A Mixed Deterministic and Stochastic Small Gain Theorem and Its Application to Networked Stabilization

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***Abstract*— Classical small gain theorems can handle closed-loop systems with either deterministic or stochastic uncertainty, but not the ones with both of them. However this is exactly the case in the stabilization problem of a networked control system (NCS) with both logarithmic quantization and fading phenomenon in the transmission channel. To solve the NCS problem, in this paper we develop a new small gain theorem which can handle closed-loop systems with such mixed uncertainty, and then use it to solve the NCS stabilization problem. Both unstructured and structured cases are worked out, leading to the solutions to the single-input and multi-input NCS stabilization problems, respectively.**

***Index Terms*— Small gain theorem, networked control system, channel resource allocation, stochastic control.**

I. INTRODUCTION

SMALL gain theorem is one of the most fundamental building blocks in the robust control theory. It suggests an important method to deal with closed-loop systems with uncertainty. By using the technique of "pulling out the uncertainty" (see e.g. [40]), we can always reformulate such a closed-loop system into the one depicted in Fig. 1 with $\boldsymbol{G}$ the knowns and $\boldsymbol{\Delta}$ the constrained unknowns. Then the robust stability, i.e. whether the closed-loop system is stable for all possible $\boldsymbol{\Delta}$ satisfying the constraint, is considered, and small gain theorem asserts that the stability holds if and only if the small gain condition is fulfilled. Such a methodology enables the designer to stay with relatively simple model of $\boldsymbol{G}$ and relatively little knowledge of $\boldsymbol{\Delta}$ while still being able to guarantee the stability. Besides, the set of possible $\boldsymbol{\Delta}$ may cover uncertainty that is unmodeled or neglected, then the robust controller can still stabilize the system in this case while other controllers may fail.

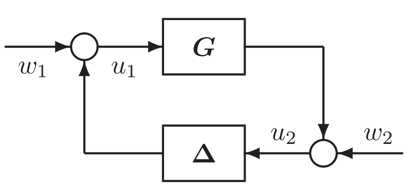

Fig. 1. A general closed-loop system with uncertainty

The theorem is first introduced by Zames in [37] as a general sufficient stability condition for deterministic systems in the sense of maximal $\mathcal{L}_2$-gain or the $\mathcal{H}_\infty$ norm. Along this line of research, it is shown in [8] that when $\boldsymbol{G}$ is linear time-invariant (LTI) and the set of $\boldsymbol{\Delta}$ contains all admissible unstructured LTI uncertainty, then the small gain condition is also necessary. Further study on structured LTI $\boldsymbol{\Delta}$ leads to μ-analysis and μ-synthesis (see e.g. [23], [40]). Several other papers [27], [28] work on structured uncertainty being possibly linear time-varying (LTV) or nonlinear time-invariant. A diagonally scaled small gain condition is proved to be the necessary and sufficient condition for stability in this case. Almost at the same time, some other researchers investigated the stability under stochastic setup and also establish the small gain theorem in this sense. The paper [35] is an early effort on this and a relatively complete solution can be found in [20]. The stability is pursued in the mean-square (MS) sense, the uncertainty is assumed to be a stochastic memoryless gain, and the $\mathcal{H}_\infty$ norm in deterministic small gain theorems is replaced by the $\mathcal{H}_2$ norm or the so-called MS norm (see e.g. [11], [31] for definition). There also exist some other small gain theorems, e.g. the one for deterministic systems in the sense of ℓ_∞-gains is also well developed and details can be found in e.g. [5], [6], etc.

We have seen so many small gain theorems in the above. However they have not covered all important cases. Take a look at these theorems again and we would notice that the mixed case is not yet handled, i.e. when there exist simultaneously deterministic and stochastic uncertainties in the closed-loop system. On the other hand, recent development on the research of networked control system (NCS) calls for a small gain theorem in such a case. In an NCS we assume that some of the signals in the closed-loop are transmitted through communication channels which introduce various types of uncertainties. For instance, in [32], the channel is assumed to introduce both sector-bounded nonlinearity and fading. The nonlinearity Δ is deterministic and satisfies an ∞-norm bound, while the fading factor Ψ is a stochastic memoryless gain with known first and second order moments. When analyzing the robust stability of this closed-loop system, small gain theorem is expected to play the same role as before. It turns out that existing small gain theorems are applicable when either Δ [25] or Ψ [31] exists in the NCS, but cannot handle the case when both exist simultaneously. This motivates us to establish a new small gain theorem which is compatible with both deterministic and stochastic uncertainties, and gives the necessary and sufficient condition of the robust stability. In particular, we are interested in extending the deterministic ℓ_2 input-output theory to the stochastic case. Besides, the theorem does not have to be

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limited to solving the NCS problem considered in this paper and could be further applicable to other general ones.

Although the desired small gain theorem has not yet appeared in the literature, some researchers have considered similar problems and reported important results which help us build the theorem. Almost at the same time as the development of small gain theorems, some researchers worked on extending the definition of deterministic ℓ_2 signals to stochastic ones, and by making use of this, reported some nice results. The paper [14] discusses the stabilization of a linear stochastic system with multiplicative noise which is achieved by optimizing a linear matrix inequality (LMI). A small gain theorem is proved as a sufficient condition for closed-loops consisting of only linear stochastic systems in [9]. A bounded real lemma is constructed in [1] for discrete-time linear systems in the sense of the extended norms and the continuous-time case is worked out in [18]. The same authors also discuss the stability radius which is more or less related to the small gain theorem for a special class of linear systems in [2]. The book [7] works on the same stability radius problem in continuous-time case. We find all these results inspiring. Moreover, some of them are very useful when deriving our result. This will be made clear in the context.

The notation in this paper is mostly standard. The symbol ":=" stands for "defined as". The expectation of random variables is denoted to be $\boldsymbol{E}\{\cdot\}$. Given a matrix, its transpose is denote by $'$. When the matrix is square, denote its ith eigenvalue by $\lambda_i(\cdot)$, its spectral radius by $\rho(\cdot)$ and its determinant by $\det(\cdot)$. Given a vector z, denote z_i to be its ith component. $\mathbb{N}$ stands for non-negative integers. Throughout this paper the log base is set to 2.

The rest part of the paper is organized as follows. In Section II we state the definitions of extended signal norm which is compatible with both deterministic and stochastic signals, and also define the induced system norm as well as the corresponding stability. With these new concepts, in Section III we work out a new small gain theorem as desired. In Section IV we apply the new theorem to solve the particular NCS stabilization problem for both single-input (SI) and multi-input (MI) cases. The paper ends with the conclusions in Section VI.

II. Norms and Stability

Some tools need to be developed before the new small gain theorem can be discussed. In this section we extend the ℓ_2 norm and the corresponding induced norm to stochastic signals and systems. The definitions are mainly conventional and used by many existing papers, e.g. [1], [9], [14], [38], etc. But in this paper we further discuss more details of them, e.g. the stability condition, the computation of induced norms, and the value of induced norms when restricting to different filtration-induced subspaces, etc.

Consider the class of linear stochastic systems $\boldsymbol{G}$ with the following type of state space representation

$$\begin{aligned} x(k+1) &= \left(A + \sum_{i=1}^{m} A_i \Psi_i(k)\right) x(k) \\ &\quad + \left(B + \sum_{i=1}^{m} B_i \Psi_i(k)\right) u(k) \\ y(k) &= Cx(k) + Du(k). \end{aligned} \tag{1}$$

Here $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$ are constant matrices, and so are $[A_i|B_i] \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ for $i = 1, \ldots, m$. The initial condition $x(0) = x_0 \in \mathbb{R}^n$ is assumed to be an unknown deterministic constant. Besides, $\{\Psi_i\}$ are real-valued i.i.d. random processes for $i = 1, \ldots, m$ on a given complete probability space $(\Omega, \mathcal{F}, P)$. The sample space Ω consists of all possible sequences of $\{\Psi_i(k)\}$, and the σ-algebra $\mathcal{F}$ is the smallest σ-algebra containing $\cup_{i=-1}^{\infty} \mathcal{F}(k)$ where

$$\mathcal{F}(k) = \sigma(\{\Psi_i(t), 0 \le t \le k, 1 \le i \le m\})$$

is the smallest σ-algebra such that $\{\Psi_i(t)\}$ is measurable for all t and i when $k \in \mathbb{N}$ and $\mathcal{F}(-1) = \{\emptyset, \Omega\}$. The probability measure is P. Assume for any $k \in \mathbb{N}$, $1 \le i, j \le m$ and $j \ne i$,

$$\boldsymbol{E}\{\Psi_i(k)\} = 0, \boldsymbol{E}\{\Psi_i(k)^2\} = \nu_i^2, \boldsymbol{E}\{\Psi_i(k)\Psi_j(k)\} = 0.$$

Then clearly the sequence of σ-algebras $\{\mathcal{F}(k)\}$ is non-decreasing and $\{\Psi_i(k)\}$ is independent of $\mathcal{F}(k)$ for all $k \in \mathbb{N}$, i.e. $\boldsymbol{E}\{\Psi_i(k)|\mathcal{F}(k-1)\} = \boldsymbol{E}\{\Psi_i(k)\}$ for all i. We denote the sequence by $\mathfrak{F}_\Psi$.

Then an $\mathbb{R}^m$-valued square integrable random variable X defined on the above $(\Omega, \mathcal{F}, P)$ is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^m$ with $\boldsymbol{E}\{\|X\|^2\}$ well-defined and finite where $\|\cdot\|$ is the normal Euclidean norm in $\mathbb{R}^m$. The set of all such random variables is an inner product space, denoted by $L_2(\Omega, \mathbb{R}^m)$ with semi-norm defined as

$$\|X\|_{L_2}^2 := \boldsymbol{E}\{\|X\|^2\}$$

and inner product

$$\langle X, Y\rangle_{L_2} := \boldsymbol{E}\{X'Y\}.$$

for $X, Y \in L_2(\Omega, \mathbb{R}^m)$.

We can then define an $\mathbb{R}^m$-valued random process on $(\Omega, \mathcal{F}, P)$ as a function $z : \mathbb{N} \to L_2(\Omega, \mathbb{R}^m)$ which can be also considered as a sequence of random variables in $L_2(\Omega, \mathbb{R}^m)$ indexed by $\mathbb{N}$. It will be also called a *stochastic signal*. In this paper we consider only the signals that are *strictly adapted* to $\mathfrak{F}_\Psi$, i.e. such a signal z satisfies that $z(k)$ is $\mathcal{F}(k-1)$-measurable for all $k \in \mathbb{N}$. A stochastic signal z is said to be an ℓ_2 *stochastic signal*, or simply to be ℓ_2 if $z(k) \in L_2(\Omega, \mathbb{R}^m)$ for all k and further

$$\sum_{k=0}^{\infty} \|z(k)\|_{L_2}^2 = \sum_{k=0}^{\infty} \boldsymbol{E}\{\|z(k)\|^2\} < \infty.$$

The set of all such ℓ_2 signals being strictly adapted to $\mathfrak{F}_\Psi$ is denoted as $\ell_2(\mathbb{N}, L_2(\Omega, \mathbb{R}^m), \mathfrak{F}_\Psi)$ and is abbreviated as

$\ell_2^m(\mathfrak{F}_\Psi)$. We will also write $z \in \ell_2(\mathfrak{F}_\Psi)$ when the dimension is not of importance. For an signal $z \in \ell_2(\mathfrak{F}_\Psi)$, its *stochastic ℓ_2 norm* is defined by

$$\|z\|_2 := \sqrt{\sum_{k=0}^{\infty} \boldsymbol{E}\{\|z(k)\|^2\}}, \tag{2}$$

which can also be induced from the following inner product

$$\langle x, y\rangle := \sum_{k=0}^{\infty} \langle x(k), y(k)\rangle_{L_2}, \quad x, y \in \ell_2^m(\mathfrak{F}_\Psi).$$

Equipped with this inner product, $\ell_2^m(\mathfrak{F}_\Psi)$ is an inner product space. Note that (2) is a semi-norm. By the following standard procedure we can convert it into a genuine norm. Define two ℓ_2 signals $z, \tilde{z}$ to be *mean square (MS) equivalent* if they satisfy

$$\|z - \tilde{z}\|_2^2 = \sum_{k=0}^{\infty} \|z(k) - \tilde{z}(k)\|_{L_2}^2 = 0.$$

A signal z and all its MS equivalent signals form a equivalent class $[z]$ and have the same ℓ_2 norm. Hence we consider the set of these equivalent classes instead but still denote it as $\ell_2^m(\mathfrak{F}_\Psi)$. We also abuse the notation a bit by writing $[z]$ simply as z. Then the ℓ_2 norm (2) is a proper norm in this sense. It is also straight forward to verify that $\ell_2^m(\mathfrak{F}_\Psi)$ is in fact also complete [9] and hence is a Hilbert space.

Note that (2) is compatible with the traditional ℓ_2 norm for deterministic signals in the sense that the stochastic ℓ_2 norm of a deterministic ℓ_2 signal coincides with its traditional ℓ_2 norm. Hence we will simply call the definition (2) to be the ℓ_2 norm in the following.

We can then induce the system norm in the ℓ_2 signal space. A system $\boldsymbol{G}$ is said to be *non-anticipative (w.r.t. $\mathfrak{F}_\Psi$)* if $\boldsymbol{G}u$ is strictly adapted to $\mathfrak{F}_\Psi$ whenever the stochastic signal u is. A non-anticipative system $\boldsymbol{G}$ is said to be *ℓ_2-BIBO stable* if for any $u \in \ell_2(\mathfrak{F}_\Psi)$, $\boldsymbol{G}u$ is also in $\ell_2(\mathfrak{F}_\Psi)$ and the following holds

$$\sup_{u\in\ell_2(\mathfrak{F}_\Psi), \|u\|_2>0} \frac{\|\boldsymbol{G}u\|_2}{\|u\|_2} < \infty.$$

The *induced norm* of an ℓ_2-BIBO stable system $\boldsymbol{G}$ is then given by the above ℓ_2 gain

$$\|\boldsymbol{G}\|_\infty := \sup_{u\in\ell_2(\mathfrak{F}_\Psi), \|u\|_2>0} \frac{\|\boldsymbol{G}u\|_2}{\|u\|_2}. \tag{3}$$

Note that like the ℓ_2 norm, this definition is also compatible with the ∞-norm for deterministic systems, i.e. they lead to the same value when $\boldsymbol{G}$ is deterministic. The set of all non-anticipative ℓ_2-BIBO stable systems from $\ell_2^m(\mathfrak{F}_\Psi)$ to $\ell_2^m(\mathfrak{F}_\Psi)$ is a normed space w.r.t. the induced norm (3), and is denoted by B^m. Since the target space is a Hilbert space, B^m is a Banach space. Denote the identity system as $\boldsymbol{I}$, i.e. $\boldsymbol{I}u = u$ for any stochastic signal $u \in \ell_2^m(\mathfrak{F}_\Psi)$. Then obviously $\boldsymbol{I} \in \mathsf{B}^m$ and $\|\boldsymbol{I}\|_\infty = 1$.

It is then natural to ask how the system stability can be asserted and how the induced norms can be calculated, as neither of their definitions provides a direct method. In general the stability does not have an efficient method to check; and if $\boldsymbol{G}$ does not have more properties other than being ℓ_2-BIBO stable and non-anticipative, the induced norm is not easy to compute and an analytic expression as desired may not exist.

But when the system $\boldsymbol{G}$ is specified as in (1), we can have better characterization of stability and norms. Obviously $\boldsymbol{G}$ is causal and non-anticipative. We start with the following bounded real lemma presented in [1].

Lemma 1 *Let $\boldsymbol{G}$ be given as in (1). Then $\boldsymbol{G}$ is ℓ_2-BIBO stable and $\|\boldsymbol{G}\|_\infty < \gamma$ for some $\gamma > 0$ if and only if there exists $P > 0$ such that*

$$\begin{bmatrix} P - A'PA - \sum_{i=1}^m \nu_i^2 A_i' P A_i - C'C & K' \\ K & L \end{bmatrix} > 0, \tag{4}$$

where

$$L = \gamma^2 I - B'PB - \sum_{i=1}^m \nu_i^2 B_i' P B_i - D'D,$$

$$K = B'PA + \sum_{i=1}^m \nu_i^2 B_i' P A_i + D'C.$$

The following sufficient condition of ℓ_2-BIBO stability can be easily induced from the above lemma by relaxing γ to positive infinity.

Corollary 1 *Let $\boldsymbol{G}$ be given as in (1). Then $\boldsymbol{G}$ is ℓ_2-BIBO stable if there exists $P > 0$ such that*

$$P > A'PA + \sum_{i=1}^m \nu_i^2 A_i' P A_i. \tag{5}$$

Proof: Easy to see that by relaxing γ to positive infinity in (4), the ℓ_2-BIBO stability of $\boldsymbol{G}$ is equivalent to require the existence of some $P > 0$ such that

$$P > A'PA + \sum_{i=0}^m \nu_i^2 A_i' P A_i + C'C. \tag{6}$$

Now assume that some $\tilde{P} > 0$ satisfies (5). Then there exists some small enough $\epsilon > 0$ such that

$$\tilde{P} > A'\tilde{P}A + \sum_i \nu_i^2 A_i' \tilde{P} A_i + \epsilon C'C.$$

Denote $P = \tilde{P}/\epsilon$ and we can see that (6) holds for this P. ■

According to [1], (5) is the necessary and sufficient condition to the MS stability of the autonomous part of (1), i.e.

$$x(k+1) = Ax(k) + \sum_{i=1}^m A_i \Psi_i(k) x(k).$$

Hence the ℓ_2-BIBO stability of (1) can be implied from the MS stability of its autonomous part. The converse is also true by adding some mild assumptions on $\boldsymbol{G}$, but we will not further discuss it here. The analysis on the relationship between ℓ_2-BIBO stability and MS stability also deserves extension to nonlinear systems in the future research.

Lemma 1 is important because it not only helps to derive the above stability condition, but also enables us to calculate the system norm indirectly. Moreover, its proof in [1] also provides some useful tool as stated below.

Lemma 2 *Let $\boldsymbol{G}$ be given as in (1) and be ℓ_2-BIBO stable in $\ell_2^m(\mathfrak{F}_\Psi)$. Suppose that $P > 0$ solves (4) and K, L are defined as in (4). Then*

$$\gamma^2\|u\|_2^2 - \|\boldsymbol{G}u\|_2^2 = \\ -x_0'Px_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{-u(k)'Kx_0 - x_0'K'u(k) + u(k)'Lu(k)\}.$$

By this we can obtain the following important corollary.

Corollary 2 *Let $\boldsymbol{G}$ be given as in (1) and is ℓ_2-BIBO stable. Then $\|\boldsymbol{G}\|_\infty < \gamma$ if and only if $\|\boldsymbol{G}u\|_2 < \gamma\|u\|_2$ for all deterministic input signals $u \in \ell_2(\mathfrak{F}_\Psi)$.*

Proof: The necessity is obvious since the space of deterministic ℓ_2 signals is a subspace of $\ell_2(\mathfrak{F}_\Psi)$. Conversely assume that $\gamma^2\|\hat{u}\|_2^2 - \|\boldsymbol{G}\hat{u}\|_2^2 > 0$ whenever $\hat{u}$ is a deterministic ℓ_2 signal. In particular, this is true when $x_0 = 0$. Then by Lemma 2,

$$\sum_{k=0}^{\infty} \hat{u}(k)'L\hat{u}(k) = \gamma^2\|\hat{u}\|_2^2 - \|\boldsymbol{G}\hat{u}\|_2^2 > 0$$

for all deterministic $\hat{u}$, which implies that $L > 0$. Now for each $u \in \ell_2(\mathfrak{F}_\Psi)$, construct $\hat{u}$ such that $\hat{u}(k) = \boldsymbol{E}\{u(k)\}$ for all k, then

$$\sum_{k=0}^{\infty} \boldsymbol{E}\{-u(k)'Kx_0 - x_0'K'u(k) + u(k)'Lu(k)\} \\ \geq \sum_{k=0}^{\infty}[-\hat{u}(k)'Kx_0 - x_0'K'\hat{u}(k) + \hat{u}(k)'L\hat{u}(k)],$$

since $\boldsymbol{E}\{u(k)'Lu(k)\} \geq \hat{u}(k)'L\hat{u}(k)$. Hence $\gamma^2\|u\|_2^2 - \|\boldsymbol{G}u\|_2^2 \geq \gamma^2\|\hat{u}\|_2^2 - \|\boldsymbol{G}\hat{u}\|_2^2 > 0$ for any $u \in \ell_2(\mathfrak{F}_\Psi)$, and thus $\|\boldsymbol{G}\|_\infty < \gamma$. ■

Remark 1 The above corollary allows us to derive the induced norm of a system described in (1) by only testing deterministic inputs.

III. Small Gain Theorems

With the preparations in the previous section, we are now ready to pursue the new small gain theorems. Consider again the closed-loop system depicted in Fig. 1 and denote it to be $(\boldsymbol{G}, \boldsymbol{\Delta})$. In the $\mathcal{H}_\infty$ small gain theorem in the deterministic setup, $\boldsymbol{\Delta}$ is a norm-bounded uncertainty while $\boldsymbol{G}$ is an LTI system. To extend this to stochastic setup, we assume that $\boldsymbol{G}$ is now given by (1), i.e. the stochastic uncertainty $\{\Psi_i\}$ is integrated within the linear plant, while $\boldsymbol{\Delta}$ is still norm-bounded, and is non-anticipative with respect to $\mathfrak{F}_\Psi$. Then the small gain theorems are stated in terms of the induced norms of $\boldsymbol{G}$ and $\boldsymbol{\Delta}$, and are proved for both unstructured and structured cases.

A. Unstructured Case

In this section we will discuss the mixed deterministic and stochastic small gain theorem for the unstructured case. We first present a general sufficient condition. The necessary and sufficient condition is then shown when more information of the closed-loop system is given.

Theorem 1 *Consider the closed-loop $(\boldsymbol{G}, \boldsymbol{\Delta})$ depicted in Fig. 1. Assume that $\boldsymbol{G} : \ell_2^m(\mathfrak{F}_\Psi) \to \ell_2^m(\mathfrak{F}_\Psi)$ is a causal non-anticipative ℓ_2-BIBO stable system and so is $\boldsymbol{\Delta}$. Then $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable if $\|\boldsymbol{\Delta}\|_\infty\|\boldsymbol{G}\|_\infty < 1$.*

Proof: The closed-loop $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable if and only if the input-output relation from $[w_1 \quad w_2]'$ to $[u_1 \quad u_2]'$ is an ℓ_2-BIBO stable system. Since

$$u_1 = w_1 + \boldsymbol{\Delta}u_2, \\ u_2 = w_2 + \boldsymbol{G}u_1,$$

it is easy to see for $w_1, w_2 \in \ell_2^m(\mathfrak{F}_\Psi)$,

$$\|u_1\|_2 \leq \|w_1\|_2 + \|\boldsymbol{\Delta}\|_\infty\|u_2\|_2, \\ \|u_2\|_2 \leq \|w_2\|_2 + \|\boldsymbol{G}\|_\infty\|u_1\|_2.$$

Hence

$$\|u_1\|_2 \leq \|w_1\|_2 + \|\boldsymbol{\Delta}\|_\infty(\|w_2\|_2 + \|\boldsymbol{G}\|_\infty\|u_1\|_2) \\ = \|w_1\|_2 + \|\boldsymbol{\Delta}\|_\infty\|w_2\|_2 + \|\boldsymbol{\Delta}\|_\infty\|\boldsymbol{G}\|_\infty\|u_1\|_2.$$

If $\|\boldsymbol{\Delta}\|_\infty\|\boldsymbol{G}\|_\infty < 1$, it follows that

$$\|u_1\|_2 \leq (1 - \|\boldsymbol{\Delta}\|_\infty\|\boldsymbol{G}\|_\infty)^{-1}(\|w_1\|_2 + \|\boldsymbol{\Delta}\|_\infty\|w_2\|_2).$$

Hence the system from $[w_1 \quad w_2]'$ to u_1 is ℓ_2-BIBO stable. Similar argument also holds for u_2, and hence the closed-loop is ℓ_2-BIBO stable if $\|\boldsymbol{\Delta}\|_\infty\|\boldsymbol{G}\|_\infty < 1$. ■

It is easy to show that for given $(\boldsymbol{G}, \boldsymbol{\Delta})$, Theorem 1 may not be necessary. Consider the SISO case with the closed-loop consisting of static systems $\boldsymbol{G} = 2$ and $\boldsymbol{\Delta} = 4$. Easy to see

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Obviously u_1 and u_2 are ℓ_2 whenever w_1 and w_2 are ℓ_2. Then $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable but $\|\boldsymbol{G}\|_\infty\|\boldsymbol{\Delta}\|_\infty = 8 > 1$.

Small gain theorems were known as only sufficient conditions of closed-loop stability for a long period of time since the first appearance fifty years ago [37]. They were used to check the stability for given $\boldsymbol{G}$ and $\boldsymbol{\Delta}$. On the other hand, the necessity which was discovered a couple of decades later [8], [27], in fact focuses on a slightly different problem. The simultaneous stability of a set of closed-loop systems is considered rather than the stability of a fixed closed-loop system, and the small gain condition is necessary only in this robust sense. This is also the case in our small gain theorem. In the following, we will consider the robust stability of $(\boldsymbol{G}, \boldsymbol{\Delta})$ parameterized by a set of $\boldsymbol{\Delta}$. Define D to be the set of all causal non-anticipative, possibly nonlinear, time-varying (TV) and dynamic uncertainties $\boldsymbol{\Delta} : \ell_2^m(\mathfrak{F}_\Psi) \to \ell_2^m(\mathfrak{F}_\Psi)$. Further define a subset of D as

$$\mathcal{B}_r := \{\boldsymbol{\Delta} \in \mathsf{D} : \|\boldsymbol{\Delta}\|_\infty \leq r\}.$$

The unstructured small gain theorem is stated as below.

Theorem 2 *Assume that $\boldsymbol{G}$ is a causal non-anticipative ℓ_2-BIBO stable linear stochastic system represented in (1). Then the closed-loop system $(\boldsymbol{G}, \boldsymbol{\Delta})$ depicted in Fig. 1 is ℓ_2-BIBO stable for all $\boldsymbol{\Delta} \in \mathcal{B}_r$ if and only if $\|\boldsymbol{G}\|_\infty < 1/r$.*

Proof: The sufficiency immediately follows from Theorem 1.

The necessity will be proved by constructing an ℓ_2 signal w_1 and a non-ℓ_2 signal u_1 with $w_2 = 0$, such that $w_1 = (\boldsymbol{I} - \boldsymbol{\Delta G})u_1$ for some $\|\boldsymbol{\Delta}\|_\infty \leq r$ when $\|\boldsymbol{G}\|_\infty \geq 1/r$. If so then $(\boldsymbol{I} - \boldsymbol{\Delta G})^{-1}$ maps w_1 to u_1 and is thus not ℓ_2-BIBO stable. This is inspired by the proof in [5], [6], [27].

Some notations are clarified first. Assume all the random processes to be $\mathbb{R}^m$-valued in the following. Consider random processes $a_1 = \{a_1(0), \ldots, a_1(l_1-1)\}$ of length l_1 and $a_2 = \{a_2(0), \ldots, a_2(l_2-1)\}$ of length l_2. Then by $\{a_1, a_2\}$ we mean their concatenation

$$\{a_1, a_2\} := \{a_1(0), \ldots, a_1(l_1-1), a_2(0), \ldots, a_2(l_2-1)\}$$

which is of length l_1+l_2. Also denote 0_l as the zero sequence of length l and write the infinite zero sequence as 0_∞. Besides, denote $\|z\|^2_{[a,b]} := \sum_{k=a}^b \boldsymbol{E}\{\|z(k)\|^2\}$ to be the truncated ℓ_2 norm of an ℓ_2 signal z.

When $\|\boldsymbol{G}\|_\infty > 1/r$, by Corollary 2, there exist some deterministic signal $\hat{f}$ of length l and an ℓ_2 signal $f = \{\hat{f}, 0_\infty\}$ in the sense of MS equivalence such that $\|\boldsymbol{G}f\|_{[0,l-1]} > (1/r + \alpha)\|f\|_{[0,l-1]} = (1/r + \alpha)\|f\|_2$ for some $l \in \mathbb{N}$ and $\alpha > 0$. Moreover there exists a sequence of numbers $\{N_i\}$ for $i \in \mathbb{N}$ such that $N_0 = 0$ and for $i > 0$, N_i is the smallest numbers such that $\|\boldsymbol{G}f\|_{[N_i,\infty)} < \epsilon^i\|f\|_2$ for some $0 < \epsilon < 1$. Denote $M_i := \sum_{j=0}^i N_j$ and the time intervals $I_i = [M_i, M_i + l - 1]$. Construct $u_1 = \{\hat{f}, 0_{N_1-l}, \hat{f}, 0_{N_2-l}, \ldots\}$. Then for all N_i,

$$\begin{aligned}\|\boldsymbol{G}u_1\|_{I_i} &\geq \|\boldsymbol{G}f\|_{[0,l-1]} - \sum_{j=1}^i \|\boldsymbol{G}f\|_{I_j} \\ &> \left(\frac{1}{r} + \alpha\right)\|f\|_2 - \sum_{j=1}^i \epsilon^j\|f\|_2 \\ &> \left(\frac{1}{r} + \alpha - \frac{\epsilon}{1-\epsilon}\right)\|f\|_2,\end{aligned}$$

By taking ϵ sufficiently small, we have $\|\boldsymbol{G}u_1\|_{I_i} > \|f\|_2/r$. We need to construct $\boldsymbol{\Delta}$ such that

$$\boldsymbol{\Delta G}u_1 = \{0_l, 0_{N_1-l}, \hat{f}, 0_{N_2-l}, \hat{f}, 0_{N_3-l}, \ldots\}$$

in the sense of MS equivalence. Then

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{\Delta G})u_1 &= \{\hat{f}, 0_{N_1-l}, \hat{f}, 0_{N_2-l}, \hat{f}, 0_{N_3-l}, \ldots\} \\ &\quad - \{0_l, 0_{N_1-l}, \hat{f}, 0_{N_2-l}, \hat{f}, 0_{N_3-l}, \ldots\} = f =: w_1.\end{aligned}$$

Hence $(\boldsymbol{I} - \boldsymbol{\Delta G})^{-1}$ maps an ℓ_2 signal w_1 into a non-ℓ_2 signal u_1, which proves the necessity.

The desired $\boldsymbol{\Delta}$ can be a nonlinear TV system leading to the result. Let $\hat{f}$ and u_1 be constructed as above. Then the output could be

$$\boldsymbol{\Delta}z(k) := \begin{cases} \dfrac{\hat{f}(k)\|z\|_{I_{j-1}}}{\|\boldsymbol{G}u_1\|_{I_{j-1}}}, & \text{if } k \geq M_1, k \in I_j; \\ 0, & \text{otherwise.}\end{cases} \quad (7)$$

Note that since $M_{j-1}+l-1 < k$ always holds, the evaluation is always possible at time k. On the other hand $\|\boldsymbol{G}u_1\|_{I_{j-1}}$ is the ℓ_2 norm. It can be calculated in advance when $\boldsymbol{G}$ and u_1 are given, and is hence a presumed constant. Therefore the constructed $\boldsymbol{\Delta}$ is causal and non-anticipative. Morever for $k \geq M_1$ and $k - M_j < l$,

$$\boldsymbol{E}\{\|u_1(k) - \boldsymbol{\Delta G}u_1(k)\|^2\} = 0$$

Then $(\boldsymbol{I} - \boldsymbol{\Delta G})u_1 = w_1$ and $\|\boldsymbol{\Delta}\|_\infty \leq r$.

It remains to show the case when $\|\boldsymbol{G}\|_\infty = 1/r$. Note that when this is true, consider a sequence of real numbers $\{\beta_n\}$ such that $\beta_n > 1$ and $\lim_{n\to\infty}\beta_n = 1$. Then for each n, $(\boldsymbol{I} - \beta_n\boldsymbol{\Delta}_n\boldsymbol{G})^{-1}$ could be unstable for some $\|\boldsymbol{\Delta}_n\|_\infty \leq r$ by the proof above. But then

$$\begin{aligned}&(\boldsymbol{I} - \beta_n\boldsymbol{\Delta}_n\boldsymbol{G})^{-1} \\ &\quad = (\boldsymbol{I} - \boldsymbol{\Delta}_n\boldsymbol{G})^{-1}[\boldsymbol{I} - (\beta_n - 1)\boldsymbol{\Delta}_n\boldsymbol{G}(\boldsymbol{I} - \boldsymbol{\Delta}_n\boldsymbol{G})^{-1}]^{-1}.\end{aligned}$$

If $(\boldsymbol{I} - \boldsymbol{\Delta}_n\boldsymbol{G})^{-1}$ is stable, then the right-hand-side of the above equality should be stable for sufficiently large n, which establishes a contradiction. Hence for $(\boldsymbol{G}, \boldsymbol{\Delta})$ to be stable for all admissible $\boldsymbol{\Delta}$, it is necessary that $\|\boldsymbol{G}\|_\infty < 1/r$. ■

In the construction of Theorem 2, the set of uncertainty $\mathcal{B}_r$ is given in the sense of the new induced norm. It contains uncertainty $\boldsymbol{\Delta}$ which could be a stochastic system. However since the only stochastic information in the closed-loop is provided by $\{\Psi_i(k)\}$, this means that $\boldsymbol{\Delta}$ is also dependent on these random noises, which is not very desirable. Besides, the NCS which motivates our research on the mixed small gain theorem, as depicted in Fig. 3, contains quantization which is also a deterministic nonlinearity. Hence we wish to further investigate when the necessity holds for deterministic $\boldsymbol{\Delta}$. Fortunately, by adding some assumption to $\boldsymbol{G}$, we have the following result as desired.

Theorem 3 *Under the same hypotheses as in Theorem 2, there exists $P > 0$ and $\gamma > 0$ such that (4) holds. Further assume that $K = 0$ in (4) for this P. Then $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable for all deterministic $\boldsymbol{\Delta} \in \mathcal{B}_r$ if and only if $\|\boldsymbol{G}\|_\infty < 1/r$.*

Proof: The sufficiency is again immediate by Theorem 1. Now we prove the necessity. Assume that $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable for all $\boldsymbol{\Delta} \in \mathcal{B}_r$ and $\|\boldsymbol{G}\|_\infty \geq 1/r$. Then set $w_1 = w_2 = 0$, by Lemma 2, for $y = \boldsymbol{G}u$,

$$\begin{aligned}\|y\|_2^2 &= x_0'Px_0 \\ &\quad + \sum_{k=0}^\infty \boldsymbol{E}\{x_0'K'u(k) + u(k)'Kx_0 + u(k)'Hu(k)\}.\end{aligned}$$

where K is defined as in (4) where

$$H = B'PB + \sum_{i=1}^m \nu_i^2 B_i'PB_i + D'D. \quad (8)$$

Since $H > 0$, its spectral radius is also its eigenvalue. Then there exists p which is a corresponding eigenvector of $\rho(H)$ with $\|p\|^2 = \rho(H)^{-1}$. It's obvious that $\rho(H) \geq 1/r^2$, otherwise $\|\boldsymbol{G}\|_\infty \geq 1/r$ would not hold. Now if $K = 0$ then the destabilizing $\boldsymbol{\Delta}$ could be the one satisfying

$$\boldsymbol{\Delta} z(k) := \|z(k)\| p. \tag{9}$$

Clearly $\boldsymbol{\Delta} \in \mathcal{B}_r$ since $\|\boldsymbol{\Delta}\|_\infty \leq r$. Hence $u(k) = \boldsymbol{\Delta} y(k) = \|y(k)\| p$ for $k \in \mathbb{N}$ and we have

$$\begin{aligned}
\|y\|_2^2 &= x_0' P x_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{\|y(k)\|^2 p' H p\} \\
&= x_0' P x_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{\|y(k)\|^2\} \\
&= x_0' P x_0 + \|y\|_2^2.
\end{aligned}$$

This implies that $P = 0$, which is a contradiction. The proof is done. ∎

Remark 2 It is worth pointing out that when $K = 0$, the norm $\|\boldsymbol{G}\|_\infty$ is also analytically computable. By the nice property in this special case, some analytic results can be obtained. The formulation in [2] makes $K = 0$ always true by assuming that in (1) there holds $B = 0, D = 0$ and either $A_i = 0$ or $B_i = 0$ for all i. It is a pleasant surprise that the NCS problem we wish to solve in later part of the paper happens to satisfy $K = 0$ when the optimal control is used. Hence the stabilizability condition of the NCS is necessary and sufficient for deterministic uncertainty.

B. Structured Case

In this section we continue to work on the structured case. Instead of considering all uncertainties in $\mathcal{B}_r$, here we consider only the ones in a subset of $\mathcal{B}_r$, namely $\hat{\mathcal{B}}_r$, such that each $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ is in the diagonal form

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_1 & & \\ & \ddots & \\ & & \boldsymbol{\Delta}_m \end{bmatrix},$$

where for each $i = 1, \ldots, m$, $\boldsymbol{\Delta}_i : \ell_2^1(\mathfrak{F}_\Psi) \to \ell_2^1(\mathfrak{F}_\Psi)$ is a causal non-anticipative uncertainty.

We need the following lemma first. Denote $\mathcal{D}$ to be the set of all $m \times m$ positive definite diagonal matrices.

Lemma 3 *For the linear stochastic system $\boldsymbol{G}$ represented in (1), the following two statements are equivalent:*

1) $\inf_{S \in \mathcal{D}} \|S^{-1} \boldsymbol{G} S\|_\infty \geq 1/r$.
2) *There exists $u \in \ell_2^m(\mathfrak{F}_\Psi)$ such that for each integer $1 \leq i \leq m$, $\|(\boldsymbol{G}u)_i\|_2 \geq r^{-1}\|u_i\|_2$.*

Proof: $2 \Rightarrow 1$: If Statement 2 holds then for arbitrary $S = \operatorname{diag}\{s_1, \ldots, s_m\} \in \mathcal{D}$, so does the following inequality for the same u:

$$\sum_{i=1}^{m} s_i^2 \|(\boldsymbol{G}u)_i\|_2^2 \geq r^{-1} \sum_{i=1}^{m} s_i^2 \|u_i\|_2^2,$$

since $d_i > 0$. Hence the above inequality can be rewritten as

$$\|S\boldsymbol{G}u\|_2^2 \geq r^{-1}\|Su\|_2^2, \quad \forall S \in \mathcal{D},$$

which implies Statement 1.

$1 \Rightarrow 2$: Assume that Statement 2 is violated, i.e. for any $u \in \ell_2^m(\mathfrak{F}_\Psi)$, $\|(\boldsymbol{G}u)_j\|_2 < r^{-1}\|u_j\|_2$ for some j. Then take $\bar{S} = \operatorname{diag}\{s_1, \ldots, s_m\}$ where $s_j = 1$ and $s_i = \epsilon$ for $i \neq j$ with ϵ an arbitrary small positive number. For this $\bar{S}$, easy to see that by taking ϵ small enough,

$$\|\bar{S}\boldsymbol{G}u\|_2^2 < r^{-1}\|\bar{S}u\|_2^2.$$

Hence Statement 1 is also violated, which finishes the proof. ∎

Now we show that Statement 2 in the above lemma is related to the existence of a destabilizing uncertainty $\boldsymbol{\Delta}$.

Lemma 4 *If Statement 2 in Lemma 3 holds then there exists an uncertainty $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ such that $(\boldsymbol{G}, \boldsymbol{\Delta})$ is not ℓ_2-BIBO stable.*

Proof: If Statement 2 in Lemma 3 holds, in light of Corollary 2, it also holds for some $f = \{\hat{f}, 0_\infty\} \in \ell_2(\mathfrak{F}_\perp)$. Then we can repeat the necessity proof of Theorem 2 except that the destabilizing uncertainty $\boldsymbol{\Delta} = \operatorname{diag}\{\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_m\}$ and each $\boldsymbol{\Delta}_i$ is

$$\boldsymbol{\Delta}_i z(k) := \begin{cases} \dfrac{\hat{f}_i(k)\|z\|_{I_{j-1}}}{\|(\boldsymbol{G}u_1)_i\|_{I_{j-1}}}, & \text{if } k \geq M_1, k \in I_j; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

with u_1, I_{j-1}, M_1 all defined in the same way as in (7). Easy to see that $\|\boldsymbol{\Delta}_i\|_\infty \leq r$ and hence $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ destabilizes $(\boldsymbol{G}, \boldsymbol{\Delta})$ as well. ∎

Theorem 4 *Assume that $\boldsymbol{G}$ is a causal non-anticipative ℓ_2-BIBO stable linear stochastic system represented in (1). The closed-loop system $(\boldsymbol{G}, \boldsymbol{\Delta})$ is ℓ_2-BIBO stable for all $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ if and only if $\inf_{S \in \mathcal{D}} \|S^{-1}\boldsymbol{G}S\|_\infty < 1/r$.*

Proof: First assume that $\inf_{S \in \mathcal{D}} \|S^{-1}\boldsymbol{G}S\|_\infty < 1/r$. Notice that for any $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$, $\|S^{-1}\boldsymbol{\Delta}S\|_\infty = \|\boldsymbol{\Delta}\|_\infty \leq r$ for all $S \in \mathcal{D}$. Hence by Theorem 2, for some $S \in \mathcal{D}$, the closed-loop system $(S^{-1}\boldsymbol{G}S, S^{-1}\boldsymbol{\Delta}S)$ is stable for all $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$. But $(S^{-1}\boldsymbol{G}S, S^{-1}\boldsymbol{\Delta}S)$ is in fact equivalent to $(\boldsymbol{G}, \boldsymbol{\Delta})$. Hence $(\boldsymbol{G}, \boldsymbol{\Delta})$ is also ℓ_2-BIBO stable.

Then we consider the necessity. By Lemma 3 and Lemma 4, if

$$\inf_{S \in \mathcal{D}} \|S^{-1}\boldsymbol{G}S\|_\infty \geq 1/r \tag{11}$$

holds then there exists some $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ such that $(\boldsymbol{G}, \boldsymbol{\Delta})$ is not stable. ∎

Then we wish to prove the following statement similar to Theorem 3.

Theorem 5 *Under the same hypotheses as Theorem 4, then we have $P > 0$ solving (4). Further assume that $K = 0$ for K defined in (4) and that H defined in (8) is diagonal. Then $(\boldsymbol{G}, \boldsymbol{\Delta})$ for all deterministic $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ if and only if $\inf_{S \in \mathcal{D}} \|S^{-1}\boldsymbol{G}S\| < 1/r$.*

Proof: The sufficiency is immediate by Theorem 1. Then we consider the necessity. Assume that $\inf_{S\in\mathcal{D}}\|S^{-1}\boldsymbol{G}S\|_\infty \geq 1/r$. For $\boldsymbol{G}$ we have the LMI (4) and H defined in (8). Then similarly we can also have the corresponding LMI for $S^{-1}\boldsymbol{G}S$ with solution P_S and define the corresponding H_S. Since $H > 0$ is diagonal, by similar argument to Proposition 2.2 in [17], H_S is also diagonal, and there exists some $S \in \mathcal{D}$ such that $H_S = hI$ for some $h > 0$. Moreover $h \geq 1/r^2$ since $\inf_{S\in\mathcal{D}}\|S^{-1}\boldsymbol{G}S\|_\infty \geq 1/r$. Then similar to the proof of Theorem 2, for $y = S^{-1}\boldsymbol{G}Su$ there holds

$$\|y\|_2^2 = x_0' P_S x_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{u(k)' H_S u(k)\}.$$

Then there exists the following destabilizing uncertainty $\boldsymbol{\Delta}_i : \ell_2^1(\mathfrak{F}_\Psi) \to \ell_2^1(\mathfrak{F}_\Psi)$ satisfying

$$\boldsymbol{\Delta}_i z(k) = h^{-1/2} z(k).$$

Then $\boldsymbol{\Delta} \in \hat{\mathcal{B}}_r$ and $u(k) = \boldsymbol{\Delta} y(k) = h^{-1/2} y(k)$. Hence

$$\begin{aligned}\|y\|_2^2 &= x_0' P_S x_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{y(k)' y(k)\} \\ &= x_0' P_S x_0 + \sum_{k=0}^{\infty} \boldsymbol{E}\{\|y(k)\|^2\} \\ &= x_0' P_S x_0 + \|y\|_2^2.\end{aligned}$$

This again implies $P_S = 0$, i.e. $(S^{-1}\boldsymbol{G}S, S^{-1}\boldsymbol{\Delta}S)$, or equivalently $(\boldsymbol{G}, \boldsymbol{\Delta})$ is unstable. ■

In the rest part of the paper, Theorem 2 and Theorem 4 will be referred to as *the mixed small gain theorems*.

Remark 3 As discussed in the introduction, there have been some papers in the literature working towards the mixed small gain theorem. A small gain theorem exists in [9] in the sense of stochastic ℓ_2 norm and ℓ_2-induced norms, however it considers only closed-loops consisting of two given linear stochastic systems, and hence only the sufficiency part of the theorem is proved. In paper [1] the same formulation as ours is considered but only a bounded real lemma is derived. A similar lemma is also shown in [38] which is then used to solve a stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, which is different from the NCS problem in this paper. The paper [2] derives the stability radius which is closely related to the small gain condition, but as discussed in Remark 2, it considers a different type of $\boldsymbol{G}$ which admits an analytic expression of stability radius but does not cover the model in this paper, hence the result cannot be applied to our problem. The mixed small gain theorems are proved in this paper for the first time, both for unstructured and structured cases, and are the main contribution of this paper.

IV. Application in NCS Stabilization

In this section we will solve a state feedback NCS stabilization problem with mixed channel uncertainties, which is also the motivating problem of the research in this paper. In [25], the solutions to the state feedback NCS stabilization problem are presented when the channel model is given by either the SER, the R-SER or the SNR channel model. The problem with mixed channel uncertainty is first studied in [30] for SI case, where the channel introduces both logarithmic quantization and packet drop. Later in [12], [32], the channel is assumed to introduce logarithmic quantization with a different scheme and fading. Both present the same equivalent condition for state feedback stabilizability for SI case, and the former further works on SI output feedback problem while the later provides a sufficient condition for MI case. The paper [36] also investigates the channel with data rate constraint and logarithmic quantization.

In this paper we continue the step of [32] and seek to solve the NCS stabilization problem when both logarithmic quantization (represented by R-SER model) and fading exist. Both SI and MI cases are solved.

We present the system setup first. Consider the discrete-time LTI system $[A|B]$ or

$$x(k+1) = Ax(k) + Bu(k),$$

where $x(k) \in \mathbb{R}^n$ is the system state and $u(k) \in \mathbb{R}^m$ is the system input. Assume $[A|B]$ to be stabilizable. Under static state feedback, $v(k) = Fx(k)$ and is sent through a communication channel. The channel sends $v(k)$ by sending each of its m components $v_i(k)$ through one of its m independent parallel subchannels. The subchannels are assumed to introduce both fading and logarithmic quantization errors during the transmission. That is, the channel output $u(k) = (\boldsymbol{\Xi}\boldsymbol{Q}v)(k)$. Here the quantization $\boldsymbol{Q} = \operatorname{diag}\{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m\}$, while the fading $\boldsymbol{\Xi} = \operatorname{diag}\{\Xi_1, \ldots, \Xi_m\}$. The channel output $u(k)$ enters the plant as control input. This is shown in Fig. 2.

The mathematical models of $\boldsymbol{Q}$ and $\boldsymbol{\Xi}$ are as follows. Each $\boldsymbol{Q}_i$ in $\boldsymbol{Q}$ in Fig. 2 is given by the alternative logarithmic quantizer in [25], i.e. for $w_i = \boldsymbol{Q}_i v_i$, on each time step k,

$$w_i(k) = \begin{cases} \rho_i^l, & \text{if } 1-\delta_i < v_i(k)/\rho_i^l \leq 1+\delta_i; \\ 0, & \text{if } v_i(k) = 0; \\ -[\boldsymbol{Q}_i(-v_i)](k), & \text{if } v_i(k) < 0, \end{cases}$$

where $0 < \rho_i < 1, \delta_i = (1-\rho_i)/(1+\rho_i)$ and $l \in \mathbb{Z}$. Notice that this is different from the one used in [10], [13], etc. Such a quantizer is known to satisfy the R-SER model [25], i.e.

$$w_i(k) = v_i(k) - \Delta_i(w_i(k)).$$

Here $\Delta_i(\cdot)$ is a deterministic causal uncertainty such that $\Delta_i(w_i(k)) = (\boldsymbol{\Delta}_i w_i)(k)$ for each k where $\boldsymbol{\Delta}_i$ is scalar and satisfies $\|\boldsymbol{\Delta}_i\|_\infty \leq \delta_i$. Denote $\boldsymbol{\Delta} = \operatorname{diag}(\boldsymbol{\Delta}_1, \cdots, \boldsymbol{\Delta}_m)$. Clearly

$$(\boldsymbol{\Delta}w)(k) = \Delta(w(k)) := \begin{bmatrix} \Delta_1(w_1(k)) \\ \vdots \\ \Delta_m(w_m(k)) \end{bmatrix}.$$

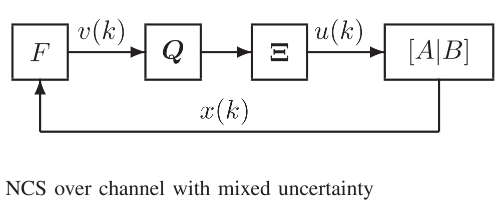

Fig. 2. NCS over channel with mixed uncertainty

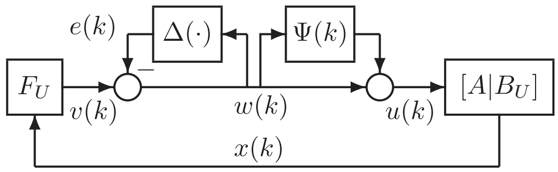

Fig. 3. NCS with the R-SER model

Meanwhile Ξ_i is given by the fading channel model with nonzero mean μ_i and variance σ_i^2, i.e. for $u_i = \Xi_i w_i$,

$$u_i(k) = \frac{1}{\mu_i}\Phi_i(k)w_i(k),$$

and Φ_i is an i.i.d. random process satisfying

$$\boldsymbol{E}\{\Phi_i(k)\} = \mu_i, \boldsymbol{E}\{|\Phi_i(k)|^2\} = \sigma_i^2, \boldsymbol{E}\{\Phi_i(k)\Phi_j(k)\} = 0$$

for $j \neq i$ and $k \in \mathbb{N}$. Ξ_i can be replaced by $\boldsymbol{I} + \boldsymbol{\Psi}_i$ and the i.i.d. random process Ψ_i corresponding to $\boldsymbol{\Psi}_i$ has zero mean and finite variance $\nu_i^2 = \sigma_i^2/\mu_i^2$ on each time step k. Denote $\Psi = \mathrm{diag}(\Psi_1, \dots, \Psi_m)$.

Finally, we need to introduce a unitary coding-decoding procedure at the ends of the transmission channel. Assume that the channel sends $U'v(k)$ instead of the input $v(k)$ itself in the transmission, and the received signal $u(k)$ is multiplied by U before sent to the system $[A|B]$ as input. Here U is a unitary matrix in $\mathbb{R}^{m\times m}$. We denote $B_U = BU$ and $F_U = U'F$, and also abuse the notation a bit by regarding $v(k)$ as the output of F_U instead of F. The closed-loop dynamics is thus

$$x(k+1) = Ax(k) + B_U[I+\Psi(k)][F_U x(k) - \Delta(w(k))], \quad (12)$$

as shown in Fig. 3. Note that (12) can be rewritten as

$$\begin{aligned} x(k+1) &= (A + B_U F_U)x(k) + B_U\Psi(k)F_U x(k) \\ &\quad - B_U[I + \Psi(k)]e(k) && (13) \\ w(k) &= F_U x(k) - e(k) && (14) \\ e(k) &= \Delta(w(k)). && (15) \end{aligned}$$

Obviously (13)-(14) gives the input-output relation from e to w and (15) gives the relation from w to e. Denote $\boldsymbol{S}$ to be the system such that $w = \boldsymbol{S}e$. Then the closed-loop can be equivalently considered to consist of $\boldsymbol{S}$ in the forward path and $\boldsymbol{\Delta}$ in the feedback path, as shown in Fig. 4. Denote the closed-loop as $(\boldsymbol{S}, \boldsymbol{\Delta})$. It turns out that $\boldsymbol{S}$ is a special case of (1) and hence the mixed small gain theorems can be applied to analyze the stability of $(\boldsymbol{S}, \boldsymbol{\Delta})$. But before doing this, we first present the following key lemma on the optimal control in the sense of $\|\boldsymbol{S}\|_\infty$. Denote $\Sigma^2 := \mathrm{diag}\{\nu_1^2, \dots, \nu_m^2\}$.

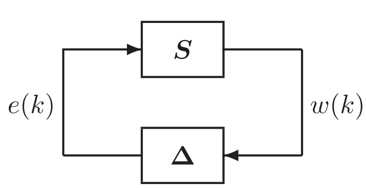

Fig. 4. The equivalent closed-loop system

Lemma 5 *There holds*

$$\inf_{F_U} \|\boldsymbol{S}\|_\infty^2 = \rho(I + B_U'XB_U + \Sigma^2 \odot B_U'XB_U),$$

and the infimum can be achieved by taking

$$F_U = -[I + B_U'XB_U + \Sigma^2 \odot B_U'XB_U]^{-1}B_U'XA \quad (16)$$

where $X \geq 0$ is the stabilizing solution to

$$X = A'XA - A'XB_U[I + B_U'XB_U + \Sigma^2 \odot B_U'XB_U]^{-1}B_U'XA, \quad (17)$$

Proof: For X being the stabilizing solution to (17). denote $\tilde{J} := I + \Sigma^2 \odot B_U'XB_U, \tilde{H} := \tilde{J} + B_U'XB_U$ and $\tilde{K} := B_U'X(A + B_U F_U) + \tilde{J}F_U$. By Lemma 1, $\|\boldsymbol{S}\|_\infty < \gamma$ if and only if there exists $X > 0$ such that $\tilde{L} = \gamma^2 I - \tilde{H} > 0$ and

$$X > (A + B_U F_U)'X(A + B_U F_U) + F_U'\tilde{J}F_U + \tilde{K}'\tilde{L}^{-1}\tilde{K}. \quad (18)$$

The latter inequality can be manipulated into the following after tedious arithmetic

$$\begin{aligned} X &> A'XA - A'XB_U\tilde{H}^{-1}B_U'XA \\ &\quad + (A'XB_U + F_U'\tilde{H})(\tilde{H}^{-1} + \tilde{L}^{-1})(B_U'XA + \tilde{H}F_U). \end{aligned}$$

Obviously the infimum of $\|\boldsymbol{S}\|_\infty$ over F can be reached by taking

$$F_U = -\tilde{H}^{-1}B_U'XA. \quad (19)$$

In this case the inequality becomes

$$X > A'XA - A'XB_U\tilde{H}^{-1}B_U'XA, \quad (20)$$

and is always satisfied when $\|\boldsymbol{S}\|_\infty$ is finite. Thus $\|\boldsymbol{S}\|_\infty < \gamma$ if and only if $\tilde{L} > 0$ or $\tilde{H} < \gamma^2 I$ for $X > 0$ solving (20). Hence to obtain $\|\boldsymbol{S}\|_\infty$ we need to minimize γ, which is equivalent to finding the infimum of all solutions to (20). However it is well known that this is given by the stabilizing solution to the corresponding ARE, which is exactly (17). See e.g. [19]. Hence it follows that

$$\inf_{F_U} \|\boldsymbol{S}\|_\infty^2 = \inf_{\tilde{L}>0} \gamma^2 = \rho(\tilde{H}),$$

where X is the stabilizing solution to (17). ■

Note that in the proof above, the selected controller F_U in (19) always makes $\tilde{K} = 0$ true for any X solving (18). Hence the assumption in Theorem 3 is always satisfied. This implies that the new small gain theorem is applicable to the NCS problem in which $\boldsymbol{\Delta}$ is assumed to be deterministic.

Now we work on to solve the stabilization problem of the NCS $(\boldsymbol{S}, \boldsymbol{\Delta})$ and consider the SI case first. Denote the *Mahler measure* of a system $[A|B]$ as

$$M(A) := \prod_{i=1}^{n} \max\{1, |\lambda_i(A)|\},$$

and its *topological entropy* as

$$h(A) := \sum_{i=1}^{n} \max\{0, \log|\lambda_i(A)|\}.$$

Clearly $h(A) = \log M(A)$. Then we can quote the following lemma presented in [32] which is useful here.

Lemma 6 *Assume that $[A|B]$ is SI and stabilizable. Then the stabilizing solution $X_\lambda \geq 0$ to the parameterized* ARE

$$X_\lambda = A'X_\lambda A - \lambda A'X_\lambda B(1 + B'X_\lambda B)^{-1}B'X_\lambda A. \quad (21)$$

exists if and only if $\lambda > 1 - M(A)^{-2}$. If $\lambda > 1 - M(A)^{-2}$, then its stabilizing solution is given by

$$X_\lambda = cX_1, \quad c = 1/[1 + (\lambda - 1)M(A)^2] \quad (22)$$

with X_1 the stabilizing solution to the same ARE (21) *at $\lambda = 1$, and the corresponding stabilizing state feedback gain satisfies*

$$F_\lambda = -\lambda(1 + B'X_\lambda B)^{-1}B'X_\lambda A \equiv F_1,$$

i.e. F_λ is the same for all such λ.

Now we present the result for SI case. Since in this case the coding-decoding procedure is trivial, we will write B_U and F_U simply as B and F. The result has appeared in authors' previous work [33], as well as in [32] with a different approach. However since it is useful to the result for MI case, we still present it here to make the paper self-contained. Define the channel capacity as

$$\mathfrak{C} := \frac{1}{2}\log\frac{1+\nu^2}{\delta^2+\nu^2}.$$

Theorem 6 *Consider the closed-loop system given in (12) when $[A|B]$ is SI and stabilizable and the channel has capacity $\mathfrak{C}$. In such a closed-loop, there exists a static state feedback F such that (12) is ℓ_2-BIBO stable if and only if*

$$\mathfrak{C} > h(A). \quad (23)$$

Proof: By Theorem 3, $(\boldsymbol{S}, \boldsymbol{\Delta})$ can be stabilized if and only if

$$\inf_F \|\boldsymbol{S}\|_\infty < 1/\delta. \quad (24)$$

In light of Lemma 5, this is in turn equivalent to

$$1 + (1+\nu^2)B'XB < \delta^{-2}, \quad (25)$$

where $X \geq 0$ solves the ARE

$$X = A'XA - A'XB[1 + (1+\nu^2)B'XB]^{-1}B'XA. \quad (26)$$

Denote $\tilde{X} := (1+\nu^2)X$, then (26) can be rewritten as

$$\tilde{X} = A'\tilde{X}A - (1+\nu^2)^{-1}A'\tilde{X}B(1 + B'\tilde{X}B)^{-1}B'\tilde{X}A.$$

By Lemma 6 we can calculate that

$$B'\tilde{X}B = \frac{M(A)^2 - 1}{\left(\frac{1}{1+\nu^2} - 1\right)M(A)^2 + 1}.$$

Hence

$$\begin{aligned} \delta^{-2} &> 1 + (1+\nu^2)B'XB \\ &= 1 + B'\tilde{X}B \\ &= \frac{\frac{1}{1+\nu^2}M(A)^2}{\left(\frac{1}{1+\nu^2} - 1\right)M(A)^2 + 1}. \end{aligned}$$

Rearranging terms gives

$$\frac{1+\nu^2}{\delta^2+\nu^2} > M(A)^2.$$

Taking log on both sides yields (23). ■

Now we are ready to solve the MI case problem and present the main result of this section. Denote the capacity of the ith subchannel to be

$$\mathfrak{C}_i := \frac{1}{2}\log\frac{1+\nu_i^2}{\delta_i^2+\nu_i^2},$$

and the overall capacity $\mathfrak{C} := \sum_{i=1}^m \mathfrak{C}_i$.

Theorem 7 *Consider the closed-loop system given in (12) when $[A|B]$ is MI and stabilizable and the channel has capacity $\mathfrak{C}$. In such a closed-loop, there exists a static state feedback F and a coding matrix U such that (12) is ℓ_2-BIBO stable if and only if*

$$\mathfrak{C} > h(A). \quad (27)$$

Proof: We show the necessity first. By Theorem 5, $(\boldsymbol{S}, \boldsymbol{\Delta})$ can be stabilized if and only if

$$\inf_{S\in\mathcal{D}} \inf_F \|S^{-1}\boldsymbol{S}SD_\delta\|_\infty < 1. \quad (28)$$

with U chosen such that $I + B_U'XB_U + \Sigma^2 \odot B_U'XB_U$ is diagonal. In light of Lemma 5, if $(\boldsymbol{S}, \boldsymbol{\Delta})$ is stable, then there exists S such that

$$D_\delta(I + B_S'XB_S + \Sigma^2 \odot B_S'XB_S)D_\delta < I, \quad (29)$$

by taking the optimal controller $F_S = S^{-1}F_U = -(J_S + B_S'XB_S)^{-1}B_S'XA$. Here $B_S = B_US = [B_1 \quad \cdots \quad B_m]$ and $X \geq 0$ is the stabilizing solution to the following ARE

$$X = A'XA - A'XB_S(J_S + B_S'XB_S)^{-1}B_S'XA, \quad (30)$$

with $J_S := I + \Sigma^2 \odot B_S'XB_S$. Taking determinant on both sides of (30) gives

$$\begin{aligned} \det(X) &= \det(A')\det(X) \\ &\quad * \det[I - B_S(J_S + B_S'XB_S)^{-1}B_S'X]\det(A). \end{aligned}$$

Here we may assume that A has only eigenvalues outside the unit circle without loss of generality. The assumption is only technical in order to simplify the proof and can be removed as argued in many papers, e.g [10], [13], [25], [32], etc. Hence $\det(A) = M(A)$ and

$$\begin{aligned} M(A)^2 &= \det[I - (J_S + B_S'XB_S)^{-1}B_S'XB_S]^{-1} \\ &= \det(J_S + B_S'XB_S)\det(J_S^{-1}) \\ &= \det(I + J_S^{-1/2}B_S'XB_SJ_S^{-1/2}) \\ &\leq \prod_{i=1}^m \left(1 + \frac{B_i'XB_i}{1 + \nu_i^2 B_i'XB_i}\right). \end{aligned}$$

The last inequality is due to Hadamard's inequality with the equality holds if and only if $B_S'XB_S$ is diagonal.

Consider (29). Since $D_\delta(I+B_S'XB_S+\Sigma^2\odot B_S'XB_S)D_\delta$ is positive semi-definite, its spectral radius is also its 2-norm. Hence

$$\begin{aligned}&\delta_i^2[1+(1+\nu_i^2)B_i'XB_i]\\ &\qquad\le \rho(D_\delta(I+B_S'XB_S+\Sigma^2\odot B_S'XB_S)D_\delta)<1.\end{aligned}$$

Hence $B_i'XB_i<(\delta_i^{-2}-1)/(1+\nu_i^2)$ for all i. Also note that $\frac{B_i'XB_i}{1+\nu_i^2B_i'XB_i}$ is an increasing function of $B_i'XB_i\ge 0$. Hence we may conclude that

$$\begin{aligned}M(A)^2&<\prod_i\left(1+\frac{(\delta_i^{-2}-1)/(1+\nu_i^2)}{1+\nu_i^2(\delta_i^{-2}-1)/(1+\nu_i^2)}\right)\\ &=\prod_i\frac{1+\nu_i^2}{\delta_i^2+\nu_i^2}.\end{aligned}$$

Taking log on both sides yields (27).

Now we turn to the sufficiency proof. We will constructively solve the synthesis problem with the technique of *channel resource allocation* first introduced in [15]. Given $\mathfrak{C}$ satisfying (23) and select $U=I$, we need to design $S\in\mathcal{D}$ and the stabilizing state feedback gain F_U such that (28) holds. Without loss of generality, the system $[A|B_U]$ is assumed to have the so-called Wonham decomposition [34]:

$$\left[\begin{array}{cccc|cccc} A_1 & * & \cdots & * & b_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots & 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m & 0 & \cdots & 0 & b_m \end{array}\right] \tag{31}$$

where A_i has dimension $n_i\times n_i$ and b_i $n_i\times 1$ with $\sum_{i=1}^m n_i=n$. Select $S=\mathrm{diag}(1,\epsilon,\cdots,\epsilon^{m-1})$ with $\epsilon>0$ close to 0. Also define $W=\mathrm{diag}(I_{n_1},\epsilon I_{n_2},\cdots,\epsilon^{m-1}I_{n_m})$. Then $S^{-1}\boldsymbol{S}S$ has the following state space realization

$$\begin{aligned}x_\epsilon(k+1)&=(A_\epsilon+B_\epsilon F_\epsilon)x_\epsilon(k)+B_\epsilon\Psi(k)F_\epsilon x_\epsilon(k)\\ &\quad-B_\epsilon[I+\Psi(k)]e_\epsilon(k)\\ w_\epsilon(k)&=F_\epsilon x_\epsilon(k)-e_\epsilon(k)\end{aligned}$$

where

$$A_\epsilon=W^{-1}AW=\begin{bmatrix} A_1 & o(\epsilon) & \cdots & o(\epsilon)\\ 0 & A_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & o(\epsilon)\\ 0 & \cdots & 0 & A_m\end{bmatrix},$$

$$B_\epsilon=W^{-1}B_US=\begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon)\\ 0 & b_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & o(\epsilon)\\ 0 & \cdots & 0 & b_m\end{bmatrix},$$

furthermore $F_\epsilon=S^{-1}FW$ and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon\to 0$. Taking $F_\epsilon=\mathrm{diag}\{f_1,\cdots,f_m\}$ yields

$$\boldsymbol{S}\to\mathrm{diag}\{\boldsymbol{S}_1,\cdots,\boldsymbol{S}_m\}$$

as $\epsilon\to 0$ where $\boldsymbol{S}_i$ is represented by

$$\begin{aligned}x_i(k+1)&=(A_i+b_if_i)x_i(k)+b_i\Psi_i(k)f_ix_i(k)\\ &\quad+b_i[1+\Psi_i(k)]e_i(k)\\ w_i(k)&=f_ix_i(k)+e_i(k)\end{aligned}$$

Hence we have m equivalent single channel problems and Theorem 6 can be applied. Set $f_i=-[1+(1+\nu_i^2)b_i'X_ib_i]^{-1}b_i'X_iA_i$ where $X_i\ge 0$ solves

$$X_i=A_i'X_iA_i-A_i'X_ib_i[1+(1+\nu_i^2)b_i'X_ib_i]^{-1}b_i'X_iA_i, \tag{32}$$

and satisfies $1+(1+\nu_i^2)b_i'X_ib_i<\delta_i^{-2}$. When (23) holds, there always exists an allocation $\{\mathfrak{C}_1,\cdots,\mathfrak{C}_m\}$ such that $\mathfrak{C}_i>h(A_i)$ for each $i=1,\cdots,m$, i.e.

$$\frac{1+\nu_i^2}{\delta_i^2+\nu_i^2}>M(A_i)^2.$$

Since $\delta_i>0$, this implies $1+\nu_i^2<[1-M(A_i)^{-2}]^{-1}$. Therefore the solution $X_i\ge 0$ to (32) always exists and f_i can thus be constructed. By Theorem 6, such f_i implies $\|\boldsymbol{S}_i\|_\infty<\delta_i^{-1}$ for all i. It follows that $\|S^{-1}\boldsymbol{S}SD_\delta\|_\infty<1$ for sufficiently small ϵ. ■

Remark 4 A recent paper [16] also considers the above NCS problem and derives the necessary and sufficient condition for MI case. The conclusion is the same as Theorem 7, however the formulation is a bit different. Instead of using the extended ℓ_2 norm and related input-output theory as in this paper, [16] uses the power norm for signals and the stability is induced accordingly. Also, the fading Ψ is not integrated into the plant but rather considered as a separated uncertainty. Hence the closed-loop consists of the LTI part and the uncertain part. Due to the special nice property of the closed-loop system as discussed in Remark 2, the optimal control can be easily seen and is applied immediately. Then the stability can be analyzed based on the $\mathcal{H}_\infty$ small gain theorem. However the solution relies on the special property of the problem and may not be easy to extend to other problems with different structures, while the method in this paper based on mixed small gain theorems may be more general and applicable to other important problems.

V. Numerical Example

In this section we give a numerical example to illustrate how an NCS is stabilized by channel/controller co-design. Consider the unstable system $[A|B]$ with

$$A=\begin{bmatrix}8&0&0\\0&2&0\\0&0&4\end{bmatrix},\ B=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}.$$

Obviously $[A|B]$ is stabilizable and is in the Wonham decomposition form with

$$A_1=\mathrm{diag}(8,2),A_2=4,b_1=[1\quad 1]',b_2=1,$$

and $h(A)=h(A_1)+h(A_2)=4+2=6$. Let the total given capacity be $\mathfrak{C}=6+2\times 10^{-2}$. Allocate $\mathfrak{C}_1=4+10^{-2}$ and $\mathfrak{C}_2=2+10^{-2}$. It is worth mentioning that for the NCS problems dealing solely with quantization [15] or fading

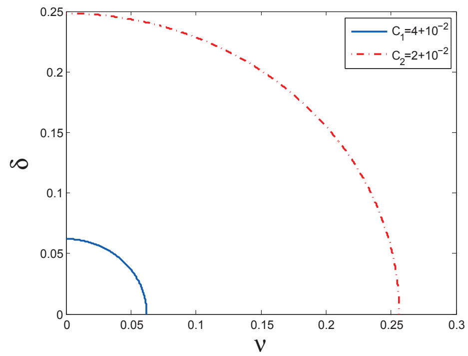

Fig. 5. All possible realizations of the allocated capacity

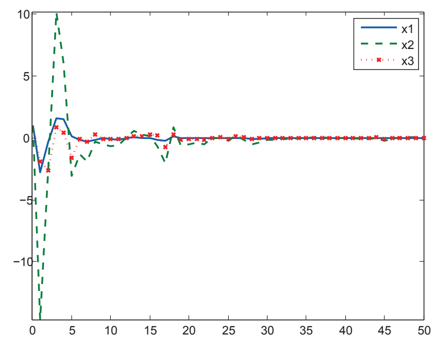

Fig. 6. $(\delta_1, \nu_1) = (0.05, 0.0368), (\delta_2, \nu_2) = (0.05, 0.251)$

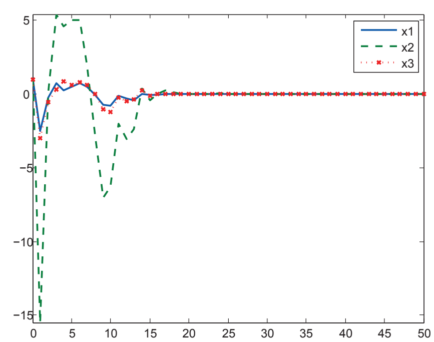

Fig. 7. $(\delta_1, \nu_1) = (0.0369, 0.05), (\delta_2, \nu_2) = (0.2435, 0.05)$

errors [31], $\{\delta_i\}$ or $\{\nu_i\}$ is determined once the channel capacity is allocated. However in this paper, there still exists trade-off between $\{\delta_i\}$ and $\{\nu_i\}$ for a fixed capacity, i.e. the realizations of the allocated capacity are not unique, nor is there any preferred choice among them. Fig. 5 shows all possible combinations of $\{\delta_i\}$ and $\{\nu_i\}$ realizing the allocated capacity.

Take $\delta_1 = \delta_2 = 0.05$ for instance, then $\nu_1 = 0.0368$ and $\nu_2 = 0.251$. Solve the optimization problem described in the proof of Lemma 5 for $[A_1|b_1]$ and $[A_2|b_2]$ respectively to obtain f_1 and f_2. By Lemma 6 they are exactly given by the standard expensive controllers, i.e. $f_1 = [-6.5625 \quad 1.3125]$ and $f_2 = -1.5$. Set

$$F = \text{diag}(f_1, f_2) = \begin{bmatrix} -6.5625 & 1.3125 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}.$$

With the above channel/controller co-design, the closed-loop evolution of the plant states starting from the initial condition $x(0) = [1 \quad 1 \quad 1]'$ is shown in Fig. 6. Clearly the system state converges to zero asymptotically.

A different pair of parameters could be $\nu_1 = \nu_2 = 0.05$ and hence $\delta_1 = 0.0369, \delta_2 = 0.2435$, while the allocated capacity to each subchannel remains the same. By Fig. 7 we can see that the system can be stabilized by the same controller as the previous example.

VI. Conclusion

In this paper we extend the classical deterministic $\mathcal{H}_\infty$ small gain theorem to the mixed case with both deterministic and stochastic uncertainties. The mixed small gain theorem is proved for both unstructured and structured uncertainties. Finally the theorem is applied to an NCS stabilization problem and the solution is provided for both SI and MI cases.

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