

A Smooth Distributed Feedback for Global Rendezvous of Unicycles

Ashton Roza, Manfredi Maggiore, Luca Scardovi

Abstract—This paper presents a solution to the rendezvous control problem for a network of kinematic unicycles in the plane, each equipped with an onboard camera measuring its relative displacement with respect to its neighbors in body frame coordinates. A smooth, time-independent control law is presented that drives the unicycles to a common position from arbitrary initial conditions, under the assumption that the sensing digraph contains a reverse-directed spanning tree. The proposed feedback is very simple, and relies only on the onboard measurements. No global positioning system is required, nor any information about the unicycles' orientations.

I. INTRODUCTION

This paper investigates rendezvous control of kinematic unicycles. The objective is to design smooth feedbacks for each robot so as to drive the group to a common position from arbitrary initial conditions. An important requisite is that the feedback be local and distributed. In other words, it is required that the feedback depend only on the relative displacement of each robot to its neighbours measured in the robot's own body frame, so that the feedback can be computed using onboard sensing devices such as cameras or laser systems.

The solution to the rendezvous control problem proposed in this paper is time-independent and it does not require any information about the orientation of the unicycles, not even their relative orientation. To the best of our knowledge, this is the first solution having the property of being local and distributed, continuously differentiable, and time-independent. As we argue below, previous solutions require either time-varying or discontinuous feedback. For simplicity of exposition, the proposed solution relies on the assumption that the sensing digraph of the unicycles is time-invariant. However, it is only required to contain a reverse directed spanning tree, which is the minimal connectivity requirement.

The main difficulty in solving the rendezvous control problem comes from the fact that the unicycles are nonholonomic, in that their velocity is restricted to be parallel to the vehicle's heading direction. To overcome this difficulty, the solution we present relies on a control structure made of two nested loops. An outer loop treats the vehicles as fully-actuated single integrators with a linear consensus controller providing a reference velocity. Here we leverage existing consensus algorithms for single integrators [1], [2], [3]. The desired

velocity computed by the outer loop becomes a reference signal for the inner loop, which assigns local and distributed feedbacks that solve the rendezvous control problem. This methodology is inspired by our previous work in [4], [5] for rendezvous of rigid bodies in three dimensions.

The rendezvous control problem for unicycles has been investigated before. In [6], the authors presented the first solution. The feedback in [6] is local and distributed, but it requires the use of time-varying feedbacks. In [7] both positions and attitudes of the unicycles are synchronized using a time-invariant distributed control. The graph is time-dependent and the authors assume an initially connected communication graph. The controller that is implemented, however, is discontinuous. In [8] a time-independent, local and distributed controller is presented. However, the authors make the assumption that whenever two vehicles get sufficiently close together they merge into a single vehicle, introducing a discontinuity in the control function. To the best of our knowledge, the solution presented in this paper is the first one involving feedbacks that are local and distributed, time-independent, and continuously differentiable. The proposed solution is of simple implementation, not even requiring any knowledge about the relative orientation of the unicycles. As we illustrate through simulations, the proposed time-independent, continuously differentiable feedback has practical advantages over the time-varying feedback in [6] and the discontinuous feedback in [7] in that it induces a more natural behaviour in the ensemble of unicycles. The feedback in [6] makes the unicycle “wobble” indefinitely, a behaviour which would be unacceptable in practice. The feedback in [7] induces instantaneous changes in direction that are impossible to achieve with realistic implementations.

The paper is organized as follows. In Section II we present the notation and review basic graph theory and stability definitions. In Section III we formulate the rendezvous control problem. The solution of the rendezvous control problem is presented in Section IV, together with an intuitive description of its operation. The proof of the main theorem is presented in Section V. Finally, in Section VI we make concluding remarks. Lemmas and claims related to the proof are in the appendix.

II. PRELIMINARIES

A. Notation

We use interchangeably the notation $v = [v_1 \cdots v_n]^T$ or (v_1, \dots, v_n) for a column vector in \mathbb{R}^n . We denote by $\mathbf{1} \in \mathbb{R}^m$ the vector $(1, \dots, 1)$. If v, w are vectors in \mathbb{R}^2 , we denote by $v \cdot w := v^T w$ their Euclidean inner product, and by $\|v\| :=$

This research was supported by the National Sciences and Engineering Research Council of Canada.

The authors are with the Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, ON, M5S 3G4, Canada. ashton.roza@mail.utoronto.ca, maggiore@ece.utoronto.ca, scardovi@scg.utoronto.ca

$(v \cdot v)^{1/2}$ the Euclidean norm of v . If $\omega \in \mathbb{R}$, we define

$$\omega^\times := \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

Let $\{e_1, e_2\}$ denote the natural basis of \mathbb{R}^2 , $\text{SO}(2) := \{M \in \mathbb{R}^{2 \times 2} : M^{-1} = M^\top, \det(M) = 1\}$ and let \mathbb{S}^1 denote the unit circle. If Γ is a closed subset of a geodesically complete Riemannian manifold \mathcal{X} , and $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is a distance metric on \mathcal{X} , we denote by $\|\chi\|_\Gamma := \inf_{\psi \in \Gamma} d(\chi, \psi)$ the point-to-set distance of $\chi \in \mathcal{X}$ to Γ . If $\varepsilon > 0$, we let $B_\varepsilon(\Gamma) := \{\chi \in \mathcal{X} : \|\chi\|_\Gamma < \varepsilon\}$ and by $\mathcal{N}(\Gamma)$ we denote an open subset of \mathcal{X} containing Γ . If $A, B \subset \mathcal{X}$ are two sets, denote by $A \setminus B$ the set-theoretic difference of A and B . If $I = \{i_1, \dots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \dots, x_{i_n})$ is denoted by $(x_j)_{j \in I}$.

Let U, W be finite-dimensional vector spaces. A function $f : U \rightarrow W$ is *homogeneous of degree r* if, for all $\lambda > 0$ and for all $x \in U$, $f(\lambda x) = \lambda^r f(x)$. A function $f : U \times V \rightarrow W$, $(x, y) \mapsto f(x, y)$, is *homogeneous of degree r with respect to x* if for all $\lambda > 0$ and for all $(x, y) \in U \times V$, $f(\lambda x, y) = \lambda^r f(x, y)$.

B. Graph Theory

We refer the reader to [9] for more details on the notions reviewed in this section. We denote a *digraph* by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is a set of nodes labelled as $\{1, \dots, n\}$ and \mathcal{E} is the set of edges. The set of *neighbors* of node i is $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

Given positive numbers $a_{ij} > 0$, $i, j \in \{1, \dots, n\}$, the associated *weighted Laplacian matrix* of \mathcal{G} is the matrix $L := D - A$, where D is a diagonal matrix whose i -th diagonal entry is the sum $\sum_{j \in \mathcal{N}_i} a_{ij}$, and A is the matrix whose element $(A)_{ij}$ is a_{ij} if $j \in \mathcal{N}_i$, and 0 otherwise.

A *directed spanning tree* is a graph consisting of $n - 1$ edges such that there exists a unique directed path from a node, called the root, to every other node. A *reverse directed spanning tree* is a graph which becomes a directed spanning tree by reversing the directions of all its edges. We identify the root of a reverse spanning tree with the root of its associated spanning tree. A graph \mathcal{G} contains a *reverse directed spanning tree* if it has a subgraph which is a reverse directed spanning tree.

Proposition 1 ([1], [6]): The following conditions are equivalent for a digraph \mathcal{G} :

- (i) \mathcal{G} contains a reverse directed spanning tree.
- (ii) For any set of positive gains $a_{ij} > 0$, $i, j \in \{1, \dots, n\}$ the associated weighted Laplacian matrix L of \mathcal{G} has rank $n - 1$, and $\text{Ker } L = \text{span}\{\mathbf{1}\}$.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *strongly connected* if for any two nodes $i, j \in \mathcal{V}$ there exists a path from i to j . A set of nodes $S \subset \mathcal{V}$ is an *isolated component* if it has no outgoing edges, i.e., for any edge $(i, j) \in \mathcal{E}$, if $i \in S$ then $j \in S$. A graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a *subgraph* of \mathcal{G} if $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{E}' \subset \mathcal{E}$. A subgraph \mathcal{G}' is an *induced subgraph* of \mathcal{G} if for any two vertices $i, j \in \mathcal{V}'$, $(i, j) \in \mathcal{E}'$ if and only if $(i, j) \in \mathcal{E}$. A *strongly connected component* \mathcal{G}' of \mathcal{G} is a maximal strongly connected induced subgraph of \mathcal{G} . In other words, there does

not exist any other strongly connected induced subgraph of \mathcal{G} containing \mathcal{G}' . Letting $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0), \dots, \mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r)$ be the strongly connected components of \mathcal{G} , the *condensation digraph* of \mathcal{G} , denoted $\mathcal{C}(\mathcal{G}) = (\mathcal{V}_\mathcal{C}(\mathcal{G}), \mathcal{E}_\mathcal{C}(\mathcal{G}))$, is defined as follows. The *vertex set* $\mathcal{V}_\mathcal{C}(\mathcal{G})$ is the set of nodes $\{v_i\}_{i \in \{0, \dots, r\}}$ where the node v_i is a *contraction* of the vertex set \mathcal{V}_i of the i -th strongly connected component \mathcal{G}_i . The *edge set* $\mathcal{E}_\mathcal{C}(\mathcal{G})$ contains an edge (v_i, v_j) if there exist vertices $i' \in \mathcal{V}_i$ and $j' \in \mathcal{V}_j$ such that $(i', j') \in \mathcal{E}$. The following properties of the condensation digraph are found in [10].

Proposition 2 ([10]): Consider a graph \mathcal{G} containing a reverse directed spanning tree. The condensation $\mathcal{C}(\mathcal{G})$ satisfies the following properties:

- (i) $\mathcal{C}(\mathcal{G})$ is acyclic, i.e., there is no path in $\mathcal{C}(\mathcal{G})$ beginning and ending at the same node.
- (ii) $\mathcal{C}(\mathcal{G})$ contains a reverse directed spanning tree \mathcal{T} with a unique root $v_0 \in \mathcal{V}_\mathcal{C}(\mathcal{G})$.
- (iii) There exists at least one vertex $v_i \in \mathcal{V}_\mathcal{C}(\mathcal{G})$ such that v_0 is the only neighbor of v_i .

An example of a digraph \mathcal{G} containing a reverse directed spanning tree is shown in Figure 1. The strongly connected components are boxed. The resulting acyclic condensation graph $\mathcal{C}(\mathcal{G})$ is shown in Figure 2. The vertex v_0 in the figure is the unique root of the reverse directed spanning tree in $\mathcal{C}(\mathcal{G})$.

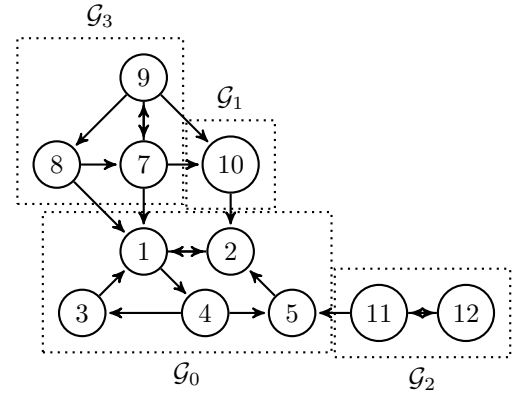


Fig. 1: Directed graph \mathcal{G} containing a reverse directed spanning tree. The strongly connected components $\mathcal{G}_0, \dots, \mathcal{G}_3$ are boxed

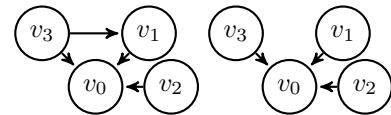


Fig. 2: Condensation digraph $\mathcal{C}(\mathcal{G})$ associated with the graph \mathcal{G} in Figure 1 (left) and reverse directed spanning tree contained in $\mathcal{C}(\mathcal{G})$ (right).

As in [10], we define the vertex set $\mathcal{L}_k \subset \mathcal{V}$ to be the union of those vertex sets \mathcal{V}_i that correspond to vertices v_i in the condensation digraph with the property that the maximal path length from v_i to the root v_0 is equal to k . By this definition, $\mathcal{L}_0 := \mathcal{V}_0$. We let $\mathcal{L}_{-1} := \emptyset$. Defining the vertex set $\mathcal{L}_k :=$

$\cup_{i=0}^k \mathcal{L}_i$, by construction, the neighbors of any vertex in \mathcal{L}_k are contained in $\bar{\mathcal{L}}_{k-1}$. Therefore each node set $\bar{\mathcal{L}}_k$ is isolated. For the example in Figure 2, we have $\mathcal{L}_0 = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{L}_1 = \{10\} \cup \{11, 12\}$ and $\mathcal{L}_2 = \{7, 8, 9\}$.

C. Stability Definitions

The following stability definitions are taken from [11]. Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a geodesically complete Riemannian manifold \mathcal{X} with Riemannian distance $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$, so that (\mathcal{X}, d) is a complete metric space. Let $\phi(t, \chi_0)$ denote the local phase flow of Σ .

Definition 1: Consider a closed set $\Gamma \subset \mathcal{X}$ that is positively invariant for Σ , i.e., for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t > 0$ for which $\phi(t, \chi_0)$ is defined.

- Γ is *stable* for Σ if for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\phi(t, \chi_0) \in B_\varepsilon(\Gamma)$, for all $t > 0$ for which $\phi(t, \chi_0)$ is defined.
- Γ is *attractive* for Σ if there exists neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\lim_{t \rightarrow \infty} \|\phi(t, \chi_0)\|_\Gamma = 0$. The *domain of attraction* of Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t \rightarrow \infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. Γ is *globally attractive* for Σ if it is attractive with domain of attraction \mathcal{X} .
- Γ is *locally asymptotically stable (LAS)* for Σ if it is stable and attractive. The set Γ is *globally asymptotically stable (GAS)* for Σ if it is stable and globally attractive.

Definition 2: Let $\Gamma_1 \subset \Gamma_2$ be two subsets of \mathcal{X} that are positively invariant for Σ . Assume that Γ_1 is compact and Γ_2 is closed.

- Γ_1 is *globally asymptotically stable relative to Γ_2* if it is GAS when initial conditions are restricted to lie in Γ_2 .
- Γ_2 is *locally stable near Γ_1* if for all $c > 0$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(\Gamma_1)$ and all $t^* > 0$, if $\phi([0, t^*], x_0) \subset B_c(\Gamma_1)$ then $\phi([0, t^*], x_0) \subset B_\varepsilon(\Gamma_2)$.
- Γ_2 is *locally attractive near Γ_1* if there exists a neighbourhood $\mathcal{N}(\Gamma_1)$ such that, for all $x_0 \in \mathcal{N}(\Gamma_1)$, $\|\phi(t, x_0)\|_{\Gamma_2} \rightarrow 0$ as $t \rightarrow \infty$.

We present a reduction theorem used to derive our main result

Theorem 1 (Reduction Theorem [11], [12]): Let Γ_1 and Γ_2 , $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed sets that are positively invariant for Σ , and suppose Γ_1 is compact. Consider the following conditions: (i) Γ_1 is LAS relative to Γ_2 ; (i') Γ_1 is GAS relative to Γ_2 ; (ii) Γ_2 is locally stable near Γ_1 ; (iii) Γ_2 is locally attractive near Γ_1 ; (iii)' Γ_2 is globally attractive; (iv) all trajectories of Σ are bounded.

Then, the following implications hold: (i) \wedge (ii) $\implies \Gamma_1$ is stable; (i) \wedge (ii) \wedge (iii) $\iff \Gamma_1$ is LAS; (i)' \wedge (ii) \wedge (iii)' $\iff \Gamma_1$ is GAS.

III. RENDEZVOUS CONTROL PROBLEM

Consider a group of n kinematic unicycles. Let $\mathcal{I} = \{i_x, i_y\}$ be an inertial frame in three-dimensional space and consider the i -th unicycle in Figure 4. Fix a body frame $\mathcal{B}_i = \{b_{ix}, b_{iy}\}$ to the unicycle, where b_{ix} is the heading axis,

and denote by $x_i \in \mathbb{R}^2$ the position of the unicycle in the coordinates of frame \mathcal{I} . The unicycle's attitude is represented by a rotation matrix R_i whose columns are the coordinate representations of b_{ix} and b_{iy} in frame \mathcal{I} . Letting $\theta_i \in \mathbb{S}^1$ be the angle between vectors i_x and b_{ix} , we have

$$R_i = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}.$$

The angular speed of robot i is denoted by ω_i . The unicycle dynamics are given by,

$$\dot{x}_i = u_i R_i e_1 \quad (1)$$

$$\dot{R}_i = R_i(\omega_i)^\times, \quad i = 1, \dots, n. \quad (2)$$

In what follows, we refer to system (1)-(2) as Σ_i . Its control inputs are the linear speed u_i and angular speed ω_i . The relative displacement of robot j with respect to robot i is $x_{ij} := x_j - x_i$. If $v \in \mathbb{R}^2$ is the coordinate representation of a vector in frame \mathcal{I} , then we denote by $v^i := R_i^{-1}v$ the coordinate representation of v in body frame \mathcal{B}_i .

We define the *sensor digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each node represents a robot, and an edge from node i to node j indicates that robot i can sense robot j . We assume that \mathcal{G} has no self-loops and is time-invariant. Given a node i , its set of neighbors \mathcal{N}_i represents the set of vehicles that robot i can sense. If $j \in \mathcal{N}_i$, then we say that robot j is a *neighbour* of robot i . If this is the case, then robot i can sense the relative displacement of robot j in its own body frame, i.e., the quantity x_{ij}^i . Define the vector $y_i := (x_{ij}^i)_{j \in \mathcal{N}_i}$. The relative displacements available to robot i are contained in the vector $y_i^i := (x_{ij}^i)_{j \in \mathcal{N}_i}$. A *local and distributed feedback* (u_i, ω_i) for robot i is a locally Lipschitz function of y_i^i . We define the *rendezvous manifold*

$$\Gamma := \{(x_i, R_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^{2n} \times \text{SO}(2)^n : x_{ij} = 0, \forall i, j\}. \quad (3)$$

We are now ready to state the rendezvous control problem.

Rendezvous Control Problem: For system (1)-(2) with sensor digraph \mathcal{G} , find local and distributed feedbacks $(u_i, \omega_i)_{i \in \{1, \dots, n\}}$ that globally asymptotically stabilize the rendezvous manifold Γ . \triangle

IV. SOLUTION OF THE RENDEZVOUS CONTROL PROBLEM

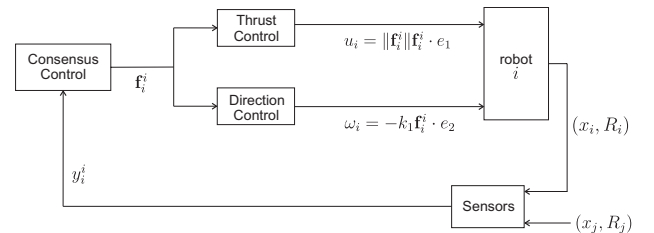


Fig. 3: Block diagram of the rendezvous control system for robot i .

In this section we present the solution of the rendezvous control problem. Consider the function,

$$\mathbf{f}_i(y_i) := \sum_{j \in \mathcal{N}_i} a_{ij} x_{ij}, \quad (4)$$

$i = 1, \dots, n$ with $a_{ij} > 0$. The function $\mathbf{f}_i(y_i)$ is a standard linear consensus controller for single integrator systems [1], [2], [3]. We use $\mathbf{f}(y_i)$ to construct the feedbacks

$$\begin{aligned} u_i &= \|\mathbf{f}_i(y_i^i)\| \mathbf{f}_i(y_i^i) \cdot \mathbf{e}_1, \\ \omega_i &= -k_1 \mathbf{f}_i(y_i^i) \cdot \mathbf{e}_2, \quad i = 1, \dots, n. \end{aligned} \quad (5)$$

The result below states that for sufficiently large k_1 , the feedbacks in (5) solve the rendezvous control problem if the network of unicycles has a sensor digraph containing a reverse directed spanning tree.

Theorem 2: The rendezvous control problem is solvable for system (1)-(2) if, and only if, the sensor graph \mathcal{G} contains a reverse directed spanning tree, in which case a solution is as follows. There exists $k_1^* > 0$ such that for any $k_1 > k_1^*$ feedback (5) with $\mathbf{f}_i(y_i)$ in (4) solves the rendezvous control problem.

The necessity portion of Theorem 2 was proved in [6]. The sufficiency part, namely the fact that the feedback (5) solves the rendezvous control problem, is proved in Section V.

The proposed control architecture is illustrated in the block diagram of Figure 3. There are two nested loops. The outer loop treats each robot as a single-integrator driven by the linear consensus controller,

$$\dot{x}_i = \mathbf{f}_i(y_i), \quad i = 1, \dots, n. \quad (6)$$

The set $\{(x_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^{2n} : x_{ij} = 0, \forall i, j\}$ is globally asymptotically stable for (6) if the sensing graph has a reverse directed spanning tree [2]. The signal $\mathbf{f}_i(y_i(t))$ is computed in the body frame \mathcal{B}_i , and used as a reference signal for the inner-loop thrust and direction controllers that assign the unicycle control inputs in (5). The intuition behind these controllers is shown in Figure 4. The speed input u_i is the dot product $u_i = \|\mathbf{f}_i(y_i^i)\| \mathbf{f}_i(y_i^i) \cdot \mathbf{e}_1$. This is the projection of the reference $\|\mathbf{f}_i(y_i)\| \mathbf{f}_i(y_i)$ onto the heading axis b_{ix} of robot i . The angular speed, on the other hand, is proportional to the dot product between the reference $\mathbf{f}_i(y_i)$ and the second body axis b_{iy} . In Figure 4, one can see that $\omega_i = -k_1 \|\mathbf{f}_i\| \sin(\phi_i)$ acts to reduce the angle ϕ_i between b_{ix} and $\mathbf{f}_i(y_i)$ with a rate proportional to the magnitude of \mathbf{f}_i . Together, these control inputs drive the robot velocity $u_i b_{ix}$ approximately to the reference $\|\mathbf{f}_i(y_i)\| \mathbf{f}_i(y_i)$. The convergence is approximate because the control inputs do not depend on the time derivative of \mathbf{f}_i . It is the difference in angle between $u_i b_{ix}$ and $\|\mathbf{f}_i(y_i)\| \mathbf{f}_i(y_i)$ as opposed to the difference in magnitude that is important for obtaining rendezvous. Since $\|\mathbf{f}_i(y_i)\| \mathbf{f}_i(y_i)$ is homogeneous of degree two, as the robots approach consensus, ω_i converges to zero slower than u_i . This allows ω_i to exert sufficient control authority even as the robots converge to consensus, closing the gap between the vectors $u_i b_{ix}$ and $\|\mathbf{f}_i(y_i)\| \mathbf{f}_i(y_i)$.

A. Simulation Results

We consider a group of five robots with sensor digraph in Figure 5. For the feedback in (5), we pick $a_{ij} = 0.05$ for all $j \in \mathcal{N}_i$. The control gain k_1 is chosen to be $k_1 = 1$. The initial conditions of the robots are shown in Table I. The simulation is presented in Figure 6(a). The proposed feedback

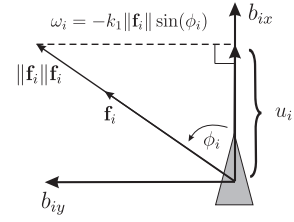


Fig. 4: Illustration of the control inputs u_i and ω_i in (5).

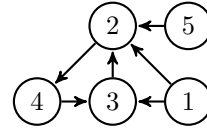


Fig. 5: Sensor digraph used in the simulation results.

TABLE I: Simulation Initial Conditions

Vehicle i	$x_i(0)$ (m)	$\theta_i(0)$ (rad)
1	(0, 10)	0
2	(-10, -10)	$2\pi/5$
3	(-50, 10)	$4\pi/5$
4	(-10, 0)	$6\pi/5$
5	(10, 0)	$8\pi/5$

V. PROOF OF THEOREM 2

This section presents the sufficiency proof of Theorem 2. The necessity was proved in [6]. The key tool in our proof is the condensation graph and the isolated node sets $\bar{\mathcal{L}}_k$ defined in Section II-B. The same tool was employed in [10] for pose synchronization (synchronization of positions and attitudes) of fully actuated vehicles.

The dynamics of unicycles associated with an isolated node set $\bar{\mathcal{L}}_k$ are independent of the nodes outside of this set because, for any robot $i \in \bar{\mathcal{L}}_k$, the feedbacks u_i and ω_i in (4), (5) depend only on states of robots within $\bar{\mathcal{L}}_k$. Therefore, the dynamics of the collection of unicycles in $\bar{\mathcal{L}}_k$,

$$\dot{x}_i = u_i R_i \mathbf{e}_1 \quad (7)$$

$$\dot{R}_i = R_i(\omega_i)^\times, \quad i \in \bar{\mathcal{L}}_k \quad (8)$$

define an autonomous dynamical system. Henceforth, the dynamics in (7), (8) are denoted by $\Sigma_{\bar{\mathcal{L}}_k}$ and we define the *reduced rendezvous manifold* $\Gamma_{\bar{\mathcal{L}}_k} := \{(x_i, R_i)_{i \in \bar{\mathcal{L}}_k} : x_{ij} = 0, \forall i, j \in \bar{\mathcal{L}}_k\}$.

Recall from Section II-B that the set $\bar{\mathcal{L}}_{-1}$ is empty, which implies that the set $\Gamma_{\bar{\mathcal{L}}_{-1}}$ is also empty. We adopt the convention that $\Gamma_{\bar{\mathcal{L}}_{-1}}$ is GAS for $\Sigma_{\bar{\mathcal{L}}_{-1}}$.

The proof of Theorem 2 relies on an induction argument on the node sets $\bar{\mathcal{L}}_k$. Key in the induction argument is the next

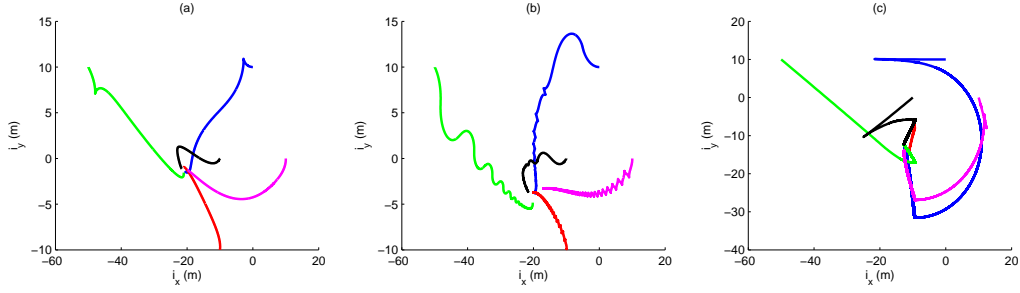


Fig. 6: Rendezvous control simulation for: (a) proposed feedback in (5), (b) feedback in [6], and (c) feedback in [7]

result stating that if the vehicles in $\bar{\mathcal{L}}_{k-1}$ achieve rendezvous, then so do the vehicles in $\bar{\mathcal{L}}_k$.

Proposition 3: Consider system (1), (2) and assume that the sensor graph \mathcal{G} contains a reverse directed spanning tree. Let u_i and ω_i be as in (5) with $\mathbf{f}_i(y_i)$ as in (4). Suppose that, for some integer $k \geq 0$, the set $\Gamma_{\bar{\mathcal{L}}_{k-1}}$ is globally asymptotically stable for the dynamics $\Sigma_{\bar{\mathcal{L}}_{k-1}}$. There exists $k_1^* > 0$ such that choosing $k_1 > k_1^*$ in (5), implies $\Gamma_{\bar{\mathcal{L}}_k}$ is globally asymptotically stable for the dynamics $\Sigma_{\bar{\mathcal{L}}_k}$.

In Section V-A, we use the above proposition to prove Theorem 2, and in Section V-B we prove Proposition 3.

In the special case when \mathcal{G} is strongly connected, we have $\bar{\mathcal{L}}_0 = \mathcal{V}$. Since, by definition, $\bar{\mathcal{L}}_{-1} = \emptyset$, the set $\Gamma_{\bar{\mathcal{L}}_{-1}}$ is GAS for $\Sigma_{\bar{\mathcal{L}}_{-1}}$, and Proposition 3 yields the following corollary.

Corollary 1: Consider system (1), (2) and assume that the sensor graph \mathcal{G} is strongly connected. Let u_i and ω_i be as in (5) with $\mathbf{f}_i(y_i)$ as in (4). There exists $k_1^* > 0$ such that choosing $k_1 > k_1^*$ solves the rendezvous control problem.

A. Proof of Theorem 2

To begin with, the feedback in (5) is local and distributed because it is a smooth function of y_i^i only. Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ containing a reverse directed spanning tree and the node sets \mathcal{L}_k and $\bar{\mathcal{L}}_k$ defined in Section II-B. By construction, the node sets $\bar{\mathcal{L}}_k$ are isolated, the subgraph $(\mathcal{V}_0, \mathcal{E}_0)$ is strongly connected, and $\bar{\mathcal{L}}_0 = \mathcal{L}_0 = \mathcal{V}_0$.

The proof is by induction. Since the subgraph $(\bar{\mathcal{L}}_0, \mathcal{E}_0)$ is strongly connected, by Corollary 1, there exists l_0 such that choosing $k_1 > l_0$ makes the set $\Gamma_{\bar{\mathcal{L}}_0}$ globally asymptotically stable for system $\Sigma_{\bar{\mathcal{L}}_0}$.

Now consider $\bar{\mathcal{L}}_k$ and suppose the reduced rendezvous manifold $\Gamma_{\bar{\mathcal{L}}_{k-1}}$ is globally asymptotically stable for system $\Sigma_{\bar{\mathcal{L}}_{k-1}}$. It holds from Proposition 3 that there exists l_k such that choosing $k_1 > l_k$ makes the isolated node set $\Gamma_{\bar{\mathcal{L}}_k}$ globally asymptotically stable for system $\Sigma_{\bar{\mathcal{L}}_k}$. By part (ii) of Proposition 2, $\mathcal{C}(\mathcal{G})$ contains a reverse directed spanning tree, so there is a path from every node of $\mathcal{C}(\mathcal{G})$ to the unique root of $\mathcal{C}(\mathcal{G})$. By part (i) of the same proposition, $\mathcal{C}(\mathcal{G})$ is acyclic, which implies that the paths connecting the nodes of $\mathcal{C}(\mathcal{G})$ to the unique root of $\mathcal{C}(\mathcal{G})$ have a maximum length, k^* . Recall that, by definition, $\bar{\mathcal{L}}_{k^*} = \sum_{i=1}^{k^*} \mathcal{L}_i$ is the union of those strongly connected components \mathcal{V}_i of \mathcal{V} that are associated with nodes v_i of the condensation digraph $\mathcal{C}(\mathcal{G})$ with the property that the maximum path length from v_i to the root v_0 is $\leq k^*$. As we argued earlier, the set of such nodes v_i equals

the entire condensation digraph, implying that $\bar{\mathcal{L}}_{k^*} = \mathcal{V}$. Let $k_1^* > \max\{l_0, \dots, l_{k^*}\}$. By induction, it must hold that choosing $k_1 > k_1^*$ makes $\Gamma_{\bar{\mathcal{L}}_{k^*}} = \Gamma$ globally asymptotically stable for system $\Sigma_{\bar{\mathcal{L}}_{k^*}} = \Sigma_{\mathcal{V}} = \Sigma$. We conclude that Γ is globally asymptotically stable. \square

B. Proof of Proposition 3

We denote $A := \bar{\mathcal{L}}_{k-1}$ and $B := \mathcal{L}_k$ and therefore $\bar{\mathcal{L}}_k = A \cup B$. By assumption, Γ_A is globally asymptotically stable for the dynamics Σ_A and the graph associated to the nodes in B is strongly connected. We need to show that $\Gamma_{A \cup B}$ is globally asymptotically stable for the dynamics $\Sigma_{A \cup B}$. The proof relies on the following coordinate transformation.

1) Coordinate Transformation: For notational convenience, we collect the position vectors x_i and rotation matrices R_i into variables $x := (x_1, \dots, x_n)$ and $R := (R_1, \dots, R_n)$. We define the spaces $\mathbf{X} := \mathbb{R}^{2n}$, $\mathbf{R} := \text{SO}(2) \times \dots \times \text{SO}(2)$ (n times), so that $x \in \mathbf{X}$ and $R \in \mathbf{R}$. For each $i \in \{1, \dots, n\}$, define

$$X_i := \mathbf{f}_i(y_i)/A_i, \quad (9)$$

where $A_i := \sum_{j \in \mathcal{N}_i} a_{ij}$, and let $X := (X_1, \dots, X_n)$. We may express X as

$$X = \text{diag}(1/A_1, \dots, 1/A_n)(L \otimes I_2)x.$$

In the above, $\text{diag}(\dots)$ is the diagonal matrix with diagonal elements inside the parenthesis; L is the weighted Laplacian matrix of the sensor digraph associated with the gains a_{ij} ; finally, \otimes denotes the Kronecker product of matrices. Since the sensor digraph contains a reverse directed spanning tree, by Proposition 1 the matrix $L \otimes I_2$ has rank $2(n-1)$, and $\text{Ker}(L \otimes I_2) = \text{span}\{\mathbf{1} \otimes e_1, \mathbf{1} \otimes e_2\}$ with $\mathbf{1} \in \mathbb{R}^n$. Let $\bar{x} := [I_2 \ \dots \ I_2]x = \sum_i x_i$, then the linear map $T : \mathbf{X} \rightarrow \mathbf{X} \times \mathbb{R}^2$, $x \mapsto (X, \bar{x})$ is an isomorphism onto its image. Under the action of T , the subspace $\{x \in \mathbf{X} : x_1 = \dots = x_n\}$ is mapped isomorphically onto the subspace $\{(X, \bar{x}) \in \text{Im } T : X = 0\}$. Since the feedbacks in (4)-(5) are local and distributed, it can be seen that the dynamics of the closed-loop unicycles in (X, \bar{x}, R) coordinates are independent of \bar{x} . Moreover, as we have seen, in these coordinates the control specification is the global stabilization of $\{(X, \bar{x}, R) \in \mathbf{X} \times \mathbb{R}^2 \times \mathbf{R} : X = 0\}$, a set whose description is independent of \bar{x} . In light of these considerations, for the stability analysis we may drop the variable \bar{x} , and show that the set $\hat{\Gamma} := \{(X, R) \in \mathbf{X} \times \mathbf{R} : X = 0\}$ is GAS for the (X, R) dynamics.

From here on we will use the hat notation to refer to quantities represented in (X, R) coordinates. Denote $\mathbf{g}_i(y_i) := \|\mathbf{f}_i(y_i)\|\mathbf{f}_i(y_i)$. Using (9), the functions \mathbf{f}_i and \mathbf{g}_i and their body frame representations are given in (X, R) coordinates by

$$\begin{aligned}\hat{\mathbf{f}}_i(X_i) &= A_i X_i, \quad \hat{\mathbf{g}}_i(X_i) = A_i^2 \|X_i\| X_i \\ \hat{\mathbf{f}}_i^i(X_i, R_i) &= A_i R_i^{-1} X_i, \quad \hat{\mathbf{g}}_i^i(X_i, R_i) = A_i^2 R_i^{-1} \|X_i\| X_i,\end{aligned}\quad (10)$$

and we can use these expressions to rewrite the feedback (5) in new coordinates as $u_i = \hat{\mathbf{g}}_i(X_i, R_i) \cdot e_1$, $\omega_i = -k_1 \hat{\mathbf{f}}_i(X_i, R_i) \cdot e_2$. We remark that $\hat{\mathbf{f}}_i$ and $\hat{\mathbf{f}}_i^i$ are homogeneous of degree one with respect to X_i . Similarly, $\hat{\mathbf{g}}_i$ and $\hat{\mathbf{g}}_i^i$ are homogeneous of degree two with respect to X_i . The closed-loop unicycle dynamics in (X, R) coordinates are given by

$$\dot{X}_i = \frac{\sum_{j \in N_i} a_{ij} ((\hat{\mathbf{g}}_j^j \cdot e_1) R_j e_1 - (\hat{\mathbf{g}}_i^i \cdot e_1) R_i e_1)}{A_i}, \quad (11)$$

$$\dot{R}_i = R_i (-k_1 \hat{\mathbf{f}}_i^i \cdot e_2)^\times. \quad (12)$$

We will refer to system (11)-(12) as $\hat{\Sigma}_i$.

In analogy with what we did earlier, for a set of nodes $S \subset \mathcal{V}$ we let $X_S := (X_i)_{i \in S} \in X_S$ and $R_S := (R_i)_{i \in S} \in R_S$. Moreover, if S is an isolated node set, the systems $\hat{\Sigma}_i, i \in S$ determine an autonomous dynamical system which we denote by $\hat{\Sigma}_S$. We also denote the reduced rendezvous manifold by $\hat{\Gamma}_S := \{(X_S, R_S) \in X_S \times R_S : X_S = 0\}$. In new coordinates, it needs to be shown that the set $\hat{\Gamma}_{AUB}$ is globally asymptotically stable for the dynamics $\hat{\Sigma}_{AUB}$ under the assumption that $\hat{\Gamma}_A$ is globally asymptotically stable for the dynamics $\hat{\Sigma}_A$.

2) *Stability analysis:* Let

$$\begin{aligned}V(X_B) &= \sum_{i \in B} \gamma_i X_i^\top X_i \\ W_{\text{tran}}(X_B) &= \sqrt{V(X_B)} \\ W_{\text{rot}}(X_B, R_B) &= \sum_{i \in B} \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_1,\end{aligned}\quad (13)$$

where $\gamma_i > 0$ are gains that will be defined later. Consider the function $W : X_B \times R_B \rightarrow \mathbb{R}$ defined as

$$W(X_B, R_B) = \alpha W_{\text{tran}}(X_B) + W_{\text{rot}}(X_B, R_B), \quad (14)$$

where $\alpha > 0$ is a design parameter.

The next two lemmas are used in the subsequent analysis.

Lemma 1: Consider the continuous function $W(X_B, R_B)$ defined in (14). There exists $\alpha^* > 0$ such that, for all $\alpha > 2\alpha^*$, the following properties hold:

- (i) $W \geq 0$ and $W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}$.
- (ii) For all $c > 0$, the sublevel set $W_c := \{(X_B, R_B) : W(X_B, R_B) \leq c\}$ is compact.
- (iii) $\alpha^* \sqrt{V(X_B)} < W(X_B, R_B) < 2\alpha \sqrt{V(X_B)}$.

The proof is in the appendix. From now on assume $\alpha > 2\alpha^*$.

Lemma 2: Consider system (11), (12). There exist gains γ_i in (13) and $k_1^* > 0$ such that choosing $k_1 > k_1^*$ implies

$$\frac{d}{dt} W(X_B, R_B) \leq -\sigma V(X_B) + \Phi(X_A, R), \quad \sigma > 0, \quad (15)$$

where $\Phi(X_A, R)$ is continuous with respect to its arguments and $\Phi(0, R) = 0$.

The proof of Lemma 2 is presented in the appendix.

We will now show that choosing $k_1 > k_1^*$ implies $\hat{\Gamma}_{AUB}$ is globally asymptotically stable for $\hat{\Sigma}_{AUB}$. The proof will make use of the reduction theorem (Theorem 1). We will first show that all solutions of the closed-loop system are bounded. The rotation matrices live in a compact set, therefore we only need to show that the states $X_{AUB} = (X_i)_{i \in AUB}$ are bounded. Since A is isolated, $\hat{\Sigma}_A$ is an autonomous subsystem and by assumption, $\hat{\Gamma}_A = \{(X_A, R_A) \in X_A \times R_A : X_A = 0\}$ (compact), is globally asymptotically stable. Therefore, X_A is bounded. From the inequality $W(X_B, R_B) \geq \alpha^* \sqrt{V(X_B)}$ in part (iii) of Lemma 1, to show boundedness of $V(X_B)$, it suffices to show that $W(X_B, R_B)$ is bounded. Boundedness of $V(X_B)$, in turn, implies boundedness of X_B . From the bound on the derivative of W in (15), and by Lemma 1 we obtain

$$\frac{d}{dt} W(X_B, R_B) \leq -\frac{\sigma W(X_B, R_B)^2}{(2\alpha)^2} + \Phi(X_A, R), \quad \sigma > 0.$$

Since X_A is bounded and $R \in \mathbb{R}$ lies on a compact set, it holds that $\Phi(X_A, R)$ is bounded and therefore W is bounded, which implies that X_B is bounded. Therefore X_{AUB} is bounded, as claimed. Now define the set, $\hat{\Lambda} := \{(X_{AUB}, R_{AUB}) \in X_{AUB} \times R_{AUB} : X_A = 0\}$. Since the set $\hat{\Gamma}_A$ is globally asymptotically stable for system $\hat{\Sigma}_A$ and X_{AUB} is bounded, it holds that $\hat{\Lambda}$ is globally asymptotically stable for $\hat{\Sigma}_{AUB}$.

To show that the set $\hat{\Gamma}_{AUB}$, which is compact, is globally asymptotically stable for the system $\hat{\Sigma}_{AUB}$, it suffices to show that $\hat{\Gamma}_{AUB}$ is globally asymptotically stable relative to $\hat{\Lambda}$. On the set $\hat{\Lambda}$, $\Phi(X_A, R)$ is equal to zero and the derivative of W is therefore given by $\frac{d}{dt} W(X_B, R_B) \leq -\frac{\sigma W(X_B, R_B)^2}{(2\alpha)^2}$, $\sigma > 0$. By Lemma 1, all level sets of $W(X_B, R_B)$ are compact and $W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}$. This implies $\hat{\Gamma}_{AUB}$ is globally asymptotically stable relative to the set $\hat{\Lambda}$. By Theorem 1, $\hat{\Gamma}_{AUB}$ is globally asymptotically stable for $\hat{\Sigma}_{AUB}$. This completes the proof.

VI. CONCLUSION

We have presented the first solution to the rendezvous control problem for a group of kinematic unicycles on the plane using continuous, time-independent feedback that is local and distributed. The solution assumes a fixed sensing digraph that contains a reverse-directed spanning tree. The control methodology is based on a control structure made of two nested loops. An outer loop produces a standard feedback for consensus of single integrators which becomes reference to an inner loop assigning the unicycle control inputs that rely only on onboard measurements. Information of the unicycle's relative orientations is not required.

REFERENCES

- [1] W. Ren and R. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
- [2] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004.
- [3] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

- [4] A. Roza, M. Maggiore, and L. Scardovi, "A class of rendezvous controllers for underactuated thrust-propelled rigid bodies," in *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, California, 2014.
- [5] —, "Local and distributed rendezvous of underactuated rigid bodies," *arXiv:1509.07022v1 [math.OC]*, 2015.
- [6] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient conditions for formation control of unicycles," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 121–127, 2005.
- [7] D. V. Dimarogonas and K. J. Kyriakopoulos, "On the rendezvous problem for multiple nonholonomic agents," *Automatic Control, IEEE Transactions on*, vol. 52, no. 5, pp. 916–922, 2007.
- [8] R. Zheng and D. Sun, "Rendezvous of unicycles: A bearings-only and perimeter shortening approach," *Systems & Control Letters*, vol. 62, no. 5, pp. 401–407, 2013.
- [9] C. Godsil and G. Royle, "Algebraic graph theory, volume 207 of graduate texts in mathematics," 2001.
- [10] T. Hatanaka, N. Chopra, M. Fujita, and M. W. Spong, *Passivity-Based Control and Estimation in Networked Robotics*. Springer, 2015.
- [11] M. El-Hawwary and M. Maggiore, "Reduction theorems for stability of closed sets with application to backstepping control design," *Automatica*, vol. 49, no. 1, pp. 214–222, 2013.
- [12] P. Seibert and J. S. Florio, "On the reduction to a subspace of stability properties of systems in metric spaces," *Annali di Matematica pura ed applicata*, vol. CLXIX, pp. 291–320, 1995.

APPENDIX

Throughout this appendix we will make use of functions μ_i and μ defined as follows. Recall that $V(X_B)$ is positive definite. Define the functions $\mu : X_B \setminus 0 \rightarrow \mu(X_B \setminus 0)$, $\mu(X_B) := X_B / \sqrt{V(X_B)}$, and $\mu_i : X_B \setminus 0 \rightarrow \mu_i(X_B \setminus 0)$, $\mu_i(X_B) := X_i / \sqrt{V(X_B)}$, $i \in B$. Since the numerator and denominator are both homogeneous of degree one, these functions are both homogeneous of degree zero with respect to X_B . Therefore, the images satisfy $\mu(X_B \setminus 0) = \mu(\mathbb{S}^k)$ and $\mu_i(X_B \setminus 0) = \mu_i(\mathbb{S}^k)$, where \mathbb{S}^k is the unit sphere in X_B . Since μ and μ_i are continuous functions and \mathbb{S}^k is a compact set, the images $\mu(X_B \setminus 0)$ and $\mu_i(X_B \setminus 0)$ are compact sets.

A. Proof of Lemma 1

Recall the definition of $W(X_B, R_B)$,

$$\begin{aligned} W &= \alpha \sqrt{V(X_B)} + \sum_{i \in B} \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_1 \\ &= \sqrt{V(X_B)} \left(\alpha + \frac{\sum_{i \in B} \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_1}{\sqrt{V(X_B)}} \right). \end{aligned}$$

Using the fact that $\hat{\mathbf{f}}_i^i(X_i, R_i)$ is homogeneous with respect to its first argument, we have $W = \sqrt{V(X_B)} \left(\alpha + \sum_{i \in B} \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_1 \right)$. Since $\hat{\mathbf{f}}_i^i$ is continuous, $\mu_i(X_B)$ is bounded, and $R_B \in R_B$, a compact set, it follows that the function $\sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_1 \right|$ has a bounded supremum. Accordingly, let $\alpha^* = \sup_{(X_B, R_B) \in X_B \times R_B} \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_1 \right|$. For all $\alpha > 2\alpha^*$, we have $W(X_B, R_B) \geq \underline{W}(X_B, R_B) := \alpha^* \sqrt{V(X_B)} \geq 0$. This inequality implies that $W \geq 0$ and $W^{-1}(0) \subset \underline{W}^{-1}(0)$. But $\underline{W} = 0$ if and only if $V(X_B) = 0$ (i.e., $X_B = 0$). Thus $W^{-1}(0) \subset \{(X_B, R_B) : X_B = 0\}$. Conversely, on the set $\{(X_B, R_B) : X_B = 0\}$, $X_B = 0$ and hence $W = 0$, and therefore $\{(X_B, R_B) : X_B = 0\} \subset W^{-1}(0)$. It follows that $W^{-1}(0) = \{(X_B, R_B) : X_B = 0\}$ proving part (i).

For part (ii), note that for all $c > 0$, $W_c \subset \{\underline{W}(X, R) \leq c\}$. Since the sublevel sets of \underline{W} are compact and $R_B \in R_B$, a compact set, the set W_c is bounded. Continuity of W implies that W_c is compact.

For part (iii), it has already been shown that $W(X_B, R_B) \geq \alpha^* \sqrt{V(X_B)}$. It also holds that $W = \sqrt{V(X_B)} \left(\alpha + \sum_{i \in B} \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_1 \right) \leq \sqrt{V(X_B)} (\alpha + \alpha) \leq 2\alpha \sqrt{V(X_B)}$. \square

B. Proof of Lemma 2

We first compute inequalities for \dot{W}_{tran} and \dot{W}_{rot} for system (11) and (12). We then combine them to derive (15). Consider unicycle $i \in B$. The dynamics of X_i in (11) are split into two terms, for neighboring robots $j \in \mathcal{N}_i \cap A$ and $j \in \mathcal{N}_i \cap B$ respectively,

$$\begin{aligned} \dot{X}_i &= \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \frac{(u_j R_j e_1 - u_i R_i e_1)}{A_i} \\ &\quad + \sum_{j \in \mathcal{N}_i \cap B} a_{ij} \frac{(u_j R_j e_1 - u_i R_i e_1)}{A_i}. \end{aligned} \quad (16)$$

For simplicity of notation, we drop the arguments of $\hat{\mathbf{g}}_i(X_i)$ and $\hat{\mathbf{g}}_i^i(X_i, R_i)$. Adding and subtracting the term,

$$\frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (\hat{\mathbf{g}}_j - \hat{\mathbf{g}}_i) - \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \hat{\mathbf{g}}_i}{A_i}$$

to (16) yields,

$$\begin{aligned} \dot{X}_i &= \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (\hat{\mathbf{g}}_j - \hat{\mathbf{g}}_i) - \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \hat{\mathbf{g}}_i}{A_i} \\ &\quad + \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (u_j R_j e_1 - u_i R_i e_1)}{A_i} - \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (\hat{\mathbf{g}}_j - \hat{\mathbf{g}}_i)}{A_i} \\ &\quad + \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} u_j R_j e_1}{A_i} + \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} (\hat{\mathbf{g}}_i - u_i R_i e_1)}{A_i} \\ &= \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (\hat{\mathbf{g}}_j - \hat{\mathbf{g}}_i) - \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \hat{\mathbf{g}}_i}{A_i} \\ &\quad + \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (u_j R_j e_1 - \hat{\mathbf{g}}_j)}{A_i} - \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (u_i R_i e_1 - \hat{\mathbf{g}}_i)}{A_i} \\ &\quad + \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} u_j R_j e_1}{A_i} + \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} (\hat{\mathbf{g}}_i - u_i R_i e_1)}{A_i}. \end{aligned}$$

Replacing u_j and u_i by the assigned feedbacks in (5) and using the identity $R_i \hat{\mathbf{g}}_i^i = \hat{\mathbf{g}}_i$ then,

$$\dot{X}_i = a_i(X_B) + b_i(X_B, R) + c_i(X_B, R) + d_i(X_A, R),$$

where,

$$\begin{aligned} a_i(X_B) &:= \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} (\hat{\mathbf{g}}_j - \hat{\mathbf{g}}_i) - \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \hat{\mathbf{g}}_i}{A_i} \\ b_i(X_B, R) &:= \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} R_j ((\hat{\mathbf{g}}_j^j \cdot e_1) e_1 - \hat{\mathbf{g}}_j^j)}{A_i} \\ &\quad - \frac{\sum_{j \in \mathcal{N}_i \cap B} a_{ij} R_i ((\hat{\mathbf{g}}_i^i \cdot e_1) e_1 - \hat{\mathbf{g}}_i^i)}{A_i} \\ c_i(X_B, R) &:= \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} R_i (\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1) e_1)}{A_i} \\ d_i(X_A, R) &:= \frac{\sum_{j \in \mathcal{N}_i \cap A} a_{ij} (\hat{\mathbf{g}}_j^j \cdot e_1) R_j e_1}{A_i}. \end{aligned}$$

The time derivative of $W_{\text{tran}} = \sqrt{V(X_B)}$ in (13) yields,

$$\begin{aligned} \dot{W}_{\text{tran}} = & \frac{1}{2\sqrt{V(X_B)}} \left[\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (a_i(X_B) + b_i(X_B, R) \right. \\ & \left. + c_i(X_B, R)) \right] + \frac{1}{2\sqrt{V(X_B)}} \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} d_i(X_A, R). \end{aligned} \quad (17)$$

The derivative of the first term is considered in Claim 1.

Claim 1: There exist gains γ_i in (13) and a negative definite function $\mathbf{r}(X_B)$, homogeneous of degree three, such that $\sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} a_i(X_B) \leq \mathbf{r}(X_B)$.

The proof of Claim 1 is presented in Section C of this Appendix. Let the gains γ_i be as in Claim 1. The derivative of the remaining terms in the square brackets of (17) satisfies,

$$\begin{aligned} & \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (b_i(X_B, R) + c_i(X_B, R)) \\ & \leq \sum_{i \in B} \frac{1}{A_i} \frac{\partial V(X_B)}{\partial X_i} \left[\sum_{j \in \mathcal{N}_i \cap B} a_{ij} \|(\hat{\mathbf{g}}_j^j \cdot e_1)e_1 - \hat{\mathbf{g}}_j^j\| \right. \\ & \quad \left. + \sum_{j \in \mathcal{N}_i \cap B} a_{ij} \|(\hat{\mathbf{g}}_i^i \cdot e_1)e_1 - \hat{\mathbf{g}}_i^i\| + \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \|\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1\| \right] \\ & \leq \sum_{i \in B} \frac{1}{A_i} \frac{\partial V(X_B)}{\partial X_i} \left[\sum_{j \in \mathcal{N}_i \cap B} a_{ij} \|(\hat{\mathbf{g}}_j^j \cdot e_1)e_1 - \hat{\mathbf{g}}_j^j\| \right. \\ & \quad \left. + \sum_{j \in \mathcal{N}_i} a_{ij} \|(\hat{\mathbf{g}}_i^i \cdot e_1)e_1 - \hat{\mathbf{g}}_i^i\| \right]. \end{aligned}$$

We claim that $\|(\hat{\mathbf{g}}_i^i(X_i, R_i) \cdot e_1)e_1 - \hat{\mathbf{g}}_i^i(X_i, R_i)\| = |\hat{\mathbf{g}}_i^i(X_i, R_i) \cdot e_2|$. Indeed, writing $\hat{\mathbf{g}}_i^i = (\hat{\mathbf{g}}_i^i \cdot e_1)e_1 + \hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1$, we have $\hat{\mathbf{g}}_i^i \cdot e_2 = (\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1) \cdot e_2$. Since the vector $\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1$ is parallel to e_2 , $|(\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1) \cdot e_2| = \|\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1\|$, so that $|\hat{\mathbf{g}}_i^i \cdot e_2| = \|\hat{\mathbf{g}}_i^i - (\hat{\mathbf{g}}_i^i \cdot e_1)e_1\|$. Then,

$$\begin{aligned} & \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} (b_i(X_B, R) + c_i(X_B, R)) \\ & \leq \sum_{i \in B} \frac{\bar{a}}{A_i} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\| \left(\sum_{j \in B} |\hat{\mathbf{g}}_j^j \cdot e_2| + n |\hat{\mathbf{g}}_i^i \cdot e_2| \right) \end{aligned}$$

where $\bar{a} = \max\{a_{ij}\}_{i,j \in \{1, \dots, n\}}$ which is homogeneous of degree three with respect to X_B since $\frac{\partial V(X_B)}{\partial X_i}$ is homogeneous of degree one and $\hat{\mathbf{g}}_i^i$ is homogeneous of degree two with

respect to X_B for all $i \in B$. The last term in (17) satisfies,

$$\begin{aligned} & \frac{1}{2\sqrt{V(X_B)}} \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} d_i(X_A, R) \\ & \leq \frac{1}{2\sqrt{V(X_B)}} \sum_{i \in B} \frac{1}{A_i} \frac{\partial V(X_B)}{\partial X_i} \sum_{j \in \mathcal{N}_i \cap A} a_{ij} (\hat{\mathbf{g}}_j^j \cdot e_1) R_j e_1 \\ & \leq \frac{1}{2\sqrt{V(X_B)}} \sum_{i \in B} \frac{1}{A_i} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\| \sum_{j \in \mathcal{N}_i \cap A} a_{ij} \|\hat{\mathbf{g}}_j^j\| \\ & \leq \sum_{i \in B} \sup_{X_B \in \mathcal{X}_B} \left\{ \frac{\bar{a}}{A_i} \frac{1}{2\sqrt{V(X_B)}} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\| \right\} \sum_{j \in \mathcal{N}_i \cap A} \|\hat{\mathbf{g}}_j^j\| := \Phi_{\text{tran}}(X_A, R). \end{aligned} \quad (18)$$

The bounded supremum of $\frac{1}{\sqrt{V(X_B)}} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\|$ exists because this term is homogeneous of degree 0 with respect to X_B . Moreover, $X_A = 0$ implies that $\|\hat{\mathbf{g}}_j^j\| = 0$ for all $j \in A$ and hence $\Phi_{\text{tran}}(0, R) = 0$. Everything together, (17) yields,

$$\begin{aligned} \dot{W}_{\text{tran}} \leq & \frac{1}{2\sqrt{V(X_B)}} \left[\mathbf{r}(X_B) + \sum_{i \in B} \frac{\bar{a}}{A_i} \left\| \frac{\partial V(X_B)}{\partial X_i} \right\| \right. \\ & \left. \left(\sum_{j \in B} |\hat{\mathbf{g}}_j^j \cdot e_2| + n |\hat{\mathbf{g}}_i^i \cdot e_2| \right) \right] + \Phi_{\text{tran}}(X_A, R). \end{aligned} \quad (19)$$

Since $\mathbf{r}(X_B)$ is homogeneous of degree three. We can write,

$$\begin{aligned} \mathbf{r}(X_B) &= \frac{\sqrt{V(X_B)} V(X_B)}{\sqrt{V(X_B)} V(X_B)} \mathbf{r}(X_B) \\ &= \sqrt{V(X_B)} V(X_B) \mathbf{r} \left(\frac{X_B}{\sqrt{V(X_B)}} \right) \\ &= \sqrt{V(X_B)} V(X_B) \mathbf{r}(\mu(X_B)). \end{aligned}$$

Analogous operations can be performed with the remaining term in the square bracket of (19) yielding,

$$\begin{aligned} \dot{W}_{\text{tran}} \leq & \frac{V(X_B)}{2} \left[\mathbf{r}(\mu(X_B)) + \sum_{i \in B} \frac{\bar{a}}{A_i} \left\| \frac{\partial V(\mu(X_B))}{\partial X_i} \right\| \right. \\ & \left. \left(\sum_{j \in B} |\hat{\mathbf{g}}_j^j(\mu_j(X_B), R_j) \cdot e_2| + n |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right) \right] \\ & + \Phi_{\text{tran}}(X_A, R). \end{aligned}$$

Since \mathbf{r} is continuous and negative definite and $\mu(X_B)$ lies on a compact set S_1 , it follows that $\mathbf{r}(\mu(X_B))/2$ has bounded maximum $-M_2 < 0$. Similarly, the function $\frac{\bar{a}}{A_i} \left\| \frac{\partial V(\mu(X_B))}{\partial X_i} \right\|$

has a maximum. Letting $M_1 := n \max_{\theta \in S^1} \frac{\bar{a}}{A_i} \left\| \frac{\partial V(\theta)}{\partial X_i} \right\|$ yields,

$$\begin{aligned} \dot{W}_{\text{tran}} &\leq V(X_B) \left[-M_2 + \frac{M_1}{2n} \sum_{i \in B} \left(\sum_{j \in B} |\hat{\mathbf{g}}_j^j(\mu_j(X_B), R_j) \cdot e_2| \right. \right. \\ &\quad \left. \left. + n |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right) \right] + \Phi_{\text{tran}}(X_A, R) \\ &\leq V(X_B) \left[-M_2 + \frac{M_1}{2n} \sum_{i \in B} (n |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right. \\ &\quad \left. + n |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2|) \right] + \Phi_{\text{tran}}(X_A, R) \\ &\leq V(X_B) \left[-M_2 + M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right] \\ &\quad + \Phi_{\text{tran}}(X_A, R). \end{aligned} \quad (20)$$

This proves the first inequality. We now turn to the second. Recall the definition of W_{rot} , $W_{\text{rot}}(X_B, R_B) = \sum_{i \in B} \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_1$. The time derivative of W_{rot} along the vector field in (11)-(12) is $\dot{W}_{\text{rot}} = \sum_{i \in B} \left(\frac{d}{dt} \hat{\mathbf{f}}_i^i \right) \cdot e_1$. To express $(d/dt) \hat{\mathbf{f}}_i^i$, recall that $\hat{\mathbf{f}}_i^i(X_i, R_i) = R_i^{-1} \hat{\mathbf{f}}_i(X_i)$. Then, $\frac{d}{dt} \hat{\mathbf{f}}_i^i = \left(\frac{d}{dt} R_i^{-1} \right) \hat{\mathbf{f}}_i + R_i^{-1} \frac{d\hat{\mathbf{f}}_i}{dt}$. We will denote the derivative of $\hat{\mathbf{f}}_i(X_i) = A_i X_i$ by,

$$\begin{aligned} \mathbf{h}_i(X, R) &:= (d/dt) \hat{\mathbf{f}}_i(X_i) \\ &= A_i (a_i(X_B) + b_i(X_B, R) + c_i(X_B, R) + d_i(X_A, R)) \end{aligned}$$

where the first three terms are homogeneous of degree two with respect to X_B and the last term is homogeneous of degree two with respect to X_A . Consistently with our notational convention, we will let $\mathbf{h}_i^i(X, R) := R_i^{-1} \mathbf{h}_i(X, R)$. Returning to the derivative of $\hat{\mathbf{f}}_i^i$, we have

$$\begin{aligned} \frac{d}{dt} \hat{\mathbf{f}}_i^i &= -(\omega_i)^\times R_i^{-1} \hat{\mathbf{f}}_i(X_i) + R_i^{-1} \mathbf{h}_i(X, R) \\ &= - \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \hat{\mathbf{f}}_i^i(X_i, R_i) + \mathbf{h}_i^i(X, R). \end{aligned}$$

We substitute the above identity in the expression for \dot{W}_{rot} ,

$$\begin{aligned} \dot{W}_{\text{rot}} &= \sum_{i \in B} \left(-e_1^\top \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \hat{\mathbf{f}}_i^i(X_i, R_i) + \mathbf{h}_i^i(X, R) \cdot e_1 \right) \\ &= \sum_{i \in B} \left((\hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_2) \omega_i + \mathbf{h}_i^i(X, R) \cdot e_1 \right). \end{aligned}$$

Substituting the feedback $\omega_i = -k_1 (\hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_2)$ and taking norms, we arrive at the inequality

$$\dot{W}_{\text{rot}} \leq \sum_{i \in B} \left[-k_1 \left| \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_2 \right|^2 + \mathbf{h}_i^i(X, R) \cdot e_1 \right].$$

This gives,

$$\dot{W}_{\text{rot}} \leq \left[-k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_2 \right|^2 + \ell(X_B, R) \right] + \Phi_{\text{rot}}(X_A, R)$$

where

$$\ell(X_B, R) := \sum_{i \in B} A_i R_i^\top (a_i(X_B) + b_i(X_B, R) + c_i(X_B, R)) \cdot e_1$$

and $\Phi_{\text{rot}}(X_A, R) := \sum_{i \in B} A_i R_i^\top d_i(X_A, R) \cdot e_1$. Note that $\sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(X_i, R_i) \cdot e_2 \right|^2$ and $\ell(X_B, R)$ are homogeneous of degree two with respect to X_B . The function $\Phi_{\text{rot}}(X_A, R)$ does not depend on X_B and $\Phi_{\text{rot}}(0, R) = 0$. This yields,

$$\begin{aligned} \dot{W}_{\text{rot}} &\leq V(X_B) \left[-k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(X_i / \sqrt{V(X_B)}, R_i) \cdot e_2 \right|^2 \right. \\ &\quad \left. + \ell(X_B / \sqrt{V(X_B)}, R) \right] + \Phi_{\text{rot}}(X_A, R) \\ &\leq V(X_B) \left[-k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right|^2 \right. \\ &\quad \left. + \ell(\mu(X_B), R) \right] + \Phi_{\text{rot}}(X_A, R). \end{aligned}$$

$|\ell(\mu(X_B), R)|$ has a bounded supremum. Letting $M_3 = \sup_{(\theta, R) \in S^1 \times R} (|\ell(\theta, R)|)$, we conclude that,

$$\begin{aligned} \dot{W}_{\text{rot}} &\leq V(X_B) \left[-k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right|^2 + M_3 \right] \\ &\quad + \Phi_{\text{rot}}(X_A, R). \end{aligned} \quad (21)$$

By using the inequalities (20) and (21) we now bound the derivative of W to derive (15). Notice that

$$\begin{aligned} \dot{W} &= \alpha \dot{W}_{\text{tran}} + \dot{W}_{\text{rot}} \\ &\leq V(X_B) \left[-\alpha M_2 + \alpha M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right. \\ &\quad \left. - k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right|^2 + M_3 \right] + \Phi(X_A, R), \end{aligned}$$

where $\Phi(X_A, R) := \alpha \Phi_{\text{tran}}(X_A, R) + \Phi_{\text{rot}}(X_A, R)$.

Choose $\alpha > 3M_3/M_2$. This implies,

$$\begin{aligned} \dot{W} &\leq V(X_B) \left[-2M_3 + \alpha M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right. \\ &\quad \left. - k_1 \sum_{i \in B} \left| \hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right|^2 \right] + \Phi(X_A, R). \end{aligned} \quad (22)$$

Since $\hat{\mathbf{f}}_i^i(X_i, R_i)$ is homogeneous with respect to X_i , we have, $\hat{\mathbf{f}}_i^i(\mu_i(X_B), R_i) = \frac{\sqrt{\|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)\|}}{\|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)\|} \hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)$. Plugging the last expression into (22) yields

$$\begin{aligned} \dot{W} &\leq V(X_B) \left[-2M_3 + \alpha M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right. \\ &\quad \left. - k_1 \sum_{i \in B} \left(\frac{\sqrt{\|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)\|}}{\|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)\|} \hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right)^2 \right] \\ &\quad + \Phi(X_A, R) \\ &\leq V(X_B) \left[-2M_3 + \alpha M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| \right. \\ &\quad \left. - k_1 \sum_{i \in B} \frac{1}{\|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)\|} \left| \hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2 \right|^2 \right] \\ &\quad + \Phi(X_A, R). \end{aligned}$$

Since $\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)$ is a continuous function of its arguments and $\mu_i(X_B)$ is compact, $|\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i)|$ has a maximum M_4 . This implies,

$$\dot{W} \leq V(X_B) \left[-2M_2 + \alpha M_1 \sum_{i \in B} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2| - k_1 \sum_{i \in B} \frac{1}{M_4} |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2|^2 \right] + \Phi(X_A, R).$$

Denote $\beta_i(\mu_i(X_B), R_i) := |\hat{\mathbf{g}}_i^i(\mu_i(X_B), R_i) \cdot e_2|$, and $\beta := (\beta_i(\mu_i(X_B), R_i))_{i \in B}$. Then,

$$\begin{aligned} \dot{W} &\leq V(X_B) \left[-2M_2 + \alpha M_1 \mathbf{1}^\top \beta - \frac{k_1}{M_4} |\beta|^2 \right] + \Phi(X_A, R) \\ &= V(X_B) [\mathbf{1}^\top \quad \beta^\top] \begin{bmatrix} \frac{-2M_2}{\alpha \frac{M_1}{2}} I & \alpha \frac{M_1}{2} I \\ \frac{\alpha \frac{M_1}{2}}{M_4} I & \frac{-k_1}{M_4} I \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \beta \end{bmatrix} + \Phi(X_A, R). \end{aligned}$$

There exists $k_1^* > 0$ such that choosing $k_1 > k_1^*$, the matrix above is negative definite and therefore the first term satisfies,

$$V(X_B) [\mathbf{1}^\top \quad \beta^\top] \begin{bmatrix} \frac{-2M_2}{\alpha \frac{M_1}{2}} I & \alpha \frac{M_1}{2} I \\ \frac{\alpha \frac{M_1}{2}}{M_4} I & \frac{-k_1}{M_4} I \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \beta \end{bmatrix} \leq -\sigma V(X_B), \quad (23)$$

$\sigma > 0$. This concludes the proof of Lemma 2. \square

C. Proof of Claim 1

Recalling that $V(X_B) = \gamma_i X_i^\top X_i$ with $X_i = \hat{\mathbf{f}}_i/A_i$ and defining $b_{ij} := \frac{\alpha_{ij}}{A_i^2}$, it holds that,

$$\begin{aligned} \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} a_i(X_B) &= 2 \sum_{i \in B} \gamma_i \frac{\hat{\mathbf{f}}_i}{A_i} \cdot a_i(X_B) \\ &\leq 2 \sum_{i \in B} \gamma_i \hat{\mathbf{f}}_i \cdot \left(\sum_{j \in \mathcal{N}_i \cap B} b_{ij} (\|\hat{\mathbf{f}}_j\| \hat{\mathbf{f}}_j - \|\hat{\mathbf{f}}_i\| \hat{\mathbf{f}}_i) - \sum_{j \in \mathcal{N}_i \cap A} b_{ij} \|\hat{\mathbf{f}}_i\| \hat{\mathbf{f}}_i \right) \\ &\leq 2 \sum_{i \in B} \gamma_i \left(\sum_{j \in \mathcal{N}_i \cap B} b_{ij} (-\|\hat{\mathbf{f}}_i\|^3 + \|\hat{\mathbf{f}}_j\| \hat{\mathbf{f}}_j \cdot \hat{\mathbf{f}}_i) - \sum_{j \in \mathcal{N}_i \cap A} b_{ij} \|\hat{\mathbf{f}}_i\|^3 \right) \\ &\leq \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap B} b_{ij} \left(-\frac{4}{3} \|\hat{\mathbf{f}}_i\|^3 + \frac{4}{3} \|\hat{\mathbf{f}}_j\|^3 \right) \\ &\quad + \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap B} b_{ij} \left(-\frac{2}{3} \|\hat{\mathbf{f}}_i\|^3 + 2 \|\hat{\mathbf{f}}_j\| \hat{\mathbf{f}}_j \cdot \hat{\mathbf{f}}_i - \frac{4}{3} \|\hat{\mathbf{f}}_j\|^3 \right) \\ &\quad - 2 \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap A} b_{ij} \|\hat{\mathbf{f}}_i\|^3. \end{aligned}$$

The first term equals $\frac{4}{3} \gamma^\top M \bar{h}$ with $\bar{h} := (\|\hat{\mathbf{f}}_i\|^3)_{i \in B}$. M is the $(r \times r)$ -matrix whose (i, j) -th component is $\sum_{k \in \mathcal{N}_i \cap B} b_{ik}$ for $i = j$, b_{ij} for $j \in \mathcal{N}_i \cap B$ and zero otherwise for $i, j \in \{1, \dots, r\}$ where it is assumed without loss of generality that $B = \{1, \dots, r\}$. Choose $\gamma = (\gamma_1, \dots, \gamma_n)$ as the left eigenvector associated to the zero eigenvalue of M . Since B corresponds to a collection of strongly connected components with no links from one to the other, the zero eigenvalue is unique and all components of γ are positive (see Proposition

D.5 in [10]). Therefore,

$$\begin{aligned} \sum_{i \in B} \frac{\partial V(X_B)}{\partial X_i} a_i(X_B) &\leq \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap B} b_{ij} \left(-\frac{2}{3} \|\hat{\mathbf{f}}_i\|^3 + 2 \|\hat{\mathbf{f}}_j\| \hat{\mathbf{f}}_j \cdot \hat{\mathbf{f}}_i - \frac{4}{3} \|\hat{\mathbf{f}}_j\|^3 \right) \\ &\quad - 2 \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap A} b_{ij} \|\hat{\mathbf{f}}_i\|^3 =: \mathbf{r}(X_B). \end{aligned}$$

The term

$$\begin{aligned} \mathbf{r}_1(X_B) &:= \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap B} b_{ij} \left(-\frac{2}{3} \|\hat{\mathbf{f}}_i\|^3 + 2 \|\hat{\mathbf{f}}_j\| \hat{\mathbf{f}}_j \cdot \hat{\mathbf{f}}_i - \frac{4}{3} \|\hat{\mathbf{f}}_j\|^3 \right) \\ &\leq \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap B} b_{ij} \left(-\frac{2}{3} \|\hat{\mathbf{f}}_i\|^3 + 2 \|\hat{\mathbf{f}}_i\| \|\hat{\mathbf{f}}_j\|^2 - \frac{4}{3} \|\hat{\mathbf{f}}_j\|^3 \right) \end{aligned}$$

is less than or equal to zero with equality only when $\hat{\mathbf{f}}_i = \hat{\mathbf{f}}_j$ for all $i, j \in B$ and as such $\mathbf{r}(X_B)$ is less than or equal to zero with equality only when $\hat{\mathbf{f}}_i = \hat{\mathbf{f}}_j$ for all $i, j \in B$.

Now we prove that $\mathbf{r}(X_B) = 0$ only if $\hat{\mathbf{f}}_i = 0$ for all robots $i \in B$. In the case that A is not empty, the inequality $\mathbf{r}(X_B) \leq -2 \sum_{i \in B} \gamma_i \sum_{j \in \mathcal{N}_i \cap A} b_{ij} \|\hat{\mathbf{f}}_i\|^3$ implies $\mathbf{r}(X_B) = 0$ only if $\hat{\mathbf{f}}_i = 0$ for any $i \in B$ with a neighbor in A . As such, by the previous arguments, $\mathbf{r}(X_B) = 0$ only if $\hat{\mathbf{f}}_i = 0$ for all $i \in B$. On the other hand, if A is empty, then B is isolated and strongly connected. Therefore $\mathbf{r}(X_B) = \mathbf{r}_1(X_B)$ is equal to zero only if $\mathbf{r}_1(X_B) = 0$ which is the case only if $\hat{\mathbf{f}}_i = \hat{\mathbf{f}}_j$ for all $i, j \in B$. This implies that $(L \otimes I_2)x \in \text{span}\{\mathbf{1} \otimes e_1, \mathbf{1} \otimes e_2\}$. Since B is a strongly connected component there exists a unique vector $\bar{\gamma}$ (with positive entries) such that $\bar{\gamma}^\top (L \otimes I_2) = 0$. Since $\bar{\gamma}^\top (L \otimes I_2)x = \bar{\gamma}^\top \mathbf{1} \otimes (\alpha e_1 + \beta e_2)$ for some $\alpha, \beta \in \mathbb{R}$, it holds that $\bar{\gamma}^\top \mathbf{1} \otimes (\alpha e_1 + \beta e_2) = 0$. Since all entries of $\bar{\gamma}$ are positive, this implies $\alpha = \beta = 0$ and $(L \otimes I_2)x = 0$. Therefore $x \in \text{span}\{\mathbf{1} \otimes e_1, \mathbf{1} \otimes e_2\}$ or, equivalently, that $\hat{\mathbf{f}}_i = 0$ for all $i \in B$.

Therefore $\mathbf{r}(X_B) = 0$ only if $X_i = 0$ for all $i \in B$ and as such $\mathbf{r}(X_B)$ is negative definite. Note that $\mathbf{r}(X_B)$ is homogeneous of degree three with respect to X_B because $\hat{\mathbf{f}}_i$ is homogeneous of degree one with respect to X_B for all $i \in B$. This completes the proof of the claim.