Sub-Optimal Boundary Control of Semilinear PDEs Using a Dyadic Perturbation Observer

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***Abstract*— In this paper, we present a sub-optimal controller for semilinear partial differential equations, with partially known nonlinearities, in the dyadic perturbation observer (DPO) framework. The dyadic perturbation observer uses a two-stage perturbation observer to isolate the control input from the nonlinearities, and to predict the unknown parameters of the nonlinearities. This allows us to apply well established tools from linear optimal control theory to the controlled stage of the DPO. The small gain theorem is used to derive a condition for the robustness of the closed loop system.**

I. Introduction

In this paper, we are concerned with the boundary control of systems of semilinear partial differential equations (PDEs) of the form $\dot{w}(t) = \mathfrak{A}w(t) + f(w),\ \mathfrak{B}w(t) = u(t), y(t) = \mathcal{C}w(t)$, where $w(t)$ denotes the system state, $u(t)$ is the control input, and $y(t)$ is the output. The operators $\mathfrak{A}$, $\mathfrak{B}$ and $\mathcal{C}$ are the drift, boundary control, and output operators, respectively. The nonlinearity $f(w)$ is described as a linear combination of known basis functions and unknown coefficients. The specific objective of this paper is to present an extension, based on the linear quadratic regulator (LQR) theory, of the dyadic perturbation observer (DPO) architecture developed by the authors in a series of recent papers [12], [13], [14].

A large body of work on the control of PDE systems has focussed on approximating the PDE system by ordinary differential equations (ODEs), and using any of the rich assortment of techniques available for controlling ODEs [1], [2], [4]. More recently, a number of control techniques have emerged which use Lyapunov-based approaches to derive controllers for PDEs directly, without resorting to finite order approximations of the PDE [5], [6], [7], [10], [16], [17], [18]. These methods tend to do away with some limitations of the ODE-based approach, such as the occasional necessity for large order approximations, and the risk of spillover instabilities. The DPO control architecture is also PDE-based, but is designed largely in the operator-theoretic framework. The motivation for the DPO framework lies in the need to accommodate unmatched nonlinearities and disturbances, such as those which arise routinely in semilinear boundary control systems.

In this paper, we investigate the inclusion of optimality in the DPO framework. We use the LQR theory in an infinite dimension setting (Chapter 6, [3]) to aid the design of the tracking law. This seemingly straight-forward idea opens up the possibility of achieving (sub-) optimality in the presence of distributed nonlinearities which could be unmatched to the control signal.

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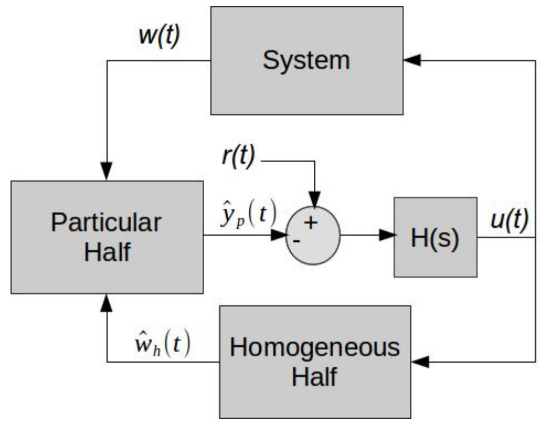


Fig. 1. A schematic of the DPO architecture.

The DPO architecture, shown in Fig. 1, uses the linear term $\mathfrak{A}w(t)$ as a pivot and decouples the system into two components, or halves. The homogeneous half contains the tracking control signal (i.e., the signal based on the tracking error) but not the nonlinearity, while the particular half contains the nonlinearity but not the tracking control signal. The control signal itself is designed to ensure that the output of the homogeneous half tracks the reference signal minus the output of the particular half, ensuring that the tracking objective is met. The small gain theorem from robust control is used to prove the stability and the robustness of the closed loop system in the sense of $\mathcal{L}_\infty$. The analysis is similar to [11], although the key difference vis-a-vis [11] is the unmatched nature of the nonlinearities.

It must be noted that the nature of the DPO architecture prevents us from applying the exact solution of the optimal control problem as the control signal. Instead, we approximate the optimal controller by a linear filter, which allows us to prove the well-posedness of the closed-loop system and its stability. The sub-optimality arises, in particular, due to the tracking objective of the homogeneous half consisting of the output of the particular half.

The paper is organized as follows. We introduce the mathematical preliminaries in Section II, and the problem formulation in Section III. The design of the DPO-based sub-optimal controller is presented in Section IV, and the stability analysis in Section V.

II. Preliminaries

Definition 1 ($\mathcal{L}_\infty$ and $\mathcal{L}_1$ norms): Given $q(t) \in \mathbb{R}^n$ with components $q_i(t)$ $(1 \le i \le n)$, we define

$$\begin{aligned} \|q(t)\|_\infty &= \max_{1\le i\le n} |q_i(t)|,\ \|q\|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{t\ge 0} \|q(t)\|_\infty \\ \|q\|_{\mathcal{L}_\infty,\tau} &= \operatorname{ess}\sup_{0\le t\le\tau} \|q(t)\|_\infty \end{aligned}$$

The 1-norm of a matrix $F : \mathbb{R}^m \to \mathbb{R}^n$ is defined as $\|F\|_1 = \sup_{\|q\|_\infty=1} \|Fq\|_\infty,\ q \in \mathbb{R}^m$. If $\|q\|_{\mathcal{L}_\infty} < \infty$, then we denote $q \in \mathcal{L}_\infty^m$. The $\mathcal{L}_1$ norm of a linear operator $\mathcal{F} : \mathcal{L}_\infty^m \mapsto \mathcal{L}_\infty^n$ is defined as $\|\mathcal{F}\|_{\mathcal{L}_1} = \sup_{\|q\|_{\mathcal{L}_\infty}=1} \|\mathcal{F}q\|_{\mathcal{L}_\infty},\ q \in \mathcal{L}_\infty^m$

The spatial domain of interest in this paper is the closed interval $[0, L]$ for some finite $L > 0$. Let $\mathbb{Z} = \mathcal{L}_2([0, L], \mathbb{R}^n)$, $n \ge 1$, denote the standard Hilbert space (in the spatial domain) with the inner product $\langle z_1, z_2\rangle_\mathbb{Z} = \int_0^L z_1^\top z_2 dx$ for any $z_1, z_2 \in \mathbb{Z}$. The standard norm on $\mathbb{Z}$ is given by $\|z\|_\mathbb{Z} = \sqrt{\int_0^L z^\top z\, dx}$.

Definition 2: We define the space $\mathbb{W}$ consisting of variables $w(t,x) \in \mathbb{R}^n$, with $x \in [0, L]$ and $t \in \mathbb{R}^+$, satisfying $w(t) \triangleq w(t,\cdot) \in \mathbb{Z},\ \forall t \ge 0$ and $\operatorname{ess\,sup}_{t\ge 0} \|w(t)\|_\mathbb{Z} < \infty$. More formally, $\mathbb{W} = \mathcal{L}_\infty(\mathbb{R}^+, \mathcal{L}_2([0, L], \mathbb{R}^n))$. The space $\mathbb{W}$ is a Banach space (and not necessarily a Hilbert space) with the norm $\|w\|_\mathbb{W} = \operatorname{ess\,sup}_{t\ge 0} \|w(t)\|_\mathbb{Z}$, and the truncated norm given by $\|w\|_{\mathbb{W},\tau} = \operatorname{ess\,sup}_{0\le t\le\tau} \|w(t)\|_\mathbb{Z}$.

Definition 3: The domain of an operator $\mathcal{V}$ is denoted by $\mathcal{D}(\mathcal{V})$. If $\mathcal{V} : X \to Y$ where X and Y are Banach spaces, (obviously, $\mathcal{D}(\mathcal{V}) \subset X$), then we denote the induced norm of $\mathcal{V}$ by $\|\mathcal{V}\|_i$.

In this paper, we will encounter operators which map $\mathbb{R}^m \to \mathbb{Z}$, with the ∞ norm used for $\mathbb{R}^m$. An expression for the induced norm of such operators, denoted by $\|\cdot\|_{(\mathbb{R}^m,\mathbb{Z})}$, may be found in [9].

Definition 4 ([15], Definition 1.1, Ch. 6): Consider a system $\dot{w} = \mathcal{A}w + f(t,w)$, $w(t=0) = w_0 \in \mathbb{Z}$, where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$. The mild solution $w(t)$ is given by

$$w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t-\tau) f(\tau, w(\tau))\, d\tau, \qquad (1)$$

Definition 5 (Convolution): Given a semigroup $\mathcal{T}(t)$, we define the operator $\mathcal{T}\star(t) : \mathbb{W} \mapsto \mathbb{W}$ so that $\forall f \in \mathbb{W}$ and $\forall t > 0$, $\mathcal{T}\star(t) f(t,w(t)) = \int_0^t \mathcal{T}(t-\tau) f(\tau, w(\tau))\, d\tau$. We further define the induced norm $\|\mathcal{T}*\|_i \triangleq \operatorname{ess\,sup}_{(t\ge 0)} \|\mathcal{T}\star(t)\|_i$

We recall the following result from Pazy [15] for solutions of initial value problems in Definition 4.

Theorem 1 (Theorems 6.1.4, 6.1.5, [15]): Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ on the Hilbert space $\mathbb{Z}$. If $f : [0, T] \times \mathbb{Z} \to \mathbb{Z}$ is continuously differentiable with respect to both arguments, for $T > 0$, then the mild solution (1) is a classical solution of the initial value problem in Definition 4 for $t \in [0, T]$. If the solution exists only up to $T_{\max} < T$, then $\|w(t)\|_\mathbb{Z} \to \infty$ as $t \to T_{\max}$.

Finally, we define the projection operator, following [8], which will be used for constructing the adaptive law in the paper. Let $\pi : \mathbb{R}^k \to \mathbb{R}$ be defined by

$$\pi(\alpha) \equiv \pi(\alpha; K, \epsilon) = \frac{\langle \alpha, \alpha\rangle - K^2}{\epsilon K^2}, \quad \alpha \in \mathbb{R}^k$$

where $K \in \mathbb{R}^+$. The number $\epsilon \in \mathbb{R}^+$ is chosen to be arbitrarily small. The Fréchet derivative of π at $\alpha_1 \in \mathbb{R}^k$ is denoted by $\pi'(\alpha_1) \in \mathbb{R}^k$ and it satisfies

$$\langle \pi'(\alpha_1), \alpha_2\rangle = \frac{2\langle \alpha_1, \alpha_2\rangle}{\epsilon K^2} \quad \forall \alpha_2 \in \mathbb{R}^k \qquad (2)$$

Definition 6: The projection operator $\mathrm{Proj} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is defined as

$$\mathrm{Proj}(\alpha_1, \alpha_2) = \begin{cases} \alpha_2, \text{ if } \pi(\alpha_1) \le 0 \text{ or } \langle \pi'(\alpha_1), \alpha_2\rangle \le 0 \\ \alpha_2 - \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2} \times \left\langle \frac{\pi'(\alpha_1)}{\|\pi'(\alpha_1)\|_2}, \alpha_2 \right\rangle \pi(\alpha_1), \\ \text{otherwise} \end{cases} \qquad (3)$$

The following property of the projection operator will be invoked in the proof of convergence of the observation error. Let Ω_0 and Ω_1 denote the convex sets satisfying

$$\Omega_0 = \{\alpha \mid \pi(\alpha) \le 0\}, \quad \Omega_1 = \{\alpha \mid \pi(\alpha) \le 1\}$$

The following result has been proved in [8]:

Lemma 1: Suppose that $\alpha_1^* \in \Omega_0$. Then, for all $\alpha_1, \alpha_2 \in \mathbb{R}^k$, $(\alpha_1 - \alpha_1^*)(\mathrm{Proj}(\alpha_1, \alpha_2) - \alpha_2) \le 0$. Moreover, the solution of the initial value problem $\dot{\alpha}_1 = \mathrm{Proj}(\alpha_1, \alpha_2)$, $\alpha_1(0) = \alpha_{10}$, has the property that if $\alpha_{10} \in \Omega_1$, then $\alpha_1(t) \in \Omega_1$ for all t.

III. Problem Formulation

This paper is concerned with boundary control of systems of semilinear partial differential equations described by

$$\dot{w}(t) = \mathfrak{A}w(t) + f(w),\ \mathfrak{B}w(t) = u(t), y(t) = \mathcal{C}w(t) \qquad (4)$$

where $w(t) \in \mathbb{Z} \subseteq \mathcal{L}_2([0, L], \mathbb{R}^n)$ and $u(t) \in U$. The operator $\mathcal{C}$ is bounded. We consider the abstract Cauchy problem

$$\begin{aligned} &\dot{v} = \mathcal{A}v + \mathfrak{A}\boldsymbol{\beta}u - \boldsymbol{\beta}\dot{u} + f(v + \boldsymbol{\beta}u),\ v(0) = v_0 \\ &y(t) = \mathcal{C}(v + \boldsymbol{\beta}u) \end{aligned} \qquad (5)$$

where $\mathcal{A}z = \mathfrak{A}z\ \forall z \in \mathbb{Z}$, $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and $\mathfrak{B}\boldsymbol{\beta}u = u$. The operators $\boldsymbol{\beta}$ and $\mathfrak{A}\boldsymbol{\beta}$ are bounded [3]. The control objective is to design $u(t)$ so that the output $y(t)$ tracks a reference signal $r(t)$, and the resulting closed-loop system is stable and robust (in a sense which will be made precise later).

Assumption 1: The nonlinearity can be expressed as a linear combination of known basis functions with unknown weights: $f(w) = \sum_{i=1}^N \alpha_i \phi_i(w)$, where $\phi_i(w)$ are C^1 functions of w, and $\alpha_i \in \mathbb{R}^n$ satisfy $|\alpha_{i,j}| < \nu_\alpha$ for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, n\}$. Note that the analysis in the paper does not require that α_i be a constant (see [13]). The assumption does simplify the presentation.

Let us reformulate (5) on an extended space $\mathbb{Z}^e = U \oplus \mathbb{Z}$:

$$\begin{aligned} \dot{w}^e(t) &= \begin{bmatrix} 0 & 0 \\ \mathfrak{A}\boldsymbol{\beta} & \mathcal{A} \end{bmatrix} w^e(t) + \begin{bmatrix} 1 \\ -\boldsymbol{\beta} \end{bmatrix} \bar{u}(t) + \begin{bmatrix} 0 \\ f(w^e) \end{bmatrix}, \\ w^e(0) &= \begin{bmatrix} w^e_{0,1} \\ w^e_{0,2} \end{bmatrix} \end{aligned} \quad (6)$$

which we denote succinctly as $\dot{w}^e(t) = \mathcal{A}^e w^e(t) + \mathcal{B}^e \bar{u}(t) + f^e(w^e)$. The output is given by $y(t) = \mathcal{C}^e w^e(t) = \mathcal{C}[\boldsymbol{\beta} \quad I_n] w^e(t)$. where I_n is the $n \times n$ identity matrix. Note that the operators $\mathcal{B}^e$ and $\mathcal{C}^e$ are bounded.

Assumption 2: The permissible initial conditions are restricted by $\|w^e_0\|_{\mathbb{Z}^e} < \rho_0$, and $w^e_0 \in \mathcal{D}(\mathfrak{A})$.

Assumption 3: The system ($\mathcal{A}^e$, $\mathcal{B}^e$) is exponentially stabilizable.

Definition 7: Corresponding to the space $\mathbb{Z}^e$, we define the Banach space $\mathbb{W}^e = \mathcal{L}_\infty(\mathbb{R}^+, \mathbb{Z}^e)$.

Lemma 2: For every $\rho > 0$, there exist positive constants $\nu_{\phi,1}(\rho)$ and $\nu_{\phi,2}(\rho)$ such that if $\|w^e(t)\|_{\mathbb{Z}^e} < \rho$ for some $t > 0$, then $\|\phi_j(w)\|_{\mathbb{Z}} \leq \nu_{\phi,1}(\rho)\|w^e(t)\|_{\mathbb{Z}^e} + \nu_{\phi,2}(\rho),\ \forall j$. In general, $\nu_{\phi,1}(\rho)$ and $\nu_{\phi,2}(\rho)$ are class $\mathcal{K}$ functions of ρ. It follows that if $\|w^e\|_{\mathbb{W}^e,\tau} < \rho$ for some $\tau > 0$, then $\|f(w^e)\|_{\mathbb{W},\tau} \leq \nu_1(\rho)\|w^e\|_{\mathbb{W}^e,\tau} + \nu_2(\rho)$, for some constants $\nu_1(\rho)$ and $\nu_2(\rho)$.

IV. Control Design

A. Optimal Control of a Subsystem

Consider the system

$$\dot{w}^e_h(t) = \mathcal{A}^e w^e_h(t) + \mathcal{B}^e \bar{u}^e(t),\ y_h = C^e w^e_h \quad (7)$$

which is found by neglecting the nonlinearity in (6). Suppose that we have to design an optimal controller to ensure that y_h tracks a reference signal $\sigma(t)$. To facilitate the design of an optimal control law, we define an extended state space $\mathbb{Z}^f = \mathbb{Z}^e \oplus \mathbb{R}$, and define

$$\dot{w}^f(t) = \begin{bmatrix} \mathcal{A}^e & 0 \\ 0 & 0 \end{bmatrix} w^f(t) + \begin{bmatrix} \mathcal{B}^e \\ 0 \end{bmatrix} \bar{u}^e(t),\ w^f(0) = \begin{bmatrix} w^e_{h,0} \\ 1 \end{bmatrix} \quad (8)$$

The control problem is stated as follows:

$$\begin{aligned} &\min_{\bar{u}^e} \int_0^T \left(\langle w^f(t),\, Q(t) w^f(t) \rangle + \langle \bar{u}^e(t),\, R\bar{u}^e(t) \rangle \right) dt \\ &Q(t) = [\mathcal{C}^e \quad -\sigma(t)]^* [\mathcal{C}^e \quad -\sigma(t)]; \\ &R > 0 \text{ is coercive} \end{aligned} \quad (9)$$

The control design mirrors the approach in ([3], Chapter 6). Ideally, we would like $T \to \infty$ in (9). However, since the reference signal $\sigma(t)$ is arbitrary, the optimal cost will be infinite as $T \to \infty$.

The solution to (9) is given by

$$\bar{u}^e(t) = -R^{-1}\mathcal{B}^{e*} \left(\Pi(t) w^e(t) + q(t) \right) \quad (10)$$

where $\Pi(t)$ is the solution of the Riccati equation

$$\begin{aligned} &\frac{d}{dt} \langle z_2,\, \Pi(t) z_1 \rangle = -\langle z_2,\, \Pi(t)\mathcal{A}^e z_1 \rangle - \langle \mathcal{A}^e z_2,\, \Pi(t) z_1 \rangle \\ &+ \langle \mathcal{C}^e z_1,\, \mathcal{C}^e z_2 \rangle + \langle \Pi(t)\mathcal{B}^e R^{-1} \mathcal{B}^{e*} \Pi(t) z_1,\, z_2 \rangle \quad (11) \\ &\Pi(T) = 0, \quad z_1,\, z_2 \in \mathcal{D}(\mathcal{A}^e) \end{aligned}$$

and $q(t)$ is the mild solution of

$$\dot{q}(t) = -\left(\mathcal{A}^e - \mathcal{B}^e R^{-1} \mathcal{B}^{e*} \Pi(t) \right)^* q(t) + C^{e*}\sigma(t),\ q(T) = 0 \quad (12)$$

It is evident that the problem has to be restricted to a finite horizon setup due to $\sigma(t)$; the first part of the control signal (10) is identical to the problem where $\sigma \equiv 0$. Therefore, we set $\Pi(t) \equiv \Pi$, the (steady state) solution to the *infinite* horizon Riccati equation. This can be justified if T is sufficiently large. This allows us to write the control signal as

$$\bar{u}^e(t) = -R^{-1}\mathcal{B}^{e*}\Pi w^e_h(t) - R^{-1}\mathcal{B}^{e*} q(t) \quad (13)$$

$$\dot{q}(t) = -\left(\mathcal{A}^e - \mathcal{B}^e R^{-1} \mathcal{B}^{e*} \Pi \right)^* q(t) + C^{e*}\sigma(t),\ q(T) = 0 \quad (14)$$

Definition 8: The operator $\mathcal{A}^e_m = \mathcal{A}^e - \mathcal{B}^e R^{-1} \mathcal{B}^{e*} \Pi$ generates an exponentially stable semigroup $\mathcal{T}(t)$ (as a consequence of Assumption 3), i.e., there exist constants $M,\ \beta > 0$ such that $\|\mathcal{T}(t)\|_i \leq M e^{-\beta t}$. Moreover, $\|\mathcal{T} * \|_i$ is bounded.

The equation for the adjoint state $q(t)$ is considerably challenging to solve when $\sigma(t)$ is not known *a priori*. This is a well-known problem in optimal tracking and there are no known exact analytical solutions to the problem. Rather than prescribing a solution based on nonlinear model predictive control, a natural option in this scenario, we make an assumption which compromises the optimality of the controller, but ensures that no future values of the states are needed.

Assumption 4: For a known $\sigma(t)$, the control signal can be represented approximately as

$$\begin{aligned} \bar{u}^e(t) &= -R^{-1}\mathcal{B}^{e*}\Pi w^e_h(t) - H_C p(t), \\ \dot{p}(t) &= H_A p(t) + H_B \sigma(t),\ p(0) = p_0 \in \mathbb{R}^{n_p} \end{aligned}$$

where $H_A \in \mathbb{R}^{n_p \times n_p}$ is Hurwitz, and $H_B,\ H_C^\top \in \mathbb{R}^{n_p}$. All of H_A, H_B and H_C may depend on r.

The motivation for Assumption 4 is that the adjoint state evolves on the stable manifold of the combined system-adjoint dynamics, and the stable eigenvalues are precisely those of $\mathcal{A}^e_m$.

B. Sub-Optimal Control of the Original System

Based on the analysis in the previous section, we propose the following control law for the system (6):

$$\bar{u}(t) = -\underbrace{R^{-1}\mathcal{B}^{e*}\Pi}_{\mathcal{K}_w} w^e(t) - H_C p(t) \quad (15)$$

$$\dot{p}(t) = H_A p(t) + H_B \sigma(t),\ p(0) = p_0 \quad (16)$$

The term $\sigma(t)$, on which $p(t)$ depends, will be defined presently. The system (6) can now be written as

$$\dot{w}^e(t) = \mathcal{A}^e_m w^e(t) - \mathcal{B}^e H_C p(t) + f^e(w^e(t)) \quad (17)$$

Using the linear term as a pivot, we decompose the system in (6) into two sub-systems

$$\dot{w}^e_p = \mathcal{A}^e_m w^e_p + f^e(w^e),\ y_p = C^e w^e_p \quad (18)$$

$$\dot{w}^e_h = \mathcal{A}^e_m w^e_h - \mathcal{B}^e H_C p(t),\ y_h = C^e w^e_h \quad (19)$$

The two systems (18) and (19) are referred to as the *particular* and *homogeneous* halves, respectively. In the next section, we will derive an observer for estimating the states; for now, we use (18) and (19) to investigate tracking.

We immediately note that (19) is identical to (7). If we choose $\sigma(t) = r(t) - y_p(t)$, it is evident that we have basically optimized the control input and obtained guaranteed bounds on the tracking error $y(t) - r(t) = y_h(t) - \sigma(t)$. In practice, we will choose

$$\sigma(t) = r(t) - \hat{y}_p(t)$$

where $\hat{y}_p(t)$ is the output of an observer which will be designed presently (see (21)). The next assumption asserts the existence of a Lyapunov function corresponding to the generator $\mathcal{A}_m$, which we need for constructing the observer.

Assumption 5: There exists a *self-adjoint coercive* operator $\mathcal{P} > 0$ and a constant $\lambda_P > 0$ such that $\forall t$,

$$\langle \mathcal{A}_m^e z(t), \mathcal{P}z(t)\rangle_{\mathbb{Z}} + \langle \mathcal{P}z(t), \mathcal{A}_m^e z(t)\rangle_{\mathbb{Z}} \leq -\lambda_P \langle z(t), \mathcal{P}z(t)\rangle_{\mathbb{Z}}, \quad \forall\ z(t) \in \mathcal{D}(\mathcal{A}_m^e) \tag{20}$$

C. Observer Design

We use the symbol "$\wedge$" to denote observer states, and the subscripts *p* and *h* to denote states of the particular and the homogeneous halves, respectively. The dynamics of the two halves are given by

$$\dot{\hat{w}}_p^e = \mathcal{A}_m^e \hat{w}_p^e + \begin{bmatrix} 0 \\ \sum_{i=1}^N \hat{\alpha}_i(t)\phi_i(w^e) \end{bmatrix},\ \hat{y}_p = \mathcal{C}^e \hat{w}_p^e \tag{21}$$

$$\dot{\hat{w}}_h^e = \mathcal{A}_m^e \hat{w}_h^e - \mathcal{B}^e H_C p(t),\ \hat{y}_h = \mathcal{C}^e \hat{w}_h^e \tag{22}$$

with the initial conditions at $t = 0$ set to $\hat{w}_h^e(0) = w^e(0)$ and $\hat{w}_p^e(0) = 0$. The predicted values $\hat{\alpha}_i(t)$ are found using the projection operator (see [8], [11] for details).

$$\dot{\hat{\alpha}}_{i,j}(t) = \gamma \operatorname{Proj}\left(\hat{\alpha}_{i,j}, -\langle \mathcal{P}\tilde{w}^e(t), [0 \quad I_n]^\top \phi_j(w) e_i\rangle_{\mathbb{Z}}\right), \quad |\hat{\alpha}_{i,j}(t)| < \nu_\alpha(1+\epsilon) \tag{23}$$

where $\epsilon \in \mathbb{R}^+$ is arbitrarily small; $\tilde{w}^e = \hat{w}_p^e + \hat{w}_h^e - w^e$; $\hat{\alpha}_{i,j}$ $(1 \leq i \leq n)$ is the j^{th} component of $\hat{\alpha}_i$, e_i denotes the i^{th} column of the $n \times n$ identity matrix, and $\gamma > 0$ is the adaptation gain.

In summary, the closed-loop system consists of the original system (6), together with the controller (15), and the dyadic observer (21), (22) and (23).

V. Closed-Loop Stability Analysis

A. Observer Error Regulation

We first show that the observer states $\hat{w}_h^e$ and $\hat{w}_p^e$ converge to w_p^e and w_h^e, respectively. Let $\hat{w}^e = \hat{w}_h^e + \hat{w}_p^e$, and let $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$ denote the error between predicted and the actual terms.

From (17), (21) and (22), the observation error dynamics are given by

$$\dot{\tilde{w}}^e = \mathcal{A}_m^e \tilde{w}^e + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \sum_{i=1}^N \tilde{\alpha}_i(t)\phi_i(w^e) \tag{24}$$

Lemma 3: Suppose that $\|w^e\|_{\mathbb{Z}^e,t} < \rho_w$ for some constant $\rho_w > 0$. Then, the observation error dynamics (24) are uniformly bounded, i.e., $\|\tilde{w}^e(t)\|_{\mathbb{Z}^e}$ is uniformly bounded. Moreover, the bound can be made arbitrarily small by increasing γ.

Proof: We consider the Lyapunov function

$$V(t) = \langle \tilde{w}^e(t),\, \mathcal{P}\tilde{w}^e(t)\rangle + \frac{1}{\gamma}\sum_{i=1}^N \tilde{\alpha}_i^\top \tilde{\alpha}_i(t)$$

Differentiating the Lyapunov function gives

$$\dot{V}(t) = \langle \mathcal{P}\tilde{w}^e,\, \mathcal{A}_m^e \tilde{w}^e\rangle + \langle \mathcal{A}_m^e \tilde{w}^e,\, \mathcal{P}\tilde{w}^e\rangle + 2\langle \mathcal{P}\tilde{w}^e,\, [0 \quad I_n]^\top \sum_{i=1}^N \tilde{\alpha}_i(t)\phi_i(w^e)\rangle + \frac{1}{\gamma}\sum_{i=1}^N \tilde{\alpha}_i(t)^\top \dot{\hat{\alpha}}_i(t)$$

Using the properties of the projection operator in Lemma 1 and Assumption 5, it follows that

$$\dot{V} \leq -\lambda_p \langle \tilde{w}^e,\, \mathcal{P}\tilde{w}^e\rangle \tag{25}$$

Using Barbalat's lemma, and the fact that $\mathcal{P}$ is coercive, we deduce that $\|\tilde{w}^e\|_{\mathbb{Z}^e} \to 0$. Furthermore, we can write (25) as $\dot{V} \leq -\lambda_p V + \frac{\lambda_p}{\gamma}\sum_{i=1}^N \tilde{\alpha}_i(t)^\top \tilde{\alpha}_i(t)$. Since $\|w^e\|_{\mathbb{Z}^e} < \rho$, and $\tilde{\alpha}_i$ is bounded, it follows that there exists a constant μ_p such that $V(t) \leq \dfrac{\mu_p}{\gamma} e^{-\lambda_p t}$. Using the coercivity of $\mathcal{P}$, we deduce that $\|\tilde{w}^e(t)\|_{\mathbb{Z}^e} \leq \frac{\nu_p}{\sqrt{\gamma}}$, where the constant ν_p depends on $\mathcal{P}$ and μ_p. This completes the proof. ■

Lemma 4: The observation errors $\tilde{w}^p$ and $\tilde{w}^h$ are uniformly bounded, and can be made arbitrarily small by increasing γ.

Proof: The observation error dynamics of the *homogeneous half* are given by $\dot{\tilde{w}}_h^e = \mathcal{A}_m^e \tilde{w}_h^e$. It is obvious that $\tilde{w}_h^e(t) \to 0$ exponentially fast (in the sense of the $\mathbb{Z}^e$ norm). The boundedness of $\tilde{w}_p^e$ follows from Lemma 3 and the triangle inequality. ■

B. Error and Control Boundedness

Lemma 5: Suppose that $\|w^e\|_{\mathbb{W}^e,t} \leq \rho_w$ for some $\rho_w > 0$. Then, there exist constants δ_0 and δ_1 such that $\|\hat{y}_p\|_{\mathcal{L}_\infty,t} \leq \delta_0 + \delta_1 \|w^e\|_{\mathbb{W}^e,t}$.

Proof: From (21), we get

$$\hat{w}_p^e(t) = \mathcal{T} \star (t) \left(\begin{bmatrix} 0 \\ \sum_{i=1}^N \hat{\alpha}_i(t)\phi_i(w) \end{bmatrix} \right)$$

Taking the truncated $\mathbb{W}^e$ norm of both sides, and using Lemma 2, gives $\|\hat{w}_p^e\|_{\mathbb{W}^e,t} \leq \|\mathcal{T} * \|_i \left(\nu_1(\rho_w)\|w^e\|_{\mathbb{W}^e,t} + \nu_2(\rho_w)\right)$. Since $\mathcal{C}^e$ is bounded, there exists a constant K such that $|y^p(t)| \leq K\|\hat{w}_p^e(t)\|_{\mathbb{W}^e}$. This leads to $\|\hat{y}_p\|_{\mathcal{L}_\infty,t} \leq K\|\mathcal{T} * \|_i \left(\nu_1(\rho_w)\|w^e\|_{\mathbb{W}^e,t} + \nu_2(\rho_w)\right)$. Set $\delta_0 = K\|\mathcal{T} * \|_i \nu_2(\rho_w)$ and $\delta_1 = K\|\mathcal{T} * \|_i \nu_1(\rho_w)$. This completes the proof. ■

To prove that the closed-loop system is well-posed, we construct the augmented vector $\boldsymbol{w}^e = [w^e,\, \hat{w}_p^e,\, \hat{w}_h^e,\, p(t)]^\top \in$

$\mathbb{V} = \mathbb{Z}^e \times \mathbb{Z}^e \times \mathbb{Z}^e \times \mathbb{R}^{n_p}$. The dynamics of $\boldsymbol{w}^e$ is given by

$$\dot{\boldsymbol{w}}^e(t) = \bar{\mathcal{A}}^e \boldsymbol{w}^e + \bar{f}(\hat{\alpha}(t), w(t), r(t)) \tag{26}$$
$$w^e(0) = \hat{w}_h^e = w_0^e,\ \hat{w}_p^e(0) = 0, \quad p(0) = p_0$$
$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_m^e & 0 & 0 & -\mathcal{B}^e k_r \\ 0 & \mathcal{A}_m^e & 0 & 0 \\ 0 & 0 & \mathcal{A}_m^e & -\mathcal{B}^e k_r \\ 0 & -H_B & 0 & H_A \end{bmatrix}$$

where the exogenous signal $\hat{\alpha}(t)$ is known to be C^1 in time. Therefore, it can be checked readily that $\bar{f}(\cdot)$ function of its arguments. Furthermore, the operator $\bar{\mathcal{A}}^e$ is the infinitesimal generator of a semigroup. We state the following result without proof, but as a direct application of Thm. 1.

Lemma 6: The system (26) has a unique classical solution $\boldsymbol{w}^e(t)$ for $t \in [0, T_{\max}]$ for some $T_{\max} > 0$. Moreover, if $T_{\max} < T$, then $\lim_{t \to T_{\max}} \|\boldsymbol{w}^e(t)\|_{\mathbb{V}} \to \infty$.

We will now prove that the control input $\bar{u}(t)$, given by (15) and (16), is bounded. Let us denote the second term in the control signal (15) by $\bar{u}_r(t) = -H_C p(t)$. Let $H(s) = H_C(sI - H_A)H_B$ denote the transfer function between $U_r(s) = H_C p(t)$ and $(R(s) - \hat{Y}_p(s))$.

Lemma 7: Let $\|w^e\|_{\mathbb{W}^e,t} < \rho_w$ for some t and $\rho_w > 0$. Then, the control input $\bar{u}(t)$ is bounded and a C^1 function of time. Moreover, there exist constants $\delta_{iw} \equiv \delta_{iw}(H(s), \rho)$, $\delta_{ir} \equiv \delta_{ir}(H(s), \rho)$ and $\delta_{iu} \equiv \delta_{iu}(H(s), \rho)$ for $i = 0,\, 1$ such that $\|u_r\|_{\mathcal{L}_\infty,\tau} \leq \delta_{0w}\|w^e\|_{\mathbb{W},\tau} + \delta_{0r}\|r\|_{\mathcal{L}_\infty,\tau} + \delta_{0u}$.

Proof: The control input $\bar{u}(t) = -\mathcal{K}_w w^e(t) - \mathcal{B}^e \bar{u}_r(t)$. Since the operators $\mathcal{K}_w$ and $\mathcal{B}^e$ are bounded, it follows from Lemma 6 that $\bar{u}(t)$ is C^1 in time. The second term of the control $\bar{u}(t)$ is found as the output of (16), where $p(t)$ is C^1 function of time (from Lemma 6). Thus, $\bar{u}(t)$ is C^1 function of time. Taking the Laplace transform of (16) gives $U(s) = H(s)(R(s) - \hat{Y}_p(s))$, and it follows that $\|\bar{u}_r\|_{\mathcal{L}_\infty,\tau} \leq \|H(s)\|_{\mathcal{L}_1}(\|r\|_{\mathcal{L}_\infty} + \|\hat{y}_p\|_{\mathcal{L}_\infty,\tau})$, where $\tau < T$, the maximum interval for the existence of the classical solution in Lemma 6. This completes the proof. The constants $\delta_{0(\cdot)}$ in the statement of the result can be found readily from the above expressions in terms of $\|H(s)\|_{\mathcal{L}_1}$ and using Lemma 5. ■

We are now ready to prove the stability of the complete closed-loop system, in the sense of $\mathbb{W}$ boundedness of signals, using the following small gain.

Assumption 6 (Small-gain condition): We assume that there exists a constant ρ_w, an arbitrarily small $\epsilon_s > 0$, and a stable strictly proper $H(s)$ such that the following inequality is satisfied:

$$\frac{M\rho_0 + \|\mathcal{T} * \|_i(\tau)(\nu_2(\rho_w) + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u})}{1 - \|\mathcal{T} * \|_i(\tau)(\nu_1(\rho_w) + \|\mathcal{B}^e\|_i(\delta_{0w})} \leq \rho - \epsilon_s$$

where the constants have been defined in Lemmas 2 and 7. In the small gain condition, the constants $\delta_{i(\cdot)}$ were derived in Lemma 7, while $\nu_i(\rho)$ were defined in Lemma 2.

Theorem 2: The closed-loop system (17), (21), (22), (23), (15) and (16) is bounded-input-bounded-state stable in the sense of $\mathcal{L}_\infty$ if Assumption 6 is satisfied. Moreover, the solution exists for all time.

Proof: We will prove the result by contradiction. Suppose that $\|w^e(\tau)\|_{\mathbb{Z}^e} = \rho_w$ for some $\tau < T_{\max}$, and $\|w(t)\|_{\mathbb{Z}^e} < \rho_w$ for all $t < \tau$. The solution to (17) is given by

$$w^e(t) = \mathcal{T}(t) w_0^e + \mathcal{T} \star (t) \left(\mathcal{B}^e \bar{u} + [0 \quad f(w)]\right)$$

Taking the $\mathbb{W}^e$ norm of both sides, we get

$$\|w^e\|_{\mathbb{W}^e,\tau} \leq M\|w_0^e\|_{\mathbb{Z}^e} + \|\mathcal{T} * \|_i \bigg(\nu_1(\rho_w)\|w\|_{\mathbb{W}^e,\tau}$$
$$+ \nu_2(\rho_w) + \|\mathcal{B}^e\|_i(\delta_{0w}\|w^e\|_{\mathbb{W}^e,\tau} + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u}) \bigg) \tag{27}$$

Isolating $\|w^e\|_{\mathbb{W}^e,\tau}$ on the left hand side, we get

$$\|w^e\|_{\mathbb{W}^e,\tau} \leq \frac{M\rho_0 + \|\mathcal{T} * \|_i(\tau)(\nu_2(\rho_w) + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u})}{1 - \|\mathcal{T} * \|_i(\tau)(\nu_1(\rho_w) + \|\mathcal{B}^e\|_i(\delta_{0w})}$$
$$< \rho_w - \epsilon_s$$

which contradicts our claim that $\|w^e\|_{\mathbb{W}^e,\tau} = \rho_w$. Thus, $\|w^e\|_{\mathbb{W}^e,\tau} < \rho_w$. This bound on w^e automatically translates into a similar bound on the other states, viz., $\hat{w}_p^e$, $\hat{w}_h^e$ and $p(t)$. It thus follows from Thm 1 that $T_{\max} = T$ because of the boundedness of the states. Notice that $T_{\max}$ would be infinite if the problem is posed on an infinite time interval. This completes the proof. ■

C. Closed Loop System in the Boundary Control Form

We link up the dynamics on the extended space $\mathbb{Z}^e$ with the dynamics on $\mathbb{Z}$ and the boundary control system. We note that the dynamics on $\mathbb{Z}$ have a classical solution $(v(t), \hat{v}_{(\cdot)}(t))$ which is directly related to that in $\mathbb{Z}^e$.

We can view $\mathcal{A}_m^e$ and $\mathcal{B}^e$ as matrix operators

$$\mathcal{A}_m^e = \begin{bmatrix} \mathcal{A}_{m,11}^e & \mathcal{A}_{m,12}^e \\ \mathcal{A}_{m,21}^e & \mathcal{A}_{m,22}^e \end{bmatrix}, \quad \mathcal{B}^e = \begin{bmatrix} 1 \\ -\boldsymbol{\beta} \end{bmatrix}$$

Consider the systems (4), (5) and (6), together with the control input given by $\dot{u} = \bar{u}$ from (15). Since we have proved in Lemma 7 that $\dot{u} = \bar{u}$ is C^1 in time, it follows that $u(t)$ is C^2 in time. This allows us to use ([3], Theorem 3.3.4) to state the following result.

Lemma 8: Consider the abstract Cauchy equation for open loop system (5) with $u(t)$ found using (15). If $v_0 \in \mathcal{D}(\mathcal{A})$ and $u \in C^2([0,T]; U)$, then (6) with $w_{0,1}^e = u(0)$, $w_{0,2}^e = \hat{v}_h(0)$, and $\bar{u} = \dot{u}$ has a unique classical solution $w^e(t) = [u(t),\, v(t)]^\top$ for $t \in [0,T]$, where $v(t)$ is the classical solution of (5). Furthermore, if $w_0 = v_0 + \boldsymbol{\beta} u(0)$, then the classical solution of (4) is given by $w(t) = [\boldsymbol{\beta} \quad I_n] w^e(t)$.

Lemma 8 establishes that the system (4) with (15) is identical to (17). This analogy allows us to construct the boundary form for the homogeneous and particular halves:

$$\dot{\hat{w}}_h = \mathfrak{A}\hat{w}_h, \quad \mathfrak{B}\hat{w}_h = \hat{u}_h$$
$$\dot{\hat{u}}_h = (\mathcal{A}_{m,11}^e - \mathcal{A}_{m,12}^e\boldsymbol{\beta})\hat{u}_h + \mathcal{A}_{m,12}^e\hat{w}_h - H_C p(t) \tag{28}$$
$$\dot{\hat{w}}_p = \mathfrak{A}\hat{w}_p + \sum_{i=1}^{N} \hat{\alpha}(t)\phi(w), \quad \mathfrak{B}\hat{w}_p = \hat{u}_p$$
$$\dot{\hat{u}}_p = (\mathcal{A}_{m,11}^e - \mathcal{A}_{m,12}^e\boldsymbol{\beta})\hat{u}_p + \mathcal{A}_{m,12}^e\hat{w}_p \tag{29}$$

This form of the controller is different from the original DPO architecture as follows. In the original DPO, the control signal was generated entirely in the homogeneous half and consisted of just $\bar{u}_r(t)$ in (15). On the other hand, in the current system, it is generated partly in the original system itself (via the feedback term $-\mathcal{K}_w w^e$ in (15)), and partly designed for the homogeneous half (the term $u_r(t)$). The modified DPO architecture thus provides a rigorous way to inject stabilizing feedback into the system. At the same time, the tracking objective is formulated entirely in the homogeneous half in the original as well as the modified architectures, and this forms the basis for isolating the nonlinearity from the tracking objective.

VI. Simulation

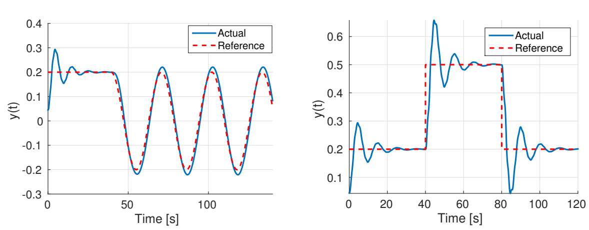
Consider the unstable forced wave equation

$$\ddot{\theta}(t,x) - 0.02\dot{\theta}_{xx}(t,x) - \theta_{xx}(t,x) = (5+\alpha)\,\theta(t,x) \tag{30}$$

$$\theta_x(t,1) = 0,\ \ \theta(t,0) = u(t), y(t) = \int_0^1 \theta(t,x)\,dx$$

where the value of $\alpha = 0.6$ is assumed to be unknown to the controller, and $\alpha \in [-2, 2]$ for the purpose of designing the projection law. We note that the uncontrolled system is unstable for $\alpha \geq -2.5$. For designing the LQR, we choose the state-dependent part of the cost function as $J_1 = \int_0^\infty \int_0^1 \left(\hat{\dot{\theta}}_h^2 + \hat{\theta}_{x,h}^2\right) dx\, dt$.

We design the DPO and the controller for the purpose of simulation in the finite dimensional space. We write $\theta = \sum_i^N \eta_i(t)\psi_i(x)$, where $\psi_i(x)$ are chosen to be the mode shapes of the unforced wave equation. We choose the operator $\boldsymbol{\beta} u = u$, $\mathfrak{A}\boldsymbol{\beta} = 0$, in (5), which allows us to write the finite dimensional system for $\eta_i(t)$ using Galerkin's method as $\dot{\boldsymbol{\eta}} = \mathbf{A}\boldsymbol{\eta} + \mathbf{B}\boldsymbol{u}$, where where $\boldsymbol{u} = \ddot{u}$ and $\boldsymbol{\eta} = [\eta^\top, u, \dot{\eta}^\top, \dot{u}]^\top$. The cost function for LQR is chosen as $J = 20\, J_1 + \int_0^\infty \boldsymbol{u}^2\, dt$. The DPO is designed for the resulting finite dimensional system exactly as described in the paper. Simulation results in Figs. 2(a) and (b), obtained by approximating the system with $N = 5$ modes, demonstrate that the steady-state tracking error is negligible, and that transient response characteristics are uniform with respect to the initial condition and the reference input.



(a) Tracking (b) Uniform transients

Fig. 2. The optimized DPO illustrated for two reference inputs.

VII. Conclusion

We derived an LQR-based tracking control law in the DPO framework for a class of semilinear partial differential equations. The LQR-based law was found as an approximation to a time-varying optimal control law, in the dyadic perturbation observer framework. The closed-loop stability was proved in the bounded-input-bounded-ouput sense using the small gain theorem. The DPO architecture was modified in the process, with the addition of a stabilizing feedback term. Future work is expected to refine the optimal law by determining the most suitable filter-based approximation to the adjoint equation.

Acknowledgement

The authors thank the reviewers for their comments and suggestions. The second author gratefully acknowledges support by the National Science Foundation (IIS-1253758; CMMI-1427111).

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