Control of Sensors of a Gaussian Stochastic Control System

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Abstract— **With which intensity to request high-quality components of the observed output of a Gaussian system driven by Brownian motions so as to minimize the trace of the error covariance when there is a cost for doing so? The optimal stochastic control problem is considered on an infinite horizon and with a discounted cost, and separately also for discounted impulse control. The optimal control law is for the scalar cases proven to be a threshold control law. The dynamic behavior of the closed-loop system is such that if the conditional variance is large then the sensor input for better quality observations is switched on while if the uncertainty is small then the input for better quality observations is switched off. The optimal control law for the drift input is identical to the classical LQG case.**

I. INTRODUCTION

The purpose of this paper is to present the optimal control laws for the problem of control of sensors of a Gaussian stochastic control system.

The problem is motivated by control of a set of underwater vehicles, see [1]. Other motivations are smart grids, medical systems, and a variety of networked control problems. A particular underwater vehicle does not know the state of other vehicles in the neighborhood. In that case it can send a sonar communication to another vehicle to request its position and speed vector. But such a communication requires electric energy which is a scarce resource. The problem is thus whether or not to request sensor information and, if so, when. A related problem arises in the competition of commercial firms where a firm can request a report about another firm at a cost.

The problem of control of the activation of sensors has been investigated but the assumptions on the models differ widely. Early publications are [2], [3], [4]. The results differ by case. In several papers it is assumed that there is a discrete-time system and that the optimal control law is periodic. With these assumptions it is then computed what the length of the period should be. This is useful for particular applications. But it leaves open the question what the form of the optimal control law actually is. A more theoretical motivation is to increase the understanding of the interaction of the control of the diffusion process and the control of the observation process

The problem of control of the observation channel was formulated by the authors in [5]. That paper differs from the current paper in that a discrete-time system was formulated. In addition this paper uses a different control objective and investigates optimality of a particular control law. Computations for the model of [5] were carried out by a student and these are described in [6].

The contribution of this paper is to show for a particular continuous-time Gaussian stochastic control system and an infinite-horizon discounted cost function that the optimal control law is a threshold control law as defined in the paper which depends on the conditional variance or the uncertainty of the state estimate. The problem is investigated for the optimal stochastic control with partial observations and dynamic programming, and for an impulse control formulation. The control law uses an information system consisting of the state estimate of the Kalman filter and of the estimation variance provided by the solution of the filter Riccati differential equation. The latter equation is a deterministic differential equation whose parameters depend on the past outputs. The optimal control law for the drift or diffusion is an ordinary LQG control law though the control law depends on the states of the filter system which in turn depend on the sensor input. The optimal control law for the sensor input is a threshold control, which is defined in this paper as a generalization of a threshold control law of control of queueing systems. Thus, the sensor is activated if the error variance exceeds a threshold. The dynamic behavior of the closed-loop system controlled with a threshold control law may be periodic or not depending on the particulars of the problem.

The contents of this paper is briefly summarized. The next section contains a detailed verbal problem formulation. Section III presents the stochastic control problem, the set of control laws, the closed-loop systems, the cost function, and a few special cases. Section IV treats only the scalar case with dynamic programming. The impulse control case is treated in Section V.

II. PROBLEM

The problem is motivated by control of underwater vehicles as described in the introduction. Another motivation is the operation of sensor networks where the available energy is finite and both sensor activation and communication require a considerable amount of energy.

A second motivation is to develop control theory with results on the interaction of observation and control. How can the input be used to obtain state estimates with smaller uncertainty? What types of control laws for the sensor input are to be expected?

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In the literature there are several variants of the problem which may not be transformed into each other. In a version, the controller may choose only one extra sensor. In another version, the controller can choose any subset of the available sensors. In a third model, the controller may choose a linear combination of all sensors. The authors have chosen the general model described in the next section.

The problem is thus to determine when and with what intensity to request additional sensor information from the control system so as to minimize a cost. The costs consist of the sum of two components, the first component of the performance contains the drift input and the controlled-behavior of the control system. The second component of the performance criterion contains the sensor input and the quality of the state estimate.

The authors have published a paper, see [5], with a discrete-time model and a finite-horizon cost function. The dynamic programming recursion was derived. Subsequently, a student working with A.C.M. Ran and Jan H. van Schuppen has carried out computations which show that the optimal control law becomes almost periodic if time progresses for the numerical cases considered. The authors had conjectured that for the infinite-horizon control problem the optimal control law is periodic. But this may not be true in general. They now prefer the viewpoint that the optimal control law is a threshold control law.

Several papers in the literature deal with what may now be called the problem of control of sensors. The paper by M. Athans, [2], formulates a problem which is solved by the matrix minimum principle but not by dynamic programming based on past observations. Willsky et al. in [7] provide a solution in another special case. See further the reference on the discrete-time infinite-horizon case [8], [9] and the references mentioned in those papers. The paper by Wu and Arapostathis [10] formulates an optimal stochastic control problem for discrete-time stochastic systems. The authors of that paper prove that the optimal control law for the drift of the control system has the same structure as the LQG control law.

In the engineering literature there are still other publications which provide algorithms to compute the optimal control law on an infinite-horizon. In those papers the assumption is made that the control law is periodic and then the best optimal periodic control law is computed. For the latter approach see for example the papers of Ling Shi and co-authors, see [11].

The heart of the problem is control of the Riccati differential equation. A paper on controllability of the Riccati differential equation is [12].

The authors of this paper have chosen to investigate in this paper the continuous-time version of the problem with a discounted cost because they expected it to yield a periodic control law which the discrete-time problem was conjectured not to have. After the derivation of the results, it turns out that the optimal control law may not necessarily lead to a periodic behavior of the closed-loop system. The structure of the control law is that of a general threshold control law. The continuous-time case has its own merits besides the discrete-time case.

III. APPROACH

A reference on continuous-time stochastic control problems with partial observations is the book [13].

Definition III.1 *Consider a time-invariant continuous-time* Gaussian stochastic control system *with representation,*

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu_d(t)dt + Mdv(t), && (1)\\ dy(t) &= C(u_s(t))x(t)dt + Ndv(t), && (2)\\ & x(0) = x_0,\ y(0) = 0, && (3)\\ u &= \begin{pmatrix} u_d \\ u_s \end{pmatrix}, && (4) \end{aligned}$$

in which (Ω, F, P) denotes the probability space, $T = [0, \infty)$ is the time index set, $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ are respectively the state space, the input space, and the output space, $x_0 : \Omega \to \mathbb{R}^n$ is the initial state which is a Gaussian random variable with mean m_0 and variance Q_0 denoted by $x_0 \in G(m_0, Q_0)$ (Gaussian random variables are denoted by membership in $G(m, Q)$ where m and Q denote respectively the mean and the variance), $v : \Omega \times T \to \mathbb{R}^{m_v}$, is a standard (variance equal to identity times time) Brownian motion process, F^{x_0} and F^v_∞ are independent σ-algebras, $u : \Omega \times T \to \mathbb{R}^m$ is a stochastic process of which the dependence is specified below, $m_d,\ m_s \in \mathbb{Z}_+$ such that $m = m_d + m_s$, $u_d : \Omega \times T \to \mathbb{R}^{m_d}$ called the drift input, $u_s : \Omega \times T \to \mathbb{R}^{m_s}$ called the sensor input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M \in \mathbb{R}^{n\times m_v}$, $C \in \mathbb{R}^{p\times n}$, $N \in \mathbb{R}^{p\times m_v}$ are matrices, $x : \Omega \times T \to X = \mathbb{R}^n$ is a stochastic process defined by the stochastic differential equation (1), and $y : \Omega \times T \to Y = \mathbb{R}^p$ is a stochastic process defined by the equation (2). Denote the filtration satisfying the usual conditions generated by the σ-algebras F^{x_0} and $\{F^v_t, t \in T\}$ by $\{F_t \subseteq F,\ t \in T\}$. Denote the filtration generated by the output process y and satisfying the usual conditions by $\{F^y_t,\ t \in T\}$. Restrict attention to the subclass of input processes which are adapted to the filtration generated by the output process, denoted by $\{u(t), F^y_t, t \in T\}$, hence for all $t \in T$, $u(t)$ is F^y_t measurable. These conditions then hold for both u_d and u_s.

It is assumed that for any nonzero input vector u_s, the matrix pair $(A, C(u_s))$ is an observable pair. This is conjectured to be a sufficient condition for a new observability condition involving both the output function and the sensor input.

Denote the identity matrix of size $n \in \mathbb{Z}_+$ by I_n. Denote the set of symmetric positive-definite matrices of size $n \times n$ by $\mathbb{R}^{n\times n}_{\mathrm{spd}}$.

Example III.2 The *case of a scalar Gaussian stochastic control system.* The model is defined in terms of the formulas,

$$n = 1,\ m_d = 1,\ m_s = 1,\ p = 1, \quad (5)$$

$$c(u) = 1 + c_1 u_s,\ c_1 \in \mathbb{R}\backslash\{0\}. \quad (6)$$

The usefulness of this example is that the calculations are straightforward.

Example III.3 The *case of two Gaussian stochastic control systems which are interacting one-way only*.

$$n = 2,\ m_d = 1,\ m_s = 1,\ p = 2, \quad (7)$$

$$c(u) = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22}u_s \end{pmatrix},\ c_{11},\ c_{22} \in \mathbb{R}\backslash\{0\}, \quad (8)$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}. \quad (9)$$

The matrix $M \in \mathbb{R}^{2\times 2}$ is a diagonal matrix. Thus the system has two subsystems of which the first one is always observed by the first output component and the second can be observed if $u_s \neq 0$. The system consists of two subsystems of which the second subsystem influences the first subsystem but not conversely. Whether or not it is useful to switch on the sensor of the second subsystem depends on the magnitude of the elements of the A matrix, in particular on the ratio $a_{12}^2/|a_{11}a_{22}|$.

Example III.4 The *case of the sensor selection per state-component*. The case is specified by the formulas,

$$p = n,\ m_d = 1,\ m_s = n,$$
$$C(u_s) = \mathrm{Diag}(u_s) = \begin{pmatrix} u_{s,1} & 0 & \dots & 0 \\ 0 & u_{s,2} & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & u_{s,n} \end{pmatrix}.$$

The model is such that it is in principle possible to sense each state component. The control objective will then determine which sensors are most relevant where the cost of the use of each sensor will influence the outcome.

Definition III.5 *Consider the Gaussian stochastic control system. Define the* class of partially-observed control laws *by the notation,*

$$\begin{aligned} G &= \{g : T \times L(T,Y) \to U | g(t,.)\ F_t^y \text{ adapted}\}, \\ L(T,Y) &= \{h : \Omega \times T \to Y | \text{measurable function}\}, \\ y^g[0,t) &= \{y^g(s) \in \mathbb{R}^p,\ \forall\ s \in [0,t)\}, \\ & y^g[0,t) \in L(T,Y), \\ & g(t, y^g[0,t)) \text{ is } F_t^y \text{ adapted, } \forall\ t \in T. \\ & g = (g_d, g_s). \end{aligned}$$

Definition III.6 *For any control law $g \in G$ define the closed-loop system by the equations,*

$$\begin{aligned} dx^g(t) &= Ax^g(t)dt + Bg_d(t, y^g[0,t))dt + Mdv(t), \\ dy^g(t) &= C(g_s(t, y^g[0,t)))x^g(t)dt + Ndv(t), \\ & x^g(0) = x_0,\ y^g(0) = 0, \\ u_d^g(t) &= g_d(t, y^g[0,t)), \\ u_s^g(t) &= g_s(t, y^g[0,t)). \end{aligned}$$

Definition III.7 *Define the discounted cost function $J : G \to \mathbb{R}_+$ with discount rate $r \in (0,\infty)$ by the equation,*

$$J(g) = E\left[\begin{array}{l} \int_0^\infty \exp(-rs)\times \\ \left[b_d(x(s), u_d^{g_d}(s)) + b_s(x^g(s), u_s^{g_s}(s))\right] ds \end{array} \right].$$

Problem III.8 *Consider the Gaussian stochastic control problem of Def. III.1 and the cost function of Def. III.7. Solve the following* optimal stochastic control problem *for this system and for a discounted cost function,*

$$J^* = \inf_{g\in G} J(g) = J(g^*); \quad (10)$$

thus, determine the value J^* and prove existence of and determine the optimal control law $g^* \in G$.

The filtering problem is to determine at any time the conditional distribution of the state at that time conditioned on the past observations. This is called the conditional Kalman filter because the C matrix depends on the current sensor input which in turn depends on the past outputs. The conditional Kalman filter is derived for discrete-time systems in [14] and for continuous-time systems in [15, Ch. 11]. It is known that the solution of that problem for the Gaussian stochastic control system equals a conditional Gaussian distribution function of the form,

$$\begin{aligned} & E[\exp(iw^T x(t)) | F_t^y] && (11) \\ &= \exp(iw^T \hat{x}(t) - \frac{1}{2} w^T Q_e(t) w),\ \forall\ w, \\ d\hat{x}(t) &= A\hat{x}(t)dt + Bu_d(t)dt + \\ & +K(t, u_s(t))[dy(t) - C(u_s(t))\hat{x}(t)dt], \\ & \hat{x}(0) = m_0, && (12) \\ dQ_e(t)/dt &= AQ_e(t) + Q_e(t)A^T + MM^T + \\ & -H(t, u_s(t))[NN^T]^{-1}H(t, u_s(t))^T, \\ & Q_e(0) = Q_0, && (13) \\ e(t) &= x(t) - \hat{x}(t),\ e : \Omega \times T \to \mathbb{R}^n, && (14) \\ H(t, u_s(t))) &= Q_e(t)C(u_s(t))^T + MN^T, && (15) \\ K(t, u_s(t)) &= [Q_e(t)C(u_s(t))^T + MN^T][NN^T]^{-1}, \\ & \hat{x} : \Omega \times T \to \mathbb{R}^n,\ Q_e : \Omega \times T \to \mathbb{R}_{\text{spd}}^{n\times n}. \end{aligned}$$

Note that the filter system (12,13) is described by its state at time $t \in T$, $(\hat{x}(t), Q_e(t))$. The variance Q_e is of the estimation error and is not equal to the variance of the state x. The control system of the conditional variance is primarily a deterministic system (there is no stochastic disturbance process driving the system) though one of its parameters, the matrix $C(u_s)$, depends on the sensor input u_s which can be a stochastic process.

By conditioning, the cost function can be transformed to the form,

$$J(g) = E\left[\begin{array}{l} \int_0^\infty \exp(-rs)\times \\ \times[b_{d,1}(\hat{x}(s), Q_e(s), u_d(s)) + \\ +b_{s,1}(Q_e(s), u_s(s))]ds \end{array} \right], \quad (16)$$

$$\begin{aligned} & b_{d,1}(\hat{x}(t), Q_e(t), u_d(t)) \\ & \quad = \quad E[b_d(x(t), u_d(t)) | F_t^y], \end{aligned} \tag{17}$$

$$\begin{aligned} & b_{s,1}(Q_e(t), u_s(t)) \\ & \quad = \quad E[b_s(x(t), u_s(t)) | F_t^y]. \end{aligned} \tag{18}$$

From now on it will be assumed that the conditional cost function $b_{s,1}$ does not depend on the filter state $\hat{x}$ but only on the filter variance Q_e and the sensor input u_s as indicated above.

The partially-observed stochastic control problem III.8 can then be transformed to the following competely-observed optimal stochastic control problem.

Problem III.9 *Consider the completely-observed stochastic control system of the equations (12,13) and the discounted cost of equation (16). Define the class of time-invariant state-based optimal control laws*

$$\begin{aligned} G_{po} &= \{g_{po} : \hat{X} \times Q_e \to U | \textit{measurable function}\}, \\ u_d(t) &= g_{d,po}(\hat{x}(t), Q_e(t)),\ g = (g_{d,po}, g_{s,po}), && (19) \\ u_s(t) &= g_{s,po}(\hat{x}(t), Q_e(t)), && (20) \\ J_{po}(g) &= J(g),\ J_{po} : G_{po} \to \mathbb{R}_+. && (21) \end{aligned}$$

Solve the optimal stochastic control problem

$$\inf_{g_{po} \in G_{po}} J(g_{po}). \tag{22}$$

The dynamic programming equation for infinite-horizon discounted-cost optimal stochastic control is known, [16, Subsec. 3.1.2, p. 56].

Of interest to control theory is primarily the structure of the control law. Statements about this follow which are argued in this section in more detail than in the next two sections.

In the literature it is often asserted that the optimal control law is such that the behavior of the closed-loop system is periodic. The authors consider this to be incorrect in general, the behavior, after a transient phase, could be almost periodic, meaning that regularly periods are followed by a period of a different length.

The sensor control problem as formulated above has the property that the value function decomposes additively into a drift term and a sensor term. The optimal control law for the drift has the same structure at the well known LQG control law though the states of the filter system depend on the sensor inputs. The control law of the sensor input is discussed in the following two sections. Because of this decomposition, several papers describe the solution as having a separation property. The authors do not favor this term for the structural result of the control law of the drift. The decomposition was also derived by Wu and Arapostathis in [10] for the discrete-time case. The authors have written about the sensor control problem for a partially observed nonlinear and non-Markov process see [17]. For that problem, there is no decomposition of the drift and of the sensor activation problem as described above.

A general threshold control law is defined next for later use.

Definition III.10 *Consider Problem III.9. A control law is called a* threshold control law *if there exists a partition of the state set of the conditional variance such that the control law has the form,*

$$\begin{aligned} \mathbb{R}_{spd}^{n \times n} &= R_1 \cup \partial R_1 \cup R_2, && (23) \\ & \quad R_1,\ R_2,\ \textit{open connected subsets,} \\ & \quad \partial R_1 = \partial R_2,\ \textit{the common boundary of } R_1\ R_2; \\ g_s(Q) &= \begin{cases} g_{s,1}(Q), & Q \in R_1, \\ g_{s,\partial R_1}(Q), & Q \in \partial R_1, \\ g_{s,2}(Q), & Q \in R_2. \end{cases} && (24) \end{aligned}$$

If the state of the conditional variance is one ($n = 1$) then the partition above has to be in two intervals and a point in between as in

$$\mathbb{R}_+ = [0, \overline{q}_e) \cup \{\overline{q}_e\} \cup (\overline{q}_e, \infty).$$

If the state-space dimension n is two or larger then the boundary can be any manifold in the state set. The above definition is a generalization of the classical threshold control law of queueing systems and of communication networks to more general spaces.

Of interest is now the dynamic behavior of the closed-loop system of the conditional variance when it is controlled by a threshold control law. First consider the scalar case. If initially the conditional variance is below the threshold then the sensor input is switched off. The conditional variance of the filtering error, produced by the Riccati differential equation then increases either till it reaches a limit below the threshold or till it reaches the threshold from below. After the variance crosses the threshold, the sensor is activated. If at any time the conditional variance is above the threshold then the sensor is activated. The conditional variance will then decrease till it reaches an equilibrium state or till it reaches the threshold from above. After the variance crosses the threshold, the sensor input is deactivated. The character of the dynamic behavior of the closed-loop system is thus that of a switched system with two modes. The conditional variance is either monotonically increasing or monotonically decreasing depending on the mode. Whether the behavior is actually periodic and what the length of the period is, are questions that require further study.

The second case, that of a multivariable control system, with $n > 1$, is more complicated than the scalar case. Consider on the set of symmetric positive-definite matrices, the partial order defined by the difference of two matrices to be positive-definite. The trajectory of the conditional variance is not necessarily monotonically increasing or decreasing depending on the mode. Due to the form of the output matrix $C(u_s)$ and the geometry of the filter Riccati differential equation, it could be that the error variance shrinks along one axis while it grows along another axis. More time is needed to investigate the dynamics of the closed-loop Riccati differential equation.

Results are provided in the following two sections, the first section deals with the regular optimal control with discounted cost and the second section deals with discounted impulse control.

The optimal control problem for the sensor input only, can be considered a deterministic optimal control problem with a special interpretation. The control system is described by the Riccati differential equation, which is not driven by a Brownian motion process. The class of control laws is such that the control law can depend on the past outputs. Hence the sensor input is a stochastic process and so is the conditional variance which is the solution of the Riccati differential equation. The optimal control law is a threshold control which is a deterministic function of the conditional variance. The closed-loop system is thus entirely a deterministic system because the dependence on the inputs has disappeared. It is for these reasons that the optimal control problem is regarded as deterministic.

IV. THE SCALAR CASE

This section treats the case of a scalar stochastic control system for the problem defined before in this paper. The usefulness of the example is that the equations are simplified and that analytic solutions are easier to obtain.

Denote the stochastic control system for the scalar system,

$$\begin{aligned}
dx(t) &= ax(t)dt + bu_d(t)dt + Mdv(t),\ x(0)=x_0, \\
dy(t) &= c(u_s(t))dt + Ndv(t),\ y(0)=0, && (25)\\
c(u_s) &= 1+u_s, && (26)\\
& \quad d<v,v>(t) = I_2 dt,\ MN^T = 0, \\
d\hat{x}(t) &= a\hat{x}(t)dt + bu_d(t)dt + \\
& \quad + k(u_s(t))[dy(t) - c(u_s(t))\hat{x}(t)dt], && (27)\\
&= a\hat{x}(t)dt + bu_d(t)dt + k(u_s(t))d\overline{v}(t), && (28)\\
dq_e(t)/dt &= 2aq_e(t) + q_v - q_e(t)^2 c(u_s(t))^2 q_w^{-1}, \\
k(u_s(t)) &= q_e(t)c(u_s(t))q_w^{-1}, && (29)\\
d\overline{v}(t) &= dy(t) - c(u_s(t))\hat{x}(t)dt, && (30)\\
& \quad d<\overline{v},\overline{v}>(t) = q_w dt = NN^T dt, && (31)\\
& \quad \overline{v} \text{ is a Brownian motion}, \\
& \quad U_d = \mathbb{R},\ U_s = [0, \overline{u}_{s,\max}], \\
& \quad b_d(x,u_d) = c_1 x^2 + c_2 u_d^2, && (32)\\
& \quad b_{s,1}(q_e,u_s) = b_3 q_e + b_4 c(u_s)^2 q_e^2 q_w^{-1}, && (33)\\
& \quad c_1 \in \mathbb{R}_+,\ c_2 \in (0,\infty).
\end{aligned}$$

Assume that the system is stochastically controllable, $b \neq 0$, and that $c(u_s) \neq 0$ for all $u_s \in U_s$. Assume also that $\text{rank}(N) = 1$. The cost function J_{po} and the dynamic programming equation for the value function v are then,

$$\begin{aligned}
& J_{po}(g_{po}) && (34)\\
&= E\Big[\int_0^\infty \exp(-rs) \times \\
& \quad \times (b_{d,1}(\hat{x}(s), q_e(s), u_d(s)) + b_{s,1}(q_e(s), u_s(s)))ds\Big],
\end{aligned}$$

$$\begin{aligned}
0 &= \inf_{(u_d,u_s)\in(U_d\times U_s)} && (35)\\
& \quad \left[\begin{array}{l} L(\hat{x},q_e,(u_d,u_s))v(\hat{x},q_e)+ \\ +b_{d,1}(\hat{x},q_e,u_d) + b_{s,1}(q_e,u_s) - rv(\hat{x},q_e) \end{array}\right].
\end{aligned}$$

Theorem IV.1 *Consider the scalar control system. Assume further that there exists a piecewise-differentiable function $v_s : \mathbb{R}_+ \to \mathbb{R}$ which is a solution of the differential equation,*

$$\begin{aligned}
\mathbb{R}_+ &= Q_1 \cup \{\overline{q}_e\} \cup Q_2, \\
Q_1 &= \{q_e \in \mathbb{R}_+ | h(q_e) > 0\}, && (36)\\
Q_2 &= \{q_e \in \mathbb{R}_+ | h(q_e) < 0\}, \\
\partial Q_1 &= \{q_e \in \mathbb{R}_+ | h(q_e) = 0\} = \{\overline{q}_e\}, \\
h(q_e) &= q_c + b_4 - \frac{dv_s(q_e)}{dq_e}, && (37)\\
0 &= c(0)^2 h(q_e) q_e^2/q_w + c_1 q_e - rv_s(q_e) + \\
& \quad + \frac{dv_s(q_e)}{dq_e},\ \forall\ q_e \in Q_1, && (38)\\
0 &= c(u_{s,\max})^2 h(q_e) q_e^2/q_w + c_1 q_e - rv_s(q_e) + \\
& \quad + \frac{dv_s(q_e)}{dq_e},\ \forall\ q_e \in Q_2, && (39)\\
& \frac{dv_s(\overline{q}_e)}{dq_e}|_{q_e+} = \frac{dv_s(\overline{q}_e)}{dq_e}|_{q_e-} = q_c + b_4. && (40)
\end{aligned}$$

Then:

(a) *the value function satisfies an additive decomposition*

$$v(\hat{x},q_e) = v_d(\hat{x},q_e) + v_s(q_e); \quad (41)$$

(b) *the optimal control law and the value function of Problem III.9 for the drift input are equal to,*

$$\begin{aligned}
g_d(\hat{x},q_e) &= -[bq_c c_2^{-1}]\hat{x}, && (42)\\
v_d(\hat{x},q_e) &= q_c \hat{x}^2, && (43)\\
0 &= (2a-r)q_c + c_1 - q_c^2 b^2 c_2^{-1}. && (44)
\end{aligned}$$

thus the optimal control law has the same structure as the well known LQG control law though the filter state $\hat{x}$ depends on the sensor input via $k(u_s)$;

(c) *the optimal control law for the sensor input is a threshold control law of the form,*

$$g_s(q_e) = \left\{ \begin{array}{ll} 0, & q_e \in Q_1, \\ \overline{u}_{s,\max}, & q_e \in Q_2. \end{array} \right. \quad (45)$$

V. IMPULSE CONTROL

An alternative way to investigate the control of sensors, and to quantify the value of observations for the reduction of the future operations cost of the system, is to apply impulse control to the sensor activation, assuming for simplicity that no continuous observations are available, but that from time to time the sensor can be activated at a cost b_n per activation. The reader finds in this section a model and an algorithm for selecting optimal sensor activation times.

For references on counting processes and on impulse control see [18], [19], [20], [21], [22].

Definition V.1 A stochastic control system for control of sensors using impulse control. *Consider a probability space and a stochastic control system like in Def. III.1 but with the following objects. Define a collection of stopping times* $\{\tau_k,\ k \in \mathbb{Z}_+\}$ *which are predictable with respect to the observation filtration* $\{F^y_{t-}, t \in T\}$ *defined below. At each stopping time* τ_k *the sensors are activated. The process,*

$$n(t) = \sum_{k=1}^{\infty} I_{\{\tau_k \le t\}},\ n : \Omega \times T \to \mathbb{R}_+,$$

counts how many observations have been taken up to the current time t. *Define the stochastic control system,*

$$dx(t) = [Ax(t) + Bu_d(t)]dt + Mdv(t),\ x(0) = x_0, \quad (46)$$
$$dy(t) = [Cx(t) + Nw(t)]dn(t),\ y(0) = 0, \quad (47)$$

where most objects are defined as in Def. III.1, except that $v : \Omega \times T \to \mathbb{R}^{m_v}$ *and* $w : \Omega \times T \to \mathbb{R}^{m_w}$ *are standard Brownian motion processes, the* σ*-algebras* F^{x_0}, F^v_∞, *and* F^w_∞ *are independent, the state process* $x : \Omega \times T \to \mathbb{R}^n$ *is the solution of the stochastic differential equation (46) and the output process is the solution of the equation (47). Assume that the drift input is adapted to the filtration of the output process while the control process* $n(t)$ *is predictable with respect to that filtration.*

The discounted cost function is defined by the equation,

$$J = \begin{array}{l} E\int_0^\infty e^{-\gamma.s}\times \\ ([b_d(x(s), u_d(s)) + b_s(Q(s))]ds + b_n dn(s)). \end{array}$$

The solution of the stochastic control problem with partial observations is carried out as described earlier in the paper. The stochastic control system is in continuous time, with sampled observations at the stopping times $\{\tau_k,\ k \in \mathbb{Z}_+\}$. Note that the difference between two successive stopping times, $\tau_{k+1} - \tau_k$, varies in general with $k \in \mathbb{Z}_+$, but that measurement update equations are the same as for the discrete-time Kalman filter. In between observations the filter equations are the same as for the continuous-time Kalman filter but with $y(t) = 0$, see [23].

The filter system is then described by the formulas,

$$\begin{aligned}
& E[\exp(ia^T x(t))|F^y_t] \\
&\quad = \exp(ia^T \hat{x}(t) - \frac{1}{2} a^T Q(t) a), \\
d\hat{x}(t) &= [A\hat{x}(t) + Bu_d(t)]dt + K(t-)dy(t) \\
&= [A\hat{x}(t) + Bu_d(t)]dt + \\
&\quad +K(t-)[y(t-) - C\hat{x}(t-)]dn(t), \\
dQ(t) &= [AQ(t) + Q(t)A^T + MM^T]dt + \\
&\quad -\phi_k Q(t-)C^T[CQ(t-)C^T + NN^T]^{-1} \times \\
&\qquad \times CQ(t-)\phi_k dn(t),\ \tau_{k-1} \le t < \tau_k, \\
&\quad Q(0) = Q_0, \\
\phi_k &= \exp(A(\tau_k - \tau_{k-1})), \\
K(t-) &= \phi_k Q(t-)C^T[CQ(t-)C^T + NN^T]^{-1}, \\
& \Delta_Q(Q(\tau_k), u_s(\tau_k)) = Q(\tau_k+) - Q(\tau_k-) \\
&= -\phi_k Q(\tau_k)C^T[CQ(\tau_k)C^T + NN^T]^{-1} \times \\
&\quad \times CQ(\tau_k)\phi_k.
\end{aligned}$$

Note the inverse terms in the definition of K and in the Riccati differential equation which are related to the discrete-time Kalman filter. That in the formula there is no cross term of the noises is due to the assumption that the Brownian motion processes v and w are independent.

From now on it is assumed that the cost on the drift is a quadratic form in (x, u_d) and that the sensor cost is a monotonically increasing function of the variance Q. The cost function can then be rewritten as,

$$\begin{aligned}
& b_d(x(t), u_d(t)) \\
&\quad = \begin{pmatrix} x(t) \\ u_d(t) \end{pmatrix}^T \begin{pmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{pmatrix} \begin{pmatrix} x(t) \\ u_d(t) \end{pmatrix}, \\
L &= \begin{pmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{pmatrix} = L^T \ge 0,\ L_{22} > 0, \\
& E[b_d(x(t), u_d(t))|F^y_t] \\
&\quad = \begin{pmatrix} \hat{x}(t) \\ u_d(t) \end{pmatrix}^T \begin{pmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ u_d(t) \end{pmatrix} + \\
&\qquad +\text{trace}(L_{11} Q(t)), \\
& b_s(Q(t), n) \\
&\quad = \text{trace}(Q(t)Q_s),\ Q_s \in \mathbb{R}^{n\times n}_{\text{spd}}, \\
& E[b_s(Q(t), n)|F^y_t] \\
&\quad = \text{trace}(Q(t)Q_s).
\end{aligned}$$

Note that, since $n(t)$ is by definition adapted to F^y_t, the sensor activation part of the cost remains unchanged. The optimal control law for the partially-observed stochastic control problem then follows from the standard dynamic programming calculations.

Theorem V.2 *For the above defined stochastic control system, cost function, and filter system, the optimal control laws are given by the following equations, provided there exists a solution to the functional BHJ equation:*

$$\begin{aligned}
g_d(\hat{x}, Q) &= -L_{22}^{-1} B^T Q_c \hat{x}, \\
0 &= AQ_c + Q_c A^T + L_{11} - Q_c B L_{22}^{-1} B^T Q_c, \\
0 &= \sum_{i,j} \left(\frac{\partial v(Q)}{\partial Q_{i,j}} (AQ + QA^T + MM^T)_{i,j} \right) + \\
&\quad -\gamma.v(Q) + \text{trace}(QQ_c) + \text{trace}(QQ_s), \\
g_s(Q) &= \begin{cases} \textit{sensor activation} \\ \quad \textit{if } v(Q) + b_n \le v(Q - \Delta Q), \\ \textit{no sensor activation} \\ \quad \textit{if } v(Q) + b_n > v(Q - \Delta Q). \end{cases}
\end{aligned}$$

Because of space limitations, not all details of the proof are provided. The existence of an optimal control law and of the value function requires further research. First one can prove that the optimal control of the drift function is the usual LQG control law which involves the algebraic control Riccati equation with solution $Q_c \in \mathbb{R}^{n\times n}$. The remaining discounted cost equals, up to constants not involving the filter states,

$$\int_0^\infty e^{-\gamma.t}\left[\mathrm{trace}(Q(t)Q_c(t)) + \mathrm{trace}(Q(t)Q_s)\right]dt + \\ + \int_0^\infty e^{-\gamma.t} b_n dn(t).$$

The resulting deterministic dynamic programming equation (the BHJ equation) and the optimal sensor control law is then,

$$0 = \sum_{i,j}\left(\frac{\partial v(Q)}{\partial Q_{i,j}}(AQ + QA^T + MM^T)_{i,j}\right) + \\ -\gamma v(Q) + \mathrm{trace}(QQ_c) + \mathrm{trace}(QQ_s).$$

Note that the control problem optimizing the behavior of $Q(t)$ via the sensor activation control actually is a deterministic control problem. Thanks to the separation theorem the cost increase due to state uncertainty is independent of the stochastically evolving states $x(t)$ and $\hat{x}(t)$. Using this deterministic character of this sensor activation control problem it is possible to show that the behavior of the optimally controlled plant is as follows. There exists a manifold $\mathrm{Switch} \subset \mathbb{R}^{n\times n}_{\mathrm{spd}}$ of positive semidefinite matrices such that, whenever $Q(t)$ reaches a value in Switch, then the sensor is activated, and the variance matrix $Q(t+)$ jumps from a value $Q \in \mathrm{Switch}$ to a value in another manifold of positive semidefinite matrices $Q - \Delta Q$ (as defined in the Kalman filter above).

Calculating these manifolds is in general quite complicated. However in the scalar case $x(t) \in \mathbb{R}$, the optimal sensor activation policy becomes very simple. The manifold Switch is reduced to a threshold, an element $q_T \in \mathbb{R}$. When no observations are taken the $q(t)$ increases upto the next sensor activation (the variance $Q(t)$ in the Kalman filter is now reduced to a scalar positive real number). Optimal sensor activation is achieved by selecting as stopping time τ_k the next time $q(t)$ reaches q_T. The value of $q(\tau_k)$ then jumps immediately to $\frac{N^2}{N^2+C.q(\tau_k-)}$. It then starts increasing again until the next observation, at stopping time τ_{k+1}, when once again the value q_T is reached. Clearly the optimal sensor activation policy leads to a-periodic behavior of the system, after a short transient period. If initially $q(0) < \frac{N^2}{N^2+C.q_T}$ then it will keep increasing until q_T is reached, and then the periodic behavior starts; if initially $q(0) \geq q_T$ then the sensor will be activated several times (almost) simultaneously, until q achieves a value in the interval $(\frac{N^2}{N^2+C.q_T}, q_T]$ and then starts increasing until it reaches q_T, at which time the periodic operation again starts. This argument also indicates that the periodic cycle will be stable. For the higher dimensional case however no periodic behavior can be expected.

For the scalar case the calculation of q_T is actually possible using $v(q_T) - v(\frac{N^2}{N^2+C.q_T}) = b_n$, and the fact that in between observations $q(t)$ satisfies the scalar linear equation $\dot{q}(t) = 2.a.q(t) + M^2$. It is therefore possible to calculate explicitly the discounted cost in between two successive observation times τ_k and τ_{k+1}, and to explicitly obtain an expression for $v(q)$. These calculations are similar to those in [10].

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