LQR performance for multi-agent systems: benefits of introducing delayed inter-agent measurements

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***Abstract*— This paper deals with the design of an optimal controller for a set of identical multi-agent systems. The problem under consideration is to examine if there is any benefit to adding to the classical local optimal control law, obtained from solving a Riccati equation, a term which depends on delayed relative information with respect to neighbouring agents. The resulting control law has a local linear feedback term (from solving the Riccati equation) and a consensus-like term which depends on a delayed version of the relative states with respect to its neighbours. The resulting closed loop system at a network level is linear and involves delayed states. A Lyapunov-Krasovskii approach is used to synthesize the gain associated with the consensus term to provide sub-optimal LQR-like performance at a network level. Situations are demonstrated when this approach provides better performance (in terms of the LQR cost) than when a traditional decentralised approach is adopted.**

I. Introduction

Research in multi-agent systems has received a great deal of attention over the past decade. One problem which is addressed in many of these papers involves ensuring a collection of multiple agents, interconnected over an information network, operate in agreement or in a synchronized manner. Often the topology of the interconnections is captured as a graph, and many researchers have obtained novel results by combining graph theory along with systems and control ideas. See [1], [6], [16], [19], [21], [26], [28], [34], [39] and the references therein for further details.

Recently progress has been made in terms of stabilization and consensus in a network of dynamical systems subject to performance guarantees such as the rate of convergence and LQR/$\mathcal{H}_2$ performance. In [43], the weights of the Laplacian matrix of the graph are optimized to attain faster convergence to a consensus value: this is posed as a convex optimization problem and solved using LMI tools. The algebraic connectivity, characterized by the second smallest eigenvalue of the Laplacian matrix, is maximized in [18] to improve the convergence performance. An optimal communication topology for multi-agent systems is sought in [5] to achieve a faster rate of convergence. A distributed control methodology ensuring LQR performance in the case of a network of linear homogenous systems is presented in [2]. A decentralized receding horizon controller with guaranteed LQR performance for coordinated problems is proposed in [17] and its efficacy is demonstrated by an application relating to coordination among a collection of unmanned air vehicles. In [20], the relationship between the interconnection graph and closed-loop performance in the design of distributed control laws is studied using an LQR cost function. In [25], decentralized static output feedback controllers are used to stabilize a homogeneous network comprising a class of dynamical systems with guaranteed $\mathcal{H}_2$ performance, where an upper bound on the collective performance is given, depending only on the node level quadratic performance. LQ optimal control laws for a wide class of systems, known as spatially distributed large scale systems, are developed in [30] by making use of an approximation method. In [4], LQR optimal algorithms for continuous as well as discrete time consensus are developed, where the agent dynamics are restricted to be single integrators. However, interesting relations between the optimality in LQR performance and the Laplacian matrix of the underlying graph are also developed. In [24] procedures to design distributed controllers with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance have been proposed for a certain class of decomposable systems. Although delays are an ubiquitous factor associated with network interconnections as a result of information exchange over a communication medium, in all the above research work ([2], [4], [17], [20], [24], [25], [30]) no attempt is made to explicitly address or exploit the effects.

Significant research efforts analyzing stability and performance of collective dynamics (at network level) in the face of different types of delays have taken place in the recent past: references [3], [23], [31]–[33], [35], [37], [40], [44] are few examples, although this list is not exhaustive.[1] Necessary and sufficient conditions for average consensus problems in networks of linear agents in the presence of communication delays have been derived in [33]. Stability criteria associated with the consensus dynamics in networks of agents in the presence of communication delays was subsequently developed in [40] using Lyapunov Krasovskii based techniques. Moreover, the strong dependency of the magnitude of delay and the initial conditions on the consensus value was also

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[1]Another research area involving the stabilization of time-delay systems is networked control systems [15], [42]. This is not the class of problems considered in this paper.

established in [40]. In [31], a network of second order dynamical systems with heterogeneously delayed exchange of information between agents is considered, where flocking or rendezvous is obtained using decentralized control. This can also be tuned locally, based only on the delays to the local neighbours. Both frequency and time domain approaches are utilized in [31] to establish delay dependent and independent collective stability. Subsequently the theory was extended in [35] to the case of a network formed from a certain class of nonlinear systems. The robustness of linear consensus algorithms and conditions for convergence subject to node level self delays and relative measurement delays were developed and reported in [32] building on the research described in [31] and [35]. 'Scalable' delay dependent synthesis of consensus controllers for linear multi agent networks making use of delay dependent conditions is proposed in [32]. Reference [44] reports an independent attempt to achieve second order consensus using delayed position and velocity information. Recently another methodology, based on a cluster treatment of characteristic roots, has been proposed in [3] to study the effect of large and uniform delays in second order consensus problems with undirected graphs. In [37] the performance of consensus algorithms in terms of providing a fast convergence rate involving communication delays, was studied for second order multi agent systems.

Previous efforts to investigate stability and robustness in the face of delays clearly emphasizes the need to account for these delays explicitly. However research in this direction is limited when compared to the available voluminous research in the case of 'delay free' consensus algorithms. The main contribution of the present paper is the idea of designing delay dependent distributed optimal LQR control laws for homogeneous linear multi agent networks. At a collective network level, a certain level of guaranteed cost is attained, which takes into account the control effort. A Lyapunov-Krasovskii functional approach is used for synthesizing the control laws in the presence of fixed delays. The efficacy of the proposed approaches are demonstrated by considering a homogeneous linear multi agent network where the node level dynamics are represented as double integrators as in [31], [37], [44].

II. Preliminaries

In this paper the set of real numbers is denoted by $\mathbb{R}$. For all positive integers n, m, the sets $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ and $\mathbb{S}_n^+$ represent the set of n dimensional vectors, the set of $n\times m$ matrices and the set of symmetric positive definite matrices of order n, respectively. For all $M \in \mathbb{R}^{n\times n}$, the notation $\mathcal{H}e\{M\}$ stands for $M + M^T$. A column vector is denoted by $\mathcal{C}ol(.)$ and a diagonal matrix is denoted by $\mathcal{D}iag(.)$. For $x \in \mathbb{R}^n$, $|x|$ is the Euclidian norm of the vector x. An identity matrix of dimension $n \times n$ is denoted by I_n. The Kronecker product is denoted by the symbol $\otimes$.

The notations for time delay systems are standard. For $\tau > 0$, the notation x_t refers to a function defined over the interval $[-\tau,\ 0]$ and such that for all $\theta \in [-\tau,\ 0]$, $x_t(\theta) = x(t+\theta)$. Finally, the norm $\|x_t\|_\tau = sup_{\theta\in[-\tau,\ 0]}|x(t+\theta)|$.

Basic concepts from graph theory are described in this section. Standard texts such as [11] can be referred to for further reading on graph theory. In this paper bidirectional communication is assumed and hence the graphs for the networks are undirected. The graph is assumed to contain no loops and no multiple edges between two nodes. The adjacency matrix for the graph $\mathcal{A}(\mathcal{G}) = [a_{ij}]$, is defined by $a_{ij} = 1$ if i and j are adjacent nodes, and $a_{ij} = 0$ otherwise. The adjacency matrix thus defined is symmetric. The degree matrix is represented by the symbol $\Delta(\mathcal{G}) = [\delta_{ij}]$. The matrix $\Delta(\mathcal{G})$ is a diagonal matrix, and each element δ_{ii} is the degree of the i^{th} vertex. The difference $\Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ defines the Laplacian of $\mathcal{G}$, written as $\mathcal{L}$. For an undirected graph, $\mathcal{L}$ is symmetric positive semidefinite. The smallest eigenvalue of the Laplacian $\mathcal{L}$ is zero and the corresponding eigenvector is given by $\mathbf{1} = \mathcal{C}ol(1,\ldots 1)$. The Laplacian is always rank deficient and the rank of $\mathcal{L}$ is $n-1$ if and only if $\mathcal{G}$ contains a spanning tree. The maximal eigenvalue of the $\mathcal{L}$ is bounded by $2\max_i \delta_{ii}$ [27].

III. Problem formulation

A. Optimal LQR control without delays

Consider a network of N identical linear systems given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

for $i = 1,\ldots,N$, where the states $x_i(t) \in \mathbb{R}^n$ and the control inputs $u_i(t) \in \mathbb{R}^m$. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant and it is assumed that the pair (A, B) is controllable. Each agent is assumed to have knowledge of its own local state information.

The objective is to achieve a suboptimal level of LQR-like closed-loop performance at each node. Specifically the control design objective can be stated as one of designing the gain matrix K such that the cost functions

$$J_i = \int_0^\infty (x_i^T(t)Qx_i(t) + u_i^T(t)Ru_i(t))dt, \tag{2}$$

are minimized for all $i = 1,\ldots,N$, where $Q \in \mathbb{S}_n^+$ and $R \in \mathbb{S}_m^+$. It is well known that optimal control for each agent is achieved by finding a symmetric positive definite matrix P_{opt} which solves the algebraic Riccati equation

$$A^TP_{opt} + P_{opt}A - P_{opt}BR^{-1}B^TP_{opt} + Q = 0.$$

The optimal control gain is then given by

$$u_i(t) = -K_{opt}x_i(t) = -R^{-1}B^T P_{opt}x_i(t),$$

and the cost function satisfies

$$J_i \leq x_i^T(0)P_{opt}x_i(0).$$

The problem under consideration in this paper is to include in the control law of each agent an additional term which uses the (delayed) measurements of the relative positions between the agent and its neighbours viewed from a graph perspective.

B. Collaborative control law using delayed measurements

Assume now that each agent has knowledge of delayed relative state information. The motivation for the introduction of delays in the relative position between agents has been considered in the literature: see for example [28], [34], [45].

In this paper the relative information communicated to each agent is given by

$$z_i(t) = \sum_{j\in\mathcal{J}_i}(x_i(t-\tau) - x_j(t-\tau)) \quad (3)$$

where τ is a fixed delay in the communication of relative information and $\mathcal{J}_i \subset \{1, 2, \ldots N\}/\{i\}$ is the index set of the neighbours to node i. Here it is assumed that the delay is fixed and known. This assumption can be justified because of the use of buffers in the communication protocols. This information will be exploited as part of the design of the control law.

The intention is to design control laws of the form

$$u_i(t) = -K_{opt}x_i(t) - Hz_i(t), \quad (4)$$

where $K_{opt} \in \mathbb{R}^{m\times n}$ results from the solution of the Riccati equation, and the relative information scaling matrix, $H \in \mathbb{R}^{m\times n}$, is to be designed to achieve consensus. Substituting from (4) into (1), the closed loop system at node level is given by

$$\dot{x}_i(t) = (A - BK_{opt})x_i(t) - BHz_i(t). \quad (5)$$

Using Kronecker products, the system in (1) at a network level is given by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t), \quad (6)$$

where the augmented state $X(t) = \mathcal{C}ol(x_1(t), \ldots, x_N(t))$ and $U(t) = \mathcal{C}ol(u_1(t), \ldots, u_N(t))$ represents the control input. The relative information in (3) at a network level can be written as

$$Z(t) = (\mathcal{L} \otimes I_n)X(t-\tau), \quad (7)$$

where $Z(t) = \mathcal{C}ol(z_1(t), \ldots, z_N(t))$ and $\mathcal{L}$ is the Laplacian matrix associated with the sets $\mathcal{J}_i$. Using (7) the control law is given by

$$U(t) = -(I_N \otimes K_{opt})X(t) - (\mathcal{L} \otimes BH)X(t-\tau). \quad (8)$$

Substituting (8) into (6), the closed loop system at a network level is given by

$$\dot{X}(t) = \big(I_N \otimes A_{opt}\big)X(t) - (\mathcal{L} \otimes BH)X(t-\tau), \quad (9)$$

where $A_{opt} = A - BK_{opt}$.

Consider the global network level performance index $J = \sum_{i=1}^{N} J_i$. This can be rewritten using the augmented vector $X(t)$ as

$$J = \int_0^\infty (X^T(t)(I_N \otimes Q)X(t) + U^T(t)((I_N \otimes R)U(t))dt. \quad (10)$$

The objective is to design the gain matrix $H \in \mathbb{R}^{n\times m}$ in a scenario in which the communication delay τ a known and fixed.

IV. Main result

The following theorem concerns the design of the gain matrix H.

Theorem 4.1: For a given symmetric $\mathcal{L}$ and a given scalar $\epsilon > 0$, assume that there exist $\bar{P}$, $\bar{S}$ and $\bar{Z}$ in $\mathbb{S}_n^+$, and a matrix $\bar{Y}$ in $\mathbb{R}^{n\times n}$, such that the following LMI optimization is performed

$$\left.\begin{array}{c} \min_{\bar{P},\bar{S},\bar{Z},\bar{Y},\epsilon}\mu, \\ \text{subject to} \\ \\ \Pi(0) \prec 0, \quad \Pi(\lambda_{max}) \prec 0, \\ \\ \Psi(\bar{P}) \succ 0, \quad \Psi(\bar{S}) \succ 0, \quad \Psi(\bar{Z}) \succ 0, \end{array}\right\} \quad (11)$$

where

$$\begin{aligned}
\Pi(\lambda) &= \begin{bmatrix} \Pi_0(\lambda) & \begin{bmatrix} \bar{Y}^T Q \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{Y}^T K_{opt}^T R \\ \lambda \bar{H}^T R \\ 0 \end{bmatrix} \\ * & -Q & 0 \\ * & * & -R \end{bmatrix}, \\
\Pi_0(\lambda) &= \begin{bmatrix} \bar{S} - \bar{Z} + \bar{Y}^T Q \bar{Y} & \bar{Z} & \bar{P} \\ \bar{Z} & -\bar{S} - \bar{Z} & 0 \\ \bar{P} & 0 & \tau^2 \bar{Z} \end{bmatrix} \\
&\quad + \mathcal{H}e\left\{ \begin{bmatrix} I \\ 0 \\ \epsilon I \end{bmatrix} \begin{bmatrix} \bar{Y}^T A_{opt}^T \\ -\lambda \bar{H}^T B^T \\ -\bar{Y}^T \end{bmatrix}^T \right\}, \\
\Psi(M) &= \begin{bmatrix} \mu I & I \\ I & \bar{Y} + \bar{Y}^T - M \end{bmatrix}, \quad \forall M \in \mathbb{S}_n^+.
\end{aligned} \quad (12)$$

Then the gain matrix H given by $H = \bar{H}\bar{Y}^{-1}$, and the associated control law

$$u_i(t) = K_{opt}x_i(t) + Hz_i(t),$$

guarantees that the closed loop system (9) is asymptotically stable for the constant delay $\tau > 0$. Moreover the cost functions (2) satisfy:

$$J_i \leq \mu \left(|x(0)|^2 + \tau \|x_0\|_\tau^2 + \tau^2/2 \|\dot{x}_0\|_\tau^2 \right). \quad (13)$$

Proof: The proof is divided into three steps. The first parts suggests a model transformation of the multi-agent system into a single generic system with parametric uncertainty. The second step deals with the stability analysis of such systems. The final step propose a method to optimize the gain matrix H.

Model transformation: Consider a set of multi-agent systems governed by (1) and connected via an undirected graph represented by a symmetric Laplacian $\mathcal{L}$. The delay τ is positive and assumed to be known. Since $\mathcal{L}$ is symmetric positive semi-definite, a spectral decomposition allows re-writing the Laplacian as $\mathcal{L} = V\Lambda V^T$ where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathcal{L}$ and $\Lambda = \mathcal{D}iag(\lambda_1, \dots \lambda_N)$ is the matrix of the eigenvalues of $\mathcal{L}$. Moreover the symmetry of the Laplacian ensures that the λ_i's are real and can be reordered as $0 < \lambda_i < \lambda_{max}(\mathcal{L})$. Define an orthogonal state transformation

$$X \mapsto (V^T \otimes I_n)X = \tilde{X}, \tag{14}$$

where $\tilde{X}$ is an element of $\mathbb{R}^{Nn}$. The closed loop system (9) in the new coordinates is given by

$$\dot{\tilde{X}}(t) = (I_N \otimes A_{opt})\tilde{X}(t) - (\Lambda \otimes BH)\tilde{X}(t-\tau) \tag{15}$$

Since Λ is a diagonal matrix, the system in (15) is equivalent to the collection of systems

$$\forall i = 1, \dots, N, \quad \dot{\tilde{x}}_i(t) = A_{opt}\tilde{x}_i(t) + \lambda_i BH\tilde{x}_i(t-\tau), \tag{16}$$

where the variables $\tilde{x}_i$ in $\mathbb{R}^n$ represent the i^{th} components of the augmented vector $\tilde{X}$. The performance index J in (10) can also be rewritten as

$$\tilde{J}_i = \int_0^\infty (\tilde{x}_i^T(t)Q\tilde{x}_i(t) + \tilde{u}_i^T(t)R\tilde{u}_i(t))dt, \tag{17}$$

where

$$\tilde{u}_i(t) = K_{opt}\tilde{x}_i(t) + \lambda_i H\tilde{x}_i(t),$$

for $i = 1 \dots N$. In order to provide an efficient stability analysis of this set of system, a polytopic representation is adopted. Indeed the matrices $\lambda_i H$ can be viewed as a set of matrices which belong to a polytope whereby a real parameter λ varies in an interval $[0, \lambda_{max}]$.

Therefore the problem has become to find the optimal gain matrix H for the delay system with a polytopic uncertainty $\mathcal{P}(\lambda)$ given by

$$\mathcal{P}(\lambda): \quad \dot{x}(t) = A_{opt}x(t) + \lambda BHx(t-\tau), \tag{18}$$

where x is a vector of $\mathbb{R}^n$ and λ is an uncertain parameter in $[0, \lambda_{max}]$ such that the performance index

$$J^* = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \tag{19}$$

is minimized.

Stability analysis: To solve this problem, a Lyapunov-Krasovskii theorem dedicated to time-delay systems will be used. Consider the Lyapunov-Krasovskii functional given by

$$\begin{array}{rcl} V(x_t) & = & x^T(t)Px(t) + \int_{t-\tau}^t x^T(s)Sx(s)\mathrm{d}s \\ & & +\tau \int_{t-\tau}^t (\tau - t + s)\dot{x}^T(s)Z\dot{x}(s)\mathrm{d}s. \end{array} \tag{20}$$

Differentiating the functional given in (20) leads to the expression

$$\begin{array}{rcl} \dot{V}(x_t) & = & 2\dot{x}^T(t)Px(t) + x^T(t)Sx(t) \\ & & -x^T(t-\tau)Sx(t-\tau) + \tau^2\dot{x}^T(t)Z\dot{x}(t) \\ & & -\tau \int_{t-\tau}^t \dot{x}^T(s)Z\dot{x}(s)\mathrm{d}s. \end{array} \tag{21}$$

Applying Jensen's inequality [13] to the last integral term in (21) leads to

$$\begin{array}{rcl} \dot{V}(x_t) & \leq & 2\dot{x}^T(t)Px(t) + x^T(t)Sx(t) \\ & - & x^T(t-\tau)Sx(t-\tau) + \tau^2\dot{x}^T(t)Z\dot{x}(t) \\ & - & (x(t) - x(t-\tau))^T Z(x(t) - x(t-\tau)). \end{array} \tag{22}$$

In order to achieve a sub-optimal control design, the cost function is manipulated in the following manner: define a new functional

$$W(x_t) = \dot{V}(x_t) + x^T(t)Qx(t) + u^T(t)Ru(t), \tag{23}$$

where the matrices Q and R are from the cost function defined in (19). Then the objective is to find the matrices P, Q and Z such that $W(\cdot)$ is negative definite. Note that for all matrices $Y \in \mathbb{R}^{3n\times n}$ the inequality

$$\xi^T(t)Y\left(A_{opt}x(t) - \lambda BHx(t-\tau) - \dot{x}(t)\right) = 0,$$

holds, where

$$\xi(t) := \begin{bmatrix} x(t) \\ x(t-\tau) \\ \dot{x}(t) \end{bmatrix}.$$

The the right hand side of (23) can be written as

$$\begin{array}{rcl} W(x_t) & \leq & \xi^T(t) \begin{bmatrix} S-Z+Q & Z & P \\ Z & -S-Z & 0 \\ P & 0 & \tau^2 Z \end{bmatrix} \xi(t) \\ & & +\xi^T(t) \begin{bmatrix} K^T \\ \lambda H^T \\ 0 \end{bmatrix} R \begin{bmatrix} K^T \\ \lambda H^T \\ 0 \end{bmatrix}^T \xi(t) \\ & & +2\xi^T(t)Y \begin{bmatrix} A_{opt} & -\lambda BH & -I \end{bmatrix} \xi(t). \end{array} \tag{24}$$

Assume that the matrix Y has the particular structure

$$Y = \begin{bmatrix} \bar{Y}^{-1} \\ 0 \\ \epsilon\bar{Y}^{-1} \end{bmatrix},$$

where ϵ is a positive scalar and where the matrix $\bar{Y}$ in $\mathbb{R}^{n\times n}$ is nonsingular. This manipulation corresponds to the use of the descriptor approach proposed in [22]. Define the vector

$$\bar{\xi}(t) = \begin{bmatrix} \bar{Y}^{-1}x(t) \\ \bar{Y}^{-1}x(t-\tau) \\ \bar{Y}^{-1}\dot{x}(t) \end{bmatrix}.$$

Then $W(\cdot)$ can be rewritten as

$$\begin{array}{rcl} W(x_t) & \leq & \bar{\xi}^T(t)\left(\begin{bmatrix} \bar{S}-Z & \bar{Z} & \bar{P} \\ \bar{Z} & -\bar{S}-\bar{Z} & 0 \\ \bar{P} & 0 & \tau^2\bar{Z} \end{bmatrix}\right. \\ & + & 2\begin{bmatrix} I \\ 0 \\ \epsilon I \end{bmatrix}\begin{bmatrix} \bar{Y}^T A_{opt}^T \\ -\lambda\bar{Y}^T H^T B^T \\ -\bar{Y}^T \end{bmatrix}^T \\ & + & \begin{bmatrix} \bar{Y}^T K_{opt}^T R \\ \lambda\bar{Y}^T H^T R \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} \bar{Y}^T K_{opt}^T R \\ \lambda\bar{Y}^T H^T R \\ 0 \end{bmatrix}^T \\ & + & \left.\begin{bmatrix} \bar{Y}^T Q \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} \bar{Y}^T Q \\ 0 \\ 0 \end{bmatrix}^T\right)\bar{\xi}(t), \end{array} \tag{25}$$

where

$$\bar{P} = \bar{Y}^T P\bar{Y},\ \bar{S} = \bar{Y}^T S\bar{Y},\ \bar{Z} = \bar{Y}^T Z\bar{Y},$$

Finally, defining the matrix variable $\bar{H} = H\bar{Y}$ and applying the Schur complement ensures that the functional W is definite negative if the LMI condition $\Pi(\lambda) \prec 0$ for all values of λ in the interval $[0,\ \lambda_{max}]$. Since the LMI is affine in the unknown parameter λ, the condition $\Pi(\lambda) \prec 0$ is equivalent to solving the two LMI $\Pi(0) \prec 0$ and $\Pi(\lambda_{max}) \prec 0$.

Then if these two conditions hold, the functional $W(\cdot)$ is negative definite. Integrating $W(\cdot)$ in (25) over the interval $[0, T]$ ensures that

$$V(x_T) - V(x_0) + \int_0^T (x^T(t)Qx(t) + u^T(t)Ru(t))dt < 0.$$

Since $V(x_T) > 0$, a bound of the performance index is given by

$$\forall T > 0, \quad \int_0^T (x^T(t)Qx(t) + u^T(t)Ru(t))dt < V(x_0).$$

By letting T tend to infinity

$$\int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))dt < V(x_0).$$

The last step consists of obtaining an expression for the upper bound $V(x_0)$ using the matrix variables of the LMI. From the definition of the Lyapunov-Krasovskii functional, $V(x_0)$ is given by

$$\begin{array}{rcl} V(x_0) & = & x^T(0)\bar{Y}^{-T}\bar{P}\bar{Y}^{-1}x(0) \\ & + & \int_{-\tau}^0 x^T(s)\bar{Y}^{-T}\bar{S}\bar{Y}^{-1}x(s)\mathrm{d}s \\ & + & \int_{-\tau}^0 (\tau+s)\dot{x}^T(s)\bar{Y}^{-T}\bar{Z}\bar{Y}^{-1}\dot{x}(s)\mathrm{d}s. \end{array} \tag{26}$$

This leads to

$$J_i \leq \mu_{\bar{P}}|x(0)|^2 + \tau\mu_{\bar{S}}\|x_0\|_\tau^2 + \mu_{\bar{Z}}\tau^2/2\|\dot{x}_0\|_\tau^2,$$

where, for $M = \bar{P}, \bar{S}, \bar{Z}$, μ_M is the largest eigenvalue $\bar{Y}^{-1}\bar{M}\bar{Y}$.

Optimization: The optimization corresponds to the minimization of $\mu_{\bar{P}}$, $\mu_{\bar{S}}$ and $\mu_{\bar{Z}}$. Consider first the matrix $\bar{P}$. Introduce the parameter μ such that $\mu_{\bar{P}} \leq \mu$ which can be re-written in the form of a matrix inequality as follows:

$$\bar{Y}^{-T}\bar{P}\bar{Y}^{-1} \preceq \mu I. \tag{27}$$

The applying the Schur complement

$$\begin{bmatrix} \mu I & I \\ I & \bar{Y}^T\bar{P}^{-1}\bar{Y} \end{bmatrix} \succ 0.$$

Finally noting that

$$(\bar{Y}^T - \bar{P})\bar{P}^{-1}(\bar{Y} - \bar{P}) \succeq 0,$$

it follows that

$$\bar{Y}^T\bar{P}^{-1}\bar{Y} \succeq \bar{Y} + \bar{Y}^T - \bar{P}. \tag{28}$$

This proves that the condition $\Psi(\bar{P}) \succ 0$ implies (27). Repeating the same procedure, it can be shown that the LMIs $\Psi(\bar{S}) \succ 0$ and $\Psi(\bar{Z}) \succ 0$ ensure that $\bar{Y}^{-T}\bar{S}\bar{Y}^{-1} \preceq \mu I$ and $\bar{Y}^{-T}\bar{Z}\bar{Y}^{-1} \preceq \mu I$ respectively.

The proof is concluded by solving the minimization of μ as suggested in (11). ■

Remark 1: Following the initialization of the consensus algorithm provided in [40], a pragmatic choice of the initial conditions for the system (9) is

$$x(t) = x(0), \quad \forall t < 0.$$

Using such initial conditions, equation (13) becomes

$$J_i \leq \mu(1+\tau)|x(0)|^2. \tag{29}$$

V. Example

In order to illustrate the result of the previous section, consider a set of 6 systems governed by

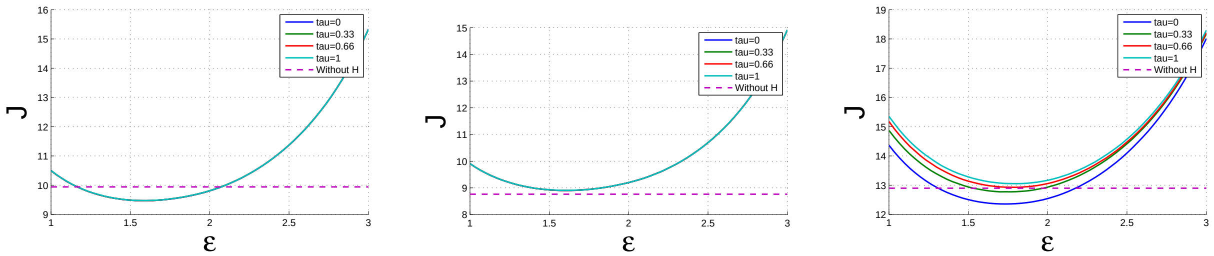
$$\dot{x}_i(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_i(t). \tag{30}$$

where the agents are connected according to the graph

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The eigenvalues of this matrix are $0, 1, 3$ and 4.

Several different pairs of matrices in the performance index are considered. The matrix R is chosen equal to



(a) Performance index with $Q = Q_1$ (b) Performance index with $Q = Q_2$ (c) Performance index with $Q = Q_3$

Fig. 1. Evolution of the performance index for different matrices Q, and different delay with respect to the parameter ϵ.

be equal to $10I_2$ and the matrix Q takes three possible value, i.e. $Q_1 = I_4$ and

$$Q_2 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \; Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{31}$$

Figure 1 represents the performance index for the different choices of Q. The performance index is computed according to the initialization setup provided in Remark 1 and with $|x(0)|^2 = 1$. This means that $J_i \leq \mu(1+\tau)$.

The first comment to make is that, depending on the choice of Q, the shape of the performance index is significantly modified.

For $Q = Q_1$, Figure 1(a) shows that the delay does not affect the performance index since the curves obtained for the four delays $0, 0.33, 0.66$ and 1 are superposed. In this case the minimal cost is obtained for $\epsilon = 1.6$. Since this minimum is below the performance index obtained without introducing interconnections among the agents, represented by the dash line, this means that in this particular situation, there is a benefit to introducing (delayed or not) relative information.

For $Q = Q_2$, Figure 1(b) shows similar results to Figure 1(a) except that the minimum is above the performance level obtained without introducing relative information. This means that in this situation the inclusion of relative information does not improve the performance index.

For $Q = Q_3$, Figure 1(c) shows that, in this situation, the delay affects the performance index. Indeed, the greater the delay, the greater the performance index. The interesting issue here is that, depending on the value of the delay τ, the minimum of the performance index can be above or below the optimal case without relative information.

Consider now a set of 4 agents connected through the Laplacian

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Since its eigenvalues are $0, 2$ and 4, the same results are directly obtained from the previous example despite the significant change to $\mathcal{L}$. This shows the strengths of our approach based on the polytopic representation in (18).

VI. Conclusions

This paper has considered the design of an optimal controller for a set of N identical multi-agent systems. The proposed control law has a local linear feedback term (from solving a local Riccati equation) and a consensus-like term which depends on a delayed version of the relative states with respect to its neighbours. The resulting closed loop system at a network level has been decomposed using spectral decomposition of the associated Laplacian into N independent systems which depend on the eigenvalues of the Laplacian. This collection of systems has been viewed from a polytopic systems perspective, and a Lyapunov-Krasovskii approach has been used to synthesize the gain associated with the consensus term to provide sub-optimal LQR-like performance at a network level. Situations are demonstrated when this approach provides better performance (in terms of the LQR cost) than when a traditional decentralised approach is adopted. The examples show that the answer to this problem depends not only on the value of the delay but also on the definition of the cost function.

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