2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Synchronization for a network of identical discrete-time agents with unknown, nonuniform constant input delay

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Abstract— **This paper studies the synchronization problem for a network of identical discrete-time agents with unknown, nonuniform constant input delays. The agents are at most critically stable and non-introspective (i.e. the agents have no access to their own states or outputs). There exists full state coupling among the agents. An upper bound for the delay tolerance is obtained which explicitly depends on agent dynamics. For any unknown delay satisfying this upper bound, a controller design methodology is proposed without relying on exact knowledge of the network topology so that synchronization in a set of unknown networks can be achieved.**

I. Introduction

The synchronization problem for a network has received substantial attention in recent decades, where the objective is to secure asymptotic agreement on a common state or output trajectory through decentralized control protocols. The analysis and design method for synchronization problem largely depends on the information available to the agents in a network. If the agents have access to their own states or outputs, they are called introspective (e.g., [22], [10]); otherwise non-introspective (e.g., [2], [9]). These works are all related to continuous-time agents. Synchronization in a network of discrete-time agents has been studied in both introspective (e.g., [18]) and non-introspective cases (e.g., [3], [4], [6], [14] and references therein). In particular, for non-introspective agents, [3] introduces the concept of a 'disc margin' in the context of the discrete-time Linear Quadratic Regulator (LQR) problem, based on which a static synchronization controller can be designed for critically unstable agents using relative information of neighboring agents. An observer-based distributed synchronization controller is constructed in [4] for general linear agents, which however requires communication between controllers using the same network topology.

All the works above assume an idealized network model. But, in practical applications, the network model is always imperfect. In particular, time-delay effects are inescapable, resulting from two main aspects: communicating limitations among agents and information processing at the input of an agent. The former reason leads to *communication delay*, while the latter one leads to *input delay*. Tremendous effort has been put into this problem. For continuous-time systems with input delay see for instance [12], [7], [13] while communication delay has been studied in for instance [1] [12]. For discrete-time systems [21] studies input and communication delays for introspective consensus problems. In [5] the focus is on communication delays in the discrete-time case.

However, the above results for systems with delays are restricted to simple agent models such as first/ second-order dynamics. Recently, in [17] and [16], the synchronization problem under uniform constant input delay is solved for both discrete- and continuous-time general linear agents that are at most critically unstable.

The objective of this paper is to extend [17] to the case of unknown, nonuniform constant input delay, which is more reasonable in practical applications. It is very normal that distributed agents with different hardware configurations may need varying time for information processing.

Nonuniform delays require an intrinsically different approach. The standard approach is using Wu and Chua's idea from [20] to convert consensus problems for a network of agents to a robust stabilization problem for a single system. However, the key step in this transformation fails if the delays are nonuniform. Our current approach can only address undirected networks while for uniform input delays, the network can be directed as well. Finally, when considering nonuniform input delay, synchronization can only be achieved in a regulation way. That is, one agent is selected as the reference and all other agents will be regulated asymptotically to the reference trajectories.

In this paper, the synchronization problem is considered for a network of identical discrete-time agents, but with nonuniform, unknown input delay. We assume the network topology is undirected and the agents can multi-input multi-output (MIMO), higher-order, and at most critically unstable, i.e., each agent has all its eigenvalues in the closed unit circle. In other words, we allow the agents to have eigenvalues on the unit circle. The agents are non-introspective and have full-state coupling where relative state information for neighboring agents are measured and can be used by the controller. We find a sufficient condition on the tolerable input delay for agents with general dynamics. This bound on the delay explicitly depends on the agent dynamics, but is independent of network topology.

Our results recover the bounds on the delay in case the delays are uniform. Moreover, in the special case where the agents only have unstable eigenvalues at 1, arbitrarily large but bounded input delay can be tolerated. This recovers many

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of the earlier results for first- and second-order systems. Finally, for any delay satisfying the proposed upper bound, a low-gain controller design methodology without exact knowledge of network topology is presented to achieve multi-agent synchronization in a set of networks.

A. Notations and preliminaries

$\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote, respectively, the sets of all complex numbers, real numbers, and natural numbers. For $\alpha, \beta \in \mathbb{R}$, (α, β) denotes the real set $\{\gamma \in \mathbb{R} : \alpha < \gamma < \beta\}$. $\mathbf{1}$ denotes a vector with all ones. $[x_1; ...; x_n]$ or col$\{x_i\}$ is the stacking column vector of $x_1, ..., x_n$. For a matrix A, A' is the conjugate, $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$ denote the smallest and the largest singular values of A, respectively.

A matrix $D = \{d_{ij}\}_{N\times N}$ is called a row stochastic matrix if

1) $d_{ij} \geq 0$ for any i, j;
2) $\sum_{j=1}^{n} d_{ij} = 1$ for $i = 1, ..., N$.

A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$. D can be associated with a graph $G = (\mathcal{N}, \mathcal{E})$. The number of nodes in $\mathcal{N}$ is the dimension of D and an edge $(j,i) \in \mathcal{E}$ if $d_{ij} > 0$. Let G be the graph associated with D. It is shown in [11] that 1 is a simple eigenvalue of D if and only if G contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disc if $d_{ii} > 0$ for all i. In the special case of undirected graphs, if $(i,j) \in \mathcal{E}$, then $(j,i) \in \mathcal{E}$, and $d_{ij} = d_{ji} > 0$. Furthermore, 1 is a simple eigenvalue of D if and only if G is connected, and all the other eigenvalues are real and located in the interval of $(-1, 1)$.

II. Multi-agent systems and problem formulation

Consider a multi-agent system (network) of N identical agents

$$\begin{cases} x_i(k+1) = Ax_i(k) + Bu_i(k - \kappa_i), \\ z_i(k) \quad = \sum_{j=1}^{N} d_{ij}[x_i(k) - x_j(k)], \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $z_i \in \mathbb{R}^n$, $\kappa_1, \ldots, \kappa_N$ are unknown constants satisfying $\kappa_i \in [0, \bar{\kappa}]$, $i = \{1, \ldots, N\}$. $D = \{d_{ij}\}_{N\times N}$ is a symmetric row stochastic matrix that satisfies $d_{ii} > 0$. In (1), each agent collects a measurement, denoted by z_i, of relative states of neighboring agents through the network.

Remark 1: The network measurement z_i is the only information that is available to each agent for controller design. The agent does not have separate observation of its own dynamics. This kind of agent is referred to as non-introspective.

The following assumptions are made throughout this paper.

Assumption 1: We assume that

- (A, B) is stabilizable, and A has all its eigenvalues in the closed unit disc,
- The communication topology represented by an undirected graph G is connected.

Remark 2: Under Assumption 1, according to [11, Corollary 3.5], D has a simple eigenvalue at 1 with the corresponding right eigenvector $\mathbf{1}$ and all the other eigenvalues are real and in the interval of $(-1, 1)$. Let $\lambda_{d_1}, \ldots, \lambda_{d_N}$ denote the eigenvalues of D such that $\lambda_{d_1} = 1$ and $-1 < \lambda_{d_i} < 1$, for $i = 2, \ldots, N$.

It should be noted that, in practice, perfect information of the communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. Next we will define a set of graphs based on some rough information of the graph. Before doing so, we first define some matrices associated with the undirected graph G. Let $M = I - D$, where $M = \{m_{ij}\}_{N\times N}$. We note that, M has a simple eigenvalue at 0, and all other eigenvalues are in the interval of $(0, 2)$.

Based on the matrix M of the network graph G, we define an associated matrix $\bar{M}$ by removing one arbitrarily selected row and its corresponding column from matrix M. The properties of the eigenvalues of $\bar{M}$ are presented in the following lemma.

Lemma 1: All the eigenvalues of $\bar{M}$ are in the open right-half real axis. Furthermore, the maximum eigenvalue of $\bar{M}$ is bounded by 2.

Proof: This is a consequence from the fact that the eigenvalues of M are in $(0, 2)$ and Cauchy's interlacing theorem [8]. ■

Now by using the smallest eigenvalue of $\bar{M}$ as a "measure" for the graph, we can introduce the following definition to characterize a set of unknown communication topologies.

Definition 1: For given $\delta \in (0, 1]$ and N, the set $\mathbb{G}_\delta^N$ is the set of undirected graphs composed of N nodes such that the eigenvalues of the associated matrix $\bar{M}$, denoted by $\lambda_1, \ldots, \lambda_{N-1}$, satisfy $\lambda_i > \delta$.

We define state synchronization as follows.

Definition 2: The agents in the network (1) achieve **state synchronization** if

$$\lim_{k\to\infty} (x_i(k) - x_j(k)) = 0, \qquad \forall i, j = \{1, \ldots, N\}.$$

The synchronization problems can be formulated as follows.

Problem 1: Consider a network of agents (1) with full-state coupling. For a given set $\mathbb{G}_\delta^N$ and a delay upper bound $\bar{\kappa}$, the state synchronization problem is to design linear static controllers $u_i = F_i z_i$ for $i \in \{1, \ldots, N\}$, such that the agents (1) achieve state synchronization in the network with any communication topology belonging to $\mathbb{G}_\delta^N$ and any $\kappa_1, \ldots, \kappa_N \leq \bar{\kappa}$.

III. State synchronization under nonuniform input delays

In this section, we consider state synchronization problems for homogeneous multi-agent systems defined in (1).

Here we achieve state synchronization in such a way that all the other agents are regulated asymptotically to the trajectory given by one arbitrarily selected agent, denoted by agent ρ. The main idea is to set the control input of agent ρ, to zero. That is, the controller for agent ρ is designed as $F_\rho = 0$. In the following, we fix the agent ρ, and then design controllers $u_i = F_i z_i$ for the other agents with $i \in \{1,\dots,N\}\backslash\rho \triangleq \mathcal{V}$.

Remark 3: Note that when agent ρ is selected to be followed by the other agents, $\bar{M}$ is constructed by removing the ρ^{th} column and ρ^{th} row from matrix M.

A decentralized local homogeneous controller for each agent $i \in \mathcal{V}$, is designed by using a low-gain feedback as follows.

$$u_i = \beta F_\varepsilon z_i, \tag{2}$$

with the design parameter β to be chosen later and

$$F_\varepsilon = -(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon A, \tag{3}$$

where for $\varepsilon \in (0,1]$, P_ε is the unique positive definite solution of the H_2 algebraic Riccati equation

$$P_\varepsilon = A'P_\varepsilon A + \varepsilon I - A'P_\varepsilon B(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon A. \tag{4}$$

The low-gain parameter ε will be chosen depending only on δ and $\bar{\kappa}$. Define

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Schur stable.} \\ \max\{\omega \in [0,\pi] | \det(e^{j\omega}I - A) = 0\}, & \text{otherwise.} \end{cases}$$

The first main result of this paper is stated in the next theorem, which solves Problem 1.

Theorem 1: For a given set $\mathbb{G}_\delta^N$ with $\delta \in (0,1)$ and $\bar{\kappa} > 0$, consider the agents (1) with any communication topology belonging to the set $\mathbb{G}_\delta^N$. In that case, Problem 1 is solvable via synchronization controller (2) if

$$\omega_{\max}\bar{\kappa} < \frac{\pi}{2}. \tag{5}$$

Specifically, for given $\mathbb{G}_\delta^N$ and $\bar{\kappa} > 0$ satisfying (5), there exist $\beta > 0$ and ε^* such that for any $\varepsilon \in (0,\varepsilon^*]$, the agents (1) with controller (2) achieve state synchronization for any communication topology in $\mathbb{G}_\delta^N$ and for any $\kappa_1,\dots,\kappa_N \in [0,\bar{\kappa}]$.

Proof: To clarify the controller design, here we introduce a delay operator R. A delay operator R_i is defined for agent i such that $(R_i u_i)(k) = u_i(k-\kappa_i)$. In the frequency domain, $\tilde{R}_i(\omega) = z^{-\kappa_i} = e^{-j\omega\kappa_i}$. In terms of the matrix M, the relative state measurement z_i in (1) can be rewritten as

$$z_i = \sum_{j=1}^{N} m_{ij}x_j.$$

Define $\bar{x}_i = x_i - x_\rho$ as the state synchronization error for agent $i \in \mathcal{V}$ and $\bar{x} = \operatorname{col}\{\bar{x}_i\}$. Then, the dynamics of $\bar{x}_i$ is governed by

$$\begin{cases} \bar{x}_i(k+1) = A\bar{x}_i(k) + BR_iu_i(k), \\ \quad z_i \quad = \sum_{j=1, j\neq\rho}^{N} m_{ij}\bar{x}_j, \end{cases} \tag{6}$$

where $i \in \mathcal{V}$ and the second equation results from $\sum_{j=1}^{N} m_{ij}x_\rho(k) = 0$. Combined with the controller (2), we obtain the full closed-loop system of the whole network as:

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \beta(R\bar{M} \otimes BF_\varepsilon)\bar{x}(k), \tag{7}$$

where

$$\bar{A} = I_{N-1} \otimes A, \qquad R = \operatorname{diag}\{R_i\}.$$

To prove Theorem 1, we need to prove that the closed-loop system (7) is Schur stable. In the following, we will have two steps to prove (7) is Schur stable.

Step 1: We will first show that the closed-loop system without delay is Schur stable. The closed-loop system without delay is

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \beta(\bar{M} \otimes BF_\varepsilon)\bar{x}(k). \tag{8}$$

Because $\bar{\kappa}$ satisfies condition (5), there exists β such that

$$\beta\delta\cos(\omega_{\max}\bar{\kappa}) > 1. \tag{9}$$

Note that β can be chosen independent of ε. Let this β be fixed.

Lemma 3 implies that (8) is Schur stable if and only if the $N-1$ systems

$$\xi_i(k+1) = (A + \beta\lambda_i BF_\varepsilon)\xi_i(k), \tag{10}$$

are asymptotically stable for $i = 1,\dots,N-1$ where $\lambda_1,\dots,\lambda_{N-1}$ are the eigenvalues of $\bar{M}$. For $\lambda_i \in (\delta,2)$ and the previous selected β, we have

$$\beta\lambda_i \in (\beta\delta, 2\beta) \subset H_1 = \left\{z \in \mathbb{C} : \operatorname{Re}(z) > \frac{1}{2}\right\} \tag{11}$$

because (9) implies that $\beta\delta > 1$. Lemma 5 implies that $(\beta\delta, 2\beta)$ is contained in a compact subset of H_1, and there exists ε_1 such that for $\varepsilon \in (0,\varepsilon_1]$, $(\beta\delta, 2\beta) \subset \Omega_\varepsilon$, where Ω_ε is the disc margin defined in (21). Lemma 5 then implies that (10) is asymptotically stable for $i = 1,\dots,N-1$. Therefore, the closed-loop system without delay (8) is Schur stable.

Step 2: We need to prove (7) is Schur stable. According to Lemma 4, system (7) is asymptotically stable if

$$\det[e^{j\omega}I - \bar{A} - (1-\alpha)\beta(\bar{M} \otimes BF_\varepsilon) - \alpha\beta(\tilde{R}(\omega)\bar{M} \otimes BF_\varepsilon)] \neq 0, \tag{12}$$

for all $\omega \in [-\pi,\pi]$, for all $\alpha \in [0,1]$, for all $\kappa_1,\dots,\kappa_N \in [0,\bar{\kappa}]$ and all possible $\bar{M}$ associated with a network graph in the set $\mathbb{G}_\delta^N$.

By (9), there exits $\eta > 0$ independent of ε such that

$$\beta\delta\cos((\omega_{\max}+\eta)\bar{\kappa}) > 1, \qquad \text{for } |\omega| < \omega_{\max}+\eta$$

Next we will split the proof of (12) in two cases where $|\omega| < \omega_{\max}+\eta$ and $\pi \geq |\omega| \geq \omega_{\max}+\eta$ respectively.

If $\pi \geq |\omega| \geq \omega_{\max}+\eta$, $\det[e^{j\omega}I - \bar{A}] \neq 0$, which yields $\underline{\sigma}(e^{j\omega}I - \bar{A}) > 0$. Because $\underline{\sigma}(e^{j\omega}I - \bar{A})$ depends continuously on ω and the set $\{\pi \geq |\omega| \geq \omega_{\max}+\eta\}$ is compact, there exists a μ such that

$$\underline{\sigma}(e^{j\omega}I - \bar{A}) > \mu, \ \forall\omega, \text{s.t. } \pi \geq |\omega| \geq \omega_{\max}+\eta. \tag{13}$$

Given β, there exists $\varepsilon_2 > 0$ independent of ε_1 such that

$$\|[(1-\alpha)\beta\bar{M} + \alpha\beta\tilde{R}(\omega)\bar{M}] \otimes BF_\varepsilon\| \leq \mu/2 \tag{14}$$

for $\varepsilon \in (0,\varepsilon_2]$. Note that μ and ε_2 can be chosen independent of $\bar{M}$ but only relying on the parameter δ. Combining (13) and (14) we obtain

$$\underline{\sigma}(e^{j\omega}I - \bar{A} - [(1-\alpha)\beta\bar{M} + \alpha\beta\tilde{R}(\omega)\bar{M}] \otimes BF_\varepsilon) \geq \mu/2, \quad \forall\omega \text{ s.t. } \pi \geq |\omega| \geq \omega_{\max}+\eta.$$

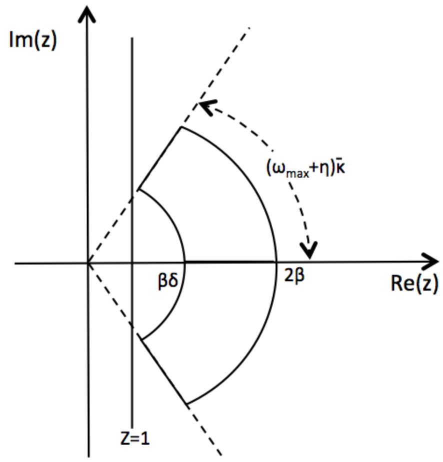


Fig. 1: Eigenvalues of $\beta\tilde{R}(\omega)\bar{M}$

Therefore, (12) holds for all $\pi \geq |\omega| \geq \omega_{\max} + \eta$.

It remains to verify condition (12) for ω with $|\omega| < \omega_{\max} + \eta$. It is clearly sufficient to prove

$$\bar{A} + S(\omega) \otimes BF_\varepsilon \tag{15}$$

is asymptotically stable for any $|\omega| < \omega_{\max} + \eta$ where

$$S(\omega) = \left[(1-\alpha)I + \alpha\tilde{R}(\omega)\right]\beta\bar{M}$$

We want to use Lemma 3 so we need to analyse the eigenvalues of $S(\omega)$.

We first note that the eigenvalues of $\beta\tilde{R}(\omega)\bar{M}$ belong to H_1 for any fixed ω satisfying $|\omega| < \omega_{\max} + \eta$.

Clearly $\beta\bar{M}$ is symmetric and $\tilde{R}(\omega) = \text{diag}\{e^{-j\omega\kappa_i}\}$, where $\omega\kappa_i$ satisfies

$$-(\omega_{\max} + \eta)\bar{\kappa} < \omega\kappa_i < (\omega_{\max} + \eta)\bar{\kappa}.$$

Moreover the real part of the eigenvalues of $\beta\bar{M}$ are larger than

$$\beta\delta\cos((\omega_{\max} + \eta)\bar{\kappa}) = 1$$

According to Lemma 2, the eigenvalues of $\beta\tilde{R}(\omega)\bar{M}$ are in the fan shaped area of Figure 1. Clearly, also the eigenvalues of $\beta\bar{M}$ are in this same region. Using [19, Theorem 1], we note that the eigenvalues of $S(\omega)$ are in this region as well. By Lemma 5, there exists ε_3 such that this region is contained in Ω_ε for all $\varepsilon \in (0, \varepsilon_3]$. We can then apply Lemma 3 to conclude that (15) is asymptotically stable for any $|\omega| < \omega_{\max} + \eta$. ■

Remark 4: The consensus controller design depends only on the agent model and parameter $\bar{\kappa}$, δ and is independent of specific network topology.

In the special case where $\omega_{\max} = 0$, i.e. the eigenvalue of A are either 1 or in the unit circle, then arbitrarily bounded input delay can be tolerated as formulated in the following corollary:

Corollary 1: For a given set $\mathbb{G}_\delta^N$ with $\delta \in (0,1)$ and $\bar{\kappa} > 0$, consider the agents (1) with any communication topology belonging to the set $\mathbb{G}_\delta^N$. Suppose $\omega_{\max} = 0$. In that case, Problem 1 is always solvable via the synchronization controller (2). Specifically, for given $\mathbb{G}_\delta^N$ and $\bar{\kappa} > 0$, there exists β and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (2) achieve synchronization for any communication topology in $\mathbb{G}_\delta^N$ and for any $\kappa_1, \ldots, \kappa_N \in [0, \bar{\kappa}]$.

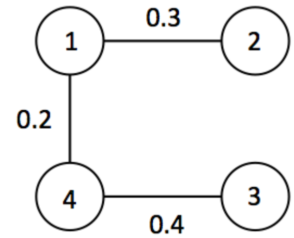


Fig. 2: The network topology

IV. Examples

We will illustrate our result on a network of four identical agents. The agent dynamics (A, B, C) are given as follows,

$$A = \begin{pmatrix} 0.5 & 1 & 1 \\ 0 & \sqrt{3}/2 & -0.5 \\ 0 & 0.5 & \sqrt{3}/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

Eigenvalues of A are 0.5, $\sqrt{3}/2 \pm 0.5$. So, $\omega_{\max} = \pi/6$. The network topology is given by Figure 2.

We choose matrix D as

$$D = \begin{pmatrix} 0.5 & 0.3 & 0 & 0.2 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.2 & 0 & 0.4 & 0.4 \end{pmatrix} \tag{16}$$

Since the graph is connected and undirected, we can choose any agent as the reference. In this example, we select agent 2 as the reference. Thus, by selecting $\delta = 0.04$, the eigenvalues of $\bar{M}$ are all larger than δ. From (5), $\bar{\kappa} < 3$ and we select $\bar{\kappa} = 2$. More specifically Agent 1 has delay $\kappa_1 = 2$, Agent 3 has delay $\kappa_3 = 1$, and Agent 4 has delay $\kappa_4 = 2$.

According to (9), we select $\beta = 55$. By choosing $\varepsilon = 1e-5$, $F_\varepsilon = [-0.000003, \ -0.0042, \ -0.0073]$. The low-gain feedback controller of the form (2) is

$$u_i = \begin{pmatrix} -0.00015 & -0.2309 & -0.4033 \end{pmatrix} z_i, \qquad i = 1, 3, 4. \tag{17}$$

Figure (3a) - (4b) show that the state synchronization is achieved for the network with D in (16).

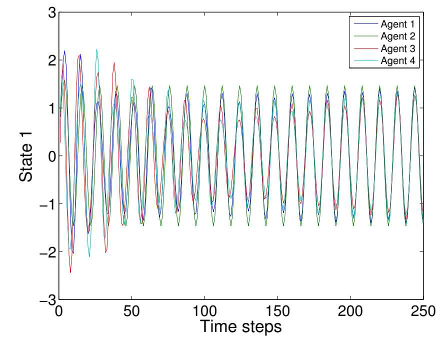
Appendix

The following lemma is adapted from [19, Corollary 3].

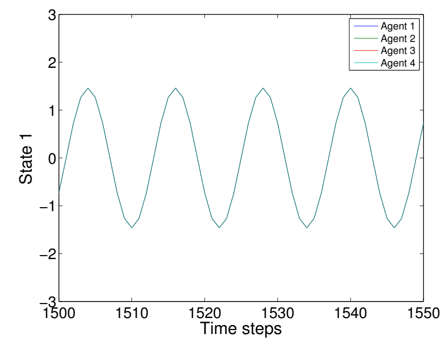
Lemma 2: Let A be unitary with eigenvalues $\alpha_1, \ldots, \alpha_n$; let B be hermitian and positive semi definitive with eigenvalues $\beta_1 \leq \ldots \leq \beta_n$ and let λ be any eigenvalue of AB or BA. Then $\beta_1 \leq |\lambda| \leq \beta_n$, and if all the α_i are contained in an arc Φ of the unit circle of length $\leq \pi$, then $\arg\lambda \in \Phi$.

Lemma 3: Let $x = [x_1; \ldots; x_N]$ and $x_i \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{N\times N}$, $M \in \mathbb{R}^{n\times n}$. Then, the system

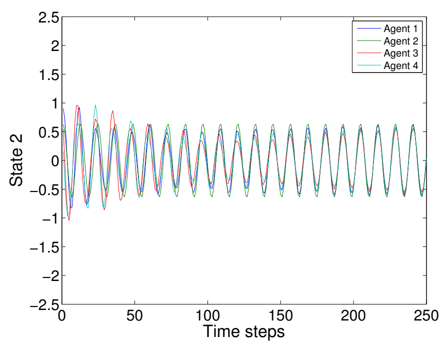
$$x(k+1) = (I \otimes A)x(k) + (S \otimes M)x(k) \tag{18}$$



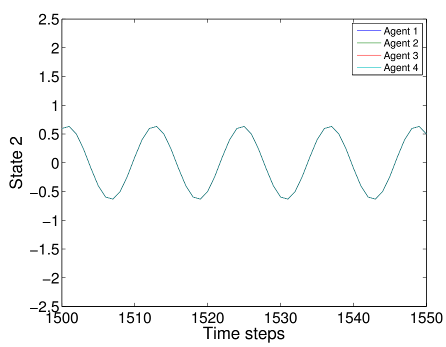
(a) State $x_{i,1}$



(b) State $x_{i,1}$

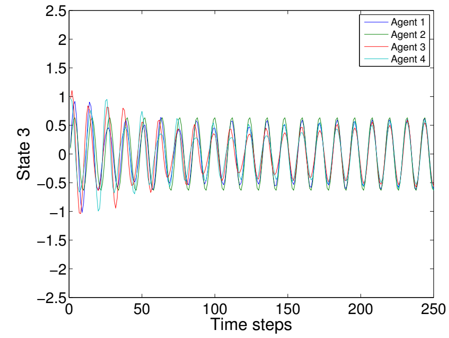


(c) State $x_{i,2}$

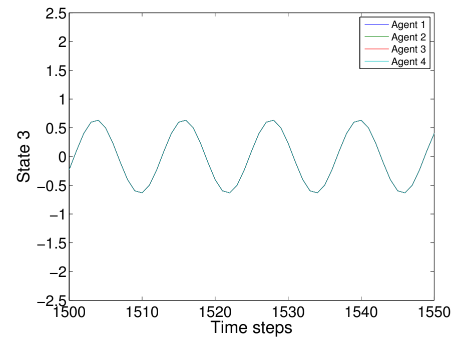


(d) State $x_{i,2}$

Fig. 3: Trajectories of State $x_{i,1}$ and $x_{i,2}$ achieve the same among agents



(a) State $x_{i,3}$



(b) State $x_{i,3}$

Fig. 4: Trajectories of State $x_{i,3}$ achieve the same among agents

is asymptotically stable if and only if the following systems

$$x(k+1) = (A + \lambda_i M)x_i(k)$$

are asymptotically stable for $i = 1, \ldots, N$, where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of the matrix S.

Proof: Let $\xi = [\xi_1; , , , ; \xi_N] = (T \otimes I_n)x$, where T is selected such that $J = TST^{-1}$ is in the Jordan canonical form. Moreover the diagonal elements of J are the eigenvalues of the matrix S. Then the system (18) is stable if and only if

$$\xi(k+1) = (I \otimes A)\xi + (J \otimes M)\xi$$

is stable. Due to the upper-triangle structure of $I \otimes A$ and $J \otimes M$, the stability of the above system is determined by the N subsystems

$$\xi_i(k+1) = A\xi_i + \lambda_i M \xi_i.$$

Thus, the result in Lemma 3 follows. ■

Lemma 4: Consider a linear time-delay system

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} A_i x(k - \kappa_i), \tag{19}$$

where $x(k) \in \mathbb{R}^n$ and $\kappa_i \in \mathbb{N}$. Suppose $A + \sum_{i=1}^{m} A_i$ is Schur stable. We have that (19) is asymptotically stable if

$$\det[e^{j\omega} I - A - (1-\alpha)\sum_{i=1}^{m} A_i - \alpha \sum_{i=1}^{m} e^{-j\omega\kappa_i} A_i] \neq 0,$$

for all $\omega \in [-\pi, \pi]$, and for all $\alpha \in [0, 1]$.

Proof: The proof follows from [15]. ■

Lemma 5: Consider a linear uncertain system

$$\left\{ \; x(k+1) = Ax(k) + \lambda Bu(k), \qquad x(0) = x_0, \right. \tag{20}$$

where $\lambda \in \mathbb{C}$ is unknown. Let Assumption 1 hold. A low-gain state feedback $u = F_\varepsilon x$ is constructed, where $F_\varepsilon = -(B' P_\varepsilon B + I)^{-1} B' P_\varepsilon A$, and P_ε is the unique positive definite solution of the H_2 algebraic Riccati equation (4).

Then, we have that $A + \lambda B F_\varepsilon$ is Schur stable if

$$\lambda \in \Omega_\varepsilon := \left\{ z \in \mathbb{C} : \left| z - \left(1 + \frac{1}{\gamma_\varepsilon}\right) \right| < \frac{\sqrt{1+\gamma_\varepsilon}}{\gamma_\varepsilon} \right\}, \tag{21}$$

where $\gamma_\varepsilon = \bar{\sigma}(B' P_\varepsilon B)$. As $\varepsilon \to 0$, Ω_ε approaches the set

$$H_1 := \{ z \in \mathbb{C} : \operatorname{Re}(z) > \tfrac{1}{2} \}$$

in the sense that any compact subset of H will be contained in Ω_ε for ε is small enough.

Proof: The proof can be found in [3]. ■

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