# Formation Control with Triangulated Laman Graphs

Xudong Chen, M.-A. Belabbas, Tamer Başar

Abstract—Formation control deals with the design of decentralized control laws that stabilize agents at prescribed distances from each other. We call any configuration that satisfies the inter-agent distance conditions a target configuration. It is well known that when the distance conditions are defined via a rigid graph, there is a finite number of target configurations modulo rotations and translations. We can thus recast the objective of formation control as stabilizing one or many of the target configurations. A major issue is that such control laws will also have equilibria corresponding to configurations which do not meet the desired inter-agent distance conditions; we refer to these as undesired equilibria. The undesired equilibria become problematic if they are also stable. Designing decentralized control laws whose stable equilibria are all target configurations in the case of a general rigid graph is still an open problem. We propose here a partial solution to this problem by exhibiting a class of rigid graphs and control laws for which all stable equilibria are target configurations.

## I. INTRODUCTION

The design of control laws stabilizing a group of mobile autonomous agents has raised a number of issues related to the number and the type of equilibria and their relations to the level of decentralization of the system. In rigidity-based formation control, one assigns agents to the vertices of a rigid graph and specifies the target distances between the pairs of agents linked by edges. We refer to any configuration of the agents that satisfies these distance requirements as a target configuration. The rigidity of the graph thus ensures that there is a finite number of target configurations up to rotations and translations of the plane. A decentralized formation control law is thus designed to either locally or globally stabilize a subset of the target configurations. However, the decentralization constraints and geometry of the state-space make the appearance of ancillary, undesired configurations inevitable [1]. We call a control law essentially stabilizing if it only stabilizes target configurations.

The relationship between the level of decentralization and the existence of essentially stabilizing control laws has been studied in [1], where it was shown that a certain pattern in the information flow of a formation control systems implied the existence of undesired yet stable equilibria. In [2], it was shown that one could not locally stabilize all target configurations for a class of directed formations. Among positive results, it was shown in [3] that the triangle formation was essentially stabilizable and in [4] that a class of acyclic directed formations was similarly essentially stabilizable. These problems are challenging, and classifying the rigid graphs for which there exists an essentially stabilizing control

law is still open. The contribution of this paper is to exhibit a class of undirected graphs, termed triangulated Laman graphs, and an associated class of essentially stabilizing control laws for which all stable equilibria are target configurations.

We now describe the model precisely. Let G = (V, E) be an undirected graph with vertex set  $V := \{1, \dots, N\}$  and edge set E. Two vertices are said to be adjacent if there is an edge joining them. We denote by  $V_i$  the set of vertices adjacent to vertex *i*. Let  $\vec{x}_i \in \mathbb{R}^2$ , i = 1, ..., N be the coordinate of agent *i*. With a slight abuse of notation, we will sometimes refer to agent *i* as agent  $\vec{x}_i$ . For every edge  $(i, j) \in E$ , we let  $d_{ij}$ be the distance between agents *i* and *j*:

$$d_{ij} := \|\vec{x}_i - \vec{x}_j\|.$$

We denote by  $\bar{d}_{ij}$  the **target distance** for  $(i, j) \in E$ . The equations of motion of the *N* agents  $\vec{x}_1, \dots, \vec{x}_N$  in  $\mathbb{R}^2$ are given by

$$\dot{\vec{x}}_i = \sum_{j \in V_i} u(d_{ij}, \bar{d}_{ij}) \cdot (\vec{x}_j - \vec{x}_i), \quad \forall i = 1, \cdots, N$$
(1)

The function  $u(d_{ij}, \bar{d}_{ij})$  is assumed to be jointly continuously differentiable in terms of both arguments. For a fixed  $\bar{d}_{ij} > 0$ , the function  $u(\cdot, \bar{d}_{ij})$  is monotonically increasing, and it has a unique zero at  $\bar{d}_{ij}$ , i.e.,

$$u(d_{ij}, \bar{d}_{ij}) = 0 \tag{2}$$

In other words, if all pairs of agents  $\vec{x}_i$  and  $\vec{x}_j$ , with  $(i, j) \in E$ , reach their target distance, then the entire formation is at an equilibrium. For simplicity of exposition, we assumed in this paper that the control function u is the same for every pair  $(i, j) \in E$ . The result however holds for the general case where different control laws  $u_{ii}$ 's are used by different pairs of adjacent agents, provided they satisfy the conditions above.

It is known that the dynamics (1) is a gradient dynamics (we introduce the potential function in the next section). We can thus rephrase our goal of obtaining an essentially stabilizing control law as designing a potential function whose local minima are all target configurations.

The undirected formation control model (1) has been investigated from various perspectives. Questions concerning the level of interaction laws for organizing such systems [5]-[7], questions about system convergence [7], and questions about local stability [5] and robustness [8]–[11] have all been treated to some degree for the case of gradient dynamics. Recently, the problem of counting the number of stable equilibria was also addressed in [12], [13]. In general, this is a hard question. For example, even counting the number of

Xudong Chen, M.-A. Belabbas, Tamer Başar are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, emails: {xdchen, belabbas, basar1}@illinois.edu

equilibria for the gradient formation control system in one dimension is challenging [12].

Following this introduction, we proceed as follows. In section II, we describe preliminary results about the gradient formation control system. In particular, we will recall some known facts about system convergence and the equivariance of the potential function. In section III, we introduce the triangulated Laman graph, and then state and prove the main results of this paper. In particular, we introduce in Section III-C a formula which can be used to compute the so-called Morse-Bott index of a critical orbit, that allows us to study the type of extremal trajectories of the potential function, which might be of independent interest. We provide concluding remarks in Section IV, and the paper ends with an Appendix.

## **II. PRELIMINARY RESULTS**

## A. The control laws and the system convergence

Let G = (V, E) be an undirected graph with N vertices. We define the *configuration space*  $P_G$  of the system as

$$P_G := \left\{ (\vec{x}_1, \cdots, \vec{x}_N) \in \mathbb{R}^{2 \times N} \middle| \vec{x}_i \neq \vec{x}_j, \forall (i, j) \in E \right\}$$
(3)

Equivalently,  $P_G$  is the set of embeddings of the graph Gin  $\mathbb{R}^2$  whose adjacent vertices have distinct positions. We call a pair (G, p) a **framework**. We now introduce the class of control laws that is studied in this paper. Let  $\mathbb{R}_+$  be the set of positive real numbers, and let  $C^1(\mathbb{R}_+,\mathbb{R})$  be the set of continuously differentiable functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ . For fixed  $\overline{d}_{ij}$ , the interaction law  $u(\cdot, \overline{d}_{ij})$  can be viewed as an element in  $C^1(\mathbb{R}_+,\mathbb{R})$ , and for convenience, we let

$$f_{ij}(d) := u(d, \bar{d}_{ij}) \tag{4}$$

Denote by  $\mathscr{U}$  the set of functions  $f \in C^1(\mathbb{R}_+, \mathbb{R})$  satisfying the next two conditions:

C1. For any x > 0, we have

$$\frac{d}{dx}(xf(x)) > 0 \tag{5}$$

and f(x) has a unique zero.

C2.  $\lim_{x\to 0} \int_x^1 tf(t)dt = -\infty.$ 

The formation control system considered in the paper is then equipped with control laws  $u(\cdot, \bar{d}_{ij}) \in \mathscr{U}$  for all  $\bar{d}_{ij}$ . An example of such a control law is:

$$u(\|\vec{x}_i - \vec{x}_j\|, \bar{d}_{ij}) = \frac{\|\vec{x}_i - \vec{x}_j\|^2 - \bar{d}_{ij}^{-2}}{\|\vec{x}_i - \vec{x}_j\|^2}.$$
 (6)

which is similar to the gradient control law [5] scaled by  $1/||x_i - x_j||^2$ .

Note that the function xf(x) appears in condition C1 because if f is an interaction law between a pair of agents, then xf(x) represents the actual attraction/repulsion between them. We impose these two conditions because the first condition implies that the interaction is a monotonically increasing function, so it is a repulsion at a short distance, and an attraction at a long distance. The second condition prevents collisions of adjacent agents along the evolution, so then the solution of system (1), with any initial condition in  $P_G$ , exists for all time. Moreover, we have shown in [7] that if each interaction law  $f_{ij}$  satisfies conditions C1 and C2, then all critical orbits of system (1) are contained in a compact subset of  $P_G$ . Moreover, by assuming  $f_{ij} \in \mathcal{U}$ , we have the global convergence of the formation control system (1) as stated below.

**Lemma 1.** If each  $f_{ij}$  is in  $\mathcal{U}$ , then the set of equilibria of system (1) is a compact subset of  $P_G$ . Furthermore, for any initial condition  $p(0) \in P_G$ , the solution p(t) of system (1) converges to the set of equilibria.

#### B. The potential function and its invariance

An important property of the class of systems (1) is that they are gradient flows. The associated potential function is given by

$$\Phi(\vec{x}_1, \cdots, \vec{x}_N) := \sum_{(i,j) \in E} \int_1^{d_{ij}} tf_{ij}(t)dt$$
(7)

Note that the potential function  $\Phi$  depends only on relative distances between agents, thus it is invariant if we translate and/or rotate the entire configuration in  $\mathbb{R}^2$ . We will now describe this property in precise terms.

The special Euclidean group SE(2) has a natural action on the configuration space. Recall that  $\gamma$  in SE(2) can be represented by a pair  $(\theta, \vec{v})$  with  $\theta$  a rotation matrix, and  $\vec{v}$ a vector in  $\mathbb{R}^2$ . With this representation, the multiplication of two elements  $\gamma_1 = (\theta_1, \vec{v}_1)$  and  $\gamma_2 = (\theta_2, \vec{v}_2)$  of SE(2) is given by

$$\gamma_2 \cdot \gamma_1 = (\theta_2 \theta_1, \theta_2 \vec{v}_1 + \vec{v}_2) \tag{8}$$

The action of SE(2) on  $P_G$  mentioned above is defined as follows: given  $\gamma = (\theta, \vec{v})$  in SE(2) and  $p = (\vec{x}_1, \dots, \vec{x}_N)$  in  $P_G$  we let

$$\gamma \cdot p := (\theta \vec{x}_1 + \vec{v}, \cdots, \theta \vec{x}_N + \vec{v}) \tag{9}$$

We denote by  $\mathcal{O}_p$  the orbit of SE(2) through  $p \in P_G$ :

$$\mathcal{O}_p := \{ q \in P_G \mid q = \gamma \cdot p \text{ for some } \gamma \in SE(2) \}.$$
(10)

The potential function  $\Phi$  keeps the same value over  $\mathscr{O}_p$ :

$$\Phi(p) = \Phi(\gamma \cdot p) \tag{11}$$

for any  $p \in P_G$  and any  $\gamma \in SE(2)$ . Denote by  $\nabla \Phi$  the gradient of  $\Phi$ . An immediate consequence of the invariance of  $\Phi$ under the group action (9) is that

$$\nabla \Phi(\gamma \cdot p) = \operatorname{diag}(\theta, \cdots, \theta) \cdot \nabla \Phi(p) \tag{12}$$

where  $\operatorname{diag}(\theta, \dots, \theta)$  is a block diagonal matrix with N copies of  $\theta$ . Since  $\operatorname{diag}(\theta, \dots, \theta)$  is invertible, when p is an equilibrium of system (1), then so is p' in  $\mathcal{O}_p$ . We thus refer to the orbit  $\mathcal{O}_p$  as a *critical orbit* if  $\nabla \Phi(p) = 0$ . Let  $\mathcal{O}_p$  be a critical orbit, and let  $H_p$  be the Hessian matrix of  $\Phi$  at p, i.e.,

$$H_p := \frac{\partial^2 \Phi(p)}{\partial p^2} \tag{13}$$

The following Lemma presents well-known facts about the Hessian matrix of an invariant function:

**Lemma 2.** Let  $\Phi : P \to \mathbb{R}$  be a function invariant under a Lie-group action over a Euclidean space. Denote by k the dimension of a critical orbit  $\mathcal{O}_p$  under the group action and denote by  $H_p$  be the Hessian of  $\Phi$  at p. Then for any  $p_1, p_2 \in \mathcal{O}_p$ , the eigenvalues of  $H_{p_1}$  and  $H_{p_2}$  are the same. In addition, the Hessian  $H_p$  has at least k zero eigenvalues. The null space of  $H_p$  at least contains the tangent space of  $\mathcal{O}_p$  at p.

In our case, each critical orbit  $\mathcal{O}_p$  for  $p \in P_G$  is of dimension 3. Let  $n_0(H_p)$  be the number of zero eigenvalues of  $H_p$ . From Lemma 2, we have  $n_0(H_p) \ge 3$ . A critical orbit  $\mathcal{O}_p$  is said to be *nondegenerate* if  $n_0(H_p) = 3$ . A potential function  $\Phi$  is said to be an **equivariant Morse function** if there are only *finitely* many critical orbits, and moreover each critical orbit is nondegenerate.

# III. TRIANGULATED LAMAN GRAPHS, INDEPENDENT PARTITIONS AND THE MORSE-BOTT INDEX FORMULA

## A. Triangulated Laman Graphs

Let G = (V, E) be an undirected graph. Let the distance function  $\rho: P_G \to \mathbb{R}^{|E|}_+$  be defined by

$$\rho: p \mapsto (\cdots, \|\vec{x}_i - \vec{x}_j\|^2, \cdots)$$
(14)

The graph *G* is called *rigid* in  $\mathbb{R}^2$  if for almost all  $d \in \mathbb{R}^{|E|}_+$ , the pre-image  $\rho^{-1}(d)$  is a finite set modulo translations and rotations. The graph *G* is called *minimally rigid* if it is not rigid after taking out any of its edges [14]. A **Laman graph** is a minimal rigid graph in  $\mathbb{R}^2$ .

It is well known that every Laman graph can be obtained via a so-called Henneberg sequence; a Henneberg sequence  $\{G_i\}$  is a sequence of minimally rigid graphs obtained via two basic operations: edge split and vertex add. Precisely, start with a graph  $G_0$  of only two vertices joined by an edge. Then the graph  $G_i$  has (i+2) vertices and is obtained from  $G_{i-1}$  by applying one of the two operations. We refer to [15] for more details about these operations. We define triangulated Laman graphs as those graphs obtained by imposing constraints on the type of operations allowed: we start with a graph  $G_0$  with two vertices connected by one edge. The graph  $G_i$  in the sequence is obtained from  $G_{i-1}$ by adding a vertex and attaching it to two *adjacent* vertices with two new edges. In other words, only the operation of vertex-add is allowed in the Henneberg construction, and in addition, the new vertex cannot be adjacent to two arbitrary vertices, but rather to two vertices connected by an existing edge. See Figure 1 for an illustration.

Let G be a triangulated Laman graph. We say that a subgraph G' of G is a 3-cycle if G' is a complete graph of three vertices. In graph theory, an *induced cycle* of a graph G is a cycle that is an induced subgraph of G. If G is a triangulated Laman graph, then all induced cycles of G are the 3-cycles. A framework (G, p) is said to be *strongly rigid* (or simply p is strongly rigid) if p satisfies the following condition: if vertices i, j and k of G form a 3-cycle of G, then the triangle formed by agents  $\vec{x}_i$ ,  $\vec{x}_j$  and  $\vec{x}_k$  is nondegenerate, i.e.,  $\vec{x}_i$ ,  $\vec{x}_j$  and  $\vec{x}_k$  are not belong to a

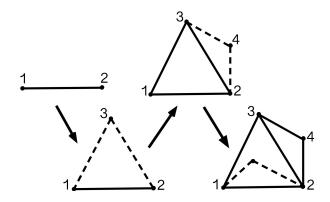


Fig. 1. An example of a triangulated Laman graph. Start with edge (1,2), then subsequently join vertices 3, 4 and 5 to two existing adjacent vertices.

one-dimensional subspace of  $\mathbb{R}^2$ . If p is strongly rigid, then so is any  $p' \in \mathcal{O}_p$ .

Let  $\rho: P_G \to \mathbb{R}^{|E|}_+$  be defined by Eq. (14). A framework (G, p) is said to be *infinitesimally rigid* [15] (or simply, p is infinitesimally rigid) if the null space of the Jacobian of  $\rho$  at p (i.e.,  $d\rho(p)/dp$ ) is of dimension three. We state below a fact without proof:

**Lemma 3.** Strongly rigid configurations are infinitesimally rigid, Moreover, they form an open and dense subset of  $P_G$ .

Let *p* be a strongly rigid configuration, and let  $d_{ij}$  be the Euclidean distance between  $\vec{x}_i$  and  $\vec{x}_j$  in *p*. Suppose vertices *i*, *j* and *k* form a 3-cycle of *G*, then

$$\begin{cases}
d_{ij} + d_{ik} > d_{jk} \\
d_{ij} + d_{jk} > d_{ik} \\
d_{ik} + d_{jk} > d_{ij}
\end{cases}$$
(15)

We say the set  $\{d_{ij}|(i,j) \in E\}$  satisfies the **triangle in**equalities associated with *G*. If the set of desired distances  $\{\bar{d}_{ij}|(i,j) \in E\}$  satisfies the triangle inequalities, then there are strongly rigid configurations satisfying the condition that  $d_{ij} = \bar{d}_{ij}$  for all  $(i, j) \in E$ ; indeed, by following a Henneberg construction, we see that there are  $2^{N-2}$  strongly rigid orbits of configuration satisfying these conditions. This exponential relation has also been explored in directed formations [16].

We now state in precise terms the main result of this paper.

**Theorem 4.** Let G = (V, E) be a triangulated Laman graph and let the target distances  $\{\bar{d}_{ij}|(i, j) \in E\}$  satisfy the triangle inequalities associated with G. Let  $u(\cdot, \bar{d}_{ij}) \in \mathcal{U}$ , for all  $(i, j) \in E$ , be such that the potential function  $\Phi$  defined in (7) is an equivariant Morse function. Then,

- 1. A critical orbit  $\mathcal{O}_p$  is (exponentially) stable if and only if it is strongly rigid. There are  $2^{N-2}$  stable critical orbits each of which satisfies the condition that  $d_{ij} = \overline{d}_{ij}$  for all  $(i, j) \in E$ .
- 2. For almost all initial conditions  $p(0) \in P_G$ , the solution p(t) of system (1) converges to one of the  $2^{N-2}$  stable critical orbits.

The implication of the above is that the control laws considered in this paper are essentially stabilizing the target configurations.

#### B. Independent Partition

We now introduce the **independent partition** associated with a framework (G, p). It is a partition of the *edge set* of G such that, roughly speaking, edges that are aligned (with respect to the embedding p) are belong to the same subset. Precisely, the independent partition associated with (G, p) can be defined via a Henneberg construction for G: given such a Henneberg sequence  $\{G'_i\}$ , we label the vertices of G with respect to the order in which they appear in the sequence. The partition is then constructed in the following way:

*Base case.* Start with the subgraph  $G'_0$  of *G* comprised of vertices  $\{1,2\}$ . Since there is only one edge (1,2), the partition is trivial.

*Inductive step.* Now let  $G'_i = (V', E')$  be the subgraph of *G* comprised of vertices  $V' = \{1, \dots, i+2\}$  and assume that we have partitioned E' into disjoint subsets as

$$E' = E'_1 \cup \cdots \cup E'_{m'}$$

Suppose that in the chosen Henneberg construction, vertex (i+3) links to vertices j and k via edges (j,i+3) and (k,i+3). Without loss of generality, we assume that  $(j,k) \in E'_1$ . Then we consider two cases:

*Case I.* If  $\vec{x}_j$ ,  $\vec{x}_k$  and  $\vec{x}_{i+3}$  are aligned, then update the partition by adding (j, i+3) and (k, i+3) into  $E'_1$ .

*Case II.* If  $\vec{x}_i$ ,  $\vec{x}_j$  and  $\vec{x}_{i+3}$  are not aligned, then update the partition as

$$E'_1 \cup \cdots \cup E'_{m'} \cup \{(j, i+3)\} \cup \{(k, i+3)\}$$

By following the Henneberg construction, we then derive the independent partition of E associated with (G, p). We note that the independent partition does not rely on the choice of the Henneberg construction [17]. We refer to Fig. 2 for an illustration.

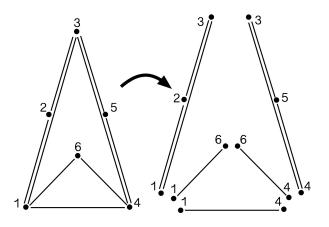


Fig. 2. An example of the independent partition. We see from the left figure that the graph *G* is a triangulated Laman graph as we label the vertices with respect to a Henneberg construction, and *p* is a planar configuration with  $\vec{x}_1, \vec{x}_2, \vec{x}_3$  aligned, and  $\vec{x}_3, \vec{x}_4, \vec{x}_5$  aligned. Then the independent partition of *E* associated with (*G*, *p*) is given by the right figure.

Let  $\{E_1, \dots, E_m\}$  be the disjoint subsets of edges associated with the independent partition for (G, p). Let  $V_i$  be the set of vertices incident to edges in  $E_i$ , let  $G_i := (V_i, E_i)$ , and let  $(G_i, p_i)$  be the corresponding framework. We summarize some properties associated with independent partitions.

**Proposition 5.** Let  $\{(G_i, p_i)\}$  be the frameworks associated with the independent partition for (G, p). Then

- 1) Each  $G_i$  is a triangulated Laman graph.
- 2) Each  $(G_i, p_i)$  is a line framework.
- If there is another partition of E satisfying conditions

   and 2), then it is a refinement of the independent
   partition. In other words, the independent partition
   contains minimal number of subgraphs satisfying con ditions 1 and 2.
- If in addition p is an equilibrium of system (1), then each p<sub>i</sub> is an equilibrium of the subsystem induced by G<sub>i</sub>.
- 5) If in addition p is strongly rigid, then each  $p_i$  is a configuration of two agents, i.e., the edge set  $E_i$  of  $G_i$  is a singleton.

More details, including proofs of these statements, can be found in [17].

#### C. The Morse-Bott Index Formula

Let  $\mathcal{O}_p$  be a critical orbit of system (1). Let  $n_+(H_p)$ ,  $n_0(H_p)$ , and  $n_-(H_p)$  be the numbers of positive, zero, and negative eigenvalues of  $H_p$ , respectively. We refer to the triplet

$$\vec{n}(H_p) = (n_+(H_p), n_-(H_p), n_0(H_p))$$
 (16)

as the *signature* of  $H_p$ . By Lemma 2, the signature of  $H_{p'}$  is invariant as p' varies over  $\mathcal{O}_p$ . Note that in terms of the signature, a critical orbit  $\mathcal{O}_p$  is *exponentially stable* if and only if

$$\vec{n}(H_p) = (0, 2N - 3, 3) \tag{17}$$

Define the **Morse-Bott index** and **co-index** of  $\mathcal{O}_p$  to be  $n_-(H_p)$  and  $n_+(H_p)$  respectively. We now show how to evaluate these two indices of a critical orbit.

Let G' = (V', E') be a subgraph of *G*. A formation control system is said to be *induced by* G' if it is comprised of agents  $\vec{x}_i$  for  $i \in V'$  together with  $f_{ij}$ 's the interaction laws for  $(i, j) \in E'$ . To be precise, the equations of motion for the subsystem induced by G' are

$$\dot{\vec{x}}_i = \sum_{j \in V'_i} u(d_{ij}, \bar{d}_{ij}) \cdot (\vec{x}_j - \vec{x}_i), \quad \forall i \in V'$$
(18)

with  $V'_i$  the neighbors of *i* in *G'*. The subsystem is a gradient flow for the potential function

$$\Phi'(p') := \sum_{(i,j)\in E'} \int_{1}^{d_{ij}} tf_{ij}(t)dt.$$
 (19)

with  $f_{ii}$  defined in (4).

**Proposition 6.** Let G be a triangulated Laman graph. Let p be an equilibrium of system (1), and let  $\{(G_i, p_i)\}_{i=1}^m$  be

the frameworks associated with the independent partition for (G, p). Let  $\Phi_i$  be the potential function of the subsystem induced by  $G_i$ . Let  $H_{p_i}$  be the Hessian of  $\Phi_i$  at  $p_i$ . Then

$$\begin{cases} n_{-}(H_{p}) = \sum_{i=1}^{m} n_{-}(H_{p_{i}}) \\ n_{+}(H_{p}) = \sum_{i=1}^{m} n_{+}(H_{p_{i}}) \end{cases}$$
(20)

*This set of expressions will be referred as the* **Morse-Bott index formula**.

We provide a sketch of the proof of Proposition 6 in the Appendix, and we refer to [17] for a complete proof. Proposition 6 is used to prove the following Corollary.

**Corollary 7.** The critical orbit  $\mathcal{O}_p$  is nondegenerate if and only if each  $\mathcal{O}_{p_i}$  is nondegenerate. Moreover, the critical orbit  $\mathcal{O}_p$  is exponentially stable if and only if each  $\mathcal{O}_{p_i}$  is exponentially stable.

*Proof.* Let  $|V_i|$  and  $|E_i|$  be the cardinalities of  $V_i$  and  $E_i$ , respectively. Since each  $G_i$  is a triangulated Laman graph,

$$|E_i| = 2|V_i| - 3. \tag{21}$$

By Lemma 2, we have  $n_0(H_{p_i}) \ge 3$ , and hence  $n_+(H_{p_i}) + n_-(H_{p_i}) \le |E_i|$ . On the other hand, we have

$$|E| = \sum_{i=1}^{m} |E_i|$$
 (22)

Thus, by Proposition 6, we know that

$$\begin{cases} n_{+}(H_{p}) + n_{-}(H_{p}) \le |E| \\ n_{0}(H_{p}) \ge 3 \end{cases}$$
(23)

The equalities hold if and only if  $n_0(H_{p_i}) = 3$  for all *i*. Thus, the critical orbit  $\mathcal{O}_p$  is nondegenerate if and only if each  $\mathcal{O}_{p_i}$  is nondegenerate. Also by Proposition 6,  $n_+(H_p) = 0$  if, and only if,  $n_+(H_{p_i}) = 0$  for all *i*. This competes the proof.  $\Box$ 

From Proposition 6 and Corollary 7, we see that it suffices to understand the Morse-Bott index of  $H_p$  for p either a strongly rigid configuration, or a line configuration. We will first focus on strongly rigid configurations, and establish the next result.

**Corollary 8.** Let G be a triangulated Laman graph. Suppose each  $f_{ij}$  is in  $\mathcal{U}$ , with  $\{\overline{d}_{ij}|(i, j) \in E\}$  satisfying the triangle inequalities associated with G. Let p be an equilibrium of system (1). If p is strongly rigid, then  $\mathcal{O}_p$  is exponentially stable. Moreover, the distance between  $\vec{x}_i$  and  $\vec{x}_j$  is the target distance  $\overline{d}_{ij}$  for all  $(i, j) \in E$ .

*Proof.* Let  $\{(G_i, p_i)\}_{i=1}^m$  be the frameworks associated with the independent partition for (G, p). Since p is strongly rigid, each  $p_i$  consists of only two agents by Proposition 5, so then m = 2N - 3. Also, by Proposition 5, each  $p_i$  is an equilibrium, and hence  $f_{ij}(d_{ij}) = 0$ , which implies  $d_{ij} = \overline{d}_{ij}$ . We will now compute the signature of  $H_{p_i}$ . Suppose  $p_i$  consists of agents  $\vec{x}_j$  and  $\vec{x}_k$ , and by Lemma 2, we may rotate and/or translate

*p* so that both  $\vec{x}_j$  and  $\vec{x}_k$  are on the *x*-coordinate. Then  $H_{p_i}$  is a 4-by-4 matrix given by

$$H_{p_i} = \bar{d}_{jk} f'_{jk} (\bar{d}_{jk}) \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
(24)

On the other hand, we have

$$\left. \frac{d}{dx}(xf_{jk}(x)) \right|_{x=\bar{d}_{jk}} = \bar{d}_{jk}f'_{jk}(\bar{d}_{jk}) > 0 \tag{25}$$

Thus,  $n_{-}(H_{p_i}) = 1$ . Since this holds for all *i*, we then have

$$n_{-}(H_{p}) = \sum_{i=1}^{m} n_{-}(H_{p_{i}}) = m = 2N - 3$$
(26)

and hence, by the argument of dimensionality, we have

$$\vec{n}(H_p) = (0, 2N - 3, 3) \tag{27}$$

Thus,  $\mathcal{O}_p$  is exponentially stable. This completes the proof.

#### D. Proof of The Main Theorem

We first focus on the case where  $p \in P_G$  is a critical line configuration, and evaluate the signature of  $H_p$ . In particular, we will establish the next result.

**Proposition 9.** Let G be a triangulated Laman graph of N vertices with N > 2. Suppose that each  $f_{ij}$  is in  $\mathcal{U}$ , with  $\{\overline{d}_{ij}|(i,j) \in E\}$  satisfying the triangle inequalities associated with G. Let  $\mathcal{O}_p$  be a nondegenerate critical orbit of line configurations. Then,  $n_+(H_p) > 0$ .

It is computationally convenient to collect the *x*-coordinates of agents  $x_1$  to  $x_N$  in the first *N* entries of a vector, and the *y*-coordinates in the last *N* entries. To this end, we let  $\vec{a}$  and  $\vec{b}$  be two vectors in  $\mathbb{R}^N$  containing *x*-coordinates and *y*-coordinates of agents respectively, i.e.,

$$\begin{cases} \vec{a} := (x_1, \cdots, x_N) \\ \vec{b} := (y_1, \cdots, y_N) \end{cases}$$
(28)

We then re-arrange entries of a configuration p so that

$$p = (\vec{a}, \vec{b}) \tag{29}$$

By Lemma 2, we can assume, without loss of generality, that the line configuration p is aligned with the *x*-axis, or equivalently that  $\vec{b} = 0$ . An advantage of re-arranging entries is that the Hessian  $H_p$  can now be expressed as a block-diagonal matrix given by

$$H_p = \begin{pmatrix} A_p & 0\\ 0 & B_p \end{pmatrix} \tag{30}$$

where  $A_p$  and  $B_p$  are *N*-by-*N* symmetric zero-row/columnsum matrices. The *ij*-th entry, for  $i \neq j$ , of  $A_p$  and  $B_p$  are given by

$$A_{ij} := \begin{cases} \frac{d}{dx} (xf_{ij}(x)) \big|_{x=d_{ij}} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$
(31)

and

$$B_{ij} := \begin{cases} f_{ij}(d_{ij}) & \text{if } (i,j) \in E\\ 0 & \text{otherwise} \end{cases}$$
(32)

the diagonal entries of  $A_p$  and  $B_p$  are then determined by the conditions that their row/column-sum are zeros.

By Lemma 2, the null space of  $H_p$  contains  $T_p \mathcal{O}_p$ , i.e., the tangent space of  $\mathcal{O}_p$  at p which is the vector space spanned by the next three vectors in  $\mathbb{R}^{2N}$ :

$$\begin{cases} \vec{t}_a := (\vec{e}, 0) \\ \vec{t}_b := (0, \vec{e}) \\ \vec{r}_p := (0, \vec{a}) \end{cases}$$
(33)

where  $\vec{e}$  is a vector of all ones in  $\mathbb{R}^N$ . The first two vectors  $\vec{t}_a$  and  $\vec{t}_b$  represent infinitesimal motions of translations of p along the *x*-coordinate and the *y*-coordinate, respectively. The third vector  $\vec{r}_p$  represents an infinitesimal motion of clockwise rotation of p around the origin. It is also straightforward to check that all three vectors are in the null space of  $H_p$ . Now suppose the critical orbit  $\mathcal{O}_p$  is nondegenerate; then by Lemma 2 the null space of  $H_p$  should only be spanned by  $\vec{t}_a$ ,  $\vec{t}_b$  and  $\vec{r}_p$ . Further, by (30) and (33), we see that the null space of  $B_p$  is spanned by  $\vec{e}$ , and the null space of  $B_p$  is spanned by  $\vec{e}$  and  $\vec{a}$ .

We are now in a position to prove Proposition 9.

*Proof of Proposition 9*: We prove the proposition by showing that  $n_+(B_p) > 0$ . The proof will be completed by induction on the number of agents. First consider the base case N = 3. Assume that p is aligned with the *x*-coordinate with  $x_2 < x_1 < x_3$ , i.e., agent  $\vec{x}_1$  lies in between  $\vec{x}_2$  and  $\vec{x}_3$ . We now show that the matrix  $n_+(B_p) > 0$ . Since p is an equilibrium, then

$$d_{12}f_{12}(d_{12}) = d_{13}f_{13}(d_{13}) = -d_{23}f(d_{23})$$
(34)

We now show that these three numbers are all negative. Suppose not, then we have

$$\begin{cases}
 d_{12} \ge \bar{d}_{12} \\
 d_{13} \ge \bar{d}_{13} \\
 d_{23} \le \bar{d}_{23}
\end{cases}$$
(35)

This holds because the function  $xf_{ij}(x)$  is strictly monotonically increasing by condition C1. On the other hand, we have

$$d_{12} + d_{13} = d_{23} \tag{36}$$

which implies that

$$\bar{d}_{12} + \bar{d}_{13} \le \bar{d}_{23} \tag{37}$$

This then violates the triangle inequality. Thus, the three numbers in (34) are all negative. In particular, both  $f_{12}(d_{12})$  and  $f_{13}(d_{13})$  are negative. Let  $\vec{e}_1 := (1,0,0)$  be a test vector. Then by computation, we have

$$\langle \vec{e}_1, B_p \vec{e}_1 \rangle = -f_{12}(d_{12}) - f_{13}(d_{13}) > 0$$
 (38)

Thus,  $n_+(B_p) > 0$ , and hence  $n_+(H_p) > 0$ .

Now apply the technique of induction: We assume the fact that if  $\mathcal{O}_p$  is nondegenerate, then  $n_+(B_p) > 0$  for any  $N \leq n$ 

with  $n \ge 3$ , and we prove for the case N = n + 1. Fix a Henneberg construction of *G*, and without loss of generality, assume that 1 is the last vertex joining *G* via edges (1,2) and (1,3) to vertices 2 and 3, respectively. We still assume that *p* is aligned with the *x*-coordinate. Then there are two cases regarding the position of agent  $\vec{x}_1$ : either  $(x_1 - x_2)(x_1 - x_3) <$ 0 or  $(x_1 - x_2)(x_1 - x_3) > 0$ , depending on whether or not agent  $\vec{x}_1$  lies in between  $\vec{x}_2$  and  $\vec{x}_3$ . For simplicity, we will only focus on the former case, and assume  $x_2 < x_1 < x_3$ . Similar analysis can be applied to the other case as well.

Let  $\vec{e}_1, \dots, \vec{e}_{n+1}$  be the standard basis of  $\mathbb{R}^{n+1}$ . Similarly, we have

$$\langle \vec{e}_1, B_p \vec{e}_1 \rangle = -f_{12}(d_{12}) - f_{13}(d_{13})$$
 (39)

We now show that if  $\langle \vec{e}_1, B_p \vec{e}_1 \rangle \ge 0$ , then  $n_+(B_p) > 0$ . Since  $\mathcal{O}_p$  is nondegenerate, the null space of  $B_p$  is spanned by  $\vec{e}$  and  $\vec{a}$  only. On the other hand, the three vectors  $\vec{e}_1$ ,  $\vec{e}$  and  $\vec{a}$  are linearly independent, so  $B_p \vec{e}_1 \neq 0$ . Let  $\lambda_1, \dots, \lambda_{n-1}$  be the non-zero eigenvalues of  $B_p$ , and let  $\vec{v}_i$  be the unit-length eigenvector of  $B_p$  corresponding to  $\lambda_i$ , then

$$\langle \vec{e}_1, B_p \vec{e}_1 \rangle = \sum_{i=1}^{n-1} \lambda_i \langle \vec{e}_1, \vec{v}_i \rangle^2 \ge 0 \tag{40}$$

Since there exists some *i* with  $\langle \vec{e}_1, \vec{v}_i \rangle \neq 0$ , there must exist at least one positive eigenvalue of  $B_p$ .

So in the rest of the proof, we only consider the case  $\langle \vec{e}_1, B_p \vec{e}_1 \rangle < 0$ . Since  $\vec{x}_1$  is balanced in *p*, we have

$$d_{12}f_{12}(d_{12}) = d_{13}f_{13}(d_{13}) \tag{41}$$

Then by expression (39), both  $f_{12}(d_{12})$  and  $f_{13}(d_{13})$  are positive. In particular, we have

$$d_{23} = d_{12} + d_{13} > \bar{d}_{12} + \bar{d}_{13} > \bar{d}_{23} \tag{42}$$

Now choose a function  $g \in C^1(\mathbb{R}_+, \mathbb{R})$  such that it satisfies the next three conditions

- 1. *g* satisfies condition C1, and  $g(\bar{d}_{23}) = 0$
- 2.  $d_{23}g(d_{23}) = d_{12}f_{12}(d_{12}) = d_{13}f_{13}(d_{13})$
- 3.  $\frac{d}{dx}(xg(x))\Big|_{x=d_{23}} = A_{12}A_{13}/(A_{12}+A_{13})$

with  $A_{ij}$  the *ij*-th entry of  $A_p$  defined in (31).

We introduce function g because of the following fact: Let G' = (V', E') be the subgraph of G induced by vertices  $V' := \{2, \dots, n+1\}$ , and let (G', p') be the corresponding framework. If we replace  $f_{23}$  with

$$\tilde{f}_{23} := f_{23} + g, \tag{43}$$

then p' is an equilibrium of the sub-system induced by G', with the modification that  $f_{23}$  is replaced by  $\tilde{f}_{23}$ . To see this, it suffices to check that agents  $\vec{x}_2$  and  $\vec{x}_3$  in p' are still balanced. But this holds because of the second condition on g. We note that the first condition on g implies that  $\tilde{f}_{23} \in \mathcal{U}$  with  $\tilde{f}_{23}(d_{23}) = 0$ . The third condition is a technical condition, and will be justified later. Also note that G' is a triangulated Laman graph, and  $\{\bar{d}_{ij} | (i, j) \in E'\}$  satisfies the triangle inequalities associated with G'. Thus, we can apply the technique of induction on the critical orbit  $\mathcal{O}_{p'}$  of the modified sub-system. Let  $\Phi'$  be the potential function associated with the modified sub-system induced by G'. Let  $H_{p'}$  be the Hessian of  $\Phi'$  at p'. Similarly, we can express  $H_{p'}$  as a block-diagonal matrix as

$$H_{p'} = \begin{pmatrix} A_{p'} & 0\\ 0 & B_{p'} \end{pmatrix} \tag{44}$$

with  $A_{p'}$  and  $B_{p'}$  defined in the same way as  $A_p$  and  $B_p$  but with respect to G'. Also we replace  $f_{23}$  and  $f'_{23}$  with  $\tilde{f}_{23}$ and  $\tilde{f}'_{23}$ , respectively. We will now introduce a formula that relates the signature of  $H_p$  to the signature of  $H_{p'}$ . First we introduce a vector-valued sign function as

$$sgn(x) := \begin{cases} (1,0,0) & \text{if } x > 0\\ (0,1,0) & \text{if } x < 0\\ (0,0,1) & \text{if } x = 0 \end{cases}$$
(45)

and recall that  $\vec{n}(H) = (n_+(H), n_-(H), n_0(H))$  is defined as the signature of H. Then,

$$\begin{cases} \vec{n}(A_p) = \vec{n}(A_{p'}) + sgn(-A_{12} - A_{13}) \\ \vec{n}(B_p) = \vec{n}(B_{p'}) + sgn(-B_{12} - B_{13}) \end{cases}$$
(46)

where  $A_{ij}$  and  $B_{ij}$  are entries of  $A_p$  and  $B_p$ , respectively (the validity of this formula requires the second and the third conditions on *g*). The proof of the formula is provided in the Appendix.

From (46), we see that if  $\mathcal{O}_p$  is nondegenerate in the original system, then so is  $\mathcal{O}_{p'}$  in the modified sub-system. Thus, by induction we have  $n_+(B_{p'}) > 0$ . Then applying (46) again, we conclude that  $n_+(B_p) > 0$ . This then completes the proof.

With the results above, we will now return to proof Theorem 4. Let p be an equilibrium of system (1). If  $\mathcal{O}_p$  is strongly rigid, then  $\mathcal{O}_p$  is (exponentially) stable as we have shown at the end of section III-C. So we assume now that  $\mathcal{O}_p$  is not strongly rigid, and we show that  $\mathcal{O}_p$  is unstable.

Let  $p_1, \dots, p_m$  be the line sub-configurations of p associated with the independent partition, and without loss of generality, we assume that  $p_1$  contains at least three agents. Since  $\mathcal{O}_p$  is a nondegenerate critical orbit, then so is  $\mathcal{O}_{p_1}$  by Corollary 7, and hence the co-index  $n_+(H_{p_1})$  must be positive by Proposition 9. We then apply the Morse-Bott index formula, i.e.,

$$n_{+}(H_{p}) = \sum_{i=1}^{m} n_{+}(H_{p_{i}})$$
(47)

to conclude that the Hessian matrix  $H_p$  also has at least one positive eigenvalue. So we have shown that a critical orbit is stable if and only if it is strongly rigid. The set of stable critical orbits is characterized by the condition that  $d_{ij} = \bar{d}_{ij}$  for all  $(i, j) \in E$ , and hence there are as many as  $2^{N-2}$ stable critical orbits in total. The convergence of system (1) is implied by Lemma 1.

#### **IV. CONCLUSIONS**

Design of control laws that only stabilize the target configurations of a formation is known to be a challenging problem. Indeed, the conjunction of the decentralization constraints and the nonlinear nature of the dynamics lead to the appearance of undesired equilibria in the system. Counting these equilibria in general is a difficult and open problem, let alone characterizing them. In this paper, we have provided a partial solution by exhibiting a class of undirected graphs and control laws for which only desired configurations are stable. We have furthermore derived results characterizing the extremal points of a class of equivariant Morse functions that might be of an independent interest.

#### REFERENCES

- [1] M.-A. Belabbas. On global stability of planar formations. *Automatic Control, IEEE Transactions on*, 58(8):2148–2153, 2013.
- [2] M. Lorenzen and M.-A. Belabbas. Distributed local stabilization in formation control. In *Control Conference (ECC)*, 2014 European, pages 2914–2919. IEEE, 2014.
- [3] B.D.O. Anderson, C. Yu, S. Dasgupta, and A.S. Morse. Control of a three-coleader formation in the plane. *Systems & Control Letters*, 56(9):573–578, 2007.
- [4] M. Cao, B.D.O. Anderson, A.S. Morse, and C. Yu. Control of acyclic formations of mobile autonomous agents. In *Decision and Control*, 2008. CDC 2008. 47th IEEE Conference on, pages 1187–1192. IEEE, 2008.
- [5] L. Krick, M.E. Broucke, and B.A. Francis. Stabilisation of infinitesimally rigid formations of multi-robot networks. *International Journal* of Control, 82(3):423–439, 2009.
- [6] V. Gazi and K.M. Passino. A class of attractions/repulsion functions for stable swarm aggregations. *International Journal of Control*, 77(18):1567–1579, 2004.
- [7] X. Chen. Gradient flows for organizing multi-agent system. In American Control Conference (ACC), 2014, pages 5109–5114. IEEE, 2014.
- [8] M.-A. Belabbas, S. Mou, A.S. Morse, and B.D.O. Anderson. Robustness issues with undirected formations. In *Conference on Decision* and Control (CDC), 2012, pages 1445–1450. IEEE, 2012.
- [9] Z. Sun, S. Mou, B.D.O. Anderson, and A.S. Morse. Formation movements in minimally rigid formation control with mismatched mutual distances. In *Conference on Decision and Control (CDC)*, 2014. IEEE, 2014.
- [10] U. Helmke, S. Mou, Z. Sun, and B.D.O Anderson. Geometrical methods for mismatched formation control. In *The 53rd Conference* on Decision and Control(CDC), 2014. IEEE, 2014.
- [11] S. Mou, A.S. Morse, M.-A. Belabbas, and B.D.O. Anderson. Undirected rigid formations are problematic. In *Conference on Decision* and Control(CDC), 2014. IEEE, 2014.
- [12] B.D.O. Anderson and U. Helmke. Counting critical formations on a line. SIAM Journal on Control and Optimization, 52(1):219–242, 2014.
- [13] U. Helmke and B.D.O. Anderson. Equivariant morse theory and formation control. In *Communication, Control, and Computing (Allerton), 2013 51st Annual Allerton Conference on*, pages 1576–1583. IEEE, 2013.
- [14] G. Laman. On graphs and rigidity of plane skeletal structures. Journal of Engineering mathematics, 4(4):331–340, 1970.
- [15] J.E. Graver, B. Servatius, and H. Servatius. *Combinatorial rigidity*, volume 2. American Mathematical Soc., 1993.
- [16] J. Baillieul. Remarks on a simple control law for point robot formations with exponential complexity. In *Conference on Decision* and Control(CDC), 2006, pages 3357–3362. IEEE, 2006.
- [17] X. Chen. Reciprocal multi-agent systems with triangulated Laman graphs Part I: Morse-Bott index formula. arXiv preprint arXiv:1412.6958, 2014.

#### APPENDIX

#### A. Sketch of the proof of Proposition 6

The Hessian matrix  $H_p$  considered here is with respect to the arrangement  $p = (\vec{x}_1, \dots, \vec{x}_N)$ . Let  $H_i$  be a 2*N*-by-2*N* symmetric matrix derived by adding zero rows and columns to  $H_{p_i}$ . The (2j-1)-th and 2j-th rows/columns of  $H_i$  are zero rows/columns if j is not a vertex of  $G_i$ , and if we remove these zero rows and columns, then we recover  $H_{p_i}$ . It should be clear that  $n_+(H_i) = n_+(H_{p_i})$  and  $n_-(H_i) = n_-(H_{p_i})$ . We then express  $H_p$  as

$$H_p = \sum_{i=1}^m H_i \tag{48}$$

It now suffices to show that

$$\begin{cases} n_{-}(H_{p}) = \sum_{i=1}^{m} n_{-}(H_{i}) \\ n_{+}(H_{p}) = \sum_{i=1}^{m} n_{+}(H_{i}) \end{cases}$$
(49)

Each  $H_{p_i}$  has at least three zero eigenvalues. Let  $\lambda_{i_1}, \dots, \lambda_{i_{l_i}}$  be the other eigenvalues of  $H_{p_i}$ , and for simplicity, assume that they are all nonzero. It should be clear that  $l_i = |E_i|$ , and hence  $\sum_{i=1}^{m} l_i = 2N - 3$ . Suppose for the moment that for each  $\lambda_{i_j}$ , we can find a vector  $u_{i_j} \in \mathbb{R}^{2N}$  so that the ensemble of these vectors satisfies the following condition:

$$\langle u_{j'_{i'}}, H_i u_{j_k} \rangle = \delta_{ij} \delta_{jj'} \delta_{kk'} \lambda_{j_k}$$
(50)

with  $\delta$  the Kronecker delta. Then we can define a matrix U with its column vectors  $u_{i_i}$ 's such that

$$U^{\top}H_{p}U = \operatorname{diag}(\Lambda_{1}, \cdots, \Lambda_{m})$$
(51)

with  $\Lambda_i := \text{diag}(\lambda_{i_1}, \dots, \lambda_{i_{l_i}})$ . Thus, Proposition 6 immediately follows from the Sylvester's Law of inertia.

We will now describe how we construct the vector  $u_{ij}$ . First consider a simple example: Suppose we have a nondegenerate triangle  $\vec{x}_1$ ,  $\vec{x}_2$  and  $\vec{x}_3$  on the plane, then for sufficiently small perturbation  $\delta \vec{x}_i$  of agent  $\vec{x}_i$  for i = 1, 2, we can find a unique displacement  $\delta \vec{x}_3$  of  $\vec{x}_3$  such that we can maintain the distances  $d_{12}$  and  $d_{23}$  by following this displacement, i.e.,

$$\|\vec{x}_3 - \vec{x}_i\| = \|(\vec{x}_3 + \delta \vec{x}_3) - (\vec{x}_i + \delta \vec{x}_i)\|, \quad \forall i = 1, 2$$
(52)

In fact, if we let  $\rho$  be the map

$$\rho: (\delta \vec{x}_1, \delta \vec{x}_2) \mapsto \delta \vec{x}_3 \tag{53}$$

then by the inverse function theorem, the map  $\rho$  is welldefined over a small neighborhood of the origin in  $\mathbb{R}^4$ . Moreover,  $\rho$  is smooth and  $\rho(0) = 0$ . Thus, we can consider the derivative map

$$d\rho_0: \mathbb{R}^4 \to \mathbb{R}^2 \tag{54}$$

at the origin, which describes the infinitesimal motion of the displacement of  $\vec{x}_3$  with respect to the infinitesimal motions of perturbations of  $\vec{x}_1$  and  $\vec{x}_2$ . This geometric fact can be generalized to an arbitrary framework (G, p) with G a triangulated Laman graph. Precisely, we let  $\{(G_i, p_i)\}_{i=1}^m$  be the frameworks associated with the independent partition for (G, p). Then we can perturb one sub-configuration  $p_i$  while preserving the shapes of the others [17], i.e., if we let  $\delta p_i$  be the perturbation of  $p_i$ , then there is a unique displacement  $\delta p_{-i}$  for the rest agents  $p_{-i}$  such that  $p_{-i} + \delta p_{-i}$  can be derived by rotating and/or translating of  $p_i$  in  $\mathbb{R}^2$ . The map

$$\rho_i: \delta p_i \mapsto \delta p_{-i} \tag{55}$$

is well defined over a small neighborhood of the origin, and similarly  $\rho$  is smooth and  $\rho(0) = 0$ . Thus, we can still consider the derivative map  $d\rho_0$  which describes the infinitesimal version of the displacement of  $p_{-i}$ .

We now return to construction of the vector  $u_{i_j}$ . Fix an *i*, and let  $v_{i_j}$  be the unit-length eigenvector of  $H_{p_i}$  corresponding to eigenvalue  $\lambda_{i_j}$ . We now treat  $v_{i_j}$  as the infinitesimal version of the perturbation of  $p_i$ , and correspondingly we let

$$w_{i_i} := d\rho_0(v_{i_i}) \tag{56}$$

be the infinitesimal version of the displacement of  $p_{-i}$ . For simplicity, we assume that  $p_i$  consists of the first k agents, then we construct  $u_{i_i}$  by concatenating  $v_{i_i}$  and  $w_{i_i}$  as

$$u_{i_i} := (v_{i_i}, w_{i_i}) \tag{57}$$

We then show in [17] that the ensemble of the vectors  $u_{ij}$  satisfies the desired condition described by (50).

#### B. Proof of formula (46)

Let  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$  be the unit-length eigenvectors of  $A_{p'}$  corresponding to eigenvalues  $\lambda_1, \dots, \lambda_n$ . We now define, for each  $\vec{v}_i$ , a vector  $\vec{v}_i^* \in \mathbb{R}^{n+1}$  as follows. Let  $v_{ij}$  be the *j*-th entry of  $\vec{v}_i$ ; then

$$\alpha_i := \frac{A_{12}v_{i1} + A_{13}v_{i2}}{A_{12} + A_{13}} \tag{58}$$

Note that this is well defined because by condition C1,  $A_{12}$  and  $A_{13}$  are always positive. Now let

$$\vec{v}_i^* := (\alpha_i, \vec{v}_i) \tag{59}$$

Then by using the third condition on g, we can get

$$A_p \vec{v}_i^* = \lambda_i(0, \vec{v}_i) \tag{60}$$

Now let  $Q_A := (\vec{e}_1, \vec{v}_1^*, \cdots, \vec{v}_n^*)$  be an (n+1)-by-(n+1) matrix; then

$$Q_A^{\top} A_p Q_A = \operatorname{diag}(-A_{12} - A_{13}, \lambda_1, \cdots, \lambda_n)$$
(61)

By Sylvester's Law of inertia, we have

$$\vec{n}(A_p) = \vec{n}(A_{p'}) + sgn(-A_{12} - A_{13})$$
(62)

The analysis for the other part is similar. Let  $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^n$  be the unit-length eigenvectors of  $B_{p'}$  corresponding to eigenvalues  $\mu_1, \dots, \mu_n$ . For each  $\vec{u}_i$ , we let

$$\beta_i := \frac{(x_3 - x_1)u_{i1} + (x_1 - x_2)u_{i2}}{x_3 - x_2} \tag{63}$$

This is also well defined because  $\vec{x}_2$  and  $\vec{x}_3$  are on the *x*-coordinate, but at different positions. Now let

$$\vec{u}_i^* := (\beta_i, \vec{u}_i) \tag{64}$$

Then by using the second condition on g, we can get

$$B_p \vec{u}_i^* = \mu_i(0, \vec{u}_i) \tag{65}$$

Letting  $Q_B := (\vec{e}_1, \vec{u}_1^*, \cdots, \vec{u}_n^*)$ , it then follows that

$$Q_B^{\dagger} B_P Q_B = \text{diag}(-B_{12} - B_{13}, \mu_1, \cdots, \mu_n)$$
 (66)

This then shows that

$$\vec{n}(B_p) = \vec{n}(B_{p'}) + sgn(-B_{12} - B_{13})$$
(67)

which completes the proof.