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Discrete-Time LQG Mean Field Games with Unreliable Communication

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Abstract— **In this paper, we consider discrete-time linear-quadratic-Gaussian (LQG) mean field games over unreliable communication links. These are dynamic games with a large number of agents where the cost function of each agent is coupled with other agents' states via a mean field term. Further, the individual dynamical system for each agent is subject to packet dropping. Under this setup, we first obtain an optimal decentralized control law for each agent that is a function of local information as well as packet drop information. We then construct a mean field system that provides the best approximation to the mean field term under appropriate conditions. We also show that the optimal decentralized controller stabilizes the individual dynamical system in the time-average sense. We prove an ϵ-Nash equilibrium property of the set of N optimal decentralized controllers, and show that ϵ can be made arbitrarily small as the number of agents becomes arbitrarily large. We note that the existence of the ϵ-Nash equilibrium obtained in this paper is primarily dependent on the underlying communication networks.**

I. INTRODUCTION

In recent years, dynamic games with a large number of agents have been studied extensively due to increasing demands of applications in engineering, economics, computer science, and social sciences [1]–[4]. One such sub-class of large population games is known as mean field games, in which there is a large number of agents, and each agent is coupled with each other via a mean field term that describes the average of all the agents' states.

Similar to other noncooperative games, the main objective of the study of mean field games is to obtain a characterization of Nash equilibria under the assumption of rationality for each player. In general, however, such a characterization may not be possible because the complexity increases with the number of players and the dimension of the state space [3], [5]. The decentralized nature of mean field games also creates another difficulty, since it requires each agent to have access to only local information [3], [5].

The Nash certainty equivalence (NCE) principle was proposed in [3] and [6] for linear and nonlinear mean field games, respectively, to overcome these issues. In general, the NCE principle consists of two steps: 1) a single optimal control problem, and 2) approximation of the mean field term. Under these steps, a deterministic function can be obtained *a priori*, by which the overall effect on each agent of the mean field term can be captured with probability one. Moreover, the set of individual optimal controllers constitutes an ϵ-Nash equilibrium for the mean field game, where ϵ converges to zero as the number of agents, say N, becomes arbitrarily large. Similar results were obtained in [7], [8] and [9] where a NCE-like methodology was used, but for different problem settings.

Other classes of mean field games were also considered in the literature. Risk-sensitive mean field games were considered in [10] where their relationship to robust mean field games was also established. A paper [11] in this conference develops decentralized optimal controllers for continuous time linear-quadratic risk-sensitive mean field games. In [12], LQG mean field games with Markov jump parameters were studied. In [13], the mean field adaptive control problem was studied, and the consensus problem was discussed in [14]. Discrete-time mean field games under the notion of oblivious equilibria were discussed in [2], [5].

In addition to the large number of agents, it is also natural that the individual dynamical system be subject to communication constraints, especially packet drops [15]. The packet drop constraint is able to capture the unreliable nature of communication from the controller to the system, or from the system to the sensor. For the single agent case ($N = 1$), such a scenario was addressed within the LQG setting in [16], [17], and a dynamic game or H^∞ setting in [18], [19]. The main objective for the single agent case is to obtain an optimal stabilizing control law and to characterize the critical value that is the maximum rate of packet losses that the controller is able to tolerate for stabilization. The packet drop problems considered so far, however, were for the single agent case, and the decentralized control problem with a large number of agents has not yet been studied.

In this paper, we study LQG mean field games when the individual dynamical system is subject to an unreliable communication link. A similar problem was considered in [20] where the individual dynamical system was constrained by measurement quantization. Here, we assume that the underlying packet dropping network has transmission control protocol (TCP); hence each player has access to information on packet losses. Under this setup, we first obtain an optimal decentralized control law for each agent that is a function of local information as well as packet drop information. We then construct a mean field system that provides the best approximation to the mean field term under appropriate conditions. We also show that the optimal decentralized controller stabilizes the individual dynamical system in the time-average sense. We prove an ϵ-Nash equilibrium property of the set of N optimal decentralized controllers, and then show that ϵ can be made arbitrarily small as the number of agents becomes arbitrarily large. We finally note that the

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existence of the ϵ-Nash equilibrium obtained in this paper depends on the underlying communication networks.

The structure of the paper is as follows. We formulate the problem in Section II. The result of a single agent optimal control problem is given in Section III. The mean field system is constructed in Section IV. The stability analysis and the approximation performance are studied in Sections V and VI, respectively. We discuss an ϵ-Nash equilibrium property in Section VII, and end the paper with the concluding remarks of Section VIII.

II. Problem Formulation

We consider the following discrete-time linear dynamical system for agent i, $1 \leq i \leq N$:

$$x_{i,k+1} = A_i x_{i,k} + \alpha_{i,k} B_i u_{i,k} + w_{i,k} \tag{1}$$

$$y_{i,k} = \beta_{i,k} C_i x_{i,k} + v_{i,k}, \tag{2}$$

where $x_{i,k} \in \mathbb{R}^n$ is the state; $y_{i,k} \in \mathbb{R}^q$ is the output; $u_{i,k} \in \mathbb{R}^m$ is the control input; $\{w_{i,k}\}$ and $\{v_{i,k}\}$ are i.i.d. Gaussian processes; A_i, B_i and C_i are time-invariant matrices with appropriate dimensions; $\alpha_{i,k}$ and $\beta_{i,k}$ describe control and measurement losses, respectively; and $k \geq 0$ is the time instance. We have the following assumption on the dynamical system (1) and (2):

Assumption 1: 1) $x_{i,0}$ is a Gaussian random vector with mean zero and $\mathbb{E}\{x_{i,0}x_{i,0}^T\} = X_0$ for all i.

2) $\{w_{i,k}\}$ and $\{v_{i,k}\}$ are i.i.d. Gaussian processes with mean zero and covariances $W \geq 0$ and $V > 0$, respectively, for all i.
3) $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$ are i.i.d. Bernoulli processes with distributions of $\mathbb{P}(\alpha_{i,k} = 1) = \alpha^i$ and $\mathbb{P}(\beta_{i,k} = 1) = \beta^i$ that model control and measurement packet losses of agent i, respectively.
4) $\{\alpha_{i,k},\ 1 \leq i \leq N\}$, $\{\beta_{i,k},\ 1 \leq i \leq N\}$ $\{x_{i,0},\ 1 \leq i \leq N\}$, $\{w_{i,k},\ 1 \leq i \leq N,\ \forall k\}$, and $\{v_{i,k},\ 1 \leq i \leq N,\ \forall k\}$ are independent for each i, and are mutually independent of each other. □

In (1) and (2), A_i, B_i and C_i are system parameters of each agent that can be identical or different depending on the game at hand. We assume that the triplet (A_i, B_i, C_i) constitutes the system parameter $\theta_i \in \Theta$, that is, $\theta_i := (A_i, B_i, C_i) \in \Theta$ where Θ is the set of θ_i. Note that θ_i is a vector that stands for the ordered (A_i, B_i, C_i) with stacked column vectors.

Assumption 2: For the first N agents, A_i, B_i, and C_i are selected from the following empirical distribution:

$$F_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\theta_i \leq \theta\}},\ \theta \in \Theta, \tag{3}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. Moreover we assume that when $\theta = \theta_i$, we have $\alpha^i := \alpha(\theta = \theta_i)$ and $\beta^i := \beta(\theta = \theta_i)$. We further assume that there is a probability distribution $F(\theta)$ such that $F_N(\theta)$ converges weakly to $F(\theta)$ as $N \to \infty$, i.e., $\lim_{N\to\infty} F_N(\theta) = F(\theta)$, and the support of $F(\theta)$, Θ, is compact. □

Now, agent i is interested in minimizing the cost function:

$$J_i^N(u_i, u_{-i}) = \limsup_{K\to\infty} \frac{1}{K}\mathbb{E}\left\{ \sum_{k=0}^{K-1} \|x_{i,k} - \rho f_k^N\|_Q^2 + \alpha_{i,k}\|u_{i,k}\|_R^2 \right\}, \tag{4}$$

where $\|z\|_S^2 := z^T S z$ for some vector z and matrix $S \geq 0$, $Q \geq 0$, $R > 0$, $\rho > 0$ is a constant, and $u_{-i} := (u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$ with $u_i := (u_{i,0}, u_{i,1}, ...)$. In (4), the mean field coupling term f_k^N describes the average behavior (as measured by their states) of the first N agents, which can be written as

$$f_k^N \triangleq \frac{1}{N}\sum_{i=1}^{N} x_{i,k}. \tag{5}$$

We define the decentralized information structure for agent i:

$$\begin{cases} \mathcal{I}_{i,0}^d &= \{y_{i,0}, \beta_{i,0}\} \\ \mathcal{I}_{i,k}^d &= \{y_{i,0:k},\ u_{i,0:k-1},\ \alpha_{i,0:k-1},\ \beta_{i,0:k}\},\ k \geq 1, \end{cases} \tag{6}$$

where $y_{i,0:k} := (y_{i,0}, ..., y_{i,k})$ and the same notation applies to $u_{i,0:k-1}$, $\alpha_{i,0:k-1}$, and $\beta_{i,0:k}$. We also define the centralized information structure:

$$\mathcal{I}_k^c = \{\mathcal{I}_{i,k}^d,\ 1 \leq i \leq N\}.$$

With these information structures, we can define the following two sets of admissible controls that will be considered throughout this paper:

$$\begin{aligned} \mathcal{U}_i^d = \{&(u_{i,0}, u_{i,1}, ...) : u_i \text{ is adapted to} \\ &\sigma(\mathcal{I}_{i,s}^d, s = 0, 1, ..., k)\} \\ \mathcal{U}_i^c = \{&(u_{i,0}, u_{i,1}, ...) : u_i \text{ is adapted to} \\ &\sigma(\mathcal{I}_s^c, s = 0, 1, ..., k)\}, \end{aligned}$$

where $\sigma(\cdot)$ is the σ-algebra generated by its argument.

Notice that while the admissible controller in $\mathcal{U}_i^d$ is a function of agent i's local state and packet loss information, the admissible controller in $\mathcal{U}_i^c$ is a function of all the agents' states and packet loss information. Therefore, we have $\mathcal{U}_i^d \subset \mathcal{U}_i^c$. We call the former a set of decentralized controllers and the latter a set of centralized controllers.

We introduce below an approximate equilibrium solution for discrete-time mean field games with unreliable communication formulated in this section.

Definition 1: The set of controllers $\{u_i \in \mathcal{U}_i^c,\ 1 \leq i \leq N\}$ constitutes an ϵ-Nash equilibrium with respect to the cost functions $\{J_i^N,\ 1 \leq i \leq N\}$, if there exists $\epsilon \geq 0$ such that for any i, $1 \leq i \leq N$,

$$J_i^N(u_i, u_{-i}) \leq \inf_{v_i \in \mathcal{U}_i^c} J_i^N(v_i, u_{-i}) + \epsilon. \quad \square$$

In what follows, we first obtain an optimal decentralized controller for a single agent, and provide appropriate conditions under which the mean field term can be approximated with arbitrarily small error. Subsequently, we analyze the equilibrium behavior of the set of N-decentralized control laws for LQG mean field games over unreliable communication links in the large population regime, i.e., the case when $N \to \infty$.

III. Optimal Tracking Control over Unreliable Communication Links

Consider the cost function

$$\bar{J}_i(u_i, g) \tag{7}$$
$$= \limsup_{K\to\infty} \frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1} \|x_{i,k} - \rho g_k\|_Q^2 + \alpha_{i,k}\|u_{i,k}\|_R^2\right\},$$

where $u_i \in \mathcal{U}_i^d$ and the deterministic function g_k can be viewed as an approximation to the mean field coupling term in (5). Although this is just an assertion now, it will be shown later that this claim is true under some specific conditions in the large population regime. In (7), we assume that $g \in \mathcal{C}_b$ where $\mathcal{C}_b$ is the set of bounded n-dimensional vector-valued functions, i.e., $\mathcal{C}_b = \{h_k \in \mathbb{R}^n : \ \|h\|_\infty := \sup_{k\geq 0}\|h_k\| < \infty\}$ where $\|\cdot\|$ denotes 2-norm of vectors. Note that $\mathcal{C}_b$ is a Banach space. The minimization of (7) can be seen as the optimal tracking problem over an unreliable communication link with respect to the given reference signal g_k [3], [7], [21].

Theorem 1: Consider the dynamical system (1) and (2), and the cost function (7) with the information structure (6). Suppose that Assumption 1 holds, (A_i, B_i) and $(A_i, W^{1/2})$ are controllable, and $(A_i, Q^{1/2})$ and (A_i, C_i) are observable. Suppose $\alpha^i > \alpha_c^i$ and $\beta^i > \beta_c^i$ where α_c^i and β_c^i are defined by

$$\alpha_c^i \triangleq \inf_{\alpha^i}\{0 \leq \alpha^i \leq 1 \ : \ Z_i = A_i^T Z_i A_i + Q - \alpha^i A_i^T Z_i B_i (B_i^T Z_i B_i + R)^{-1} B_i^T Z_i A_i, \ Z_i > 0\} \tag{8}$$

$$\beta_c^i \triangleq \inf_{\beta^i}\{0 \leq \beta^i \leq 1 \ : \ \bar{P}_i = A_i \bar{P}_i A_i^T + W - \beta^i A_i \bar{P}_i C_i^T (C_i \bar{P}_i C_i^T + V)^{-1} C_i \bar{P}_i A_i^T, \ \bar{P}_i > 0\}. \tag{9}$$

Then:

(i) The optimal decentralized control law for agent i can be written as

$$u_{i,k}^* = -L_i \hat{x}_{i,k} - U_i s_{i,k+1}, \tag{10}$$

where Z_i is a solution of the Riccati equation defined in the set (8), $U_i \triangleq (R + B_i^T Z_i B_i)^{-1} B_i^T$, and $L_i \triangleq U_i Z_i A_i$. In (10), $s_{i,k}$ satisfies the following linear difference equation:

$$s_{i,k} = H_i^T s_{i,k+1} - \rho Q g_k, \tag{11}$$

where $H_i \triangleq A_i - \alpha^i B_i L_i$.

(ii) In (i), the estimated state $\hat{x}_{i,k} := \mathbb{E}\{x_{i,k}|\mathcal{I}_k^d\}$ is generated by the stochastic Kalman filter

$$\hat{x}_{i,k+1} = A_i \hat{x}_{i,k} + \alpha_{i,k} B_i u_{i,k} + \beta_{i,k} K_{i,k}(y_{i,k} - C_i \hat{x}_{i,k}),$$

where $K_{i,k} = A_i P_{i,k} C_i^T (C_i P_{i,k} C_i^T + V_i)^{-1}$ is the estimator gain that is determined by the following stochastic Riccati equation:

$$P_{i,k+1} = A_i P_{i,k} A_i^T + W - \beta_{i,k} A_i P_{i,k} C_i^T (C_i P_{i,k} C_i^T + V)^{-1} C_i P_{i,k} A_i^T,$$

where $P_{i,k} = \mathbb{E}\{e_{i,k} e_{i,k}^T|\mathcal{I}_k^d\}$ is the error covariance matrix with $e_{i,k} := x_{i,k} - \hat{x}_{i,k}$.

(iii) $\mathbb{E}\{P_{i,k}\}$ satisfies $\tilde{P}_i \leq \mathbb{E}\{P_{i,k}\} \leq \bar{P}_i$ for all k where $\tilde{P}_i = (1-\beta^i) A \tilde{P}_i A^T + W$ and $\bar{P}_i$ is defined in (9).

(iv) $s_{i,k}$ is bounded for all k, i.e., $s_i \in \mathcal{C}_b$, if its initial condition satisfies $s_{i,0} = -\sum_{j=0}^{\infty}(H_i^j)^T \rho Q g_j$. Moreover, with this initial condition, $s_{i,k}$ can be written as

$$s_{i,k} = -\sum_{j=k}^{\infty}(H_i^{-k+j})^T \rho Q g_j. \tag{12}$$

(v) H_i is Hurwitz and $\tilde{A}_{i,k} := A_i - \alpha_{i,k} B_i L_i$ is mean-square stable[1].

(vi) The optimal cost is bounded above by

$$\begin{aligned}
&\bar{J}(u_i^*, g) \\
&\leq \mathrm{Tr}(Z_i W) + \mathrm{Tr}\Big((A_i^T Z_i A_i + Q - Z_i) \\
&\qquad \times (\bar{P}_i - \beta^i \bar{P}_i C_i^T (C_i \bar{P}_i C_i^T + V)^{-1} C_i \bar{P}_i)\Big) \\
&\quad + \limsup_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \rho^2 g_k^T Q g_k - \alpha^i s_{k+1}^T B_i U_i s_{k+1},
\end{aligned}$$

where $\mathrm{Tr}(\cdot)$ is the trace operator.

Proof. Parts (i), (ii), (iii), and (vi) can be shown by using the results in [17]. For part (iv), consider

$$s_{i,0} = (H_i^k)^T s_{i,k} - \sum_{j=0}^{k-1}(H_i^j)^T \rho Q g_j.$$

Then, it is easy to see that $s_{i,0}$ is the only initial condition that leads to $s_i \in \mathcal{C}_b$ in (12) that proceeds forward in k. The second statement of part (v) is shown in [16]. The first statement of (v) follows from Proposition 3.6 in [22]. This completes the proof. ☐

Remark 1: (i) Since H_i is Hurwitz, there are constants $c_i \geq 1$ and $\lambda_i \in (0,1)$ such that $\|H_i\|^k \leq c_i \lambda_i^k$ where $\|\cdot\|$ denotes 2-norm of matrices. Given the system and the cost matrices, the exact values of c_i and λ_i are dependent on the packet drop rate α^i. Note that due to Proposition 3.6 in [22], $\alpha^i < \alpha_c^i$ does not imply that H_i is not Hurwitz.

(ii) For the single agent case, α_c^i and β_c^i are known as the critical values of control and measurement losses that guarantee stability of the system [17].

(iii) The critical value of control packet drops for agent i satisfies $\alpha_{\min}^i \leq \alpha_c^i \leq \alpha_{\max}^i$ where

$$\alpha_{\min}^i \triangleq 1 - \frac{1}{\max_s |\lambda_s(A_i)|^2}$$
$$\alpha_{\max}^i \triangleq 1 - \frac{1}{\prod_s |\lambda_s(A_i)|^2},$$

where $\lambda_s(A_i)$ are the unstable eigenvalues of A_i [17]. Furthermore, $\alpha_c^i = \alpha_{\min}^i$ and $\alpha_c^i = \alpha_{\max}^i$ when B_i is square and invertible, and B_i is rank one, respectively.

(iv) Notice that the Kalman filter in Theorem 1 is time-varying and random due to the nature of the measurement

[1] See [22] for the definition of the mean-square stability.

process and the information structure. Moreover, $P_{i,k}$ does not converge unless $\beta^i = 1$. Detailed analysis of properties of the error covariance matrix can be found in [17]. $\square$

IV. Mean Field System

In this section, we construct a mean field system via state aggregation, which provides a good approximation to the mean field term in a sense to be made precise later.

We consider the dynamical system (1) and the decentralized controller (10). Let $\bar{x}_{i,k} := \mathbb{E}\{x_{i,k}\}$. By substituting (10) into (1) and taking expectation, the resulting system can be written as

$$\bar{x}_{i,k+1} = -\alpha^i \sum_{j=0}^{k} H_i^{k-j} B_i U_i s_{i,j+1}. \tag{13}$$

Now, by using (13) and the empirical distribution of the agents introduced in Assumption 2, the mean field system can be written as

$$\begin{aligned}\mathcal{T}(g)(k) =: \int_\Theta \alpha(\theta) \sum_{j=0}^{k-1} H^{k-1-j}(\theta) B(\theta) U(\theta) \\ \times \left(\sum_{s=j+1}^{\infty} (H^{-(j+1)+s}(\theta))^T \rho Q g_s \right) dF(\theta), \end{aligned} \tag{14}$$

where $dF(\theta)$ denotes the measure induced by the distribution function $F(\theta)$. Note that in (13) and (14), we use θ instead of i to emphasize the point that we have a continuum of agents under the parameter set and its empirical distribution in Assumption 2. Further note that (14) describes the average behavior of all the agents within Θ; hence it must be consistent with the mean field coupling term in (5) when N is sufficiently large.

Assumption 3: (i) $(A(\theta), B(\theta))$ and $(A(\theta), W^{1/2})$ are controllable, and $(A(\theta), Q^{1/2})$ and $(A(\theta), C(\theta))$ are observable for all $\theta \in \Theta$.

(ii) Communication links satisfy $\alpha(\theta) > \alpha_c(\theta)$ and $\beta(\theta) > \beta_c(\theta)$ for all $\theta \in \Theta$.

(iii) We have

$$|\rho| \|Q\| \int_\Theta |\alpha(\theta)| \|B(\theta)\| \|U(\theta)\| \Big(\sum_{s=0}^{\infty} \|H(\theta)\|^s \Big)^2 dF < 1 \square$$

We now show the existence of a fixed point of (14).

Theorem 2: Suppose that Assumptions 1, 2, and 3 hold. Then:

(i) The operator $\mathcal{T}(g)$ is in $\mathcal{C}_b$, i.e., $\mathcal{T}(g) \in \mathcal{C}_b$, for all $g \in \mathcal{C}_b$.

(ii) There is a unique $g^* \in \mathcal{C}_b$ such that $\mathcal{T}(g^*) = g^*$.

Proof. For Part (i), we have $\sup_{k\geq 0} \|\mathcal{T}(g)(k)\| < \infty$ since H_i is Hurwitz and $s_i \in \mathcal{C}_b$. For the second part, from (i), we have

$$\begin{aligned}\|\mathcal{T}(g) - \mathcal{T}(h)\|_\infty \leq \|g-h\|_\infty |\rho| \|Q\| \\ \times \int_\Theta |\alpha(\theta)| \|B(\theta)\| \|U(\theta)\| \left(\sum_{s=0}^{\infty} \|H(\theta)\|^s \right)^2 dF(\theta).\end{aligned}$$

Then by the Banach fixed point theorem and (i), the operator $\mathcal{T}(g)$ has a unique fixed point in $\mathcal{C}_b$. This completes the proof. $\square$

V. Stability Analysis of the Closed-loop System

In this section, we show closed-loop system stability with the decentralized optimal controller of Theorem 1 and the approximated mean field term, g_k^*, in Theorem 2.

We use the following controller for agent i:

$$u_{i,k}^* = -L_i \hat{x}_{i,k} - U_i s_{i,k+1}^*, \tag{15}$$

where $s_{i,k}^*$ is the bias term in (12) obtained by the fixed point of the mean field system (i.e. g^*) in Theorem 2. Therefore, the controller in (15) is now related to the mean field coupling term due to $s_{i,k}^*$. We denote the closed-loop system (1) with (15) by $x_{i,k}^*$. We also denote the mean field term and the mean field dynamical system under (15) by $(f_k^N)^*$ and $\bar{x}_{i,k}^*$, respectively.

Theorem 3: Let Assumptions 1, 2, and 3 hold. Then the decentralized controller (15) stabilizes the system (1) in the time-average sense:

$$\sup_{N\geq 1} \max_{1\leq i\leq N} \limsup_{K\to\infty} \frac{1}{K} \mathbb{E}\Big\{ \sum_{k=0}^{K-1} \|x_{i,k}^*\|^2 \Big\} < \infty.$$

Proof. We drop $*$ to simplify the notation. We write the closed-loop system with the decentralized control law (15)

$$x_{i,k+1} = \tilde{A}_{i,k} x_{i,k} + \alpha_{i,k} B_i L_i e_{i,k} - \alpha_{i,k} B_i U_i s_{i,k+1} + w_{i,k},$$

where $\tilde{A}_{i,k}$ is defined in Theorem 1. Now, we have the following relation:

$$\begin{aligned}&\mathbb{E}\{\|x_{i,k}\|^2\} \\ &\leq 4\mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,0}x_{i,0}\|^2\} \\ &\quad + 4\mathbb{E}\Big\{\sum_{j=0}^{k-1} \|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iL_ie_{i,j}\|^2\Big\} \\ &\quad + 4\mathbb{E}\Big\{\Big\|\sum_{j=0}^{k-1} \tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iU_is_{i,j+1}\Big\|^2\Big\} \\ &\quad + 4\mathbb{E}\Big\{\sum_{j=0}^{k-1} \|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}w_{i,j}\|^2\Big\},\end{aligned}$$

where the second and the last parts in the inequality follow from the unbiasedness of the Kalman filter in Theorem 1 and the independence of $\{w_{i,k}\}$, respectively. Since $\tilde{A}_{i,k}$ is mean-square stable by Theorem 1, we have from Theorem 3.9 in [22] that there exist constants $m_i \in (0,1)$ and $l_i \geq 1$ for all i such that

$$\begin{aligned}\mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,0}x_{i,0}\|^2\} \leq l_i m_i^k \operatorname{Tr}(X_0) \\ \mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}w_{i,j}\|^2\} \leq l_i m_i^{k-1-j} \operatorname{Tr}(W).\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iL_ie_{i,j}\|^2\} \\ &\leq \alpha^i l_i m_i^{k-1-j} \|B_i\| \|L_i\| \operatorname{Tr}(\bar{P}_i),\end{aligned}$$

where we used the fact that $\mathbb{E}\{\|e_{i,k}\|^2\} \leq \mathrm{Tr}(\bar{P}_i)$ due to Theorem 1. Moreover,

$$\begin{aligned}
&\mathbb{E}\left\{\left\|\sum_{j=0}^{k-1} \tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iU_is_{i,j+1}\right\|^2\right\} \\
&\leq \sum_{j=0}^{k-1}\mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iU_is_{i,j+1}\|^2\} \\
&\quad + \sum_{j\neq q}^{k-1}\mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,j+1}\alpha_{i,j}B_iU_is_{i,j+1}\|^2\}^{1/2} \\
&\qquad \times \mathbb{E}\{\|\tilde{A}_{i,k-1}\cdots\tilde{A}_{i,q+1}\alpha_{i,q}B_iU_is_{i,q+1}\|^2\}^{1/2} \\
&\leq \sum_{j=0}^{k-1} l_i m_i^{k-1-j}V_i^2 + \sum_{j\neq q}^{k-1} l_i m_i^{k-1-j/2-q/2}V_i^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality to get the first inequality, and V_i is the estimate of $\|s_i\|_\infty$. This shows that

$$\begin{aligned}
&\limsup_{K\to\infty}\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}\{\|x_{i,k}\|^2\} \\
&\leq \frac{4l_i\,\mathrm{Tr}(X_0)}{(1-m_i)} + \frac{4l_i\,\mathrm{Tr}(W)}{(1-m_i)} + \frac{4l_i\alpha^i\|B_i\|^2\|L_i\|^2\,\mathrm{Tr}(\bar{P}_i)}{(1-m_i)} \\
&\quad + \frac{4l_iV_i^2}{(1-m_i)} + \frac{8l_im_i^{1/2}V_i^2}{1-m_i} < \infty.
\end{aligned}$$

Finally, since Θ is compact, we have the desired result. □

VI. Approximation Performance

We now establish the consistency between g_k^* and the mean field term $(f_k^N)^*$ in the large population regime.

Lemma 1: Under Assumptions 1, 2, and 3, we have:

$$\lim_{N\to\infty}\limsup_{K\to\infty}\frac{1}{K}\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\bar{x}_{i,k}^* - g_k^*\right\|^2 = 0.$$

Proof. The proof is omitted due to space limitation. □

Lemma 2: Under Assumptions 1, 2, and 3, we have:

$$\lim_{N\to\infty}\limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}x_{i,k}^* - \bar{x}_{i,k}^*\right\|^2\right\} = 0.$$

Proof. Let $\tilde{x}_{i,k}^* := x_{i,k}^* - \bar{x}_{i,k}^*$. Then it can be written as

$$\begin{aligned}
\tilde{x}_{i,k+1}^* &= \tilde{A}_{i,k}\tilde{x}_{i,k}^* + (-\alpha_{i,k}+\alpha^i)B_iL_i\bar{x}_{i,k}^* \\
&\quad + (-\alpha_{i,k}+\alpha^i)B_iU_is_{i,k+1} + \alpha_{i,k}B_iL_ie_{i,k} + w_{i,k},
\end{aligned}$$

where $\tilde{x}_{i,0}^* = x_{i,0}$. Then, since the Kalman filter in Theorem 1 is unbiased, $\tilde{A}_{i,k}$ is mean-square stable, $\bar{x}_i^*, s_i \in \mathcal{C}_b$ are uniformly bounded on Θ, and Θ is compact from Assumption 2, it can be shown by following a similar line of reasoning as in Theorem 3 that there exists a constant $M > 0$, independent of K and N, such that

$$\limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\tilde{x}_{i,k}^*\right\|^2\right\} \leq \frac{M}{N}.$$

Hence, the result follows. □

We now state the main theorem for this section. The result below indicates that in the large population regime, we have the mean-square sense consistency between g_k^* and $(f_k^N)^*$.

Theorem 4: Suppose that Assumptions 1, 2, and 3 hold. Then,

$$\lim_{N\to\infty}\limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\left\|(f_k^N)^* - g_k^*\right\|^2\right\} = 0.$$

Proof. Note that

$$\begin{aligned}
&\limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}x_{i,k}^* - g_k^*\right\|^2\right\} \\
&\leq \limsup_{K\to\infty}\frac{2}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}x_{i,k}^* - \bar{x}_{i,k}^*\right\|^2\right\} \\
&\quad + \limsup_{K\to\infty}\frac{2}{K}\sum_{k=0}^{K-1}\left\|\frac{1}{N}\sum_{i=1}^{N}\bar{x}_{i,k}^* - g_k^*\right\|^2.
\end{aligned}$$

Then by Lemmas 1 and 2, the result follows. □

VII. ϵ-Nash Equilibria of the LQG Mean Field Game over Unreliable Communication Links

In this section, we show that the set of N-decentralized control laws constitutes an ϵ-Nash equilibrium for the LQG mean field game over unreliable communication links.

Let

$$\begin{aligned}
&J_i^N(u_i^*, u_{-i}^*) \qquad (16)\\
&= \limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\|x_{i,k}^* - \rho(f_k^N)^*\|_Q^2 + \alpha_{i,k}\|u_{i,k}^*\|_R^2\right\} \\
&J_i(u_i^*, g^*) \qquad (17)\\
&= \limsup_{K\to\infty}\frac{1}{K}\mathbb{E}\left\{\sum_{k=0}^{K-1}\|x_{i,k}^* - \rho g_k^*\|_Q^2 + \alpha_{i,k}\|u_{i,k}^*\|_R^2\right\} \\
&J_i^N(u_i, u_{-i}^*) \qquad (18)\\
&= \limsup_{K\to\infty}\frac{1}{K} \\
&\quad\times\mathbb{E}\left\{\sum_{k=0}^{K-1}\|x_{i,k}|_{u_i} - \rho(f_k^N)^*|_{u_i}\|_Q^2 + \alpha_{i,k}\|u_{i,k}\|_R^2\right\},
\end{aligned}$$

where g_k^* is obtained from Theorem 4. In (18), $x_{i,k}|_{u_i}$ is the state of agent i under $u_i \in \mathcal{U}_i^c$, and $(f_k^N)^*|_{u_i} = \frac{1}{N}\sum_{i=1}^{N}x_{i,k}^*|_{u_i}$ indicates that all agents except i use the optimal decentralized controller (15), and agent i is under any full state control $u_{i,k} \in \mathcal{U}_i^c$. Note that in (16) and (18), the dependency of N is required in order to emphasize the effect of the mean field coupling term in (5). We next state two lemmas, without proofs, which will lead to the main theorem of this section.

Lemma 3: Under Assumptions 1, 2, and 3, we have

$$J_i^N(u_i^*, u_{-i}^*) \leq J_i(u_i^*, g^*) + O((\limsup_{K\to\infty}\epsilon_K^N)^{1/2}),$$

where

$$\epsilon_K^N = \frac{1}{K}\mathbb{E}\Big\{\sum_{k=0}^{K-1} \|(f_k^N)^* - g_k^*\|^2\Big\}. \qquad \square$$

Lemma 4: Suppose that Assumptions 1, 2, and 3 hold. Then:

$$J_i(u_i^*, g^*) \leq \inf_{u_i \in \mathcal{U}_i^c} J_i^N(u_i, u_{-i}^*) + O((\limsup_{K\to\infty} \epsilon_K^N)^{1/2}) + O(\frac{1}{N}). \qquad \square$$

Theorem 5: Suppose that Assumptions 1, 2, and 3 hold. Then, the set of N decentralized controllers, $\{u_i^*,\ 1 \leq i \leq N\}$, where $u_i^* = (u_{i,0}^*, u_{i,1}^*, ..., u_{i,k}^*, ...)$ and $u_{i,k}^*$ is defined in (15), constitutes an ϵ-Nash equilibrium for the LQG mean field game over unreliable communication links. That is, for any i, $1 \leq i \leq N$,

$$J_i^N(u_i^*, u_{-i}^*) \leq \inf_{u_i \in \mathcal{U}_i^c} J_i^N(u_i, u_{-i}^*) + O((\limsup_{K\to\infty} \epsilon_K^N)^{1/2}) + O(\frac{1}{N}).$$

Moreover, $\epsilon := O((\limsup_{K\to\infty} \epsilon_K^N)^{1/2}) + O(\frac{1}{N})$ converges to zero as $N \to \infty$.

Proof. The result follows from Lemmas 3 and 4. $\square$

Before concluding this section, it is worth mentioning that if the packet drop rate of a particular agent, say i, fails to satisfy Assumption 3(ii), then the corresponding decentralized controller does not exist; therefore, the set of optimal control laws in (15) is no longer able to constitute an ϵ-Nash equilibrium. Moreover, the dynamical system for agent i cannot be stabilized. This implies that the existence of the ϵ-Nash equilibrium obtained in this paper is very much dependent on the underlying communication network.

VIII. Concluding Remarks

In this paper, we have studied LQG mean field games with unreliable communication links. We have obtained decentralized optimal controllers for each agent, which uses their local measurements received over unreliable communication channels. Using these controllers, we have constructed the mean field system, and then shown that it provides an asymptotically accurate approximation to the mean field term under appropriate conditions. We have also shown closed-loop system stability under the individual decentralized optimal controller. We have further shown that the set of N-decentralized optimal controllers possesses an ϵ-Nash equilibrium property, and ϵ can be made arbitrarily close to zero as $N \to \infty$. One important aspect is that the nature of communication conditions and particularly the extent of reliability of the communication links is the most important factor to guarantee the existence of an ϵ-Nash equilibrium.

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