Fast, Accurate Second Order Methods for Network Optimization

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Abstract-Dual descent methods are commonly used to solve network flow optimization problems, since their implementation can be distributed over the network. These algorithms, however, often exhibit slow convergence rates. Approximate Newton methods which compute descent directions locally have been proposed as alternatives to accelerate the convergence rates of conventional dual descent. The effectiveness of these methods, is limited by the accuracy of such approximations. In this paper, we propose an efficient and accurate distributed second order method for network flow problems. The proposed approach utilizes the sparsity pattern of the dual Hessian to approximate the the Newton direction using a novel distributed solver for symmetric diagonally dominant linear equations. Our solver is based on a distributed implementation of a recent parallel solver of Spielman and Peng (2014). We analyze the properties of the proposed algorithm and show that, similar to conventional Newton methods, superlinear convergence within a neighborhood of the optimal value is attained. We finally demonstrate the effectiveness of the approach in a set of experiments on randomly generated networks.

I. INTRODUCTION

Conventional methods for distributed network optimization are based on sub-gradient descent in either the primal or dual domains, see [8], [9], [10], [13]. For a large class of problems, these techniques yield iterations that can be implemented in a distributed fashion by only using local information. Their applicability, however, is limited by increasingly *slow* convergence rates. Second order Newton methods [3], [4] are known to overcome this limitation leading to improved convergence rates.

Unfortunately, computing *exact* Newton directions based only on local information is challenging. Specifically, to determine the Newton direction, the inverse of the dual Hessian is needed. Determining this inverse, however, requires global information. Consequently, authors in [5], [6] proposed approximate algorithms for determining these Newton iterates in a distributed fashion. Accelerated Dual Descent (ADD) [6], for instance, exploits the fact that the dual Hessian is the weighted Laplacian of the network and performs a truncated Neumann expansion of the inverse to determine a local approximate to the exact direction. ADD allows for a tradeoff between accurate Hessian approximations and communication costs through the N-Hop design, where increased N allows for more accurate inverse approximations arriving at increased cost, and lower values of N reduce accuracy but improve computational times. Though successful, the effectiveness of these approaches highly depend on the accuracy of the truncated Hessian inverse which is used to approximate the Newton direction. As shown in Section VI, the approximated iterate can resemble high variation to the real Newton direction, decreasing the applicability of these techniques.

Exploiting the sparsity pattern of the dual Hessian, in this paper we tackle the above problem and propose a Newton method for network optimization that is both faster and more accurate. Using recently-developed solvers for symmetric diagonally dominant (SDDM) linear equations, we approximate the Newton direction up-to any arbitrary precision $\epsilon > 0$. The solver is a distributed implementation of [11] constructing what is known as an inverse chain. We analyze the properties of the proposed algorithm and show that, similar to conventional Newton methods, superlinear convergence within a neighborhood of the optimal value is attained. We finally demonstrate the effectiveness of the approach in a set of experiments on randomly generated networks. Namely, we show that our method is capable of significantly outperforming state-of-the-art methods in both the convergence speeds and in the accuracy of approximating the Newton direction.

The remainder of the paper is organized as follows. Section II draws upon background material needed for the remainder of the paper. Section III defines the network flow optimization problem targeted in this paper. Section IV details our proposed distributed solver for SDDM linear systems. Section V introduces the approximate Newton method and rigorously analyzes its theoretical guarantees. Section VI presents the experimental results. Finally, Section VII concludes pointing-out interesting directions for future research.

II. BACKGROUND

A. SDDM Linear Systems

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To determine the Newton direction, we need to solve a symmetric diagonally dominant system of linear equations, defined as:

$$\boldsymbol{M}_0 \boldsymbol{x} = \boldsymbol{b}_0 \tag{1}$$

where M_0 is a Symmetric Diagonally Dominant M-Matrix (SDDM). Namely, M_0 is symmetric positive definite with non-positive off diagonal elements, such that for all i = 1, 2, ..., n:

$$oldsymbol{M}_0]_{ii} \geq -\sum_{j=1,j
eq i}^n oldsymbol{[M_0]}_{ij}$$

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The system of Equations in 1 can be interpreted as representing an undirected weighted graph, \mathcal{G} , with M_0 being its Laplacian. Namely, $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathbf{W})$, with \mathcal{N} representing the set of nodes, \mathcal{E} denoting the edges, and \mathbf{W} representing the weighted graph adjacency. Nodes v_i and v_j are connected with an edge e = (i, j) iff $W_{ij} > 0$, where:

$$\boldsymbol{W}_{ij} = \left[\boldsymbol{M}_0 \right]_{ii}$$
 (if $i = j$), or $\boldsymbol{W}_{ij} = -\left[\boldsymbol{M}_0 \right]_{ij}$, otherwise.

Following [11], we seek ϵ -approximate solutions to x^* , being the exact solution of $M_0 x = b_0$, defined as:

Definition 1: Let $x^* \in \mathbb{R}^n$ be the solution of $Mx = b_0$. A vector $\tilde{x} \in \mathbb{R}^n$ is called an ϵ - approximate solution, if:

$$||\boldsymbol{x}^{\star} - \tilde{\boldsymbol{x}}||_{\boldsymbol{M}_0} \leq \epsilon ||\boldsymbol{x}^{\star}||_{\boldsymbol{M}_0}, \quad ext{where } ||\boldsymbol{u}||_{\boldsymbol{M}_0}^2 = \boldsymbol{u}^{\mathsf{T}} \boldsymbol{M}_0 \boldsymbol{u}.$$

The R-hop neighbourhood of node v_k is defined as $\mathbb{N}_r(v_k) = \{v \in \mathcal{N} : \operatorname{dist}(v_k, v) \leq r\}$. We also make use of the diameter of a graph, \mathcal{G} , defined as diam $(\mathcal{G}) = \max_{v_i, v_j \in \mathcal{N}} \operatorname{dist}(v_i, v_j)$.

Definition 2: A matrix $A \in \mathbb{R}^{n \times n}$ is said to have a sparsity pattern corresponding to the R-hop neighborhood if $A_{ij} = 0$ for all $i = 1, \ldots, n$ and for all j such that $v_j \notin \mathbb{N}_r(v_i)$.

We will denote the spectral radius of a matrix A by $\rho(A) = \max |\lambda_i|$, where λ_i represents an eigenvalue of the matrix A. Furthermore, we will make use of the condition number¹, $\kappa(A)$ of a matrix A defined as $\kappa = \left| \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \right|$. In [?] it is shown that the condition number of the graph Laplacian is at most $\mathcal{O}\left(n^3 \frac{W_{\max}}{W_{\min}}\right)$, where W_{\max} and W_{\min} represent the largest and the smallest edge weights in \mathcal{G} . Finally, the condition number of a sub-matrix of the Laplacian is at most $\mathcal{O}\left(n^4 \frac{W_{\max}}{W_{\min}}\right)$, see [11].

B. Standard Splittings & Approximations

For determining the Newton direction, we propose a fast distributed solver for symmetric diagonally dominant linear equations. Our approach is based on a distributed implementation of the parallel solver of Spielman and Peng [11]. Before detailing the parallel solver, however, we next provide basic notions and notations required.

Definition 3: The standard splitting of a symmetric matrix M_0 is:

$$M_0 = D_0 - A_0.$$
 (3)

Here, D_0 is a diagonal matrix such that $[D_0]_{ii} = [M_0]_{ii}$ for i = 1, 2, ..., n, and A_0 representing a non-negative symmetric matrix such that $[A_0]_{ij} = -[M_0]_{ij}$ if $i \neq j$, and $[A_0]_{ii} = 0$.

We also define the Loewner ordering:

Definition 4: Let $S_{(n)}$ be the space of $n \times n$ -symmetric matrices. The Loewner ordering \leq is a partial order on $S_{(n)}$ such that $Y \leq X$ if and only if X - Y is positive semidefinite.

Finally, we define the " \approx_{α} " operation used in the sequel to come as:

Definition 5: Let X and Y be positive semidefinite symmetric matrices. Then $X \approx_{\alpha} Y$ if and only iff

$$e^{-\alpha} X \preceq Y \preceq e^{\alpha} X$$
 (4)

with $A \leq B$ meaning B - A is positive semidefinite.

Based on the above definitions, the following lemma represents the basic characteristics of the \approx_{α} operator:

Lemma 1: [11] Let X, Y, Z and, Q be symmetric positive semi definite matrices. Then

(1) If $X \approx_{\alpha} Y$, then $X + Z \approx_{\alpha} Y + Z$, (2) If $X \approx_{\alpha} Y$ and $Z \approx_{\alpha} Q$, then $X + Z \approx_{\alpha} Y + Q$ (3) If $X \approx_{\alpha} Y$ and $Z \approx_{\alpha} Q$, then $X + Z \approx_{\alpha} Y + Q$, (4) If $X \approx_{\alpha_1} Y$ and $Y \approx_{\alpha_2} Z$, then $X \approx_{\alpha_1+\alpha_2} Z$ (5) If X, and Y are non singular and $X \approx_{\alpha} Y$, then $X^{-1} \approx_{\alpha} Y^{-1}$, (6) If $X \approx_{\alpha} Y$ and V is a matrix, then $V^{\mathsf{T}} X V \approx_{\alpha} V^{\mathsf{T}} Y V$

The next lemma shows that good approximations of M_0^{-1} guarantee good approximated solutions of $M_0 x = b_0$.

Lemma 2: Let $Z_0 \approx_{\epsilon} M_0^{-1}$, and $\tilde{x} = Z_0 b_0$. Then \tilde{x} is $\sqrt{2^{\epsilon}(e^{\epsilon}-1)}$ approximate solution of $M_0 x = b_0$.

Proof: The proof can be found in the appendix.

C. The Parallel SDDM Solver

The parallel SDDM solver proposed in [11] is a parallelized technique for solving the problem of Section II-A. It makes use of inverse approximated chains (see Definition 6) to determine \tilde{x} and can be split in two steps. In the first step, denoted as Algorithm 1, a "crude" approximation, x_0 , of \tilde{x} is returned. x_0 is driven to the ϵ -close solution, \tilde{x} , using Richardson Preconditioning in Algorithm 2. Before we proceed, we start with the following two Lemmas which enable the definition of inverse chain approximation.

Lemma 3: [11] If M = D - A is an SDDM matrix, with D being positive diagonal, and A denoting a non-negative symmetric matrix, then $D - AD^{-1}A$ is also SDDM.

Lemma 4: [11] Let M = D - A be an SDDM matrix, where D is positive diagonal and, A a symmetric matrix. Then

$$(D - A)^{-1} = \frac{1}{2} \Big[D^{-1} + (I + D^{-1}A) (D - AD^{-1}A)^{-1}$$
(5)
$$(I + AD^{-1}) \Big].$$

Given the results in Lemmas 3 and 4, we now can consider inverse approximated chains of M_0 :

Definition 6: Let $C = \{M_0, M_1, \ldots, M_d\}$ be a collection of SDDM matrices such that $M_i = D_i - A_i$, with D_i a positive diagonal matrix, and A_i denoting a non-negative symmetric matrix. Then C is an inverse approximated chain if there exists positive real numbers $\epsilon_0, \epsilon_1, \ldots, \epsilon_d$ such that: (1) For $i = 1, \ldots, d$: $D_i - A_i \approx_{e_{i-1}} D_{i-1} - A_{i-1} D_{i-1}^{-1} A_{i-1}$, (2) $D_i \approx_{\epsilon_{i-1}} D_{i-1}$, and (3) $D_d \approx_{\epsilon_d} D_d - A_d$.

The quality of the "crude" solution returned by Algorithm 1 is quantified in the following lemma:

Lemma 5: [11] Let $\{M_0, M_1, \ldots, M_d\}$ be the inverse approximated chain and denote Z_0 be the operator defined by

¹Please note that in the case of the graph Laplacian, the condition number is defined as the ratio of the largest to the smallest nonzero eigenvalues.

Algorithm 1 ParallelRSolve $(M_0, M_1, \ldots, M_d, b_0)$

1:	Input:	Inverse	approximated	chain,
	$\{M_0, M_1,$	\ldots, M_d , and	l \boldsymbol{b}_0 being	
2:	Output: Th	ne "crude" app	proximation, x_0 , of x^{\star}	
3:	for $i = 1$ to	o d do		
4:	$oldsymbol{b}_i = ig(oldsymbol{I})$	$+ A_{i-1} D_{i-1}^{-1})$	$oldsymbol{b}_{i-1}$	
5:	end for			
6:	$x_d = D_d^{-1}$	$oldsymbol{b}_d$		
7:	for $i = \overline{d}$ –	- 1 to 0 do		
8:	$x_i = rac{1}{2}$ [$oldsymbol{D}_i^{-1}oldsymbol{b}_i + ig(oldsymbol{I}+$	$+ oldsymbol{D}_i^{-1} oldsymbol{A}_i ig) oldsymbol{x}_{i+1} ig]$	
9:	end for			
10:	return x_0			

ParallelRSolve $(M_0, M_1, \dots, M_d, b_0)$, namely, $x_0 = Z_0 b_0$. Then

$$Z_0 pprox_{\sum_{i=0}^d \epsilon_i} M_0^{-1}$$
 (6)

Algorithm 1 returns a "crude" solution to $M_0 x = b$. To obtain arbitrary close solutions, Spielman *et. al* [11] introduced the *preconditioned Richardson iterative scheme*, summarized in Algorithm 2. Following their analysis, Lemma 6 provides the iteration count needed by Algorithm 2 to arrive at \tilde{x} .

Algorithm 2 ParallelESolve $(M_0, M_1, \ldots, M_d, b_0, \epsilon)$

- 1: Input: Inverse approximated chain $\{M_0, M_1, \ldots, M_d\}$, b_0 , and ϵ .
- 2: **Output**: ϵ close approximation, \tilde{x} , of x^* 3: **Initialize**: $y_0 = 0$; χ = ParallelRSolve $(M_0, M_1, \dots, M_d, b_0)$ (i.e., Algorithm 1) 4: **for** k = 1 to q **do** 5: $u_k^{(1)} = M_0 y_{k-1}$ 6: $u_k^{(2)}$ = ParallelRSolve $(M_0, M_1, \dots, M_d, u_k^{(1)})$ 7: $y_k = y_{k-1} - u_k^{(2)} + \chi$ 8: **end for** 9: $\tilde{x} = y_q$ 10: **return** \tilde{x}

Lemma 6: [11] Let $\{M_0, M_1 \dots M_d\}$ be an inverse approximated chain such that $\sum_{i=1}^{d} \epsilon_i < \frac{1}{3} \ln 2$. Then ParallelESolve $(M_0, M_1, \dots, M_d, b_0, \epsilon)$ arrives at an ϵ close solution of x^* in $q = \mathcal{O}(\log \frac{1}{\epsilon})$ iterations.

III. NETWORK FLOW OPTIMIZATION

We consider a network represented by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with node set $\mathcal{N} = \{1, \ldots, N\}$ and edge set $\mathcal{E} = \{1, \ldots, E\}$. The flow vector is denoted by $\boldsymbol{x} = [x^{(e)}]_{e \in \mathcal{E}}$, with $x^{(e)}$ representing the flow on edge e. The flow conservation conditions at nodes can be compactly represented as

$$Ax = b$$
,

where \boldsymbol{A} is the $N \times E$ node-edge incidence matrix of \mathcal{G} defined as

$$A_{i,j} = \begin{cases} 1 & \text{if edge } j \text{ leaves node } i \\ -1 & \text{if edge } j \text{ enters node } i \\ 0 & \text{otherwise,} \end{cases}$$

and the vector $\mathbf{b} \in \mathbf{1}^{\perp}$ denotes the external source, i.e., $b^{(i)} > 0$ (or $b^{(i)} < 0$) indicates $b^{(i)}$ units of external flow enters (or leaves) node *i*. A cost function $\Phi_e : \mathbb{R} \to \mathbb{R}$ is associated with each edge *e*. Namely, $\Phi_e(x^{(e)})$ denotes the cost on edge *e* as a function of the edge flow $x^{(e)}$. We assume that the cost functions Φ_e are strictly convex and twice differentiable. Consequently, the minimum cost networks optimization problem can be written as

$$\min_{\boldsymbol{x}} \sum_{e=1}^{E} \boldsymbol{\Phi}_{e}(\boldsymbol{x}^{(e)})$$
(7)
s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

Our goal is to investigate Newton type methods for solving the problem in 7 in a distributed fashion. Before diving into these details, however, we next present basic ingredients needed for the remainder of the paper.

A. Dual Subgradient Method

The dual subgradient method optimizes the problem in Equation 7 by descending in the dual domain. The Lagrangian, $l : \mathbb{R}^E \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$l(\boldsymbol{x}, \boldsymbol{\lambda}) = -\sum_{e=1}^{E} \boldsymbol{\Phi}_{e}(\boldsymbol{x}^{(e)}) + \boldsymbol{\lambda}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}).$$

The dual function $q(\lambda)$ is then derived as

$$\begin{split} q(\boldsymbol{\lambda}) &= \inf_{\boldsymbol{x} \in \mathbb{R}^{E}} l(\boldsymbol{x}, \boldsymbol{\lambda}) \\ &= \inf_{\boldsymbol{x} \in \mathbb{R}^{E}} \left(-\sum_{e=1}^{E} \boldsymbol{\Phi}_{e}(\boldsymbol{x}^{(e)}) + \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{A} \boldsymbol{x} \right) - \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{b} \\ &= \sum_{e=1}^{E} \inf_{\boldsymbol{x}^{(e)} \in \mathbb{R}} \left(-\boldsymbol{\Phi}_{e}(\boldsymbol{x}^{(e)}) + \left(\boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{A} \right)^{(e)} \boldsymbol{x}^{(e)} \right) - \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{b}. \end{split}$$

Hence, it can be clearly seen that the evaluation of the dual function $q(\lambda)$ decomposes into E one-dimensional optimization problems. We assume that each of these optimization problems have an optimal solution, which is unique by the strict convexity of the functions Φ_e . Denoting the solutions by $x^{(e)}(\lambda)$ and using the first order optimality conditions, it can be seen that for each edge, e, $x^{(e)}(\lambda)$ is given by²

$$x^{(e)}(\boldsymbol{\lambda}) = [\dot{\boldsymbol{\Phi}}_e]^{-1} \left(\lambda^{(i)} - \lambda^{(j)} \right), \tag{8}$$

where $i \in \mathcal{N}$ and $j \in \mathcal{N}$ denote the source and destining nodes of edge e = (i, j), respectively (see [6] for details). Therefore, for an edge e, the evaluation of $x^{(e)}(\lambda)$ can be performed based on local information about the edge's cost function and the dual variables of the incident nodes, i and j.

²Note that if the dual is not continuously differentiable, the a generalized Hessian can be used.

The dual problem is defined as $\max_{\lambda \in \mathbb{R}^N} q(\lambda)$. Since the dual function is convex, the optimization problem can be solved using gradient descent according to

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k - \alpha_k \boldsymbol{g}_k \quad \text{for all } k \ge 0, \tag{9}$$

with k being the iteration index, and $g_k = g(\lambda_k) = \nabla q(\lambda_k)$ denoting the gradient of the dual function evaluated at $\lambda = \lambda_k$. Importantly, the computation of the gradient can be performed as $g_k = Ax(\lambda_k) - b$, with $x(\lambda_k)$ being a vector composed of $x^{(e)}(\lambda_k)$ as determined by Equation 8. Further, due to the sparsity pattern of the incidence matrix A, the i^{th} element, $g_k^{(i)}$, of the gradient g_k can be computed as

$$g_k^{(i)} = \sum_{e=(i,j)} x^{(e)}(\lambda_k) - \sum_{e=(j,i)} x^{(e)}(\lambda_k) - b^{(i)}.$$
 (10)

Clearly, the algorithm in Equation 9 can be implemented in a distributed fashion, where each node, *i*, maintains information about its dual, $\lambda_k^{(i)}$, and primal, $x^{(e)}(\lambda_k)$, iterates of the outgoing edges e = (i, j). Gradient components can then be evaluated as per 10 using only local information. Dual variables can then be updated using 9. Given the updated dual variables, the primal variables can be computed using 8.

Although the distributed implementation avoids the cost and fragility of collecting all information at centralized location, practical applicability of gradient descent is hindered by slow convergence rates. This motivates the consideration of Newton methods discussed next.

B. Newton's Method for Dual Descent

Newton's method is a descent algorithm along a scaled version of the gradient. Its iterates are typically given by

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \alpha_k \boldsymbol{d}_k \quad \text{for all } k \ge 0, \tag{11}$$

with d_k being the Newton direction at iteration k, and α_k denoting the step size. The Newton direction satisfies

$$\boldsymbol{H}_k \boldsymbol{d}_k = -\boldsymbol{g}_k, \tag{12}$$

with $H_k = H(\lambda_k) = \nabla^2 q(\lambda_k)$ being the Hessian of the dual function at the current iteration k.

1) Properties of the Dual and Assumptions: Here, we detail some assumptions needed by our approach. We also derive essential Lemmas quantifying properties of the dual Hessian.

Assumption 1: The graph, \mathcal{G} , is connected, non-bipartite and has algebraic connectivity lower bound by a constant ω .

Assumption 2: The cost functions, $\Phi_e(\cdot)$, in Equation 7 are

1) twice continuously differentiable satisfying

$$\gamma \leq \ddot{\mathbf{\Phi}}_e(\cdot) \leq \Gamma,$$

with γ and Γ are constants; and

2) Lipschitz Hessian invertible for all edges $e \in \mathcal{E}$

$$\left|rac{1}{oldsymbol{\Phi}_e(oldsymbol{x})} - rac{1}{oldsymbol{\Phi}_e(\hat{oldsymbol{x}})}
ight| \leq \delta \left|oldsymbol{x} - \hat{oldsymbol{x}}
ight|$$

The following two lemmas [5], [6] quantify essential properties of the dual Hessian which we exploit through our algorithm to determine the approximate Newton direction.

Lemma 7: The dual objective $q(\lambda) = \lambda^{\mathsf{T}}(Ax(\lambda) - b) - \sum_{e} \Phi_{e}(x(\lambda))$ abides by the following two properties [?]:

 The dual Hessian, H(λ), is a weighted Laplacian of G:

$$\boldsymbol{H}(\boldsymbol{\lambda}) = \nabla^2 q(\boldsymbol{\lambda}) = \boldsymbol{A} \left[\nabla^2 f(\boldsymbol{x}(\boldsymbol{\lambda})) \right]^{-1} \boldsymbol{A}^{\mathsf{T}}.$$

The dual Hessian *H*(λ) is Lispshitz continuous with respect to the Laplacian norm (i.e., || · ||_L) where *L* is the unweighted laplacian satisfying *L* = *AA*^T with *A* being the incidence matrix of *G*. Namely, ∀λ, λ̄:

$$||\boldsymbol{H}(\boldsymbol{\lambda}) - \boldsymbol{H}(\boldsymbol{\lambda})||_{\mathcal{L}} \leq B||\boldsymbol{\lambda} - \boldsymbol{\lambda}||_{\mathcal{L}},$$

with $B = \frac{\mu_n(\mathcal{L})\delta}{\gamma\sqrt{\mu_2(\mathcal{L})}}$ where $\mu_n(\mathcal{L})$ and $\mu_2(\mathcal{L})$ denote the largest and second smallest eigenvalues of the Laplacian \mathcal{L} .

Proof: See Appendix. The following lemma follows from the above and is needed in the analysis later:

Lemma 8: If the dual Hessian $H(\lambda)$ is Lipschitz continuous with respect to the Laplacian norm $||\cdot||_{\mathcal{L}}$ (i.e., Lemma 7), then for any λ and $\hat{\lambda}$ we have

$$\begin{aligned} ||\nabla q(\hat{\boldsymbol{\lambda}}) - \nabla q(\boldsymbol{\lambda}) - \boldsymbol{H}(\boldsymbol{\lambda})(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda})||_{\mathcal{L}} &\leq \frac{B}{2} ||\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}||_{\mathcal{L}}^{2}. \\ Proof: \text{ See Appendix.} \end{aligned}$$

As detailed in [6], the exact computation of the inverse of the Hessian needed for determining the Newton direction can not be attained exactly in a distributed fashion. Authors in [5], [6] proposed approximation techniques for computing this direction. The effectiveness of these algorithms, however, highly depend on the accuracy of such an approximation. In this work, we propose a distributed approximator for the Newton direction capable of acquiring ϵ -close solutions for any arbitrary ϵ . Our results show that this new algorithm is capable of significantly surpassing others in literature where its performance accurately traces that of the standard centralized Newton approach. Next, we detail our distributed SDD solver being at the core of our approximator.

IV. SDD DISTRIBUTED SOLVERS

We propose a distributed solver for SDDM systems which can be used to determine an approximation to the Newton direction up to any arbitrary $\epsilon > 0$ (see Section V). Our method is based on a distributed implementation of the parallel solver of Section II-C. Similar to [11], we first introduce an approximate inverse chain which can be computed in a distributed fashion. This leads us to a distributed version of the "crude" solver (i.e., Algorithm II-C). Contrary to [11], however, we then generalize the "crude" distributed solver to acquire *exact* solutions to an SDDM system. For a generic SDDM system of linear equations, our main results for determining an ϵ -close solution (i.e., $||\tilde{x} - x^*||_{M_0} \leq \epsilon ||x^*||_{M_0}$) is summarized by³:

³The complete proofs can be found at https://db.tt/MbBW15Zx

Lemma 9: For the system of equations represented by $M_0 x = b$, there is a distributed algorithm that uses only R-Hop information and computes the ϵ -close solution, \tilde{x} , in $T(n, \epsilon) = \mathcal{O}\left(\left(\frac{\beta\kappa(M_0)}{R} + \beta d_{\max}R\right)\log\left(\frac{1}{\epsilon}\right)\right)$ time steps, with $\kappa(M_0)$ being the condition number of M_0 , $\beta = \min\left\{n, \frac{d_{\max}^{n-1}-1}{d_{\max}-1}\right\}$ representing the upper bound on the size of the R-Hop neighborhood, d_{\max} the maximal degree of \mathcal{G} , and $\epsilon \in (0, \frac{1}{2}]$ being the precision parameter.

Analogous to [11], we will develop and analyze two distributed solvers for SDDM systems (i.e., "crude" R-Hop solver and "exact" R-Hop solver) leading to the proof of the above lemma.

A. "Crude" R-Hop SDDM Solver

Algorithm 3 presents the "crude" R-Hop solver for SDDM systems. Each node receives the k^{th} row of M_0 , k^{th} component, $[b_0]_k$ of b_0 , the length of the inverse chain, d, and the local communication bound⁴ R as inputs, and outputs the k^{th} component of the "rude" approximation of x^* .

Analysis of Algorithm 3 The following Lemma shows that RDistRSolve computes the k^{th} component of the "crude" approximation of x^* and provides the algorithm's time complexity

Lemma 10: Let $M_0 = D_0 - A_0$ be the standard splitting and let Z'_0 be the operator defined by RDistRSolve, namely, $x_0 = Z'_0 b_0$. Then, $Z'_0 \approx_{\epsilon_d} M_0^{-1}$. RDistRSolve requires $\mathcal{O}\left(\frac{2^d}{R}\beta + \beta R d_{max}\right)$, where $\beta = \min\left\{n, \frac{(d_{max}^{R+1}-1)}{(d_{max}-1)}\right\}$, to arrive at x_0 .

Proof: See Appendix.

B. "Exact" Distributed R-Hop SDDM Solver

Next, we provide the exact R-Hop solver. Similar to RDistRSolve, each node v_k receives the k^{th} row M_0 , $[b_0]_k$, d, R, and a precision parameter ϵ as inputs, and outputs the k^{th} component of the ϵ close approximation of vector x^* .

Analysis of Algorithm 6: The following Lemma shows that EDistRSolve computes the k^{th} component of the ϵ close approximation to x^* and provides the time complexity analysis.

Lemma 11: Let $M_0 = D_0 - A_0$ be the standard splitting. Further, let $\epsilon_d < \frac{1}{3} \ln 2$. Then Algorithm 6 requires $\mathcal{O}\left(\log \frac{1}{\epsilon}\right)$ iterations to return the k^{th} component of the ϵ close approximation to \mathbf{x}^* .

Proof: See Appendix.

Next, the following Lemma provides the time complexity analysis of EDistRSolve.

Lemma 12: Let $M_0 = D_0 - A_0$ be the standard splitting and let $\epsilon_d < \frac{1}{3} \ln 2$, then EDistRSolve requires $\mathcal{O}\left(\left(\frac{2^d}{R\beta} + \beta R d_{max}\right) \log\left(\frac{1}{\epsilon}\right)\right)$ time steps. Moreover, for each node v_k , EDistRSolve only uses information from the R-hop neighbors.

The complexity of the proposed algorithms depend on the length of the inverse approximated chain, d. Here, we provide

Algorithm 3 RDistRSolve ({ $[M_0]_{k1}, \ldots, [M_0]_{kn}$ }, $[b_0]_k, d, R$)

rart One:
$\{[A_0D_0^{-1}]_{k1},\ldots,[A_0D_0^{-1}]_{kn}\}=\left\{rac{[A_0]_{k1}}{[D_0]_{11}},\ldots,rac{[A_0]_{kn}}{[D_0]_{nn}} ight\}$
$\{[\boldsymbol{D}_0^{-1}\boldsymbol{A}_0]_{k1},\ldots,[\boldsymbol{D}_0^{-1}\boldsymbol{A}_0]_{kn}\}=\{\frac{[\boldsymbol{A}_0]_{k1}}{[\boldsymbol{D}_0]_{kk}},\ldots,\frac{[\boldsymbol{A}_0]_{kn}}{[\boldsymbol{D}_0]_{kk}}\}$
$[C_0]_{k1},\ldots,[C_0]_{kn} = \operatorname{Comp}_0([M_0]_{k1},\ldots,[M_0]_{kn},R)$
$[C_1]_{k1}, \ldots, [C_1]_{kn} = \operatorname{Comp}_1([M_0]_{k1}, \ldots, [M_0]_{kn}, R)$

Part Two:

for i = 1 to d do if $i - 1 < \rho$ $[u_1^{(i-1)}]_k = [A_0 D_0^{-1} b_{i-1}]_k$ for j = 2 to 2^{i-1} do $[u_j^{(i-1)}]_k = [A_0 D_0^{-1} u_{j-1}^{(i-1)}]_k$ end for $[b_i]_k = [b_{i-1}]_k + [u_{2^{i-1}}^{(i-1)}]_k$ if $i - 1 \ge \rho$ $l_{i-1} = 2^{i-1}/R$ $[u_1^{(i-1)}]_k = [C_0 b_{i-1}]_k$ for j = 2 to l_{i-1} do $[u_j^{(i-1)}]_k = [C_0 u_{j-1}^{(i-1)}]_k$ end for $[b_i]_k = [b_{i-1}]_k + [u_{l_{i-1}}^{(i-1)}]_k$ end for

Part Three:

$$\begin{split} & [\mathbf{x}_{d}]_{k} = [\mathbf{b}_{d}]_{k}/[\mathbf{D}_{0}]_{kk} \\ & \mathbf{for} \ i = d - 1 \ \mathbf{to} \ 1 \ \mathbf{do} \\ & \mathbf{if} \ i < \rho \\ & [\boldsymbol{\eta}_{1}^{(i+1)}]_{k} = [\mathbf{D}_{0}^{-1} \mathbf{A}_{0} \mathbf{x}_{i+1}]_{k} \\ & \mathbf{for} \ j = 2 \ \mathbf{to} \ 2^{i} \ \mathbf{do} \\ & [\boldsymbol{\eta}_{j}^{(i+1)}]_{k} = [\mathbf{D}_{0}^{-1} \mathbf{A}_{0} \boldsymbol{\eta}_{j-1}^{(i+1)}]_{k} \\ & \mathbf{end} \ \mathbf{for} \\ & [\mathbf{x}_{i}]_{k} = \frac{1}{2} \left[\frac{[\mathbf{b}_{i}]_{k}}{[\mathbf{D}_{0}]_{kk}} + [\mathbf{x}_{i+1}]_{k} + [\boldsymbol{\eta}_{2^{i}}^{i+1}]_{k} \right] \\ & \mathbf{if} \ i \ge \rho \\ & l_{i} = 2^{i}/R \\ & [\boldsymbol{\eta}_{1}^{(i+1)}]_{k} = [\mathbf{C}_{1} \mathbf{x}_{i+1}]_{k} \\ & \mathbf{for} \ j = 2 \ \mathrm{to} \ l_{i} \ \mathbf{do} \\ & [\boldsymbol{\eta}_{j}^{(i+1)}]_{k} = [\mathbf{C}_{1} \boldsymbol{\eta}_{j-1}^{(i+1)}]_{k} \\ & \mathbf{end} \ \mathbf{for} \\ & [\mathbf{x}_{i}]_{k} = \frac{1}{2} \left[\frac{[\mathbf{b}_{i}]_{k}}{[\mathbf{D}_{0}]_{kk}} + [\mathbf{x}_{i+1}]_{k} + [\boldsymbol{\eta}_{l_{i}}^{i+1}]_{k} \right] \\ & \mathbf{end} \ \mathbf{for} \\ & [\mathbf{x}_{0}]_{k} = \frac{1}{2} \left[\frac{[\mathbf{b}_{0}]_{k}}{[\mathbf{D}_{0}]_{kk}} + [\mathbf{x}_{1}]_{k} + [\mathbf{D}_{0}^{-1}\mathbf{A}_{0}\mathbf{x}_{1}]_{k} \right] \\ & \mathbf{return} \ [\mathbf{x}_{0}]_{k} \end{split}$$

an analysis to determine the value of d which guarantees $\epsilon_d < \frac{1}{3} \ln 2$ in $C = \{A_0, D_0, A_1, D_1, \dots, A_d, D_d\}$. These results are summarized the following lemma

Lemma 13: Let $M_0 = D_0 - A_0$ be the standard splitting and let κ denote the condition number of M_0 . Consider the inverse approximated chain $\mathcal{C} = \{A_0, D_0, A_1, D_1, \dots, A_d, D_d\}$ with length $d = \lceil \log \left(2 \ln \left(\frac{\sqrt[3]{2}}{\sqrt[3]{2}-1}\right) \kappa\right) \rceil$, then $D_0 \approx_{\epsilon_d} D_0 - D_0 \left(D_0^{-1}A_0\right)^{2^d}$, with $\epsilon_d < \frac{1}{3} \ln 2$.

⁴For simplicity, R is assumed to be in the order of powers of 2, i.e., $R = 2^{\rho}$.

Algorithm 4 Comp₀ ([
$$M_0$$
]_{k1},..., [M_0]_{kn}, R)
for $l = 1$ to $R - 1$ do
for j s.t. $v_j \in \mathbb{N}_{l+1}(v_k)$ do
 $[(A_0D_0^{-1})^{l+1}]_{kj} = \sum_{r:v_r \in \mathbb{N}_1(v_j)} \frac{[D_0]_{jr}}{[D_0]_{jj}} [(A_0D_0^{-1})^l]_{kr} [A_0D_0^{-1}]_{jr}$
end for
end for
return $c_0 = \{[(A_0D_0^{-1})^R]_{k1}, \dots, [(A_0D_0^{-1})^R]_{kn}\}$

Algorithm 5 Comp₁([
$$M_0$$
]_{k1},..., [M_0]_{kn}, R)
for $l = 1$ to $R - 1$ do
for j s.t. $v_j \in \mathbb{N}_{l+1}(v_k)$ do
 $[(D_0^{-1}A_0)^{l+1}]_{kj}$ =
 $\sum_{r:v_r \in \mathbb{N}_1(v_j)} \frac{[D_0]_{jj}}{[D_0]_{rr}} [(D_0^{-1}A_0)^l]_{kr} [D_0^{-1}A_0]_{jr}$
end for
end for
return $c_1 = \{[(D_0^{-1}A_0)^R]_{k1}, ..., [(D_0^{-1}A_0)^R]_{kn}\}$

Proof: See Appendix.

Combining the above results finalizes the proof of Lemma 9. The usage of this distributed solver to approximate the Newton direction, as detailed in the next section, enables fast and accurate distributed Newton methods capable of approximating centralized Newton directions up to any arbitrary ϵ .

V. FAST & ACCURATE DISTRIBUTED NEWTON METHOD

Our approach only requires R-Hop communication for the distributed approximation of the Newton direction. Given the results of Lemma 7, we can determine the approximate Newton direction by solving a system of linear equations represented by an SDD matrix⁵ according to Section IV, with $M_0 = H_k = H(\lambda_k)$.

Formally, we consider the following iteration scheme:

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{\lambda}_k + \alpha_k \tilde{\boldsymbol{d}}_k, \tag{13}$$

with k representing the iteration number, α_k the step-size, and \tilde{d}_k denoting the approximate Newton direction. We

⁵Due to space constraints, we refrain some of the proofs to the appendix.

Algorithm 6 EDistRSolve ({
$$[M_0]_{k1}, \ldots, [M_0]_{kn}$$
}, $[b_0]_k, d, R, e$

 Initialize:
 $[y_0]_k$
 =
 0, and $[\chi]_k$
 =

 RDistRSolve({ $[M_0]_{k1}, \ldots, [M_0]_{kn}$ }, $[b_0]_k, d, R$)
 for $t = 1$ to q do
 [$u_t^{(1)}]_k = [D_0]_{kk}[y_{t-1}]_k - \sum_{j:v_j \in \mathbb{N}_1(v_k)} [A_0]_{kj}[y_{t-1}]_j$
 $[u_t^{(2)}]_k = \text{RDistRSolve}({[M_0]_{k1}, \ldots, [M_0]_{kn}}, [u_t^{(1)}]_k, d, R)$
 $[y_t]_k = [y_{t-1}]_k - [u_t^{(2)}]_k + [\chi]_k$

 end forend for

 return $[\tilde{x}]_k = [y_a]_k$

determine d_k by solving $H_k d_k = -g_k$ using Algorithm 6. It is easy to see that our approximation of the Newton direction, \tilde{d}_k , satisfies

$$egin{aligned} || d_k - d_k ||_{oldsymbol{H}_k} &\leq \epsilon || d_k ||_{oldsymbol{H}_k} \ ext{with} \quad \widetilde{d}_k &= - oldsymbol{Z}_k oldsymbol{g}_k, \end{aligned}$$

where Z_k approximates H_k^{\dagger} according to the routine of Algorithm 6. The accuracy of this approximation is quantified in the following Lemma

Lemma 14: Let $H_k = H(\lambda_k)$ be the Hessian of the dual function, then for any arbitrary $\epsilon > 0$ we have

$$e^{-\epsilon^{2}} \boldsymbol{v}^{\mathsf{T}} \boldsymbol{H}_{k}^{\dagger} \boldsymbol{v} \leq \boldsymbol{v}^{\mathsf{T}} \boldsymbol{Z}_{k} \boldsymbol{v} \leq e^{\epsilon^{2}} \boldsymbol{v}^{\mathsf{T}} \boldsymbol{H}_{k}^{\dagger} \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \mathbf{1}^{\perp}.$$
Proof: See Appendix.

Given such an accurate approximation, next we analyze the iteration scheme of our proposed method showing that similar to standard Newton methods, we achieve superlinear convergence within a neighborhood of the optimal value. We start by analyzing the change in the Laplacian norm of the gradient between two successive iterations

Lemma 15: Consider the following iteration scheme $\lambda_{k+1} = \lambda_k + \alpha_k \tilde{d}_k$ with $\alpha_k \in (0, 1]$, then, for any arbitrary $\epsilon > 0$, the Laplacian norm of the gradient, $||g_{k+1}||_{\mathcal{L}}$, follows:

$$||\boldsymbol{g}_{k+1}||_{\mathcal{L}} \leq \left[1 - \alpha_k + \alpha_k \epsilon \frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right] ||\boldsymbol{g}_k||_{\mathcal{L}}$$
(14)
$$+ \frac{\alpha_k^2 B \Gamma^2 (1+\epsilon)^2}{2\mu_2^2(\mathcal{L})} ||\boldsymbol{g}_k||_{\mathcal{L}}^2,$$

with $\mu_n(\mathcal{L})$ and $\mu_2(\mathcal{L})$ being the largest and second smallest eigenvalues of \mathcal{L} , Γ and γ denoting the upper and lower bounds on the dual's Hessian, and $B \in \mathbb{R}$ is defined in Lemma 8.

Proof: See Appendix. At this stage, we are ready to present the main results quantifying the convergence phases exhibited by our approach:

Theorem 1: Let γ , Γ , B be the constants defined in Assumption 2 and Lemma 7, $\mu_n(\mathcal{L})$ and $\mu_2(\mathcal{L})$ representing the largest and second smallest eigenvalues of the normalized laplacian \mathcal{L} , $\epsilon \in \left(0, \frac{\mu_2(\mathcal{L})}{\mu_n(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right)$ the precision parameter for the SDDM (Section IV) solver, and letting the optimal stepsize parameter $\alpha^* = \frac{e^{-\epsilon^2}}{(1+\epsilon)^2} \left(\frac{\gamma}{\Gamma} \frac{\mu_2(\mathcal{L})}{\mu_n(\mathcal{L})}\right)^2$. Then the proposed algorithm given by the $\lambda_{k+1} = \lambda_k + \alpha^* \tilde{d}_k$ exhibits the following three phases of convergence:

1) Strict Decreases Phase: While $||g_k||_{\mathcal{L}} \ge \eta_1$:

$$q(\boldsymbol{\lambda}_{k+1}) - q(\boldsymbol{\lambda}_k) \le -\frac{1}{2} \frac{e^{-2\epsilon^2}}{(1+\epsilon)^2} \frac{\gamma^3}{\Gamma^2} \frac{\mu_2^2(\mathcal{L})}{\mu_n^4(\mathcal{L})} \eta_1^2.$$

2) Quadratic Decrease Phase: While $\eta_0 \leq ||g_k||_{\mathcal{L}} \eta_1$:

$$||\boldsymbol{g}_{k+1}||_{\mathcal{L}} \leq rac{1}{\eta_1}||\boldsymbol{g}_k||_{\mathcal{L}}^2.$$

3) Terminal Phase: When $||\boldsymbol{g}_k||_{\mathcal{L}} \leq \eta_0$:

$$||\boldsymbol{g}_{k+1}||_{\mathcal{L}} \leq \sqrt{\left[1 - \alpha^* + \alpha^* \epsilon \frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right]} ||\boldsymbol{g}_k||_{\mathcal{L}},$$

where $\eta_0 = \frac{\boldsymbol{\xi}(1-\boldsymbol{\xi})}{\zeta}$ and $\eta_1 = \frac{1-\boldsymbol{\xi}}{\zeta}$, with $\boldsymbol{\xi} = \sqrt{\left[1 - \alpha^* + \alpha^* \epsilon \frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right]}$ (15)
$$\begin{split} \zeta &= \frac{B(\alpha^* \Gamma(1+\epsilon))^2}{2\mu_2^2(\mathcal{L})}\\ \textit{Proof:} \quad \text{We will proof the above theorem by handling} \end{split}$$

each of the cases separately. We start by considering the case when $||g_k||_{\mathcal{L}} > \eta_1$ (i.e., Strict Decrease Phase). We have:

$$q(\lambda_{k+1}) = q(\lambda_k) + \boldsymbol{g}_k^{\mathsf{T}}(\boldsymbol{\lambda}_{k+1} - \boldsymbol{\lambda}_k) \\ + \frac{1}{2}(\boldsymbol{\lambda}_{k+1} - \boldsymbol{\lambda}_k)^{\mathsf{T}}\boldsymbol{H}(\boldsymbol{z})(\lambda_{k+1} - \boldsymbol{\lambda}_k) \\ = q(\boldsymbol{\lambda}_k) + \alpha_k \boldsymbol{g}_k^{\mathsf{T}} \tilde{\boldsymbol{d}}_k + \frac{\alpha_k^2}{2} \tilde{\boldsymbol{d}}_k^{\mathsf{T}} \boldsymbol{H}(\boldsymbol{z}) \tilde{\boldsymbol{d}}_k \\ \leq q(\boldsymbol{\lambda}_k) + \alpha_k \boldsymbol{g}_k^{\mathsf{T}} \tilde{\boldsymbol{d}}_k + \frac{\alpha_k^2}{2\gamma} \tilde{\boldsymbol{d}}_k^{\mathsf{T}} \mathcal{L} \tilde{\boldsymbol{d}}_k,$$

where the last steps holds since $H(\cdot) \leq \frac{1}{\gamma} \mathcal{L}$. Noticing that $||\tilde{d}_k||_{\mathcal{L}}^2 \leq \frac{\Gamma^2(1+\epsilon)^2}{\mu_2^2(\mathcal{L})}||g_k||_{\mathcal{L}}^2$ (see Appendix), the only remaining step needed is to evaluate $g_k^{\mathsf{T}} \tilde{d}_k$. Knowing that $d_k = -Z_k g_k$, we recognize

$$\begin{split} \boldsymbol{g}_{k}^{\mathsf{T}} \tilde{\boldsymbol{d}}_{k} &= -\boldsymbol{g}_{k}^{\mathsf{T}} \boldsymbol{Z}_{k} \boldsymbol{g}_{k} \leq e^{-\epsilon^{2}} \boldsymbol{g}_{k}^{\mathsf{T}} \boldsymbol{H}_{k}^{\dagger} \boldsymbol{g}_{k} \text{ (Lemma 14)} \\ &\leq -\frac{e^{-\epsilon^{2}}}{\mu_{n}(\boldsymbol{H}_{k})} \boldsymbol{g}_{k}^{\mathsf{T}} \boldsymbol{g}_{k} \leq -\frac{e^{-\epsilon}}{\mu_{n}(\mathcal{L})} \boldsymbol{g}_{k}^{\mathsf{T}} \boldsymbol{g}_{k} \\ &\leq -\frac{e^{-\epsilon^{2}} \gamma}{\mu_{n}(\mathcal{L})} \frac{\boldsymbol{g}_{k}^{\mathsf{T}} \mathcal{L} \boldsymbol{g}_{k}}{\mu_{n}(\mathcal{L})} = \frac{e^{-\epsilon^{2}} \gamma}{\mu_{n}^{2}(\mathcal{L})} ||\boldsymbol{g}_{k}||_{\mathcal{L}}^{2}, \end{split}$$

where the last step follows from the fact that $orall v \in \mathbb{R}^n$: $v^{\mathsf{T}}v \geq \frac{v^{\mathsf{T}}\mathcal{L}v}{\mu_n(\mathcal{L})}$. Therefore, we can write

$$q(\boldsymbol{\lambda}_{k+1}) - q(\boldsymbol{\lambda}_k) \leq -\left[\alpha_k \frac{e^{-\epsilon^2} \gamma}{\mu_n^2(\mathcal{L})} - \alpha_k^2 \frac{\Gamma^2(1+\epsilon)^2}{2\gamma \mu_2^2(\mathcal{L})}\right] ||\boldsymbol{g}_k||_{\mathcal{L}}^2.$$

It is easy to see that $\alpha_k = \alpha^* = \frac{e^{-\epsilon^2}}{(1+\epsilon)^2} \left(\frac{\gamma}{\Gamma} \frac{\mu_2(\mathcal{L})}{\mu_n(\mathcal{L})}\right)^2$ minimizes the right-hand-side of the above equation. Using $||g_k||_{\mathcal{L}}$ gives the constant decrement in the dual function between two successive iterations as

$$q(\boldsymbol{\lambda}_{k+1}) - q(\boldsymbol{\lambda}_k) \le -\frac{1}{2} \frac{e^{-2\epsilon^2}}{(1+\epsilon)^2} \frac{\gamma^3}{\Gamma^2} \frac{\mu_2^2(\mathcal{L})}{\mu_n^4(\mathcal{L})} \eta_1^2.$$

Considering the case when $\eta_0 \leq ||\boldsymbol{g}_k||_{\mathcal{L}}^2 \eta_1$ (i.e., Quadratic Decrease Phase), Equation 14 can be rewritten as

$$||\boldsymbol{g}_{k+1}||_{\mathcal{L}} \leq \boldsymbol{\xi}^2 ||\boldsymbol{g}_k||_{\mathcal{L}} + \boldsymbol{\zeta} ||\boldsymbol{g}_k||_{\mathcal{L}}^2,$$

with $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ defined as in Equation 15. Further, noticing that since $||\boldsymbol{g}_k||_{\mathcal{L}} \geq \eta_0$ then $||\boldsymbol{g}_k||_{\mathcal{L}} \leq \frac{1}{\eta_0} ||\boldsymbol{g}_k||_{\mathcal{L}}^2 = \frac{\zeta}{\boldsymbol{\xi}(1-\boldsymbol{\xi})} ||\boldsymbol{g}_k||_{\mathcal{L}}^2$. Consequently the quadratic decrease phase is finalized by

$$\begin{split} ||\boldsymbol{g}_{k+1}||_{\mathcal{L}} &\leq \boldsymbol{\zeta} \left(\frac{\boldsymbol{\xi}}{1-\boldsymbol{\xi}} + 1 \right) ||\boldsymbol{g}_{k}||_{\mathcal{L}}^{2} = \frac{\boldsymbol{\zeta}}{1-\boldsymbol{\xi}} ||\boldsymbol{g}_{k}||_{\mathcal{L}}^{2} \\ &= \frac{1}{\eta_{1}} ||\boldsymbol{g}_{k}||_{\mathcal{L}}^{2}. \end{split}$$

Finally, we handle the case where $||\boldsymbol{g}_k||_{\mathcal{L}} \leq \eta_0$ (i.e., **Terminal Phase**). Since $||g_k||_{\mathcal{L}}^2 \leq \eta_0 ||g_k||_{\mathcal{L}}$, it is easy to see that

$$||\boldsymbol{g}_{k+1}||_{\mathcal{L}} \leq (\boldsymbol{\xi}^2 + \boldsymbol{\zeta}\eta_0)||\boldsymbol{g}_k||_{\mathcal{L}} = (\boldsymbol{\xi}^2 + \boldsymbol{\xi}(1 - \boldsymbol{\xi}))||\boldsymbol{g}_k||_{\mathcal{L}}$$
$$\boldsymbol{\xi}||\boldsymbol{g}_k||_{\mathcal{L}} = \sqrt{\left[1 - \alpha^* + \alpha^* \epsilon \frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right]}||\boldsymbol{g}_k||_{\mathcal{L}}.$$

Having proved the three convergence phases of our algorithm, we next analyze the number of iterations needed by each phase. These results are summarized in the following lemma:

Lemma 16: Consider the algorithm given by the following iteration protocol: $\lambda_{k+1} = \lambda_{k+1} + \alpha^* d_k$. Let λ_0 be the initial value of the dual variable, and q^* be the optimal value of the dual function. Then, the number of iterations needed by each of the three phases satisfy:

1) The strict decrease phase requires the following number iterations to achieve the quadratic phase:

$$N_{1} \leq C_{1} \frac{\mu_{n}(\mathcal{L})^{2}}{\mu_{2}^{2}(\mathcal{L})} \left[1 - \epsilon \frac{\mu_{n}(\mathcal{L})}{\mu_{2}(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}} \right]^{-2},$$

where $C_{1} = C_{1}(\epsilon, \gamma, \Gamma, \delta, q(\lambda_{0}), q^{\star}) = 2\delta^{2}(1 + \epsilon)^{2} \left[q(\lambda_{0}) - q^{\star}\right] \frac{\Gamma^{2}}{2}.$

2) The quadratic decrease phase requires the following number of iterations to terminate:

$$N_2 = \log_2\left[\frac{\frac{1}{2}\log_2\left(\left[1 - \alpha^*\left(1 - \epsilon\frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})}\sqrt{\frac{\Gamma}{\gamma}}\right)\right]\right)}{\log_2(r)}\right],$$

where $r = \frac{1}{\eta_1} || g_{k'} ||_{\mathcal{L}}$, with k' being the first iteration of the quadratic decrease phase.

3) The radius of the **terminal phase** is characterized by:

$$p_{\text{terminal}} \leq \frac{2\left[1 - \epsilon \frac{\mu_n(\mathcal{L})}{\mu_2(\mathcal{L})} \sqrt{\frac{\Gamma}{\gamma}}\right]}{e^{-\epsilon^2 \gamma \delta}} \mu_n(\mathcal{L}) \sqrt{\mu_2(\mathcal{L})}.$$

See Appendix.

Proof: above the result, Given the total

1

message complexity can then be derived as $\mathcal{O}\left(\left(N_1+N_2\right)n\beta\left(\kappa(\boldsymbol{H}_k)\frac{1}{R}+Rd_{\max}\right)\log\left(\frac{1}{\epsilon}\right)\right).$

VI. EXPERIMENTS AND RESULTS

We evaluated our approach on two randomly generated networks. The first consisted of 30 nodes and 70 edges, while the second contained 90 nodes with 200 edges. The edges were chosen uniformly at random. The flow vectors, **b**, were chosen to place source and sink nodes $diam(\mathcal{G})$ away from each other. An ϵ of $\frac{1}{10,000}$, a gradient threshold of 10^{-10} , and an R-Hop of 1 were provided to our SDDM solver for determining the approximate Newton direction. We compared the performance of our algorithm, referred to SDDM-ADD hereafter, to ADD, standard gradient descent, and the exact Newton method (i.e., centralized Newton iterations). The values of the primal objective and feasibility were chosen as performance metric.



(a) ||Ax - b|| on a random network (b) $f(x_k)$ on a random network with (c) $||Ax_k - b||$ on a random network(d) $f(x_k)$ on a random network with with 30 nodes and 70 edges 30 nodes and 70 edges with 90 nodes and 200 edges 90 nodes and 200 edges

Fig. 1. Performance metrics on two randomly generated networks, showing the primal objective, $f(x_k)$, and feasibility $||Ax_k - b||$ as a function of the number of iterations k. On a relatively small network (i.e., 30 nodes and 70 edges) we outperform ADD and gradient descent by approximately an order of magnitude. On larger networks (i.e., 90 nodes and 200 edges), SDDM-ADD is superior to both ADD and gradient descent, where the primal objective of the latter two algorithms converges to 10^5 after 3000 iterations. It is also worth noting that we perform closely to the exact Newton method computed according to a centralized approach.

Figure 1 shows these convergence metrics comparing SDDM-ADD, to ADD [6], standard gradient descent, and the exact Newton method (i.e., centralized Newton iteration). On relatively small networks, 30 nodes and 70 edges, our approach converges approximately an order of magnitude faster compared to both ADD and gradient descent as demonstrated in Figures 1(a) and 1(b). It is also clear that on such networks, SDDM-ADD is capable of closely tracing the exact Newton method where convergence to the optimal primal objective is achieved after ≈ 200 iterations compared to ≈ 500 for ADD and ≈ 2000 for gradient descent.

In the second set of experiments that goal was to evaluate the performance of SDDM-ADD on large networks where both ADD and gradient descent underperform. Results reported in Figures 1(c) and 1(d) on the larger 90 nodes and 200 edges network clearly demonstrate the effectiveness of our approach. Benefiting from the approximation accuracy of the Newton direction, SDDM-ADD is capable of significantly outperforming state-of-the-art methods. As shown in Figure 1(d) convergence to the optimal solution (as computed by exact Newton iterations) is achieved after 3000 iterations, while ADD and gradient descent underperform by converging to a primal value of 10^5 .

VII. CONCLUSIONS

In this paper we proposed a fast and accurate distributed Newton method for network flow optimization problems. Our approach utilizes the sparsity pattern of the dual Hessian to approximate the Newton direction using only local information. We achieve ϵ -close approximations by proposing a novel distributed solver for symmetric diagonally dominant systems of linear equations involving M-matrices. Our solver provides a distributed implementation of the algorithm of Spielam and Peng by considering an approximate inverse chain that can be computed in a distributed fashion.

The proposed approximate Newton method utilizes the distributed solver to obtain ϵ -close approximations to the exact Newton direction up-to any arbitrary $\epsilon > 0$. We further analyzed the properties of the resulting approximate algorithm showing that, similar to conventional Newton methods, superlinear convergence within a neighborhood of

the optimal value can be attained. Finally, we demonstrated the effectiveness of our method in a set of experiments on randomly generated networks. Results showed that on both small and large networks, our algorithm, outperforms stateof-the-art techniques in a variety of convergence metrics.

Possible extensions include applications to network utility maximization [7], general wireless communication optimization problems [14], and stochastic settings [15].

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APPENDIX

The complete proofs can be found at: https://db.tt/MbBW15Zx