Distributed Stochastic Optimization under Imperfect Information

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problem that requires minimizing a sum of misspecified agentspecific expectation-valued convex functions over the intersection of a collection of agent-specific convex sets. This misspecification is manifested in a parametric sense and may be resolved through solving a distinct stochastic convex learning problem. Our interest lies in the development of distributed algorithms in which every agent makes decisions based on the knowledge of its objective and feasibility set while learning the decisions of other agents by communicating with its local neighbors over a time-varying connectivity graph. While a significant body of research currently exists in the context of such problems, we believe that the misspecified generalization of this problem is both important and has seen little study, if at all. Accordingly, our focus lies on the simultaneous resolution of both problems through a joint set of schemes that combine three distinct steps: (i) An alignment step in which every agent updates its current belief by averaging over the beliefs of its neighbors; (ii) A projected (stochastic) gradient step in which every agent further updates this averaged estimate; and (iii) A learning step in which agents update their belief of the misspecified parameter by utilizing a stochastic gradient step. Under an assumption of mere convexity on agent objectives and strong convexity of the learning problems, we show that the sequences generated by this collection of update rules converge almost surely to the solution of the correctly specified stochastic convex optimization problem and the stochastic learning problem, respectively.

Abstract-We consider a stochastic convex optimization

I. INTRODUCTION

Distributed algorithms have grown enormously in relevance for addressing a broad class of problems in arising in network system applications in control and optimization, signal processing, communication networks, power systems, amongst others (c.f. [1], [2], [3]). A crucial assumption in any such framework is the need for precise specification of the objective function. In practice however, in many engineered and economic systems, agent-specific functions may be misspecified from a parametric standpoint but may have access to observations that can aid in resolving this misspecification. Yet almost all of the efforts in distributed algorithms obviate the question of misspecification in the agent-specific problems, motivating the present work.

In seminal work by Tsitsiklis [4], decentralized and distributed approaches to decision-making and optimization were investigated in settings complicated by partial coordination, delayed communication, and the presence of noise. In subsequent work [5], the behavior of general distributed gradient-based algorithms was examined. In related work on parallel computing [6], iterative approaches and their convergence rate estimates were studied for distributing computational load amongst multiple processors.

Consensus-based extensions to optimization with linear constraints were considered in [7], while convergent algorithms for problems under settings of general agent specific convex constraints were first proposed in [8], as an extension of distributed multi-agent model proposed in [9], and further developed in [10]. In [11] and [12], a problem with common (global) inequality and equality constraints is considered and distributed primal-dual projection method is proposed. A more general case with agents having only partial information with respect to shared and nonsmooth constraints is studied in [13]. Recent work [14] compares and obtains rate estimates for Newton and gradient based schemes to solve distributed quadratic minimization, a form of weighted least squares problem for networks with time varying topology. In [15], a distributed dual-averaging algorithm is proposed combining push-sum consensus and gradient steps for constrained optimization over a static graph, while in [16], a subgradient method is developed using push-sum algorithm on time-varying graphs. Distributed algorithms that combine consensus and gradient steps have been recently developed [17], [18] for stochastic optimization problems. In recent work [19], the authors consider a setting of asynchronous gossip-protocol, while stochastic extensions to asynchronous optimization were considered in [20], [21], [22], [23], convergent distributed schemes were proposed, and error bounds for finite termination were obtained. All aforementioned work assumes that the functions are either known exactly or their noisy gradients are available.

While misspecification poses a significant challenge in the resolution of optimization problems, general purpose techniques for the resolution of misspecified optimization problems through the joint solution of the misspecified problem and a suitably defined learning problem have been less studied. Our framework extends prior work on deterministic [24] and stochastic [25], [26] gradient schemes. Here, we consider a networked regime in which agents are characterized by misspecified expectation-valued convex objectives and convex feasibility sets. The overall goal lies in minimizing the sum of the agent-specific objectives over the intersection of the agent-specific constraint sets. In contrast with traditional models, agents have access to the stochastic convex learning metric that allows for resolving the prescribed misspecification. Furthermore, agents only have access to their objectives and their feasibility sets and

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may observe the decisions of their local neighbors as defined through a general time-varying graph. In such a setting, we considered distributed protocols that combine three distinct steps: (i) An alignment step in which every agent updates its current belief by averaging over the beliefs of its neighbors based on a set of possibly varying weights; (ii) A projected (stochastic) gradient step in which every agent further updates this averaged estimate; and (iii) A learning step where agents update their belief of the misspecified parameter by utilizing a stochastic gradient step. We show that the produced sequences of agent-specific decisions and agent-specific beliefs regarding the misspecified parameter converge in an almost sure sense to the optimal set of solutions and the optimal parameter, respectively under the assumption of general time-varying graphs and note that this extends the results in [8].

The paper is organized as follows. In Section II, we define the problem of interest and provide a motivation for its study. In Section III, we outline our algorithm and the relevant assumptions. Basic properties of the algorithm are investigated in Section IV and the almost sure convergence of the produced sequences is established in Section V. We conclude the paper with some brief remarks in Section VI.

II. PROBLEM FORMULATION AND MOTIVATION

We consider a networked multi-agent setting with timevarying undirected connectivity graphs, where the graph at time t is denoted by $\mathcal{G}^t = \{\mathcal{N}, \mathcal{E}^t\}, \mathcal{N} \triangleq \{1, \dots, m\}$ denotes the set of nodes and \mathcal{E}^t is the set of edges at time t. Each node represents a single agent and the problem of interest is

minimize
$$\sum_{i=1}^{m} \mathbb{E}[\varphi_i(x, \theta^*, \xi)]$$
subject to $x \in \bigcap_{i=1}^{m} X_i$, (1)

where $\theta^* \in \mathbb{R}^p$ represents the (misspecified) vector of parameters, $\mathbb{E}[\varphi_i(x, \theta^*, \xi)]$ denotes the local cost function of agent *i*, the expectation is taken with respect to a random variable ξ , defined as $\xi : \Omega \to \mathbb{R}^d$, and $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the associated probability space. The function $\varphi_i : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be convex and continuously differentiable in *x* for all $\theta \in \Theta$ and all $\xi \in \Omega$.

(i) **Local information**. Agent *i* has access to its objective function $\mathbb{E}[\varphi_i(x, \theta^*, \xi)]$ and its set X_i but is unaware of the objectives and constraint sets of the other agents. Furthermore, it may communicate at time *t* with its local neighbors, as specified by the graph \mathcal{G}^t ;

(ii) **Objective misspecification**. The agent objectives are parametrized by a vector θ^* unknown to the agents.

We assume that the true parameter θ^* is a solution to a distinct convex problem, accessible to every agent:

$$\min_{\theta \in \Theta} \mathbb{E}[g(\theta, \chi)], \tag{2}$$

where $\Theta \subseteq \mathbb{R}^p$ is a closed and convex set, $\chi : \Omega_{\theta} \to \mathbb{R}^r$ is a random variable with the associated probability space given by $(\Omega_{\theta}, \mathcal{F}_{\theta}, \mathbb{P}_{\theta})$, while $g : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}$ is a strongly convex

and continuously differentiable function in θ for every χ . Our interest lies in the joint solution of (1)–(2):

$$x^* \in \underset{x}{\operatorname{argmin}} \left\{ \sum_{i=1}^m \mathbb{E}[\varphi_i(x, \theta^*, \xi)] \mid x \in \bigcap_{i=1}^m X_i \right\}, \qquad (3)$$
$$\theta^* \in \underset{\theta}{\operatorname{argmin}} \left\{ \mathbb{E}[g(\theta, \chi)] \mid \theta \in \Theta \right\}.$$

A sequential approach: Traditionally, such problems are approached sequentially: (1) an accurate approximation of θ^* is first obtained; and (2) given θ^* , standard computational schemes are then applied. However, this avenue is inadvisable when the learning problems are stochastic and accurate solutions are available via simulation schemes, requiring significant effort. In fact, if the learning process is terminated prematurely, the resulting solution may differ significantly from θ^* and this error can only be captured in an expected-value sense. Thus, such approaches can only provide approximate solutions and, consequently, cannot generally provide asymptotically exact solutions. Inspired by recent work on learning and optimization in a centralized regime [25], we consider the development of schemes for distributed stochastic optimization. We build on the distributed projection-based algorithm [8], which combines local averaging with a projected gradient step for agent-based constraints. In particular, we introduce an additional layer of a learning step to aid in resolving the misspecification.

Motivating applications: Consensus-based optimization problems arise in a range of settings including the dispatch of distributed energy resources (DERs) [2], signal processing [], amongst others. Such settings are often complicated by misspecification; for instance, there are a host of charging, discharging and efficiency parameters associated with storage resources that often require estimation.

III. ASSUMPTIONS AND ALGORITHM

We begin by presenting a distributed framework for solving the problem in (3). To set this up more concretely, for all $i \in \mathcal{N}$, we let $f_i(x, \theta) \triangleq \mathbb{E}[\varphi_i(x, \theta, \xi)]$ for all x and $\theta \in \Theta$ and $h(\theta) \triangleq \mathbb{E}[g(\theta, \chi)]$ for all $\theta \in \Theta$. Then, problem (3) assumes the following form:

$$x^{*} \in \underset{x \in \bigcap_{i=1}^{m} X_{i}}{\operatorname{argmin}} f(x, \theta^{*}), \text{ where } f(x, \theta) \triangleq \sum_{i=1}^{m} f_{i}(x, \theta^{*}),$$

$$\theta^{*} \in \underset{\theta \in \Theta}{\operatorname{argmin}} h(\theta).$$
(4)

We consider a distributed algorithm where agent *i* knows f_i and the set X_i , while all agents have access to *h*. We further assume that *i*th agent has access to oracles that produce random samples $\nabla_x \varphi_i(x, \theta, \xi)$ and $\nabla_\theta g(\theta, \chi)$. The information needed by agents to solve the optimization problem is acquired through local sharing of the estimates over a time-varying communication network. Specifically, at iteration *k*, the *i*th agent has estimates $x_i^k \in X_i$ and $\theta_i^k \in \Theta$ and at the next iteration, constructs a vector v_i^k , as an average

of the vectors x_i^k obtained from its local neighbors, given by:

$$v_i^k := \sum_{j=1}^m a_i^{j,k} x_j^k$$
 for all $i = 1, \dots, m$ and $k \ge 0$, (5)

where the weights $a_i^{j,k}$ are nonnegative scalars satisfying $\sum_{j=1}^{m} a_i^{j,k} = 1$ and are related to the underlying connectivity graph \mathcal{G}^k over which the agents communicate at time k. Then, for $i = 1, \ldots, N$, the *i*th agent updates its x- and θ -variable as follows:

$$x_i^{k+1} := \Pi_{X_i} \left(v_i^k - \alpha_k \left(\nabla_x f_i(v_i^k, \theta_i^k) + w_i^k \right) \right), \quad (6)$$

$$\theta_i^{k+1} := \Pi_{\Theta} \left(\theta_i^k - \gamma_k \left(\nabla h(\theta_i^k) + \beta_i^k \right) \right), \tag{7}$$

where $w_i^k \triangleq \nabla_x \varphi_i(v_i^k, \theta_i^k, \xi_i^k) - \nabla_x f_i(v_i^k, \theta_i^k)$ with $\nabla_x f_i(x, \theta) = \mathbb{E}[\nabla_x \varphi_i(x, \theta, \xi)]$, and $\beta_i^k \triangleq \nabla_\theta g(\theta_i^k, \chi_i^k) - \nabla h(\theta_i^k)$ with $\nabla h(\theta_i^k) = \mathbb{E}[\nabla_\theta g(\theta_i^k, \chi)]$ for all $i \in \mathcal{N}$ and all $k \geq 0$. The parameters $\alpha_k > 0$ and $\gamma_k > 0$ represent stepsizes at epoch k, while the initial points $x_i^0 \in X_i$ and $\theta_i^0 \in \Theta$ are randomly selected for each agent i. The ith agent has access to $(\nabla_x f_i(v_i^k, \theta_i^k) + w_i^k)$ and not $(\nabla_x f_i(v_i^k, \theta_i^k))$. The same is the case with the learning function. At time epoch k, agent i proceeds to average over its neighbors' decisions by using the weights in (5) and employs this average to update its decision in (6). Furthermore, agent i makes a subsequent update in its belief regarding θ^* , by taking a similar (stochastic) gradient update, given by (7).

The weight $a_i^{j,k}$ used by agent *i* for the iterate of agent *j* at time *k* is based on the connectivity graph \mathcal{G}^k . Specifically, letting \mathcal{E}_i^k be the set of neighbors of agent *i*:

$$\mathcal{E}_i^k = \{j \in \mathcal{N} \mid \{j, i\} \in \mathcal{E}^k\} \cup \{i\}, i \in \mathcal{E}^k\} \cup \{i\}, j \in \mathcal{E}^k\} \cup \{j\}, j \in \mathcal{E}^k\} \cup \{j\}, j \in \mathcal{E}^k\} \cup \{j\}, j \in \mathcal{E}^k\}$$

the weights $a_i^{j,k}$ are compliant with the neighbor structure:

$$a_i^{j,k} > 0 \text{ if } j \in \mathcal{E}_i^k \text{ and } a_i^{j,k} = 0 \text{ if } j \notin \mathcal{E}_i^k.$$

We assume that each graph \mathcal{G}^k is connected and that matrices are doubly stochastic, as given in the following assumption.

Assumption 1 (Graph and weight matrices):

(a) The matrix A(k) (whose (ij)th entry is denoted by $a_i^{j,k}$) is doubly stochastic for every k, i.e., $\sum_{i=1}^{m} a_i^{j,k} = 1$ for every j and $\sum_{j=1}^{m} a_i^{j,k} = 1$ for every i.

(b) The matrices A(k) have positive diagonal entries, and all positive entries in every A(k) are uniformly bounded away from zero, i.e., there exists $\eta > 0$ such that, for all i, j, and k, we have $a_i^{j,k} \ge \eta$ whenever $a_i^{j,k} > 0$.

(c) The graph
$$\mathcal{G}^k$$
 is connected for every $k \ge 0$.

The instantaneous connectivity assumption on the graphs \mathcal{G}^k can be relaxed by requiring that the union of these graphs is connected every T units of time, for instance. The analysis of this case is similar to that given in this paper. We choose to work with connected graphs in order to keep the analysis somewhat simpler and to provide a sharper focus on the learning aspect of the problem.

Next, we define $\mathcal{F}_0 \triangleq (x_i^0, \theta_i^0)$, $i \in \mathcal{N}$ and $\mathcal{F}_k = \{(\xi_i^t, \chi_i^t), i \in \mathcal{N}, t = 0, 1, \dots, k-1\}$ for all $k \ge 1$ and make the following assumptions on the conditional first and

second moments of the stochastic errors w_i^k and β_i^k . These assumptions are relatively standard in the development of stochastic gradient schemes.

Assumption 2 (Conditional first and second moments): (a) $\mathbb{E}[w_i^k | \mathcal{F}_k] = 0$ and $\mathbb{E}[\beta_i^k | \mathcal{F}_k] = 0$ for all k and $i \in \mathcal{N}$. (b) $\mathbb{E}[||w_i^k||^2 | \mathcal{F}_k] \le \nu^2$ and $\mathbb{E}[||\beta_i^k||^2 | \mathcal{F}_k] \le \nu_{\theta}^2$ for all k and $i \in \mathcal{N}$.

(c) $\mathbb{E}[\|\theta_i^0\|^2]$ is finite for all $i \in \mathcal{N}$.

We now discuss the assumptions on agent objectives, the learning metric and the underlying set constraints.

Assumption 3 (Feasibility sets): (a) For every $i \in \mathcal{N}$, the set $X_i \subset \mathbb{R}^n$ is convex and compact.

(b) The intersection set $\bigcap_{i=1}^{m} X_i$ is nonempty.

(c) The set $\Theta \subseteq \mathbb{R}^p$ is convex and closed.

Note that under the compactness assumption on the sets X_i , we have $\mathbb{E}[||x_i^0||^2] < \infty$ for all $i \in \mathcal{N}$. Furthermore we have

$$\max_{x_i, y_i \in X_i} \|x_i - y_i\| \le D \tag{8}$$

for some scalar D > 0 and for all *i*. Next, we consider the conditions for the agent objective functions.

Assumption 4 (Agent objectives): For every $i \in \mathcal{N}$, the function $f_i(x,\theta)$ is convex in x for every $\theta \in \Theta$. Furthermore, for every $i \in \mathcal{N}$, the gradients $\nabla_x f_i(x,\theta)$ are uniformly Lipschitz continuous functions in θ for all $x \in X_i$: $\|\nabla_x f_i(x,\theta^a) - \nabla_x f_i(x,\theta^b)\| \leq L_{\theta} \|\theta^a - \theta^b\|$ for all $\theta^a, \theta^b \in \Theta$, all $x \in X_i$, and all $i \in \mathcal{N}$.

Assumption 5 (Learning metric): The function h is strongly convex over Θ with a constant $\kappa > 0$, and its gradients are Lipschitz continuous with a constant R_{θ} , i.e., $\|\nabla h(\theta^a) - \nabla h(\theta^b)\| \le R_{\theta} \|\theta^a - \theta^b\|$ for all $\theta^a, \theta^b \in \Theta$.

By the strong convexity of h, the problem (2) has a unique solution denoted by θ^* . From the convexity of the functions f_i in x (over \mathbb{R}^n) for every $\theta \in \Theta$, as given in Assumption 4, these functions are continuous. Thus, when $\bigcap_{i=1}^m X_i$ is nonempty and each X_i is compact (Assumption 3), the problem $\min_{x \in \bigcap_{i=1}^m X_i} \sum_{i=1}^m f_i(x, \theta^*)$ has a solution.

IV. BASIC PROPERTIES OF THE ALGORITHM

In this section, we provide some basic relations for the algorithm (5)–(7) that are fundamental to establishing the almost sure convergence of the sequences produced by the algorithm. The proofs of all the results can be found in [27].

A. Iterate Relations

We start with a simple result for weighted averages of a finitely many points.

Lemma 1: Let $y_1, \ldots, y_m \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, with $\lambda_i \geq 0$ for all i and $\sum_{i=1}^m \lambda_i = 1$. Then, for any $c \in \mathbb{R}^n$, we have

$$\left\|\sum_{i=1}^{m} \lambda_{i} y_{i} - c\right\|^{2} = \sum_{i=1}^{m} \lambda_{i} \|y_{i} - c\|^{2} - \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \lambda_{j} \lambda_{\ell} \|y_{j} - y_{\ell}\|^{2}.$$

Proof: By using the fact that λ_i are convex weights, we have we write

$$\left\|\sum_{i=1}^{m} \lambda_i y_i - c\right\|^2 = \left\|\sum_{i=1}^{m} \lambda_i (y_i - c)\right\|^2$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \lambda_j (y_i - c)^T (y_j - c).$$

Noting that $2a^Tb = ||a||^2 + ||b||^2 - ||a - b||^2$, valid for any $a, b \in \mathbb{R}^n$, and applying it to each inner product, we obtain

$$\begin{aligned} \left\| \sum_{i=1}^{m} \lambda_{i} y_{i} - c \right\|^{2} \\ &= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i} \lambda_{j} \left(\|y_{i} - c\|^{2} + \|y_{j} - c\|^{2} - \|y_{i} - y_{j}\|^{2} \right) \\ &= \sum_{i=1}^{m} \lambda_{i} \|y_{i} - c\|^{2} - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i} \lambda_{j} \|y_{i} - y_{j}\|^{2}, \end{aligned}$$

where the second equality follows by noting that $\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \lambda_j \left(||y_i - c||^2 + ||y_j - c||^2 \right) = \sum_{p=1}^{m} \lambda_p ||y_p - c||^2$, which can be seen by using $\sum_{j=1}^{m} \lambda_j = 1$.

We use the following lemma that provides a bound on the difference between consecutive x-iterates of the algorithm and an analogous relation for consecutive θ -iterates.

Lemma 2: Let Assumptions 1–4 hold. Also, let $X = \bigcap_{i=1}^{m} X_i$ and let h be strongly convex over Θ . Let the iterates x_i^k be generated according to (5)–(7). Then, almost surely, we have for all $x \in X$ and $k \ge 0$,

$$\begin{split} &\sum_{i=1}^{m} \mathbb{E} \left[\|x_{i}^{k+1} - x\|^{2} \mid \mathcal{F}_{k} \right] \leq \sum_{j=1}^{m} \|x_{j}^{k} - x\|^{2} \\ &- \eta^{2} \sum_{\{s,\ell\} \in \mathcal{T}^{k}} \|x_{s}^{k} - x_{\ell}^{k}\|^{2} + m\alpha_{k}^{2}(2S^{2} + \nu^{2}) + m\alpha_{k}^{2-\tau}L_{\theta}^{2}D \\ &+ \left(\alpha_{k}^{\tau} + 2\alpha_{k}^{2}L_{\theta}^{2}\right) \sum_{i=1}^{m} \|\theta_{i}^{k} - \theta^{*}\|^{2} \\ &- 2\alpha_{k} \sum_{i=1}^{m} \left(f_{i}(v_{i}^{k}, \theta^{*}) - f_{i}(x, \theta^{*})\right), \end{split}$$

where \mathcal{T}^k is a spanning tree in the graph \mathcal{G}^k , $\tau \in (0,2)$ is an arbitrary but fixed scalar, $\theta^* = \operatorname{argmin}_{\theta \in \Theta} h(\theta)$, and $S = \max_i \max_{x \in \bar{X}} \|\nabla_x f_i(x, \theta^*)\|$, with \bar{X} being the convex hull of the union $\bigcup_{i=1}^m X_i$.

Proof: First, we note that by the strong convexity of h, the point $\theta^* \in \Theta$ minimizing h over Θ exists and it is unique. Next, we use the projection property for a closed convex set Y, according to which we have, $\|\Pi_Y[x] - y\|^2 \le \|x - y\|^2$ for all $y \in Y$ and all x. Therefore, for all i, and any $x \in X$,

$$\begin{aligned} \|x_{i}^{k+1} - x\|^{2} \\ &= \left\|\Pi_{X_{i}}\left(v_{i}^{k} - \alpha_{k}\left(\nabla_{x}f_{i}(v_{i}^{k}, \theta_{i}^{k}) + w_{i}^{k}\right)\right) - x\right\|^{2} \\ &\leq \left\|v_{i}^{k} - \alpha_{k}\left(\nabla_{x}f_{i}(v_{i}^{k}, \theta_{i}^{k}) + w_{i}^{k}\right) - x\right\|^{2} \\ &\leq \left\|v_{i}^{k} - x\right\|^{2} + T_{k}^{k} + T_{D}^{k}, \end{aligned}$$
(9)

where
$$T_A^k \triangleq \alpha_k^2 \| \nabla_x f_i(v_i^k, \theta_i^k) + w_i^k \|^2$$
, (10)
 $T_B^k \triangleq -2\alpha_k (v_i^k - x)^T (\nabla_x f_i(v_i^k, \theta_i^k) + w_i^k)$. (11)

Expanding
$$T_A^k$$
, we obtain that

$$T_{A}^{k} = \alpha_{k}^{2} \|\nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}) + w_{i}^{k}\|^{2} = \alpha_{k}^{2} \|\nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}) + w_{i}^{k}\|^{2}$$

= $\alpha_{k}^{2} \|\nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k})\|^{2} + \alpha_{k}^{2} \|w_{i}^{k}\|^{2}$
+ $2\alpha_{k}^{2} (w_{i}^{k})^{T} \nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}).$

Taking the conditional expectations on both sides of (12) with respect to the past and using Assumption 2 on the stochastic gradients, we have that almost surely,

$$\mathbb{E}\left[T_{A}^{k} \mid \mathcal{F}_{k}\right] = \alpha_{k}^{2} \|\nabla_{x} f_{i}(v_{i}^{k}, \theta^{*}) + \nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}) - \nabla_{x} f_{i}(v_{i}^{k}, \theta^{*})\|^{2} \\
+ \alpha_{k}^{2} \underbrace{\mathbb{E}\left[\left\|w_{i}^{k}\right\|^{2} \mid \mathcal{F}_{k}\right]}_{\leq \nu^{2}} + 2\alpha_{k}^{2} \underbrace{\mathbb{E}\left[(w_{i}^{k})^{T} \nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}) \mid \mathcal{F}_{k}\right]}_{= 0} \\
\leq \alpha_{k}^{2}(\|\nabla_{x} f_{i}(v_{i}^{k}, \theta^{*})\| + \|\nabla_{x} f_{i}(v_{i}^{k}, \theta_{i}^{k}) - \nabla_{x} f_{i}(v_{i}^{k}, \theta^{*})\|)^{2} \\
+ \alpha_{k}^{2} \nu^{2} \\
\leq \alpha_{k}^{2}(2S^{2} + 2L_{\theta}^{2} \|\theta_{i}^{k} - \theta^{*}\|^{2} + \nu^{2}),$$
(12)

where in the last inequality we use $(a + b)^2 \leq 2a^2 + 2b^2$ valid for all $a, b \in \mathbb{R}$, and the Lipschitz property of $\nabla_x f_i(x, \theta)$ (cf. Assumption 4). Furthermore, since the sets X_i are compact by Assumption 3, the convex hull \bar{X} of $\cup_{i=1}^m X_i$ is also compact, implying by continuity of the gradients that $\max_i \max_{k\geq 0} \|\nabla_x f_i(v_i^k, \theta^*)\| \leq \max_i \max_{x\in \bar{X}} \|\nabla_x f_i(x, \theta^*)\| = S$, with $S < \infty$.

Next, we consider the term T_B^k . By taking the conditional expectation with respect to \mathcal{F}_k and using $\mathbb{E}\left[(v_i^k - x)^T w_i^k \mid \mathcal{F}_k\right] = 0$, we obtain

$$\begin{split} & \mathbb{E}\left[T_B^k \mid \mathcal{F}_k\right] \\ &= -2\alpha_k (v_i^k - x)^T (\nabla_x f_i(v_i^k, \theta_i^k) - \nabla_x f_i(v_i^k, \theta^*)) \\ &- 2\alpha_k (v_i^k - x)^T \nabla_x f_i(v_i^k, \theta^*) \\ &\leq 2\alpha_k \|v_i^k - x\| \|\nabla_x f_i(v_i^k, \theta_i^k) - \nabla_x f_i(v_i^k, \theta^*)\| \\ &- 2\alpha_k (v_i^k - x)^T \nabla_x f_i(v_i^k, \theta^*). \end{split}$$

By using the Lipschitz property of $\nabla_x f_i(x,\theta)$ (cf. Assumption 4), the relation $2\alpha ab = 2(\sqrt{\alpha^{2-\tau}a})(\sqrt{\alpha^{\tau}b})$ valid for any $a, b \in \mathbb{R}$ and any $\tau > 0$, and the Cauchy-Schwarz inequality,

we further obtain

$$\mathbb{E}\left[T_{B}^{k} \mid \mathcal{F}_{k}\right] \\
\leq 2\alpha_{k}L_{\theta}\|v_{i}^{k} - x\|\|\theta_{i}^{k} - \theta^{*}\| - 2\alpha_{k}(v_{i}^{k} - x)^{T}\nabla_{x}f_{i}(v_{i}^{k}, \theta^{*}) \\
\leq \alpha_{k}^{2-\tau}L_{\theta}^{2}\|v_{i}^{k} - x\|^{2} + \alpha_{k}^{\tau}\|\theta_{i}^{k} - \theta^{*}\|^{2} \\
- 2\alpha_{k}(v_{i}^{k} - x)^{T}\nabla_{x}f_{i}(v_{i}^{k}, \theta^{*}) \\
\leq \alpha_{k}^{2-\tau}L_{\theta}^{2}D^{2} + \alpha_{k}^{\tau}\|\theta_{i}^{k} - \theta^{*}\|^{2} \\
- 2\alpha_{k}\left(f_{i}(v_{i}^{k}, \theta^{*}) - f_{i}(x, \theta^{*})\right),$$
(13)

where in the last inequality we also employ the convexity of f_i and boundedness of sets X_i , together with the fact that $v_i^k, x \in X_i$ for all *i* (cf. Assumption 3).

Now, we take the conditional expectation in relation (9) and we substitute estimates (12) and (13), which yields almost surely, for all i, all $x \in X$ and all k,

$$\mathbb{E} \left[\|x_i^{k+1} - x\|^2 | \mathcal{F}_k \right] \leq \|v_i^k - x\|^2 + \alpha_k^2 (2S^2 + \nu^2) + \alpha_k^{2-\tau} L_{\theta}^2 D^2 + \left(\alpha_k^{\tau} + 2\alpha_k^2 L_{\theta}^2 \right) \|\theta_i^k - \theta^*\|^2 - 2\alpha_k \left(f_i(v_k^k, \theta^*) - f_i(x, \theta^*) \right).$$

Summing the preceding relations over i = 1, ..., m, we have the following inequality almost surely, for all $x \in X$ and all $k \ge 0$,

$$\sum_{i=1}^{m} \mathbb{E} \left[\|x_{i}^{k+1} - x\|^{2} | \mathcal{F}_{k} \right]$$

$$\leq \sum_{i=1}^{m} \|v_{i}^{k} - x\|^{2} + m\alpha_{k}^{2}(2S^{2} + \nu^{2}) + m\alpha_{k}^{2-\tau}L_{\theta}^{2}D^{2}$$

$$+ \left(\alpha_{k}^{\tau} + 2\alpha_{k}^{2}L_{\theta}^{2}\right)\sum_{i=1}^{m} \|\theta_{i}^{k} - \theta^{*}\|^{2}$$

$$- 2\alpha_{k}\sum_{i=1}^{m} \left(f_{i}(v_{i}^{k}, \theta^{*}) - f_{i}(x, \theta^{*})\right).$$
(14)

We now focus on the term $||v_i^k - x||^2$. Noting that $\sum_{j=1}^m a_i^{j,k} = 1$, by Lemma 1, it follows that for all $x \in X$ and all $k \ge 0$,

$$\|v_i^k - x\|^2 = \left\|\sum_{j=1}^m a_i^{j,k} x_j^k - x\right\|^2$$
$$= \sum_{j=1}^m a_i^{j,k} \|x_j^k - x\|^2 - \frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m a_i^{j,k} a_i^{\ell,k} \|x_j^k - x_\ell^k\|^2.$$

By summing these relations over *i*, exchanging the order of summations, and using $\sum_{i=1}^{m} a_i^{j,k} = 1$ for all *j* and *k* (cf. Assumption 1(a)), we obtain for all $x \in X$ and all $k \ge 0$,

$$\begin{split} &\sum_{i=1}^m \|v_i^k - x\|^2 \\ &= \sum_{j=1}^m \|x_j^k - x\|^2 - \frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m \sum_{i=1}^m a_i^{j,k} a_i^{\ell,k} \|x_j^k - x_\ell^k\|^2. \end{split}$$

By using the connectivity assumption on the graph \mathcal{G}^k and the assumption on the entries in the matrix A(k) (cf. Assumptions 1(b) and (c)), we can see that there exists a spanning tree $\mathcal{T}^k \subseteq \mathcal{G}^k$ such that

$$\frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \sum_{i=1}^{m} a_i^{j,k} a_i^{\ell,k} \|x_j^k - x_\ell^k\|^2 \ge \sum_{\{s,\ell\} \in \mathcal{T}^k} \eta^2 \|x_s^k - x_\ell^k\|^2.$$

Therefore, for all $x \in X$ and all $k \ge 0$,

$$\sum_{i=1}^m \|v_i^k - x\|^2 \le \sum_{j=1}^m \|x_j^k - x\|^2 - \eta^2 \sum_{\{s,\ell\} \in \mathcal{T}^k} \|x_s^k - x_\ell^k\|^2,$$

and the stated relation follows by substituting the preceding relation in equation (14).

Our next lemma provides a relation for the iterates θ_i^k related to the learning scheme of the algorithm.

Lemma 3: Let Assumptions 2 and 5 hold, and let the iterates θ_i^k be generated by the algorithm (5)–(7). Then, almost surely, we have for all $k \ge 0$,

$$\sum_{i=1}^{m} \mathbb{E}\left[\|\theta_i^{k+1} - \theta^*\|^2 |\mathcal{F}_k \right]$$

$$\leq \left(1 - 2\gamma_k \kappa + \gamma_k^2 R_\theta^2 \right) \sum_{i=1}^{m} \|\theta_i^k - \theta^*\|^2 + m\gamma_k^2 \nu_\theta^2,$$

with $\theta^* = \operatorname{argmin}_{\theta \in \Theta} h(\theta)$.

Proof: By using the nonexpansivity of the projection operator, the strong monotonicity and Lipschitz continuity of $\nabla_{\theta} h(\theta)$, and by recalling the relation $\theta^* = \Pi_{\theta} \left[\theta^* - \gamma_k \nabla_{\theta} h(\theta^*) \right]$, we obtain the following relation

$$\begin{aligned} \|\theta_i^{k+1} - \theta^*\|^2 \\ &\leq \|\theta_i^k - \gamma_k (\nabla h(\theta_i^k) + \beta_i^k) - \theta^* + \gamma_k \nabla h(\theta^*)\|^2 \\ &= \|\theta_i^k - \theta^*\|^2 + \gamma_k^2 \|\nabla h(\theta_i^k) - \nabla h(\theta^*)\|^2 + \gamma_k^2 \|\beta_i^k\|^2 \\ &- 2\gamma_k (\nabla h(\theta_i^k) - \nabla h(\theta^*))^T (\theta_i^k - \theta^*) \\ &- 2\gamma_k (\beta_i^k)^T (\theta_i^k - \theta^* - \gamma_k (\nabla h(\theta_i^k) - \nabla h(\theta^*))) \\ &\leq (1 - 2\gamma_k \kappa + \gamma_k^2 R_\theta^2) \|\theta_i^k - \theta^*\|^2 + \gamma_k^2 \|\beta_i^k\|^2 \\ &- 2\gamma_k (\beta_i^k)^T (\theta_i^k - \theta^* - \gamma_k (\nabla h(\theta_i^k) - \nabla h(\theta^*))). \end{aligned}$$

Taking conditional expectation with respect to the past \mathcal{F}_k , we see that almost surely for all *i* and *k*,

$$\begin{split} & \mathbb{E}\left[\|\theta_i^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k\right] \leq (1 - 2\gamma_k \kappa + \gamma_k^2 R_\theta^2) \|\theta_i^k - \theta^*\|^2 + \gamma_k^2 \nu_\theta^2, \\ & \text{since } \mathbb{E}[(\beta_i^k)^T (\theta_i^k - \theta^* - \gamma_k (\nabla h(\theta_i^k) - \nabla h(\theta^*))) \mid \mathcal{F}_k] = 0 \\ & \text{and } \mathbb{E}[\|\beta_i^k\|^2 \mid \mathcal{F}_k] \leq \nu_\theta^2 \text{ (by Assumption 2). By summing the preceding relations over } i, we obtain the stated result. \blacksquare \end{split}$$

The following lemma gives a key result that combines the decrease properties for x- and θ -iterates established in Lemmas 2 and 3.

Lemma 4: Let Assumptions 1–5 hold, and let $X = \bigcap_{i=1}^{m} X_i$. Let the sequences $\{x_i^k\}, \{\theta_i^k\}, i \in \mathcal{N}$, be generated according to (5)–(7), and define

$$V(x^{k}, \theta^{k}; x) := \sum_{i=1}^{m} \left(\|x_{i}^{k} - x\|^{2} + \|\theta_{i}^{k} - \theta^{*}\|^{2} \right) \text{ for all } x \in X.$$

Then, for all $x \in X$, all $k \ge 0$, and all $\ell \in \mathcal{N}$, the following relation holds almost surely

$$\begin{split} \mathbb{E}[V(x^{k+1}, \theta^{k+1}; x) \mid \mathcal{F}_k] &\leq V(x^k, \theta^k; x) + m\alpha_k^{2-\tau} L_{\theta}^2 D^2 \\ &- \eta^2 \sum_{\{j,s\} \in \mathcal{T}^k} \|x_j^k - x_s^k\|^2 + m\alpha_k^2 (2S^2 + \nu^2) \\ &+ 2\alpha_k G \sum_{j=1}^m \|x_j^k - z^k\| - 2\alpha_k \left(f(z^k, \theta^*) - f(x, \theta^*)\right) \\ &- \gamma_k \left(2\kappa - \gamma_k R_{\theta}^2 - \frac{\alpha_k^{\tau} + 2\alpha_k^2 L_{\theta}^2}{\gamma_k}\right) \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2 \\ &+ m\gamma_k^2 \nu_{\theta}^2, \end{split}$$

where

$$z^k = \Pi_X[y^k]$$
 with $y^k = \frac{1}{m} \sum_{j=1}^m x_j^k$ for all $k \ge 0$,

 $\begin{array}{l} \mathcal{T}^k \text{ denotes a spanning tree in } \mathcal{G}^k, \text{ while } S, \bar{X} \text{ and } G \\ \text{are defined as } S \triangleq \max_i \max_{x \in \bar{X}} \|\nabla_x f_i(x, \theta^*)\|, \ \bar{X} \triangleq \\ \operatorname{conv}(\cup_{i=1}^m X_i), \text{ and } G \triangleq \max_{i \in \mathcal{N}} \max_{z \in X} \|\nabla_x f_i(z, \theta^*)\|. \end{array}$

Proof: By Lemma 2 we have almost surely for some $\tau > 0$ and for all $x \in X$ and all $k \ge 0$,

$$\begin{split} &\sum_{i=1}^{m} \mathbb{E} \left[\|x_{i}^{k+1} - x\|^{2} \mid \mathcal{F}_{k} \right] \\ &\leq \sum_{j=1}^{m} \|x_{j}^{k} - x\|^{2} - \eta^{2} \sum_{\{s,\ell\} \in \mathcal{T}^{k}} \|x_{s}^{k} - x_{\ell}^{k}\|^{2} \\ &+ m\alpha_{k}^{2} (2S^{2} + \nu^{2}) + m\alpha_{k}^{2-\tau} L_{\theta}^{2} D^{2} \\ &+ \left(\alpha_{k}^{\tau} + 2\alpha_{k}^{2} L_{\theta}^{2}\right) \sum_{i=1}^{m} \|\theta_{i}^{k} - \theta^{*}\|^{2} \\ &- 2\alpha_{k} \sum_{i=1}^{m} \left(f_{i}(v_{i}^{k}, \theta^{*}) - f_{i}(x, \theta^{*})\right), \end{split}$$

where \mathcal{T}^k is a spanning tree in the graph \mathcal{G}^k and $\theta^* = \operatorname{argmin}_{\theta \in \Theta} h(\theta)$, which exists and it is unique in view of the strong convexity of h. By Lemma 3 almost surely we have for all $k \geq 0$,

$$\sum_{i=1}^{m} \mathbb{E}\left[\|\theta_i^{k+1} - \theta^*\|^2 |\mathcal{F}_k\right]$$

$$\leq \left(1 - 2\gamma_k \kappa + \gamma_k^2 R_\theta^2\right) \sum_{i=1}^{m} \|\theta_i^k - \theta^*\|^2 + m\gamma_k^2 \nu_\theta^2.$$

By combining Lemmas 2 and 3 and using the notation V, after regrouping some terms, we see that for all $x \in X$ and all $k \ge 0$, the following holds almost surely:

$$\begin{split} & \mathbb{E}\left[V(x^{k+1}, \theta^{k+1}; x) \mid \mathcal{F}_k\right] \leq V(x^k, \theta^k; x) + m\alpha_k^{2-\tau} L_{\theta}^2 D^2 \\ & -\eta^2 \sum_{\{s,\ell\} \in \mathcal{T}^k} \|x_s^k - x_{\ell}^k\|^2 + m\alpha_k^2 (2S^2 + \nu^2) \\ & + \left(\alpha_k^\tau + 2\alpha_k^2 L_{\theta}^2\right) \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2 \end{split}$$

$$-2\alpha_k \sum_{i=1}^m \left(f_i(v_i^k, \theta^*) - f_i(x, \theta^*) \right)$$
$$-\gamma_k \left(2\kappa - \gamma_k R_\theta^2 \right) \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2 + m\gamma_k^2 \nu_\theta^2.$$
(15)

Next, we work with the term involving the function values. We consider the summand $f_i(v_i^k, \theta^*) - f_i(x, \theta^*)$. Define

$$y^{k} = \frac{1}{m} \sum_{j=1}^{m} x_{j}^{k}, \quad z^{k} = \Pi_{X}[y^{k}] \quad \text{for all } k \ge 0.$$

By adding and subtracting $f_i(z^k)$, and by using the convexity of $f_i(\cdot, \theta^*)$ (see Assumption 4), we can show that

$$f_{i}(v_{i}^{k},\theta^{*}) - f_{i}(x,\theta^{*})$$

$$\geq f_{i}(z^{k},\theta^{*})^{T}(v_{i}^{k}-z^{k}) + f_{i}(z^{k},\theta^{*}) - f_{i}(x,\theta^{*})$$

$$\geq -\|\nabla_{x}f_{i}(z^{k},\theta^{*})\|\|v_{i}^{k}-z^{k}\| + f_{i}(z^{k},\theta^{*}) - f_{i}(x,\theta^{*}).$$

Since X_{ℓ} is bounded for every ℓ and $z^k \in X = \bigcap_{i=1}^m X_i$, it follows that

$$\max_{k\geq 0} \|\nabla_x f_i(z^k, \theta^*)\| \le G \triangleq \max_{i\in\mathcal{N}} \left(\max_{y\in X} \|\nabla_x f_i(y, \theta^*)\| \right).$$

Thus, $\|\nabla_x f_i(z^k, \theta^*)\| \|v_i^k - z^k\| \le G \|v_i^k - z^k\|$, implying
 m

$$\sum_{i=1}^{m} \left(f_i(v_i^k, \theta^*) - f_i(x, \theta^*) \right)$$

$$\geq -G \sum_{i=1}^{m} \|v_i^k - z^k\| + f(z^k, \theta^*) - f(x, \theta^*),$$

where we also use notation $f(\cdot, \theta) = \sum_{i=1}^{m} f_i(\cdot, \theta)$. Recalling the definition of v_i^k , and by using the doublystochastic property of the weights and the convexity of the Euclidean norm, we can see that $\sum_{i=1}^{m} ||v_i^k - z^k|| \le \sum_{i=1}^{m} \sum_{j=1}^{m} a_i^{j,k} ||x_j^k - z^k|| = \sum_{j=1}^{m} ||x_j^k - z^k||$. Hence,

$$\sum_{i=1}^{m} \left(f_i(v_i^k, \theta^*) - f_i(x, \theta^*) \right)$$

$$\geq -G \sum_{j=1}^{m} \|x_j^k - z^k\| + f(z^k, \theta^*) - f(x, \theta^*). \quad (16)$$

Using (16) in inequality (15) yields the stated result.

B. Averages and Constraint Sets Intersection

Now, we focus on developing a relation that will be useful for providing a bound on the distance of the iterate averages $y^k = \frac{1}{m} \sum_{j=1}^m x_j^k$ and the intersection set $X = \bigcap_{i=1}^m X_i$. Specifically, the goal is to have a bound for $\sum_{j=1}^m ||x_j^k - z^k||$, which will allow us to leverage on Lemma 4 and prove the almost sure convergence of the method. We provide such a bound for generic points x_1, \ldots, x_m taken from sets X_1, \ldots, X_m , respectively. For this, we strengthen Assumption 3(b) on the sets X_i by requiring that the interior of Xis nonempty. This assumption has also been used in [8] to ensure that the iterates $x_i^k \in X_i$ have accumulation points in X. This assumption and its role in such set dependent iterates has been originally illuminated in [28]. Assumption 6: There exists a vector $\bar{x} \in int(X)$, i.e., there exists a scalar $\delta > 0$ such that $\{z \mid ||z - \bar{x}|| \le \delta\} \subset X$. By using Lemma 2(b) from [8] and boundedness of the sets X_i , we establish an upper bound for $\sum_{j=1}^m ||x_j - \prod_X [\frac{1}{m} \sum_{\ell=1}^m x_\ell]||$ for arbitrary points $x_i \in X_i$, as given in the following lemma.

Lemma 5: Let Assumptions 3 and 6 hold. Then, for the vector $\hat{x} = \frac{1}{m} \sum_{\ell=1}^{m} x_{\ell}$, with $x_{\ell} \in X_{\ell}$ for all ℓ , we have

$$\sum_{j=1}^{m} \|x_j - \Pi_X[\hat{x}]\| \le m \left(1 + \frac{mD}{\delta}\right) \max_{j,\ell \in \mathcal{N}} \|x_j - x_\ell\|.$$

Under the interior-point assumption, we provide a refinement of Lemma 4, which will be the key relation for establishing the convergence.

Lemma 6: Let Assumptions 1–6 hold, and let $X = \bigcap_{i=1}^{m} X_i$. Let the sequences $\{x_i^k\}, \{\theta_i^k\}$ be generated according to (5)–(7). Then, almost surely, we have for all $x \in X$, all $k \ge 0$, and all $\ell \in \mathcal{N}$,

$$\begin{split} & \mathbb{E}[V(x^{k+1}, \theta^{k+1}; x) \mid \mathcal{F}_k] \leq V(x^k, \theta^k; x) \\ & - \left(\frac{\eta^2}{m-1} - \alpha_k^{\sigma}\right) \max_{j,s \in \mathcal{N}} \|x_j^k - x_s^k\|^2 \\ & + m\alpha_k^2 (2S^2 + \nu^2) + m\alpha_k^{2-\tau} L_\theta^2 D^2 \\ & + \alpha_k^{2-\sigma} G^2 m^2 \left(1 + \frac{mD}{\delta}\right)^2 - 2\alpha_k \left(f(z^k, \theta^*) - f(x, \theta^*)\right) \\ & - \gamma_k \left(2\kappa - \gamma_k R_\theta^2 - \frac{\alpha_k^\tau + 2\alpha_k^2 L_\theta^2}{\gamma_k}\right) \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2 \\ & + m\gamma_k^2 \nu_\theta^2, \end{split}$$

where $\sigma > 0$, while z^k , y^k and other variables and constants are the same as in Lemma 4.

V. Almost sure convergence

We now prove the almost sure convergence of the sequences produced by the algorithm for suitably selected stepsizes α_k and γ_k . In particular, we impose the following requirements on the stepsizes.

Assumption 7 (Stepsize sequences): The steplength sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ satisfy the following conditions :

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \qquad \sum_{k=0}^{\infty} \gamma_k^2 < \infty, \qquad \sum_{k=0}^{\infty} \alpha_k = \infty,$$

and for some $\tau \in (0, 2)$,

$$\sum_{k=0}^{\infty} \alpha_k^{2-\tau} < \infty, \qquad \lim_{k \to \infty} \frac{\alpha_k^{\tau}}{\gamma_k} = 0.$$

Example for the stepsizes: A set of choices satisfying the above assumptions are $\gamma_k = k^{-a_1}$ and $\alpha_k = k^{-a_2}$, where

- $1 > a_2 > a_1 > \frac{1}{2};$
- $a_2(2-\tau) > 1 \implies \tau < 2 1/a_2;$
- $a_1 < \tau a_2 \implies \tau > a_2/a_1$.

There is an infinite set of choices for (a_1, a_2, τ) that satisfy these conditions; a concrete example is $(a_1, a_2, \tau) =$ (0.51, 0.9, 0.75). Note that $a_2 > a_1$ implies that the steplength sequence employed in computation decays faster than the corresponding sequence of the learning updates. To analyze the behavior of the sequences $\{\theta_i^k\}, i \in \mathcal{N}$, we leverage the following super-martingale convergence result from [29, Lemma 10, page 49].

Lemma 7: Let $\{v_k\}$ be a sequence of nonnegative random variables adapted to σ -algebra $\tilde{\mathcal{F}}_k$ and such that almost surely

$$\mathbb{E}[v_{k+1} \mid \hat{\mathcal{F}}_k] \le (1 - u_k)v_k + \psi_k \quad \text{for all } k \ge 0,$$

where $0 \le u_k \le 1, \psi_k \ge 0, \sum_{k=0}^{\infty} u_k = \infty, \sum_{k=0}^{\infty} \psi_k < \infty$, and $\lim_{k\to\infty} \frac{\psi_k}{u_k} = 0$. Then, almost surely $\lim_{k\to\infty} v_k = 0$.

Next, we establish a convergence property for the θ -iterates of the algorithm.

Proposition 1 (Almost sure convergence of $\{\theta_i^k\}$): Let Assumptions 2 and 5 hold. Also, let γ_k satisfy the conditions of Assumption 7. Let the iterates θ_i^k be generated according to (5)–(7). If $\theta^* = \operatorname{argmin}_{\theta \in \Theta} h(\theta)$, then $\theta_i^k \to \theta^*$ as $k \to \infty$ in an almost sure sense for $i = 1, \ldots, N$.

Proof: We provide a brief proof. By Lemma 3 almost surely for all $k \ge 0$,

$$\sum_{i=1}^{m} \mathbb{E} \left[\|\theta_i^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k \right]$$

$$\leq \left(1 - 2\gamma_k \kappa + \gamma_k^2 R_\theta^2\right) \sum_{i=1}^{m} \|\theta_i^k - \theta^*\|^2 + m\gamma_k^2 \nu_\theta^2.$$

Using Assumption 7, we can show that for all $k \ge k$, $\gamma_k \le \frac{\kappa}{B^2}$. Then, we have almost surely

$$\sum_{i=1}^{m} \mathbb{E}\left[\|\theta_i^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k \right] \le (1 - \gamma_k \kappa) \sum_{i=1}^{m} \|\theta_i^k - \theta^*\|^2 + m \gamma_k^2 \nu_\theta^2.$$

To invoke Lemma 7, we define $v_k = \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2$. Furthermore, $u_k = \gamma_k \kappa$ and $\psi_k = m \gamma_k^2 \nu_\theta^2$ for all $k \ge 0$. We note that $\sum_{k\ge 0} u_k = \infty$, $\sum_{k=0}^\infty \psi_k < \infty$, and $\lim_{k\to\infty} \frac{\psi_k}{u_k} = 0$ by Assumption 7. Thus, Lemma 7 applies to a shifted sequence $\{v_k\}_{k\ge \hat{k}}$ and we conclude that $v_k \to 0$ almost surely.

Now, we analyze the behavior of x-sequences, where we leverage the following super-martingale convergence theorem from [29, Lemma 11, page 50].

Lemma 8: Let v_k, u_k, ψ_k and δ_k be nonnegative random variables adapted to a σ -algebra $\tilde{\mathcal{F}}_k$. If almost surely $\sum_{k=0}^{\infty} u_k < \infty, \sum_{k=0}^{\infty} \psi_k < \infty$, and

$$\mathbb{E}[v_{k+1} \mid \tilde{\mathcal{F}}_k] \le (1+u_k)v_k - \delta_k + \psi_k \text{ for all } k \ge 0,$$

then almost surely V_k is convergent and $\sum_{k=0}^{\infty} \delta_k < \infty$.

As observed in Section III, under continuity of the functions $f_i(\cdot, \theta)$ (in view of Assumption 4) and the compactness of each X_i , the problem $\min_{x \in \bigcap_{i=1}^m X_i} \sum_{i=1}^m f_i(x, \theta^*)$ has a solution. We denote the set of solutions by X^* . Therefore, under Assumptions 3, 4, and 5, the problem (4) has a nonempty solution set, given by $X^* \times \{\theta^*\}$.

We have the following convergence result.

Proposition 2 (Almost sure convergence of $\{x_i^k\}$): Let Assumptions 1–7 hold, and let $X = \bigcap_{i=1}^m X_i$. Let the sequences $\{x_i^k\}, \{\theta_i^k\}$ be generated according to (5)–(7). Then, the sequences $\{x_j^k\}$ converge almost surely to the same solution point, i.e., there exists a random vector $z^* \in X^*$ such that almost surely

$$\lim_{k \to \infty} x_j^k = z^* \qquad \text{for all } j \in \mathcal{N}$$

Proof: In Lemma 4, we let x be an optimal solution for the problem $\min_{x \in \bigcap_{i=1}^{m} X_i} \sum_{i=1}^{m} f_i(x, \theta^*)$, i.e., $x = x^*$ with $x^* \in X^*$. Thus, by Lemma 4 we obtain almost surely for any $x^* \in X^*$, all $k \ge 0$, and all $\ell \in \mathcal{N}$,

$$\begin{split} & \mathbb{E}[V(x^{k+1}, \theta^{k+1}; x^*) \mid \mathcal{F}_k] \\ & \leq V(x^k, \theta^k; x^*) - \left(\frac{\eta^2}{m-1} - \alpha_k^{\sigma}\right) \max_{j,s \in \mathcal{N}} \|x_j^k - x_s^k\|^2 \\ & + m\alpha_k^2 (2S^2 + \nu^2) + m\alpha_k^{2-\tau} L_{\theta}^2 D^2 \\ & + \alpha_k^{2-\sigma} G^2 m^2 \left(1 + \frac{mD}{\delta}\right)^2 - 2\alpha_k \left(f(z^k, \theta^*) - f(x^*, \theta^*)\right) \\ & - \gamma_k \left(2\kappa - \gamma_k R_{\theta}^2 - \frac{\alpha_k^{\tau} + 2\alpha_k^2 L_{\theta}^2}{\gamma_k}\right) \sum_{i=1}^m \|\theta_i^k - \theta^*\|^2 \\ & + m\gamma_k^2 \nu_{\theta}^2. \end{split}$$

Next, since σ is an arbitrary positive scalar, we let $\sigma = \tau$ where $\tau \in (0, 2)$ is obtained from Assumption 7. Furthermore, let ψ_k be defined as follows:

$$\begin{split} \psi_k &\triangleq m\alpha_k^2(2S^2 + \nu^2) + m\gamma_k^2\nu_\theta^2 \\ &+ \alpha_k^{2-\tau} \left(mL_\theta^2 D^2 + G^2 m^2 \left(1 + \frac{mD}{\delta} \right)^2 \right). \end{split}$$

Using the assumptions on the stepsizes we can show that for all $k \ge k_0$, we have $\frac{\eta^2}{m-1} - \alpha_k^{\tau} \ge \epsilon$ and

$$2\kappa - \gamma_k R_\theta^2 - \frac{\alpha_k^\tau + 2\alpha_k^2 L_\theta^2}{\gamma_k} \ge 0.$$

Therefore, almost surely for all $x^* \in X^*$, all $k \ge k_0$, and all $\ell \in \mathcal{N}$,

$$\mathbb{E}[V(x^{k+1}, \theta^{k+1}; x^*) \mid \mathcal{F}_k] \le V(x^k, \theta^k; x^*) - \epsilon \max_{j,s \in \mathcal{N}} \|x_j^k - x_s^k\|^2 - 2\alpha_k \left(f(z^k, \theta^*) - f(x^*, \theta^*) \right) + \psi_k.$$

Recall that $z^k = \prod_X [y^k]$ with $y^k = \frac{1}{m} \sum_{\ell=1}^m x_\ell^k$. In view of optimality of x^* , we have $f(z^k, \theta^*) - f(x^*, \theta^*) \ge 0$ for all k and $x^* \in X^*$. Furthermore, the conditions on the stepsizes in Assumption 7 Then, we verify that the conditions of Lemma 8 are satisfied for the sequence $\{V(x^k, \theta^k; x^*)\}_{k\ge k_0}$ for an arbitrary $x^* \in X^*$. By Lemma 8 it follows that $V(x^k, \theta^k; x^*)$ is convergent almost surely for every $x^* \in X^*$, and the following hold almost surely:

$$\sum_{k=0}^{\infty} \max_{j,s \in \mathcal{N}} \|x_j^k - x_s^k\|^2 < \infty,$$
(17)

$$\sum_{k=0}^{\infty} \alpha_k \left(f(z^k, \theta^*) - f(x^*, \theta^*) \right) < \infty.$$
(18)

By Proposition 1, we have that $\theta_i^k \to \theta^*$ almost surely for all $i \in \mathcal{N}$. Since $V(x^k, \theta^k; x^*) = \sum_{i=1}^m \left(\|x_i^k - x^*\|^2 + \|\theta_i^k - \theta^*\|^2 \right)$ and the assertion that

 $\{V(x^k, \theta^k; x^*)\}$ is convergent almost surely for every $x^* \in X^*$, we can conclude that

$$\left\{\sum_{i=1}^{m} \|x_i^k - x^*\|^2\right\}$$
 is convergent a.s. $\forall x^* \in X^*$. (19)

Since $\sum_{k=0}^{\infty} \alpha_k = \infty$, the relation (18) implies that

$$\liminf_{k \to \infty} f(z^k, \theta^*) = f^*, \tag{20}$$

where f^* is the optimal value of the problem, i.e., $f^* = f(x^*, \theta^*)$ for any $x^* \in X^*$. The set X is bounded (since each X_j is bounded by assumption), so the sequence $\{z^k\} \subset X$ is also bounded. Let \mathcal{K} denote the index set of a subsequence along which the following holds almost surely:

$$\lim_{k \to \infty, k \in \mathcal{K}} f(z^k, \theta^*) = \liminf_{k \to \infty} f(z^k, \theta^*),$$
$$\lim_{k \to \infty, k \in \mathcal{K}} z^k = z^* \quad \text{with } z^* \in X^*.$$
(21)

We note that \mathcal{K} is a random sequence and z^* is a randomly specified vector from X^* . Further, relation (17) implies that all the sequences $\{x_j^k\}, j = 1, \ldots, m$, have the same accumulation points (which exist since the sets X_j are bounded). Moreover, since $\{x_j^k\} \subset X_j$ for each $j \in \mathcal{N}$, it follows that the accumulation points of the sequences $\{x_j^k\}, j = 1, \ldots, m$, must lie in the set $X = \bigcap_{j=1}^m X_j$. Without loss of generality we may assume that the limit $\lim_{k\to\infty,k\in\mathcal{K}} x_j^k$ exists a.s. for each j, so that in view of the preceding discussion we have almost surely

$$\lim_{k \to \infty, k \in \mathcal{K}} x_j^k = \tilde{x}, \quad \text{with } \tilde{x} \in X,$$
$$\lim_{k \to \infty, k \in \mathcal{K}} y^k = \lim_{k \to \infty, k \in \mathcal{K}} \frac{1}{m} \sum_{\ell=1}^m x_\ell^k = \tilde{x}.$$

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Then, by the continuity of the projection operator $v \mapsto \Pi_X[v]$ and the fact $z_k = \Pi_X[y^k]$, we have almost surely

$$\lim_{k \to \infty, k \in \mathcal{K}} z^k = \lim_{k \to \infty, k \in \mathcal{K}} \Pi_X[y^k] = \tilde{x}.$$

The preceding relation and (21) yield $\tilde{x} = z^*$, implying that for all j almost surely

$$\lim_{k \to \infty, k \in \mathcal{K}} x_j^k = z^*, \quad \text{with } z^* \in X^*.$$
 (22)

Then, we can use $x^* = z^*$ in relation (19) to conclude that $\sum_{i=1}^{m} \|x_i^k - z^*\|^2$ is convergent almost surely. This and the subsequential convergence in (22) imply that $\sum_{i=1}^{m} \|x_i^k - z^*\|^2 \to 0$ almost surely.

Special cases: We note two special cases of relevance which arise as a consequence of Propositions 1 and 2.

(i) **Deterministic optimization and learning:** First, note that if the functions $f_i(x,\theta)$ and $h(\theta)$ are deterministic in that the gradients $\nabla_x f(x,\theta)$ and $\nabla h(\theta)$ may be evaluated at arbitrary points x and θ , then the results of Propositions 1 and 2 show that $\lim_{k\to\infty} \theta_i^k = \theta^*$ and $\lim_{k\to\infty} x_i^k = x^*$ for some $x^* \in X^*$ and for all $i = 1, \ldots, m$.

(ii) **Correctly specified problems:** Second, now suppose that the parameter θ^* is known to every agent, so there

is no misspecification. This case can be treated under algorithm (5)–(7) where the iterates θ_i^k are all fixed at θ^* . Formally this can be done by setting the initial parameters to the correct value, i.e., $\theta_i^0 = \theta^*$ for all *i*, and by using the fact that the function $h(\theta^*)$ is known, in which case the algorithm reduces to: for all i = 1, ..., m and $k \ge 0$,

$$v_i^k := \sum_{j=1}^m a_i^{j,k} x_j^k$$
(23)

$$x_i^{k+1} := \Pi_{X_i} \left(v_i^k - \alpha_k \left(\nabla_x f_i(v_i^k, \theta^*) + w_i^k \right) \right).$$
 (24)

By letting $F_i(x) = f_i(x, \theta^*)$ we see that, by Proposition 2, the iterates of the algorithm (23)–(24) converge almost surely to a solution of problem $\min_{x \in \bigcap_{i=1}^m X_i} \sum_{i=1}^m F_i(x)$. Thus, the algorithm solves this problem in a distributed fashion, where both functions and the sets are distributed among the agents. In particular, this result when reduced to a deterministic case (i.e., noiseless gradient evaluations) extends the convergence results established in [8] where two special cases have been studied; namely, the case when $X_i = X$ for all i, and the case when the underlying graph is a complete graph and all weights are equal (i.e., $a_i^{j,k} = \frac{1}{m}$ for all i, j and $k \ge 0$).

Rate of convergence: While standard stochastic gradient methods achieve the optimal rate of convergence in that $\mathbb{E}[f(x_k, \theta^*)] - \mathbb{E}[f(x^*, \theta^*)] \leq \mathcal{O}(1/k)$ in the correctly specified regime, it remains to establish similar rates in this instance particularly in the context of time-varying connectivity graphs. Such rate bounds will aid in developing practical implementations.

VI. CONCLUDING REMARKS

Traditionally, optimization algorithms have been developed under the premise of exact information regarding functions and constraints. As systems grow in complexity, an a priori knowledge of cost functions and efficiencies is difficult to guarantee. One avenue lies in using observational information to learn these functions while optimizing the overall system. We consider precisely such a question in a networked multi-agent regime where an agent does not have access to the decisions of the entire collective, and are furthermore locally constrained by their own feasibility sets. Generally, in such regimes, distributed optimization can be carried out by combining a local averaging step with a projected gradient step. We overlay a learning step where agents update their belief regarding the misspecified parameter and examine the associated schemes in this regime. It is shown that when agents are characterized by merely convex, albeit misspecified, problems under general timevarying graphs, the resulting schemes produce sequences that converge almost surely to the set of optimal solutions and the true parameter, respectively.

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