Time minimum synthesis for a kinematic drone model

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***Abstract*— In this paper, we consider a (rough) kinematic model for a UAV flying at constant altitude moving forward with positive lower and upper bounded linear velocities and positive minimum turning radius. For this model, we consider the problem of minimizing the time travelled by the UAV starting from a general configuration to connect a specified target being a fixed circle of minimum turning radius. The time-optimal synthesis, presented as a partition of the state space, defines a unique optimal path such that the target can be reached optimally.**

I. INTRODUCTION

The purpose of this study is to determine the fastest way (in time) to steer a kinematic UAV (or drone) flying at a constant altitude from some starting point to a fixed horizontal circle of minimum turning radius.

The problem is only described from a kinematic point of view. In particular, we do not take into account the inertia of the drone. We consider that the drone velocities are controlled parameters. In consequence, they are allowed to vary arbitrarily fast.

From the kinematic point of view, a rough drone that flies at a constant altitude is governed by the standard Dubins equations (see e.g. [2], [7]):

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta\,, \\ \dot{\theta} = u \end{cases} \tag{1}$$

with $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ being the state (where $(x, y) \in \mathbb{R}^2$ is the UAV's coordinates in the constant altitude plane, and θ the yaw angle), and $u \in [-u_{\max}, u_{\max}]$, $v \in [v_{\min}, v_{\max}]$ being the control variables. Note that the yaw angle θ is the angle between the aircraft direction and the x-axis.

These equations express that the drone moves on a perfect plane (perfect constant altitude) in the direction of its velocity vector and is able to turn right and left.

We assume that the controls on the drone kinematics are its angular velocity u and its linear velocity v.

Moreover, we make the assumptions that the linear velocity v has a positive lower bound $v_{\min}$ and a positive upper bound $v_{\max}$ and that the time derivative u of the drone yaw angle is constrained by an upper positive bound $u_{\max}$.

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The above assumptions imply in particular that no stationary or quasi-stationary flights are allowed and that the drone is kinematically restricted by its minimum turning radius $r_{\min} = v_{\min}/u_{\max} > 0$.

A similar problem with a constant linear velocity has already been addressed in [8]. The purpose of this paper is to study the influence of a non-constant linear velocity.

Due to space limitations, most of the proofs of the results will be reported elsewhere.

II. MINIMUM TIME PROBLEM UNDER CONSIDERATION

A. Optimal control problem

We aim to steer a UAV driven by system (1) in minimum time from any given initial position point to the target manifold $\mathcal{C}$ which is defined to be the counterclockwise-oriented circular trajectory of minimum turning radius centered at the origin. In the (x, y, θ)-coordinates, $\mathcal{C}$ is given by

$$\mathcal{C} = \{(x, y, \theta) \mid x = r_{\min}\sin\theta,\ y = -r_{\min}\cos\theta\}\,.$$

More precisely, we consider the following optimal control problem :

($\mathbf{P_0}$) For every $(x_0, y_0, \theta_0) \in \mathbb{R}^2 \times \mathbb{S}^1$ find a pair trajectory-control joining (x_0, y_0, θ_0) to $\mathcal{C}$, which is time-optimal for the control system (1).

B. Existence of solutions

The following two propositions are well-known and stated without proof (see, e.g. [1]).

Proposition 2.1 (Controlability): System (1) is controllable provided that $u \in [-u_{\max}, u_{\max}]$ and $v \in [v_{\min}, v_{\max}]$ for any choice of $0 < u_{\max} \leqslant +\infty$ and $0 < v_{\min} \leqslant v_{\max} \leqslant +\infty$.

Also, Filippov's theorem gives:

Proposition 2.2 (Existence of minimizers): For any point $(x_0, y_0, \theta_0) \in \mathbb{R}^2 \times \mathbb{S}^1$, there exists a time-optimal trajectory joining (x_0, y_0, θ_0) to $\mathcal{C}$.

C. Reduction of the system

To solve problem **($\mathbf{P_0}$)** it is convenient to work with a reduced system in dimension two. Indeed, in dimension two, a complete theory for finding time-optimal synthesis exists and will be described in Section III.

For this purpose, we introduce the UAV-based coordinates $(\tilde{x}, \tilde{y}, \theta)$ with $\tilde{x}$ and $\tilde{y}$ defined by the transformation (in $SO(2)$):

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We also define the control set $\mathcal{U} = [-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] \subset \mathbb{R}^2$.

The main advantage of this UAV-based coordinate system is that it decouples the variable θ and projects the final manifold $\mathcal{C}$ to the point $\tilde{X}_0 = (0, -r_{\min})$ so that the original time-optimal control problem can be equivalently reformulated in the reduced state space $(\tilde{x}, \tilde{y})$ as the following minimum-time problem :

(P₁) For every $(\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^2$ find a pair trajectory-control joining $(\tilde{x}_0, \tilde{y}_0)$ to $\tilde{X}_0 = (0, -r_{\min})$, which is time-optimal for the control system

$$\begin{cases} \dot{\tilde{x}} = v + u\tilde{y} \\ \dot{\tilde{y}} = -u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}. \tag{2}$$

Remark 2.3: One can read, in equation (2) that $(0, -r_{\min})$ is an equilibrium point corresponding to the control values $u = u_{\max}$ and $v = v_{\min}$.

The collection of all solutions to problem $\mathbf{(P_1)}$ for every $(\tilde{x}_0, \tilde{y}_0)$ is called the *time-optimal synthesis*.

Following a standard approach for time optimal control syntheses, it is convenient to rephrase problem $\mathbf{(P_1)}$ backward in time. Hence, the sign of the dynamics is changed and the equivalent problem of finding the *time-optimal synthesis* issued from $\tilde{X}_0$ is considered:

(P₂) For every $(\tilde{x}_f, \tilde{y}_f) \in \mathbb{R}^2$ find a pair trajectory-control joining $\tilde{X}_0 = (0, -r_{\min})$ to $(\tilde{x}_f, \tilde{y}_f)$, which is time-optimal for the control system

$$\begin{cases} \dot{\tilde{x}} = -v - u\tilde{y} \\ \dot{\tilde{y}} = u\tilde{x} \end{cases}, \quad (u, v) \in \mathcal{U}. \tag{3}$$

Once problem $\mathbf{(P_2)}$ is solved, then the time-optimal synthesis (solution of problem $\mathbf{(P_1)}$) is obtained simply by following the travelled trajectories backward.

Remark 2.4: Note that up to a dilation in the (x, y)-plane and a dilation of time (a time-reparametrization with constant derivative), we may assume that $[-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] = [-1, 1] \times [1, \mu]$ ($\mu = v_{\max}/v_{\min}$). This normalization could be used to simplify the treatment. We do this in Sections IV and V.

III. Time-optimal synthesis on $\mathbb{R}^2$

In this section, following the same ideas as those developed by Boscain, Bressan, Piccoli and Sussmann in [4], [5], [6], [9], [10] for optimal syntheses on two-dimensional manifolds for single input control systems, we introduce important definitions and develop basic facts about optimal syntheses on $\mathbb{R}^2$ for control-affine systems with two bounded controls of the form (4) (which are different from those studied in [3]). This part is widely inspired by the book [5] and extends some of its results.

The definitions and results given in the next Subsections III-A, III-B and III-C are given in $\mathbb{R}^n$.

A. Pontryagin Maximum Principle

Let F and G be two smooth and complete vector fields on $\mathbb{R}^n$. Define the control variable $U = (u, v)$ and the control set $\mathcal{U} = [-u_{\max}, u_{\max}] \times [v_{\min}, v_{\max}] \subset \mathbb{R}^2$, with $u_{\max}$, $v_{\min}$, $v_{\max}$ assumed to be positive real numbers. Consider the following general control-affine time-optimal problem:

(P) For every X_0 and X_f in $\mathbb{R}^n$ find the pair trajectory-control joining X_0 to X_f, which is time-optimal for the control system

$$\dot{X} = vF(X) + uG(X), \quad X \in \mathbb{R}^n, \quad (u, v) \in \mathcal{U}. \tag{4}$$

Definition 3.1 (admissible control/trajectory): An *admissible control* for system (4) is an essentially bounded function $U(\cdot) : [t_0, t_1] \to \mathcal{U}$. An *admissible trajectory* is a solution to $\dot{X}(t) = v(t)F(X(t)) + u(t)G(X(t))$ a.e. for some admissible control $U(\cdot)$.

Thanks to the compactness of the set of controls, the convexity of the set of velocities, and the completeness of the vector fields, Filippov's theorem (see, for instance, [1]) gives:

Proposition 3.2: For any pair of points in $\mathbb{R}^n$, there exists a time-optimal trajectory joining them.

The main tool to compute time-optimal trajectories is the Pontryagin Maximum Principle (PMP). A general version of PMP can be found in [1]. The following theorem is a version of PMP for control systems of the form (4) that we state in our own context only.

Theorem 3.3 (PMP): Consider the control system (4). For every $(P, X, U) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{U}$, define the Hamiltonian function

$$H(P, X, U) = v \langle P, F(X) \rangle + u \langle P, G(X) \rangle . \tag{5}$$

Let $U(\cdot)$ be a time-optimal control and $X(\cdot)$ the corresponding trajectory defined on $[t_0, t_1]$. Then, there exist a never vanishing Lipschitz covector (or adjoint vector) $P(\cdot) : t \in [t_0, t_1] \mapsto P(t) \in \mathbb{R}^n$ and a non negative constant λ such that for almost all $t \in [t_0, t_1]$:

i. $\dot{X}(t) = \dfrac{\partial H}{\partial P}(P(t), X(t), U(t))$,

ii. $\dot{P}(t) = -\dfrac{\partial H}{\partial X}(P(t), X(t), U(t))$,

iii. $H(P(t), X(t), U(t)) = \max_{W \in \mathcal{U}} H(P(t), X(t), W)$,

iv. $H(P(t), X(t), U(t)) = \lambda \geqslant 0$.

A pair trajectory-control $(X(\cdot), U(\cdot))$ (resp. a triplet $(P(\cdot), X(\cdot), U(\cdot))$) satisfying the conditions given by the PMP is said to be an *extremal trajectory* (resp. an *extremal*). An extremal corresponding to $\lambda = 0$ is said to be an *abnormal extremal*, otherwise we call it a *normal extremal*.

Remark 3.4: Notice that, up to change $U(\cdot)$ on a set of measure zero, an extremal control can always be chosen so that the function $t \mapsto H(P(t), X(t), U(t))$ is continuous. Consequently, we may always assume (without loss of generality) that condition iv of PMP is valid everywhere.

B. Basic definitions

Definition 3.5 (Switching functions): Let $X(\cdot)$ be an extremal trajectory. The corresponding u-switching and v-switching functions are (the C^1 functions) respectively defined as

$$\phi_u(t) = \langle P(t), G(X(t))\rangle \text{ and } \phi_v(t) = \langle P(t), F(X(t))\rangle .$$

Switching functions are very important since their analysis determines when the corresponding control may change.

In the following three definitions (Definions 3.6-3.8), $X(\cdot)$ is an extremal trajectory defined on the time interval $[t_0, t_1]$ and $U(\cdot) : [t_0, t_1] \to \mathcal{U}$ is the corresponding control.

Definition 3.6 (Bang): $U(\cdot)$ is said to be a *u-bang* (resp. *v-bang*) control if, for a.e. $t \in [t_0, t_1]$, $u(t) = -u_{\max}$ (or $u(t) = u_{\max}$) (resp. $v(t) = v_{\min}$ (or $v(t) = v_{\max}$)). $U(\cdot)$ is a *bang* control if, for a.e. $t \in [t_0, t_1]$, it is *u-bang* and *v-bang*. A finite concatenation of bang controls is called a *bang-bang* control.

Definition 3.7 (Singular): We say that $U(\cdot)$ is a *u-singular* control (resp. *v-singular*) if the corresponding switching function ϕ_u (resp. ϕ_v) vanishes identically on $[t_0, t_1]$. If ϕ_u and ϕ_v both vanish identically on $[t_0, t_1]$, we say that $U(\cdot)$ is *totally singular*.

Definition 3.8 (Switching times): A *u-switching time* of $U(\cdot)$ is a time $\tau \in (t_0, t_1)$ such that, for a sufficiently small $\varepsilon > 0$, $u(t) = u_{\max}$ for a.e. $t \in (\tau - \epsilon, \tau]$ and $u(t) = -u_{\max}$ for a.e. $t \in (\tau, \tau + \epsilon]$ or vice-versa. Similarly, we define a v-switching time. A (u, v)-switching time is a time that is both a u- and a v-switching time. If τ is a switching time, the corresponding point $X(\tau)$ on the trajectory $X(\cdot)$ is called a *switching point*.

C. Abnormal trajectories

The following lemma characterizes abnormal extremals of system (4).

Lemma 3.9: Let $\gamma(\cdot) = (P(\cdot), X(\cdot), U(\cdot))$ be an abnormal extremal defined on $[t_0, t_1]$. Let $\tau \in [t_0, t_1]$. Then, there exists $\epsilon > 0$ such that in restriction to each (non empty) interval $(\tau, \tau + \epsilon) \bigcap [t_0, t_1]$ and $(\tau - \epsilon, \tau) \bigcap [t_0, t_1]$, $\gamma(\cdot)$ is either totally singular or bang with $v(t) = v_{\min}$.

Remark 3.10: According to Remark 3.13, generically, on $\mathbb{R}^2$, an abnormal extremal is not totally singular.

D. Singular trajectories

First, let us introduce the functions[1]

$$\begin{aligned}
\Delta_A(X) &= \det(F(X), G(X)),\\
\Delta_{Bu}(X) &= \det(G(X), [F,G](X)),\\
\Delta_{Bv}(X) &= \det(F(X), [F,G](X)),
\end{aligned}$$

whose zero sets are *fundamental loci* (see [5]) in the construction of the optimal synthesis.

The following lemma which is a direct generalization of [5, Theorem 12 page 47].

Lemma 3.11: u-singular (resp. v-singular) trajectories are contained in the set $\Delta_{Bu}^{-1}(0)$ (resp. $\Delta_{Bv}^{-1}(0)$).

[1]If F_1 and F_2 are two vector fields, $[F_1, F_2]$ denotes their Lie bracket.

Lemma 3.12 ((u, v)-singular trajectories): (u, v)-singular trajectories are contained in the set $\Delta_A^{-1}(0) \bigcap \Delta_{Bu}^{-1}(0) \bigcap \Delta_{Bv}^{-1}(0)$.

Remark 3.13: Although it is not addressed here, it can be proved that the intersection $\Delta_A^{-1}(0) \bigcap \Delta_{Bu}^{-1}(0) \bigcap \Delta_{Bv}^{-1}(0)$ is generically empty. In other words, generically, on $\mathbb{R}^2$, there is no totally singular trajectories.

The next lemma describes the kind of switches that may occur along singular arcs.

Lemma 3.14: Along a u-singular trajectory which is not totally singular, v is a.e. equal to $v_{\max}$. Along a v-singular trajectory which is not totally singular, a u-switching cannot occur.

Remark 3.15: When a trajectory enters or exits a u-singular piece v is equal to $v_{\max}$. In a similar way, the control u is the same at the entrance and the exit of a v-singular piece which is not totally singular.

E. Switchings

1) u-switchings:

Lemma 3.16: Along a normal extremal trajectory, a u-switching can occur only if $v = v_{\max}$.

Remark 3.17: The previous lemma implies in particular that, along a normal trajectory, a v-switching from $v_{\max}$ to $v_{\min}$ is necessarily followed by another v-switching from $v_{\min}$ to $v_{\max}$ before a u-switching can occur.

2) (u, v)-switchings:

Lemma 3.18 ((u, v)-switchings): A (u, v)-switching cannot occur along an extremal trajectory.

F. Special domains

Using the sets $\Delta_A^{-1}(0)$, $\Delta_{Bu}^{-1}(0)$ and $\Delta_{Bv}^{-1}(0)$ defined in Section III-D we can define domains in which a control can switch at most one time. Consider a point $X \notin \Delta_A^{-1}(0)$. Then $F(X)$ and $G(X)$ are linearly independent and form a basis. An easy computation shows that

$$[F, G](X) = f(X)F(X) + g(X)G(X), \tag{6}$$

with

$$f(X) = -\frac{\Delta_{Bu}(X)}{\Delta_A(X)} \qquad \text{and} \qquad g(X) = \frac{\Delta_{Bv}(X)}{\Delta_A(X)}.$$

Lemma 3.19: A normal and non-singular trajectory along which $f > 0$ (resp. $f < 0$) admits at most one u-switching and necessarily from $-u_{\max}$ to $u_{\max}$ (resp. from $u_{\max}$ to $-u_{\max}$). Similarly, a normal and non-singular trajectory along which $g > 0$ (resp. $g < 0$) admits at most one v-switching and necessarily from $v_{\max}$ to $v_{\min}$ (resp. from $v_{\min}$ to $v_{\max}$).

IV. Construction of time optimal synthesis for the reduced system

In this section, we apply the results obtained in the previous sections to solve problem $(\mathbf{P_2})$. Although this application is similar to the work done in [8], the resolution is more complicated.

In this section, for the sake of clarity and without loss of generality, we assume that $\mathcal{U} = [-1, 1] \times [v_{\min}, v_{\max}]$.

According to Remark 2.4, it is even possible to rewrite this set with only one parameter $\mu = v_{\max}/v_{\min}$, however we preferred to keep $v_{\min}$ and $v_{\max}$ to facilitate the reading. Note moreover that, in this case, $r_{\min} = v_{\min}$.

A. Pontryagin Maximum Principle

First of all, notice that system (3) is of the form (4) with

$$\tilde{X} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad F(\tilde{X}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ and } \quad G(\tilde{X}) = \begin{pmatrix} -\tilde{y} \\ \tilde{x} \end{pmatrix}.$$

We apply the PMP to $(\mathbf{P_2})$. The Hamiltonian function of PMP is

$$H\big(\tilde{X}, P, U\big) = -vp + u\,(q\tilde{x} - p\tilde{y})\,,$$

with $P = (p, q) \in \mathbb{R}^2$ being the covector. The adjoint system is thus given by

$$\begin{cases} \dot{p}(t) = -u(t)q(t) \\ \dot{q}(t) = u(t)p(t) \end{cases} \tag{7}$$

and the switching functions are

$$\begin{aligned} \phi_u(t) &= q(t)\tilde{x}(t) - p(t)\tilde{y}(t), \\ \phi_v(t) &= -p(t). \end{aligned}$$

The maximality condition of the PMP reads :

$$H\big(\tilde{X}(t), P(t), U(t)\big) = \max_{(u,v)\in\mathcal{U}} \big(u\phi_v(t) + v\phi_u(t)\big) = \lambda$$

and leads to the controls

$$u(t) = \begin{cases} -1 & \text{if } \phi_u(t) < 0 \\ 1 & \text{if } \phi_u(t) > 0 \end{cases}, \; v(t) = \begin{cases} v_{\min} & \text{if } \phi_v(t) < 0 \\ v_{\max} & \text{if } \phi_v(t) > 0 \end{cases}.$$

Remark 4.1: The cases where the switching functions vanish identically is addressed in the next subsection.

B. Singular trajectories

Let us compute the quantities

$$\Delta_A(\tilde{x}, \tilde{y}) = -\tilde{x}, \quad \Delta_{Bu}(\tilde{x}, \tilde{y}) = \tilde{y} \quad \Delta_{Bv}(\tilde{x}, \tilde{y}) = 1,$$
$$f(\tilde{x}, \tilde{y}) = \frac{\tilde{y}}{\tilde{x}}, \quad g(\tilde{x}, \tilde{y}) = -\frac{1}{\tilde{x}}.$$

Lemmas 3.11, 3.12 and 3.14 imply that

- there exists no v-singular trajectory (and consequently no totally singular trajectory) since $\Delta_{Bv}^{-1}(0) = \emptyset$;
- u-singular trajectories are contained in the set[2]

$$\Delta_{Bu}^{-1}(0) = \big\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} = 0\big\}\,.$$

To compute the corresponding control, we differentiate w.r.t. t the function ϕ_u (which is identically zero). A straightforward calculation yields

$$\begin{aligned} \dot{\phi_u}(t) &= v_{\max}\,\langle P(t), [F, G]\,(X(t))\rangle = -v_{\max}q(t) = 0, \\ \ddot{\phi_u}(t) &= u(t)\,\langle P(t), [G, [F, G]]\,(X(t))\rangle = u(t)p(t) = 0, \end{aligned}$$

which implies that, along u-singular trajectories, $u(\cdot)$ vanishes identically. Note that this is quite intuitive since $u = 0$ is the only control that allows the trajectory to stay on the x-axis.

[2]According to [5], the set $\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} = 0\}$ is a turnpike

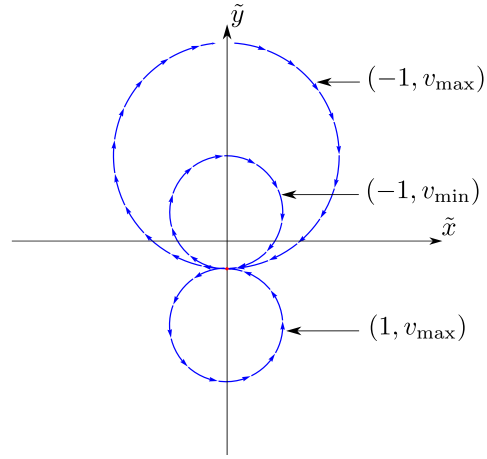

Fig. 1. Candidate extremal trajectories of problem $(\mathbf{P_2})$ issued from $\tilde{X}_0$

C. Optimal synthesis algorithm

Since $\tilde{X}_0$ is an equilibrium point for the control $(1, v_{\min})$ and since $\tilde{X}_0 \notin \Delta_{Bu}^{-1}(0)$, there are, a priori, three possible candidates as optimal starting trajectories corresponding to the bang controls $(1, v_{\max})$, $(-1, v_{\max})$ and $(-1, v_{\min})$ (see Fig. 1).

The time-optimal synthesis is constructed following the 3 steps :

1) For each bang trajectory starting from $\tilde{X}_0$, compute the last time at which the trajectory is extremal (or has lost its optimality by intersecting itself) and study which kind of extremal trajectories can bifurcate from it.
2) For each bang or u-singular trajectory bifurcating from one of the starting trajectories, compute the last time at which it is extremal. If there are intersections among trajectories, we cancel those parts that are not optimal (among trajectories already computed up to this step).
3) For each trajectory computed at the previous step that did not loose its optimality, prolong it with the next bang or singular trajectory up to the last time at which it is extremal. If there are intersections among trajectories, cancel those parts that are not optimal (among trajectories already computed up to this step).

Then, the synthesis is built recursively, repeating step 3) until no new trajectories are generated. In our case, four applications of step 3) are necessary.

D. Expression of the adjoint vector

The starting control u being either 1 or -1, the solution to system (7) with the normalization $|P(0)| = 1$ is

$$\begin{cases} p(t) = \cos{(\alpha + t)} \\ q(t) = \pm\sin{(\alpha + t)}\,, \quad \text{for } u = \pm 1 \end{cases} \tag{8}$$

where α is defined by $P(0) = (\cos\alpha, \pm\sin\alpha)$. The condition PMP iv written at the initial point implies that

$$\left(-v(0) + u(0)v_{\min}\right)p(0) \geqslant 0.$$

Since $(u(0), v(0)) \neq (1, v_{\min})\,,\ (-v(0) + u(0)v_{\min})$ is negative. Hence, $p(0) = \cos\alpha \leqslant 0$ and $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

The following study consists in analyzing the behaviour of the trajectories depending on the value of α.

E. Starting trajectories

This section details the first step (and only this one) of the algorithm described in Section IV-C.

1) Starting with the control $(-1, v_{\max})$*:* The trajectory starting from $\tilde{X}_0$ with the control $(-1, v_{\max})$ has coordinates

$$\begin{cases} \tilde{x}(t) = -\left(v_{\max} + v_{\min}\right)\sin t \\ \tilde{y}(t) = v_{\max} - \left(v_{\max} + v_{\min}\right)\cos t, \end{cases}$$

and from (8), the coordinates of the adjoint vector are

$$\begin{cases} p(t) = \cos\left(t + \alpha\right) \\ q(t) = -\sin\left(t + \alpha\right), \end{cases}$$

with $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. It follows that the switching functions are

$$\begin{aligned} \phi_u(t) &= \left(v_{\max} + v_{\min}\right)\cos\alpha - v_{\max}\cos\left(t + \alpha\right), \\ \phi_v(t) &= -p(t) = -\cos\left(t + \alpha\right). \end{aligned}$$

Recall that the sign of each of these functions determines the value of the corresponding control. It is thus fundamental to study when does a switching function change sign. The analysis of the switching functions yields the following.

- The cases $\alpha = \pi/2$ and $\alpha = 3\pi/2$ cannot occur.
- For $\alpha_{\text{sing}} = \arccos\left(-v_{\max}/(v_{\max} + v_{\min})\right)$, the trajectory reaches the $\tilde{x}$-axis at time $t = \pi - \alpha_{\text{sing}}$ and becomes u-singular.
- For every $\alpha \in (\pi/2, \alpha_{\text{sing}})$, ϕ_u vanishes before ϕ_v and there is a u-switching at time $t^u_{\text{sw}}(\alpha) = -\alpha + \arccos\left(\cos\alpha(v_{\max} + v_{\min})/v_{\max}\right)$.
- For every $\alpha \in (\alpha_{\text{sing}}, 3\pi/2)$, ϕ_v vanishes before ϕ_u and there is a v-switching at time $t^v_{\text{sw}}(\alpha) = -\alpha + 3\pi/2$.

The three possible cases in the previous analysis give three different families of trajectories to study at step 2) of the algorithm.

2) Starting with control $(1, v_{\max})$*:* A similar analysis to the previous one shows that there is no optimal trajectory starting from $\tilde{X}_0$ with the controls $(u, v) = (1, v_{\max})$.

3) Starting with the controls $(-1, v_{\min})$*:* The trajectory starting from $\tilde{X}_0$ with the control $(-1, v_{\min})$ has coordinates

$$\begin{cases} \tilde{x}(t) = -2v_{\min}\sin t \\ \tilde{y}(t) = v_{\min} - 2v_{\min}\cos t, \end{cases}$$

and from (8), the coordinates of the adjoint vector are

$$\begin{cases} p(t) = \cos\left(t + \alpha\right) \\ q(t) = -\sin\left(t + \alpha\right). \end{cases}$$

It follows that the switching functions are

$$\begin{aligned} \phi_u(t) &= 2v_{\min}\cos\left(\alpha\right) - v_{\min}\cos\left(t + \alpha\right), \\ \phi_v(t) &= -\cos\left(t + \alpha\right). \end{aligned}$$

The maximization condition PMP iv implies

$$\begin{aligned} \phi_u(0) = v_{\min}\cos\left(\alpha\right) \leqslant 0, \\ \phi_v(0) = -\cos\left(\alpha\right) \leqslant 0. \end{aligned}$$

TABLE I

COLOR CODE OF THE OPTIMAL SYNTHESIS

$(-1, v_{\min})$-bang arc	Blue
$(1, v_{\min})$-bang arc	Orange
$(-1, v_{\max})$-bang arc	Purple
$(1, v_{\max})$-bang arc	Red
u-singular arc	Black
u-switching curves	Dashed black
v-switching curves	Gray
Cut Locus	Green
Abnormal Cut Locus	Cyan

It follows that $\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ (in particular, the trajectory is abnormal). Note that ϕ_u and ϕ_v must be negative on a (small) open interval of the form $(0, \epsilon)$ since the starting control is the bang control $(-1, v_{\min})$. Moreover, since $\phi_u(0) = \phi_v(0) = 0$ and

$$\begin{aligned} \dot{\phi}_u(0) &= v_{\min}\sin\alpha, \\ \dot{\phi}_v(0) &= \sin\alpha, \end{aligned}$$

ϕ_u and ϕ_v will be both negative on $(0, \epsilon)$ if and only if $\alpha = 3\pi/2$. Consequently, there is only one extremal (corresponding to $\alpha = 3\pi/2$) starting from $\tilde{X}_0$ with control $(-1, v_{\min})$. The two switching functions along this extremal

$$\begin{aligned} \phi_u(t) &= -v_{\min}\sin t, \\ \phi_v(t) &= -\sin t, \end{aligned}$$

change their sign at $t = \pi$ and since a (u, v)-switching cannot occur (see Lemma 3.18) the trajectory loses its optimality at time π.

V. NUMERICAL SIMULATIONS

A. Synthesis in the UAV-based coordinates

For the numerical simulations, $v_{\min} = 1$ and $v_{\max} = 2$ and for the sake of brevity, the whole construction of the synthesis is not detailed.

During the construction of the optimal synthesis some special curves appears, namely

- *switching curves*, i.e., curves made of switching points;
- the *cut locus*, i.e., the set of points where the extremal curves of problem $\mathbf{(P_2)}$ lose global optimality)

In practice, these points can be very difficult to compute. In the following, some of them were computed numerically. Following the algorithm described in Section IV-C, the time-optimal synthesis corresponding to the problem $\mathbf{(P_1)}$ has been solved. The corresponding (discontinuous) state-feedback is given in Fig. 2 and Table I as a partition of the reduced state space.

Remark 5.1: Notice that the minimum time function (as a function of $\tilde{x}$ and $\tilde{y}$) is not continuous along the abnormal trajectory.

B. Correspondence with problem $\mathbf{(P_0)}$

The solutions of problem $\mathbf{(P_0)}$ can be deduced from the solutions of problem $\mathbf{(P_1)}$. In this section, we display pairs of figures (Fig. 3 and 4) showing two solutions of problem $\mathbf{(P_1)}$ (that start from the same point in the cut locus) and the

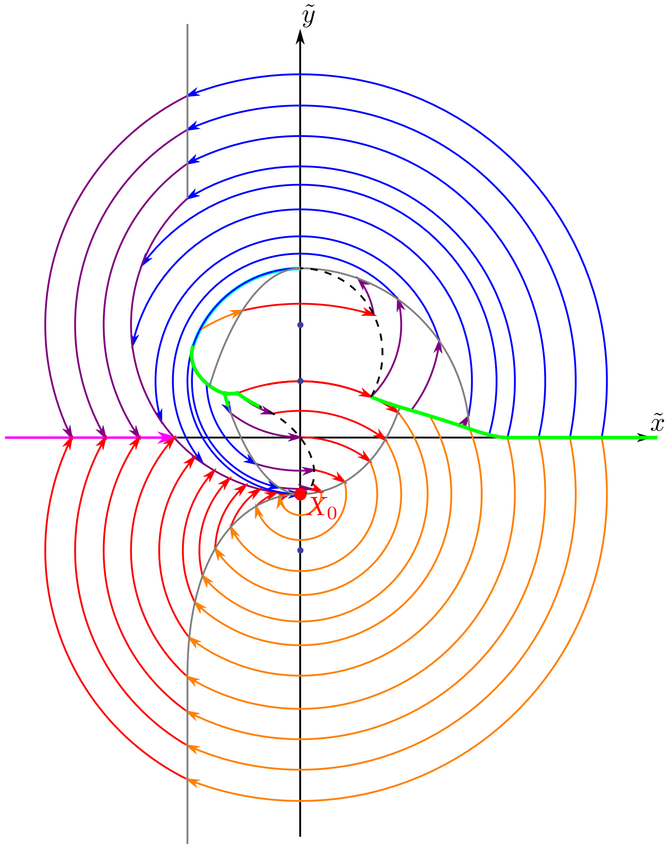

Fig. 2. Time-optimal synthesis for the problem $(\mathbf{P_1})$

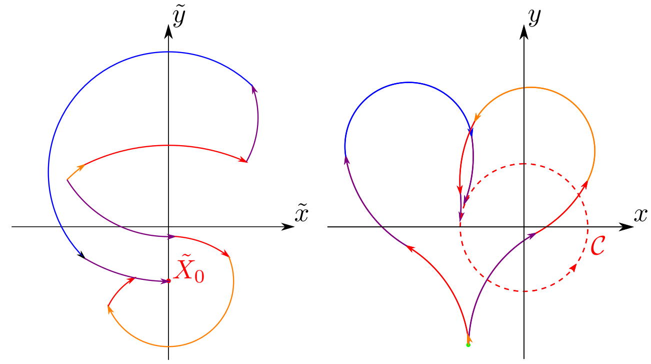

Fig. 4. Two bang-bang-bang-bang-bang optimal trajectories. Two optimal trajectories solutions of problem $(\mathbf{P_1})$ starting from the same point in the cut locus (left) and the corresponding optimal trajectories solutions of problem $(\mathbf{P_0})$ (right)

corresponding lifted solutions of problem $(\mathbf{P_0})$. Notice that the singular trajectory go straight to the center of the target.

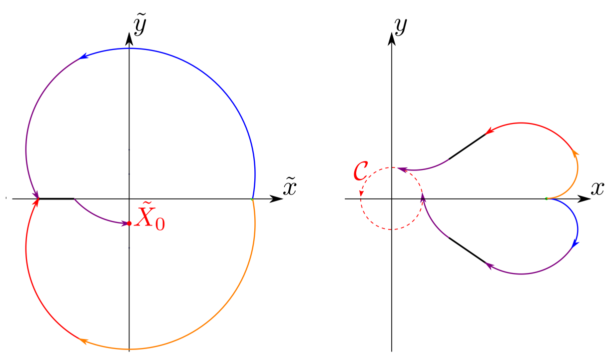

Fig. 3. Two bang-bang-singular-bang optimal trajectories. Two optimal trajectories solutions of problem $(\mathbf{P_1})$ starting from the same point in the cut locus (left) and the corresponding optimal trajectories solutions of problem $(\mathbf{P_0})$ (right)

VI. Conclusion

In this paper we have solved a time-minimal problem modeling a UAV flying at constant altitude with controls on the steering angle and on the linear velocity. Thanks to a change of coordinates applied to the three-dimensional Dubins system, we could simplify our problem and use (and extend) the existing theory of time-optimal syntheses for two-dimensional single input affine control systems to two-dimensional affine control systems with two inputs. We gave the time-optimal synthesis as a state-feedback law such that the target is reached optimally in finite time. Although the state-feedback law is not written explicitly as analytic expressions of the form $u(\tilde{x}, \tilde{y})$ and $v(\tilde{x}, \tilde{y})$, such formulas can be straightforwardly deduced from the partition of the reduced state space given by the optimal synthesis. Following this study, we could implement in simulation a controller based on our results.

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