}_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$. Also let $x'_0 = y_0$, $y'_0 = x_0$, $y'_3 = y_1$, $x'_1 = x_1$, $x'_2 = y_2$ and finally $y'_1 = y'_0 - y'_3 = x_0 - y_1$. The resulting system is shown in Figure 6.6.

The corresponding nominal system is shown in Figure 6.7.

From the two systems shown in Figure 6.6 and Figure 6.7 it follows that $P_1 = P'_1$ and $P_2 = P'_2$.

These configurations correspond to those of Figures 6.4 and 6.5, respectively, except for the presence of d_1, d_2 and d_3. Figures 6.6 and 6.7 correspond exactly to the forms shown in Figures 6.8 and 6.9, respectively, which in turn have identical structure to that of Figure 4.5. Hence, our stability condition will be applied to the systems of Figures 6.8 and 6.9.

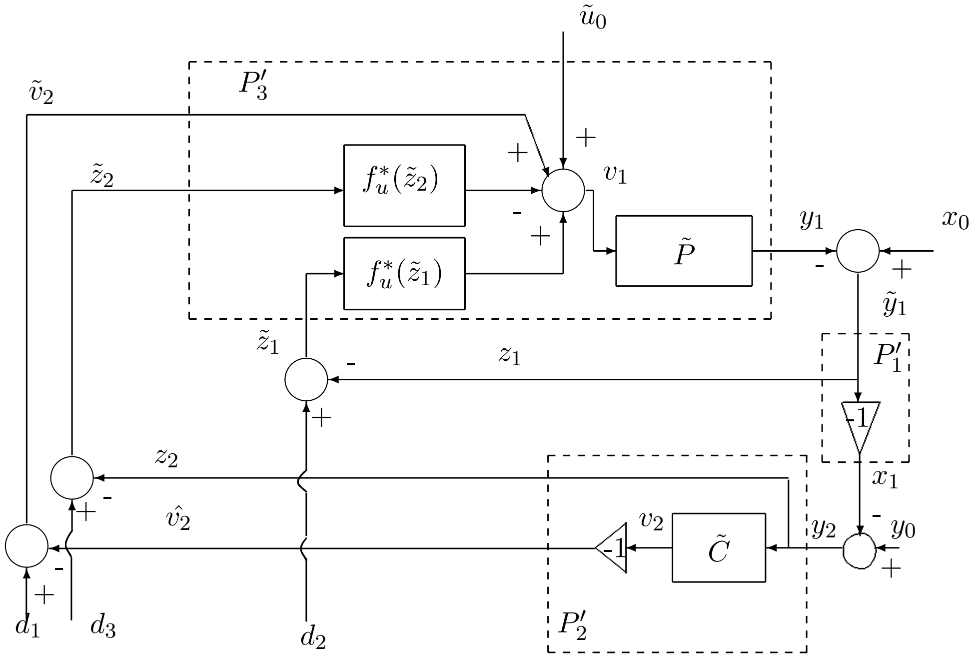

FIGURE 6.6: Augmented nonlinear system with plant nonlinearity

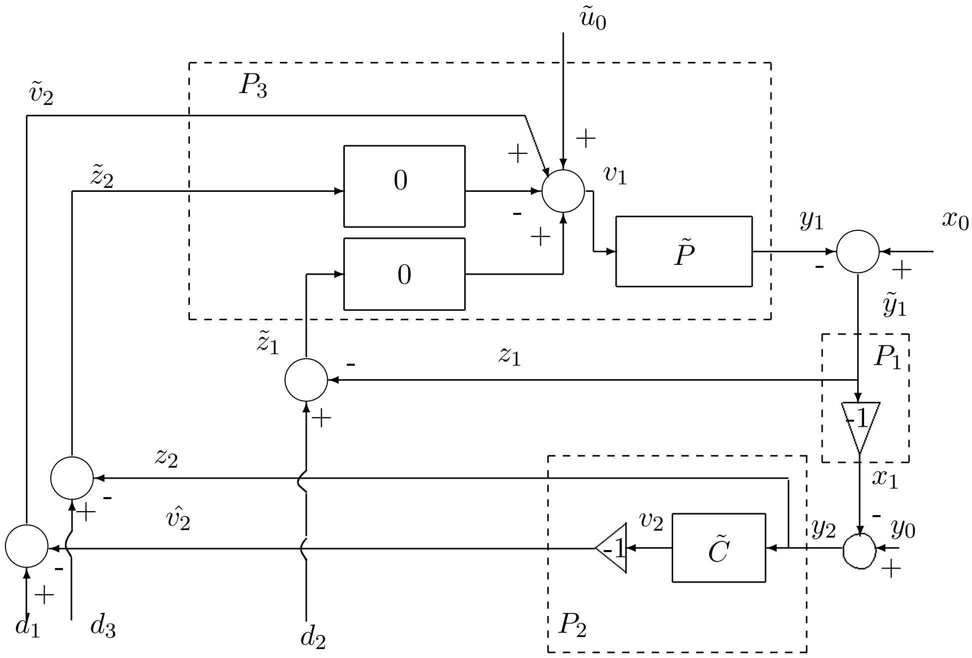

FIGURE 6.7: Nominal system with plant nonlinearity

Since $P_1 = P_1'$ and $P_2 = P_2'$, then

$$\vec{\delta}(P_1, P_1') = 0, \vec{\delta}(P_2, P_2') = 0.$$

Using Theorem 2.13, the robust stability condition is given as:

$$\sum_{i=1}^{3} \vec{\delta}(P_i, P_i') < \|\Pi_{(i)}\|^{-1},$$

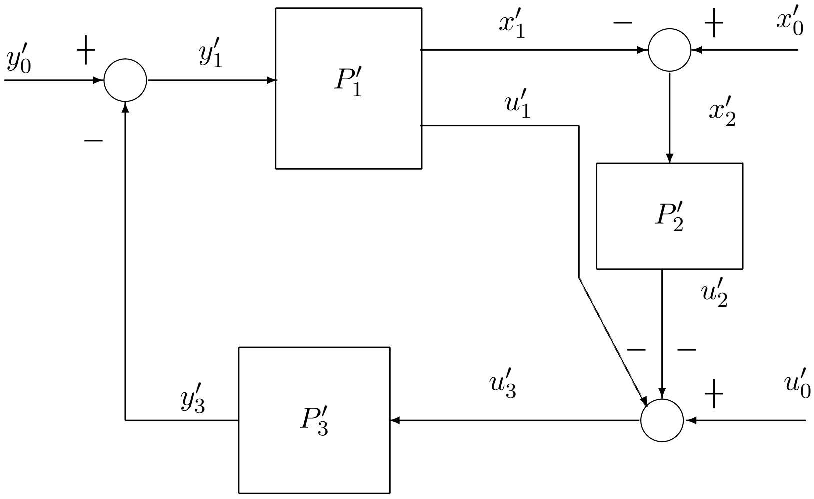

FIGURE 6.8: Nonlinear configuration of affine system with plant nonlinearity

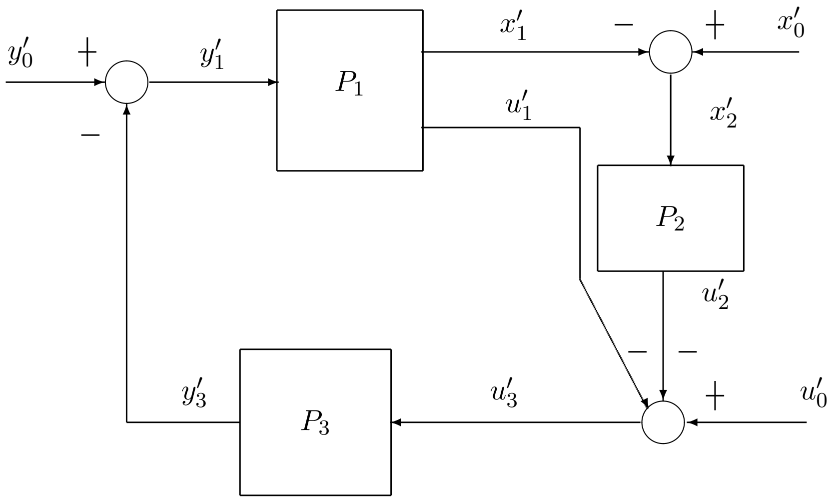

FIGURE 6.9: Nominal configuration of affine system with unstable nonlinearity

For this system the stability condition is:

$$\vec{\delta}(P_3, P'_3) < \|\Pi_{(3)}\|^{-1}. \tag{6.26}$$

Then the gap metric measures the difference between the nominal plant $P_3 : u'_3 \mapsto y'_3, y'_3 = \tilde{P}(\tilde{u}_0 - \tilde{v}_2)$ and the perturbed plant $P'_3 : u'_3 \mapsto y'_3, y'_3 = \tilde{P}(f^*_u(\tilde{z}_1) - f^*_u(\tilde{z}_2) + (\tilde{u}_0 - \tilde{v}_2))$. The plants P_3 and P'_3 are shown in Figure 6.10.

Before providing a complete description of the operators P'_1, P'_2 and P'_3 and P_1, P_2 and P_3 shown in Figures 6.8 and 6.9, respectively, we also briefly state the motivation for the proceeding manipulations (as was done in Chapter 4).

The stability condition (6.26) can be related to the original system configuration shown in Figure 6.1 as follows: It will be shown later in the proof of Theorem 6.3 that the

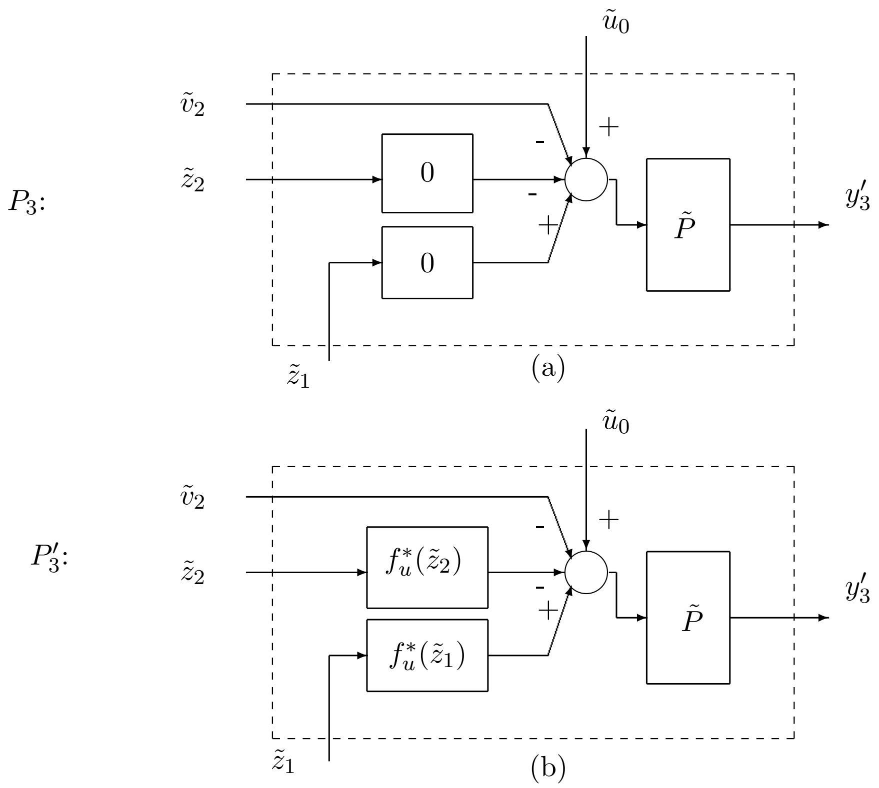

FIGURE 6.10: Nonlinear plant mapping:(a) unperturbed, (b) perturbed

stability margin for the system shown in Figure 6.6 is less than or equal to the stability margin corresponding to the system shown in Figure 6.4 which in turn is less than or equal to the stability margin corresponding to the original system shown in Figure 6.1. This is because for each pair the latter is a special case of the former.

The closed loop operators P_1', P_2' and P_3' shown in Figure 6.8 are given by

$$\begin{aligned} P_1' \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{2n}_{\infty,e} : y_1' \mapsto (x_1', u_1'), x_1' = -y_1', \\ & \quad u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1', \end{aligned}$$

where $y_1' = \tilde{y}_1$, and :

$$\begin{aligned} P_2' \quad &: \quad \mathcal{L}^n_{\infty,e} \to \mathcal{L}^{n+1}_{\infty,e} : x_2' \mapsto u_2', u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, \\ & \quad z_2 = x_2', \hat{v}_2 = -\tilde{C}x_2', \end{aligned}$$

and

$$\begin{aligned}
P_3' &: \mathcal{L}_{\infty,e}^{2n+2} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\
y_3' &= P_3' u_3', \\
&= \tilde{P} v_1, \\
&= \tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + (\tilde{u}_0 - \tilde{v}_2)).
\end{aligned} \tag{6.27}$$

The configuration shown in Figure 6.9 comprises the subsystems:

$$\begin{aligned}
P_1 &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{2n} : y_1' \mapsto (x_1', u_1') \\
& x_1' = -y_1', u_1' = \begin{pmatrix} 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1',
\end{aligned}$$

$$\begin{aligned}
P_2 &: \mathcal{L}_{\infty,e}^{n} \to \mathcal{L}_{\infty,e}^{n+1} : \quad x_2' \mapsto u_2' \\
& u_2' = \begin{pmatrix} 0 & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, z_2 = x_2', \hat{v}_2 = -\tilde{C} x_2',
\end{aligned}$$

and

$$\begin{aligned}
P_3 &: \mathcal{L}_{\infty,e}^{2n+2} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\
y_3' &= P_3 u_3', \\
&= \tilde{P}(\tilde{u}_0 - \tilde{v}_2),
\end{aligned} \tag{6.28}$$

From the above definitions given for P_1, P_2 and P_3, we conclude that with $\tilde{P} = P_{Linear}$ the resulting linear configuration of the system shown in Figure 6.9 is equivalent to the configuration shown in Figure 4.10. This leads to the conclusion that the linear gain $\|\Pi_{(3)}\|$ calculated for these systems is the same.

Similarly to the approach taken in Chapter 4, to apply Theorem 2.13 to this system, we must satisfy inequality (6.26). In the following two subsections, the two sides of this inequality will be evaluated, namely the linear gain $\|\Pi_{(3)}\|$ and the gap value $\vec{\delta}(P_3, P_3')$.

6.3.1 Finding $\|\Pi_{(3)}\|$ for an Affine Nonlinear System with Stable and Unstable Plant Nonlinearity

We start with the RHS of inequality (6.26), with $f_s^*(z_1)$ removed, P_{Linear} given by (6.23) – (6.24) corresponds to $\tilde{P}$ given by (4.10) – (4.11). Then the linear gain $\|\Pi_{(3)}\|$ calculated for the system shown in Figure 6.9 is the same as the linear gain $\|\Pi_{(3)}\|$ for the system shown in Figure 4.10. In this subsection the reader is also referred to Subsection 4.4.1 for the procedure followed to calculate this value. In Subsection 4.4.1 it was found that:

$$\|\Pi_3\| \quad \leq \quad \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\|Q\| \left\| \begin{array}{ccc} u_0' & x_0' & y_0' \end{array} \right\|^\top}{\|u_0',x_0',y_0'\|} = \|Q\|.$$

where $Q = \begin{pmatrix} \Lambda \\ c \end{pmatrix}$, with Λ, c matrices of dimension 4×6 and 1×6 respectively, their terms comprising closed loop functions of system $[P_{Linear}, \tilde{C}]$. Hence from (6.26) the gap between perturbed and unperturbed plants must satisfy:

$$\vec{\delta}(P_3, P_3') < \frac{1}{\|Q\|}. \tag{6.29}$$

6.3.2 Finding the Gap Metric for a Nonlinear System with Stable and Unstable Plant Nonlinearity

In this subsection the LHS, $\vec{\delta}(P_3, P_3')$, of the inequality (6.26) is considered. To find $\vec{\delta}(P_3, P_3')$ an analogous approach to that in Subsection 4.4.2 is used. Consider the plants P_3 and P_3' shown in Figure 6.10, the graphs for $\tilde{P}$, P_3 and P_3' are defined to be:

$$\mathcal{G}_{\tilde{P}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : y = \tilde{P}u, \|u\| < \infty, \|y\| < \infty \right\}, \tag{6.30}$$

$$\mathcal{G}_{P_3} := \left\{ \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{array}{c} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{array} \right\| < \infty, y_3' = \tilde{P}(\tilde{u}_0 - \tilde{v}_2) \right\}, \tag{6.31}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} : \left\| \begin{array}{c} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{array} \right\| < \infty, y_3' = \tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + (\tilde{u}_0 - \tilde{v}_2)) \right\}. \tag{6.32}$$

In order to find a bound on the gap between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$, a surjective map Φ is required between these graphs. The following lemma is used to define this map. First, consider the nonlinear part of the plant P_3' shown in Figure 6.10b, for this component the following lemma is used.

Lemma 6.4. *Let f_u^* satisfy Assumption 6.2, and consider the following equation:*

$$v_1 = f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2. \tag{6.33}$$

Then:

$$\|\tilde{v}_2\| < \infty, \|\tilde{u}_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

and

$$\|v_1\| < \infty, \|\tilde{u}_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Proof. We will first prove that:

$$\|\tilde{v}_2\| < \infty, \|\tilde{u}_0\| < \infty \Rightarrow \|v_1\| < \infty.$$

Let $\|\tilde{v}_2\| < \infty, \|\tilde{u}_0\| < \infty$, and Assumption 6.2 since f_u^* is a bounded function, we have $\|f_u^*(\tilde{z}_1)\| < \infty$ and $\|f_u^*(\tilde{z}_2)\| < \infty$, since $\|\tilde{v}_2\| < \infty$ and $\|\tilde{u}_0\| < \infty$,

$$\begin{aligned}
&\|v_1\| = \|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2\|, \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|f_u^*(\tilde{z}_2)\| + \|\tilde{v}_2\| + \|\tilde{u}_0\|, \\
&< \infty.
\end{aligned}$$

as required.

Next we will prove that:

$$\|v_1\| < \infty, \|\tilde{u}_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Let $\|v_1\| < \infty, \|\tilde{u}_0\| < \infty$, here $\tilde{v}_2$ can be obtained from (6.33):

$$\tilde{v}_2 = f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - v_1.$$

Also since f_u^* is a bounded function, and since $\|v_1\| < \infty$, $\|\tilde{u}_0\| < \infty$, then:

$$\begin{aligned}
\|\tilde{v}_2\| \quad &= \quad \|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - v_1\|, \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|f_u^*(\tilde{z}_2)\| + \|v_1\| + \|\tilde{u}_0\|, \\
&< \infty.
\end{aligned}$$

as required. □

In this analysis Φ is defined to be the map between stable P_3 and P_3'. The plants P_3 and P_3' are stable if the plant $\tilde{P}$ is stable, as shown by the following lemma.

Lemma 6.5. *Suppose $\tilde{P}$ is BIBO stable and let f_u^* satisfy Assumption 6.2. Then P_3 and P_3' given by Figure 6.10 and (6.27) and (6.28), respectively, are stable.*

Proof. First we prove that if $\tilde{P}$ is stable then P_3 is stable. In order to do that we must prove that if $\|u_3'\| < \infty$ then $\|P_3 u_3'\| < \infty$. So, let $\|u_3'\| < \infty$. While $u_3' = \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$, so $\|\tilde{u}_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$.

Then by definition:

$$\begin{aligned} \|y_3'\| &= \|P_3 u_3'\|, \\ &= \|\tilde{P}(\tilde{u}_0 + y_0 - \tilde{v}_2)\|, \\ &\leq \|\tilde{P}\|(\|\tilde{u}_0\| + \|\tilde{v}_2\|), \\ < \infty. \end{aligned}$$

Hence P_3 is stable.

Similarly to prove that if $\tilde{P}$ is stable then P_3' is stable we must prove that if $\|u_3'\| < \infty$ then $\|P_3' u_3'\| < \infty$. Let $\|u_3'\| < \infty$. While $u_3' = \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$, then $\|\tilde{u}_0\|, \|\tilde{v}_2\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$.

By definition:

$$\begin{aligned} y_3' &= P_3' u_3' = \tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2), \\ \|y_3'\| &= \|\tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2)\|, \end{aligned}$$

Using Lemma 6.4, since $\|\tilde{u}_0\| < \infty$ and $\|\tilde{v}_2\| < \infty$ then:

$$\|v_1\| = \|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2\| < \infty.$$

Since $\tilde{P}$ is stable, it follows that:

$$\|y_3'\| = \|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0\| < \infty.$$

Then P_3' is stable.

$\square$

Since P_3 and P_3' are stable, the graphs for P_3 and P_3' can be written in the form given in the following proposition:

Proposition 6.6. *Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2. Let P_3 and P_3' be given by Figure 6.10 and (6.27) and (6.28), respectively. Then the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$*

satisfy:

$$\mathcal{G}_{P_3} := \left\{ \left(\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3 \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) : \left\| \begin{matrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{matrix} \right\| < \infty \right\}, \tag{6.34}$$

$$\mathcal{G}_{P_3'} := \left\{ \left(\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3' \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) : \left\| \begin{matrix} u_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{matrix} \right\| < \infty \right\}. \tag{6.35}$$

Proof. To show that if $\tilde{P}$ is stable and f_u^* satisfies Assumption 6.2 then $\mathcal{G}_{P_3'}$ given in (6.32) can be written as that given in (6.35), let us denote the set given in (6.35) as $\mathcal{A}$. Let $\left(\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}, P_3' \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \right) \in \mathcal{A}$, i.e $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$, $\tilde{P}$ is stable and f_u^* satisfy Assumption 6.2, then using Lemma 6.5, P_3' is stable. Since $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ and P_3' is stable then $\|y_3'\| = \|P_3'(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_3'}$.

Next we prove that $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Let $\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix} \in \mathcal{G}_{P_1'}$, i.e $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2, y_3')\| < \infty$ and $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2)$. Then $\|(u_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ and $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2) + \tilde{u}_0 - \tilde{v}_2) = P_3'(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$.

This leads to $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_3'} = \mathcal{A}$.

To show that $\mathcal{G}_{P_3}$ given by (6.34) is equivalent to that given by (6.31), set $f_u^*(z_1) = f_u^*(z_2) = 0$. In this case $\mathcal{G}_{P_3}$ follows as a special case, as required. □

The map Φ between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ is defined using the following proposition:

Proposition 6.7. *Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2. Let P_3 and P_3' be given by Figure 6.10 and (6.27) and (6.28), respectively. Then there exists a map*

$\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by:

$$\Phi \begin{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ P_3 \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \\ P_3' \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \end{pmatrix}, \tag{6.36}$$

Furthermore this map is surjective.

Proof. First we prove that if

$$x = \begin{pmatrix} (u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2') & P_3(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2') \end{pmatrix}^\top \in \mathcal{G}_{P_3},$$

then $\Phi(x) \in \mathcal{G}_{P_3'}$. Since $x \in \mathcal{G}_{P_3}$ then $\|(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')\|, \|y_3''\| = \|P_3(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')\| < \infty$. Let $y = \begin{pmatrix} (\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & P_3'(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) \end{pmatrix}^\top = \Phi(x)$. We need to show that $y_3' = P_3'(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$, $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$. It follows from (6.36) that $(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) = (u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')$ and $y_3' = P_3'(u_0'', \tilde{v}_2', \tilde{z}_1', \tilde{z}_2')$, then $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$.

Using Proposition 6.6 since $\tilde{P}$ is stable and f_u^* satisfies Assumption 6.2 and $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$, then $\|y_3'\| = \|P_3'(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$, and hence:

$$\begin{aligned} y &= \begin{pmatrix} (\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & P_3'((\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) \end{pmatrix}^\top \\ &= \begin{pmatrix} (\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2) & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}. \end{aligned}$$

as required.
Next, to prove that Φ is surjective, let $u = (\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)$ and using Proposition 2.14 since P_3 and P_3' are stable and since $\|(\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2)\| < \infty$ then the map given in 6.36 is surjective, as required.

□

Using the previous results, a bound on the gap between P_3 and P_3' is given using the following theorem

Proposition 6.8. *Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2. Let P_3 and P_3' be given by Figure 6.10 and (6.27) and (6.28), respectively. Then a bound on the gap* *between P_3 and P_3' is*

$$\vec{\delta}(P_3, P_3') \leq \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\|}{\|\tilde{z}_1, \tilde{z}_2\|}. \tag{6.37}$$

Proof. Since $\tilde{P}$ is stable and since f_u^* satisfies Assumption 6.2 then using Proposition 6.7 there exists a surjective map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (6.36). Then the gap between P_3 and P_3' is given by:

$$\begin{aligned}
\vec{\delta}(P_3, P_3') &\leq \sup_{x \in \mathcal{G}_{P_3}\setminus\{0\}} \frac{\|(\Phi - I)x\|}{\|x\|}, \\
&= \sup_{\left\|\begin{smallmatrix}\tilde{u}_0\\ \tilde{v}_2\\ \tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}((f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)) + \tilde{u}_0) - \tilde{v}_2) - \tilde{P}(\tilde{u}_0 - \tilde{v}_2)\|}{\|\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2\|}, \\
&\leq \sup_{\left\|\begin{smallmatrix}\tilde{u}_0\\ \tilde{v}_2\\ \tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\|}{\|\tilde{u}_0, \tilde{v}_2, \tilde{z}_1, \tilde{z}_2\|}.
\end{aligned}$$

Hence:

$$\vec{\delta}(P_3, P_3') \leq \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\| \neq 0} \frac{\|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\|}{\|\tilde{z}_1, \tilde{z}_2\|}.$$

as required.

□

The bound on the gap given in (6.37) is related to the Lipschitz condition given in the following lemma:

Lemma 6.9. *Let $f : R^n \to R^n$ be a continuous function. Suppose that $[\partial f/\partial x]$ exists and is continuous. Suppose there is a constant $L \geq 0$ and that*

$$\left\|\frac{\partial f}{\partial x}(x)\right\|_\infty \leq L \quad \forall x.$$

Then

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for all x and y.

Proof. See Khalil (2002). □

where L is called the Lipschitz constant.

Using the previous results, a bound on the gap between P_3 and P_3' is given using the following theorem

Theorem 6.10. *Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2. Let P_3 and P_3' be given by Figure 6.10 and (6.27) and (6.28), respectively. Then a bound on the gap between P_3 and P_3' is*

$$\vec{\delta}(P_3, P_3') \quad \leq \quad \|\tilde{P}\|\sqrt{3}L. \tag{6.38}$$

Proof. Using Proposition 6.8 we have:

$$\vec{\delta}(P_3, P_3') \quad \leq \quad \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\|\neq 0} \frac{\|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\|}{\|\tilde{z}_1, \tilde{z}_2\|}.$$

This result can be related to the Lipschitz condition. Using Lemma 6.9, we can write:

$$\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\| \leq L\|\tilde{z}_1 - \tilde{z}_2\|, \quad \forall(\tilde{z}_1 - \tilde{z}_2) \in \mathcal{W}.$$

Then a gap bound can be further simplified as:

$$\begin{aligned}
\vec{\delta}(P_3, P_3') \quad &\leq \quad \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\|\neq 0} \frac{\|\tilde{P}\|\|f_u^*(\tilde{z}_1) - f_u^*(\tilde{z}_2)\|}{\|\tilde{z}_1, \tilde{z}_2\|}, \\
&\leq \quad \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\|\neq 0} \frac{\|\tilde{P}\|L\|\tilde{z}_1 - \tilde{z}_2\|}{\|\tilde{z}_1, \tilde{z}_2\|}, \\
&\leq \quad \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\|\neq 0} \frac{\|\tilde{P}\|L(\|\tilde{z}_1\| + \|\tilde{z}_2\|)}{\|\tilde{z}_1, \tilde{z}_2\|}
\end{aligned}$$

$$\begin{aligned}
&\leq \quad \sup_{\left\|\begin{smallmatrix}\tilde{z}_1\\ \tilde{z}_2\end{smallmatrix}\right\|\neq 0} \frac{\|\tilde{P}\|\sqrt{3}L\|\tilde{z}_1, \tilde{z}_2\|}{\|\tilde{z}_1, \tilde{z}_2\|} \\
&\leq \quad \|\tilde{P}\|\sqrt{3}L.
\end{aligned}$$

as required.

□

Then according to the following proposition robust stability is preserved for the system shown in Figure 6.6.

Proposition 6.11. *Consider the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 6.6. Suppose $\tilde{P}$ is stable and let f_u^* satisfy Assumption 6.2. Then $[P_1', P_2', P_3']$ has a robust stability margin.*

Proof. Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2, then by Lemmas 6.4, 6.5, and using Proposition 6.6 for the systems P_3 and P_3' given by Figure 6.10 and equations (6.27) and (6.28), respectively, the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ can be given by (6.34) and (6.35), respectively. Using Proposition 6.7, then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (6.36). This leads to the presence of a finite gap value between the linear and nonlinear configurations of this system given by the inequality (6.38). Then the system $[P_1', P_2', P_3']$ given by Figure 6.6 and (6.17)-(6.22) has a robust stability margin.

$\square$

The main result Theorem 6.3 easily follows from Theorem 6.12 which we establish next.

Theorem 6.12. *Consider the nonlinear closed loop system shown in Figure 6.1 and given by (6.17)-(6.22). Suppose $\tilde{P}$ is stable and let f_u^* satisfy Assumption 6.2. Then this system has a robust stability margin b_{P_1,C_1} which satisfies the inequality*

$$b_{P_1,C_1} \geq \|Q\|^{-1}. \tag{6.39}$$

Proof. Let $\frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 6.6, let $\frac{1}{\|\Pi'_{(3)}\|}$ be a stability margin for the system $[P_1', P_2', P_3']$ shown in Figure 6.4, finally let $b_{P_1,C_1} = \frac{1}{\|\Pi_{P_{Linear}//\tilde{C}}\|}$ be a stability margin for the system shown in Figure

6.1. Then

$$\begin{aligned}
\|Q\| = \|\Pi_{(3)}\| &= \sup_{\|u_0',x_0',y_0'\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}\right\|}{\|u_0',x_0',y_0'\|}, \\
&= \sup_{\|u_0,d_1,d_2,d_3,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}\begin{pmatrix} \tilde{u}_0 & d_1 & d_2 & d_3 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,d_1,d_2,d_3,y_0,x_0\|}, \\
&\geq \sup_{\|u_0,0,0,0,y_0,x_0\|\neq 0} \frac{\left\|\Pi_{(3)}'\begin{pmatrix} \tilde{u}_0 & 0 & 0 & 0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,0,0,0,y_0,x_0\|} = \|\Pi_{(3)}'\|, \\
\|\Pi_{(3)}'\| &= \sup_{\|u_0,y_0,x_0\|\neq 0} \frac{\left\|\Pi_3'\begin{pmatrix} \tilde{u}_0 & y_0 & x_0 \end{pmatrix}^\top\right\|}{\|u_0,y_0,x_0\|}, \\
&\geq \sup_{\|u_0,y_0,0\|\neq 0} \frac{\left\|\Pi_{P_{Linear}//\tilde{C}}\begin{pmatrix} \tilde{u}_0 & y_0 & 0 \end{pmatrix}^\top\right\|}{\|\tilde{u}_0,y_0,0\|}, \\
&= \|\Pi_{P_{Linear}//\tilde{C}}\|.
\end{aligned}$$

This leads us to

$$b_{P_1,C_1} = \frac{1}{\|\Pi_{P_{Linear}//\tilde{C}}\|} \geq \frac{1}{\|\Pi_{(3)}'\|} \geq \frac{1}{\|\Pi_{(3)}\|} = \|Q\|^{-1}.$$

Then the existence of a stability margin for the system shown in Figure 6.6 guarantees the existence of a stability margin for the system $[P_1, C_1]$ shown in Figure 6.1. Also, since $\tilde{P}$ is stable and f_u^* satisfies Assumption 6.2, then by Proposition 6.11, the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 6.6, has a robust stability margin. This leads to the conclusion that the system $[P_1, C_1]$ given by Figure 6.1 and (6.17)-(6.22) also has a robust stability margin. □

Based on Theorems 6.10 and 6.12 we can write the following corollary:

Corollary 6.13. *Consider the nonlinear closed loop system $[P_1, C_1]$ shown in Figure 6.1 and given by (6.17)-(6.22). Let $\tilde{P}$ be stable and let f_u^* satisfy Assumption 6.2. Then this system is stable if*

$$\|\tilde{P}\|\sqrt{3}L < \|Q\|^{-1}.$$

Proof. Using Theorem 6.10 inequality (6.38), since:

$$\vec{\delta}(P_3, P_3') \leq \|\tilde{P}\|\sqrt{3}L,$$

and using Theorem 6.12 inequality (6.39), since:

$$b_{P_1,C_1} \geq \|Q\|^{-1},$$

It follows that if

$$\|\tilde{P}\|\sqrt{3}L < \|Q\|^{-1},$$

we have:

$$\vec{\delta}(P_3, P_3') \leq \|\tilde{P}\|\sqrt{3}L < \|Q\|^{-1} \leq b_{P_1,C_1},$$

then $\vec{\delta}(P_3, P_3') < b_{P_1,C_1}$ and the conditions hold from Theorem 2.11, hence stability.

as required. □

6.4 Summary

The gap analysis in this chapter was carried out for a special class of the unstable affine systems considered in Section 5.3, where the linear part of the plant is assumed to be unstable. Moreover, the system is assumed to have only a single nonlinear component, $f^*(x^*)$ which includes a stable component, $f_s^*(x^*)$, and an unstable component, $f_u^*(x^*)$. The configuration used in the analysis undertaken in this chapter has employed the linear part of the controller, as a stabilizing feedback to the underlying coprime factors of the plant to produce a new stabilized plant with a stable linear part. The results showed that the stability for this system relies on the norm of the stabilized nominal plant and a Lipschitz constant which represents an upper bound on the difference between the nonlinear components of the plant and the controller (equation (6.38)).

Chapter 7

Conclusions and Future Work

7.1 Summary

This thesis focuses on control analysis that is based on robust stability theory and uses the feedback linearization method to control a nonlinear model. The feedback linearization approach is a method based on linearizing the input-output relation of a nonlinear system. However, to cancel the nonlinearity in the system perfect knowledge of the state equation of the system is needed. Since perfect knowledge is not available in practice, the degree to which modelling error affects performance becomes an important issue. In the presence of model uncertainty, the limitations associated with feedback lineaization has prompted the work done in this thesis.

Two main approaches were undertaken to investigate the robustness of feedback control designs and study the robust stability conditions for these systems. One approach is using the small gain theorem which despite its simplicity forms a fundamental basis for many robustness results, but unfortunately does not give desirable results in the case of having an unstable plant in the system.

The other approach is to use the gap metric introduced in Georgiou and Smith (1997) to study the robustness stability for nonlinear systems. In this approach, the presence of the nonlinear elements in multiple blocks in a nonlinear system leaded to a significant conservatism problem. To solve this problem, the procedure carried out included using new configurations for the considered systems to allow the nonlinear parts of the plant and controller to be taken into account in calculating a bound for the gap value. Using these new configurations, minimizing the gap value makes the nonlinear system resemble its nominal model. In this approach, robustness and performance margins were established for these systems. This thesis introduced the nonlinear gap metric and employed a 'network' result (Theorem (10)) Georgiou and Smith (1997) to undertake stability analysis for nonlinear systems.

In this thesis, new tools that have potential to provide reduced conservatism were developed, for example, the gap metric approach undertaken in this thesis provided better stability conditions than the small gain theorem in the case of a small gap value for stable systems (as demonstrated in Example 4.5) and unlike the small gain theorem the gap metric approach provided stability conditions for unstable systems.

Robustness analysis using the gap metric for unstable affine systems was carried out in Chapter 5. This analysis involved using coprime factors to represent these systems. Here, two cases of affine systems were considered. The first case introduced a control law which followed the feedback linearzation approach and carried out an inverting action to cancel all the nonlinear terms in the system, including the inherently stabilizing nonlinearities which can be used to stabilize the plant. The stability condition for this case showed that the stability for this system depends on the nonlinear input part of the controller and on how exact the inversion of the plant nonlinearity is, within the nonlinear part of the controller.

Conversely, an improved control law which classified the nonlinearity in the system into stable and unstable components was introduced in the second case. This controller preserved the stabilizing rule of the inherently stabilizing nonlinearities in the plant instead of aiming to cancel them and cancels only the unstable nonlinear part of the plant. The stability condition for this system showed that the stability for this system depends on the difference between the input nonlinear component and the stable part of the input nonlinear component of the controller and on how exact the inversion of the unstable nonlinear of the plant is within the nonlinear part of the controller.

In the case of having only a plant nonlinear part in the system, the linear stabilising component of the controller was used to stabilize the linear unstable part of the plant. The stability condition for the stabilized system showed that the stability for this system depends on the gain of the stabilized linear part of the system and a Lipschitz constant which represents a bound on the difference between the nonlinear components of the plant and the controller.

Using the gap metric approach it was shown that the existence of a stability margin for a transformed configuration of a nonlinear system guarantees the existence of a stability margin for the original configuration of this system.

Using this approach, more general cases were considered, such as robustness analysis for feedback linearizing controllers in the presence of output unstructured uncertainties (inverse muliplicative uncertainties).

7.2 Future Work

The work done in this thesis can be further extended in the following directions:

- Improving the gain bounds for the results given in this thesis which are obtained under strong assumptions on boundedness of nonlinearities to give the simplest global results, but these results should be generalisable to local and semi-global results in the absence of such assumptions. Gain bounds can be lightened by using normalized nonlinear coprime factors to represent the affine nonlinear systems in the robustness analysis. This can lead to more complicated analysis, however, using this kind of representation for nonlinear systems has been discussed in many publications (for example, see Scherpen and Van der Schaft (1994)) and can lead to improved stability conditions.

- Further generalization for the robustness analysis undertaken in this thesis to cover more system classes. Since the feedback linearizable systems considered in this thesis are fully linearizable systems (where the relative degree equals the state dimension of the system), the robustness analysis undertaken can be generalized to include partially linearizable systems (where the relative degree is less than the state dimension of the system), including many important topics related to the feedback linearization method such as internal states and zero dynamics. Since the relative degree and the zero dynamics cannot be altered by feedback, systems with unstable zero dynamics are harder to control than systems with stable zero dynamics. Moreover, in the presence of modelling errors these concepts may be non-robust Kokotović and Arcak (2001). Therefore, it is important that feedback linearizing controllers for these systems should be designed with the help of analytical tools (such as the gap metric) to guarantee robust stability for these systems.

- To confirm the utility of the tools developed in this thesis, the gap metric robust stability analysis carried out in Chapter 3 can be applied to design a linearizing feedback controller for an isometric stimulated muscle model and find a robust stability condition based on the nonlinear gap given in Georgiou and Smith (1997) (Theorem 1). This model represents an accurate model of the response of muscle to electrical stimulation and is structured as a Hammerstein model, and is well known in the literature Le et al. (2010). This is likely to require a biased notion of stability (see Bradley (2010)). Moreover, These analytical results should be confirmed by implementing the controller designed using this approach into robot models which are used to assist the rehabilitation of patients after suffering a stroke.

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