# Discrete-Time Models Resulting From Dynamic Continuous-Time Perturbations In Phase-Amplitude Modulation-Demodulation Schemes\*

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#### **Abstract**

We consider discrete-time (DT) systems S in which a DT input is first tranformed to a continuous-time (CT) format by phase-amplitude modulation, then modified by a non-linear CT dynamical transformation F, and finally converted back to DT output using an ideal de-modulation scheme. Assuming that F belongs to a special class of CT Volterra series models with fixed degree and memory depth, we provide a complete characterization of S as a series connection of a DT Volterra series model of fixed degree and memory depth, and an LTI system with special properties. The result suggests a new, non-obvious, analytically motivated structure of digital compensation of analog nonlinear distortions (for example, those caused by power amplifiers) in digital communication systems. Results from a MATLAB simulation are used to demonstrate effectiveness of the new compensation scheme, as compared to the standard Volterra series approach.

**Key Words:** communication system nonlinearities, nonlinear systems, modeling, phase modulation, amplitude modulation

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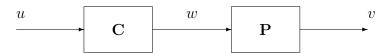
## 1 Notation and Terminology

- j a fixed square root of -1
- $\mathbb{R}$  real numbers
- $\mathbb{Z}$  integers
- $\mathbb{N}$  positive integers
- [a:b] all integers from a to b
- $\mathcal{L}$  bounded square integrable functions  $\mathbb{R} \to \mathbb{R}$
- $\ell(X)$  square summable functions  $\mathbb{Z} \to X \subset \mathbb{C}^n$

CT signals are elements of  $\mathcal{L}$ , DT signals are elements of  $\ell(X)$  for some  $X \subset \mathbb{C}^n$ . For  $w \in \ell(X)$ , w[n] denotes the value of w at  $n \in \mathbb{Z}$ . In contrast, x(t) refers to the value of  $x \in \mathcal{L}$  at  $t \in \mathbb{R}$ . Systems are viewed as functions  $\mathcal{L} \to \mathcal{L}$ ,  $\mathcal{L} \to \ell(X)$ ,  $\ell(X) \to \mathcal{L}$ , or  $\ell(X) \to \ell(Y)$ . Gf denotes the response of system G to signal f (even when G is not linear), and the series composition  $K = \mathbb{Q}$ G of systems Q and G is the system mapping f to  $\mathbb{Q}(\mathbb{G}f)$ .

### 2 Introduction and Motivation

Digital compensation offers an attractive approach to designing electronic devices with superior characteristics [1, 2, 3]. In this paper, a digital compensator is viewed as a system  $\mathbf{C}: \ell(\mathbb{R}) \to \ell(\mathbb{R})$ . More specifically, a *pre*-compensator  $\mathbf{C}: \ell(\mathbb{R}) \to \ell(\mathbb{R})$  designed for a device modeled by a system  $\mathbf{P}: \ell(\mathbb{R}) \to \mathcal{L}$  (or  $\mathbf{P}: \ell(\mathbb{R}) \to \ell(\mathbb{R})$ ) aims to make the composition  $\mathbf{PC}$ , as shown on the block diagram below,



conform to a set of desired specifications. (In the simplest scenario, the objective is to make PC as close to the identity map as possible, in order to cancel the distortions introduced by P.)

A common element in digital compensator design algorithms is selection of *compensator structure*, which usually means specifying a finite sequence  $\tilde{\mathbf{C}} = (\mathbf{C}_1, \dots \mathbf{C}_N)$  of systems  $\mathbf{C}_i : \ell(\mathbb{R}) \to \ell(\mathbb{R})$ , and restricting the actual compensator  $\mathbf{C}$  to have the form

$$\mathbf{C} = \sum_{i=1}^{N} a_i \mathbf{C}_i, \qquad a_i \in \mathbb{R},$$

i.e., to be a linear combination of the elements of  $\tilde{\mathbf{C}}$ . Once the *basis* sequence  $\tilde{\mathbf{C}}$  is fixed, the design usually reduces to a straightforward *least squares optimization* of the coefficients  $a_i \in \mathbb{R}$ .

A popular choice is for the systems  $C_k$  to be some Volterra monomials, i.e. to map their input u = u[n] to the outputs  $w_k = w_k[n]$  according to the polynomial formulae

$$w_k[n] = \prod_{i=1}^{i=d_k} u[n - n_{k,i}]$$

(where the integers  $d_k$ ,  $n_{k,i}$  will be referred to, respectively, as the *degrees* and *delays*), which makes every linear combination C of  $C_i$  a *DT Volterra series* [4], i.e., a DT system mapping signal inputs  $u \in \ell(\mathbb{R})$  to outputs  $w \in \ell(\mathbb{R})$  according to the polynomial expression

$$w[n] = \sum_{k=1}^{N} a_k \prod_{i=1}^{d_k} u[n - n_{k,i}].$$

Selecting a proper *compensator structure* is a major challenge in compensator design: a basis which is too simple will not be capable of cancelling the distortions well, while a form that is too complex will consume excessive power and space. Having an insight into the compensator basis selection can be very valuable. For an example (cooked up outrageously to make the point), consider the case when the ideal compensator  $C: u \mapsto w$  is given by

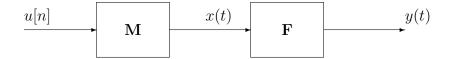
$$w[n] = \rho u[n] + \delta \left( \sum_{i=-50}^{50} u[n-j] \right)^5$$

for some (unknown) coefficients  $\rho$  and  $\delta$ . One can treat  $\mathbf{C}$  as a generic Volterra series expansion with fifth order monomials with delays between -50 and 50, and the first order monomial with delay 0, which leads to a basis sequence  $\tilde{\mathbf{C}}$  with  $1 + \binom{105}{5} = 96560647$  elements (and the same number of multiplications involved in implementing the compensator). Alternatively, one may realize that the two-element structure  $\tilde{\mathbf{C}} = \{\mathbf{C}_1, \mathbf{C}_2\}$ , with  $w_i = \mathbf{C}_i u$  defined by

$$w_1[n] = u[n], \qquad w_2[n] = \left(\sum_{j=-50}^{50} u[n-j]\right)^5$$

is good enough.

In this paper we establish that a certain special structure is good enough to compensate for imperfect modulation. We consider systems represented by the block diagram



where  $\mathbf{M}:\ell(\mathbb{C})\to\mathcal{L}$  is the *ideal* modulator with fixed sampling interval length T>0 and modulation-to-sampling frequency ratio  $M\in\mathbb{N}$ , converting complex DT signals  $w\in\ell(\mathbb{C})$  to CT signals  $x\in\mathcal{L}$  according to

$$x(t) = \sum_{n \in \mathbb{Z}} \frac{1}{T} p\left(\frac{t}{T} - n\right) \operatorname{Re} \left\{ \exp\left(j\frac{2\pi M}{T}t\right) u[n] \right\}, \tag{1}$$

with

$$p(t) = \begin{cases} 1, & t \in [0, 1), \\ 0, & t \notin [0, 1), \end{cases}$$

and  $F: \mathcal{L} \to \mathcal{L}$  is a CT dynamical system used to represent linear and nonlinear distortion in the modulator and power amplifier circuits. In particular, we are interested in the case where the relation between  $x(\cdot)$  and  $y(\cdot)$  is described by the CT Volterra series model

$$y(t) = b_0 + \sum_{k=1}^{N_b} b_k \prod_{i=1}^{\beta_k} x(t - t_{k,i}),$$
(2)

where  $N_b \in \mathbb{N}$ ,  $b_k \in \mathbb{R}$ ,  $\beta_k \in \mathbb{N}$ ,  $t_{k,i} \geq 0$  are parameters. (In a similar fashion, it is possible to consider input-output relations in which the finite sum in (2) is replaced by an integral, or an infinite sum). One expects that the memory of F is not long, compared to T, i.e., that  $\max t_{k,i}/T$  is not much larger than 1.

As a rule, the spectrum of the DT input  $u \in \ell(\mathbb{C})$  of the modulator is carefully shaped at a preprocessing stage to guarantee desired characteristics of the modulated signal  $x = \mathbf{M}u$ . However, when the distortion  $\mathbf{F}$  is not linear, the spectrum of the  $y = \mathbf{F}x$  could be damaged substantially, leading to violations of EVM and spectral mask specifications [5].

Consider the possibility of repairing the spectrum of y by pre-distorting the digital input  $u \in \ell(\mathbb{C})$  by a compensator  $\mathbf{C} : \ell(\mathbb{C}) \to \ell(\mathbb{C})$ , as shown on the block diagram below:



The desired effect of inserting C is cancellation of the distortion caused by  $\mathbf{F}: \mathcal{L} \to \mathcal{L}$ . Naturally, since C acts in the baseband (i.e., in discrete time), there is no chance that C will achieve a complete correction, i.e., that the series composition FMC of F, M, and C will be identical to M. However, in principle, it is sometimes possible to make the frequency contents of  $\mathbf{M}u$  and  $\mathbf{FMC}u$  to be identical within the CT frequency band  $(f_c - f_N, f_c + f_N)$  Hz, where  $f_c = M/T$  is the carrier frequency (Hz), and  $f_N = 0.5/T$  is the Nyquist frequency (Hz) for the sampling rate used [6]. To this end, let  $\mathbf{H}: \mathcal{L} \to \mathcal{L}$  denote the ideal band-pass filter with frequency response

$$H(f) = \begin{cases} 1, & |f| \in (f_c - f_N, f_c + f_N), \\ 0, & |f| \notin (f_c - f_N, f_c + f_N). \end{cases}$$

Let  $D: \mathcal{L} \to \ell(\mathbb{C})$  be the ideal de-modulator relying on the band selected by  $\mathbf{H}$ , i.e. the linear system for which the series composition  $\mathbf{D}\mathbf{H}\mathbf{M}$  is the identity function. Let  $\mathbf{S} = \mathbf{D}\mathbf{H}\mathbf{F}\mathbf{M}$  be the series composition of  $\mathbf{D}$ ,  $\mathbf{H}$ ,  $\mathbf{F}$ , and  $\mathbf{M}$ , i.e. the DT system with input w = w[n] and output v = v[n] shown on the block diagram below:

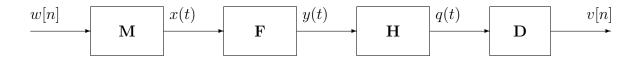


Figure 1:

By construction, the ideal compensator C should be the inverse  $C = S^{-1}$  of S, as long as the inverse does exist.

A key question answered in this paper is "what to expect from system S?" If one assumes that the continuous-time distortion subsystem F is simple enough, what does this say about S?

This paper provides an explicit expression for S in the case when F is given in the CT Volterra series form (2) with degree  $d = \max \beta_k$  and depth  $t_{max} = \max t_{k,i}$ . The result reveals that, even though S tends to have infinitely long memory (due to the ideal band-pass filter H being involved in the construction of S), it can be represented as a series composition S = LV, where  $V : \ell(\mathbb{C}) \to \ell(\mathbb{R}^N)$  maps scalar complex input  $w \in \ell(\mathbb{C})$  to real vector output  $g \in \ell(\mathbb{R}^N)$  in such a way that the k-th scalar component  $g_k[n]$  of  $g[n] \in \mathbb{R}^N$  is given by

$$g_k[n] = \prod_{i=0}^m (\operatorname{Re} w[n-i])^{\alpha_i} \prod_{i=0}^m (\operatorname{Im} w[n-i])^{\beta_i},$$
 
$$\alpha_i, \beta_i \in \mathbb{Z}_+, \qquad \sum_{i=0}^m \alpha_i + \sum_{i=0}^m \beta_i \le d,$$

m is the minimal integer not smaller than  $t_{max}/T$ , and  $\mathbf{L}:\ \ell(\mathbb{R}^N)\to\ell(\mathbb{C})$  is an LTI system.

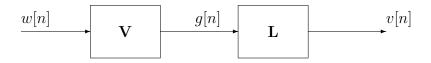


Figure 2: Block diagram of the structure of S

Moreover, L can be shown to have a good approximation of the form  $L \approx XL_0$ , where X is a static gain matrix, and  $L_0$  is an LTI model which does not depend on  $b_k$  and  $t_{k,i}$ . In other words, S can

be well approximated by combining a Volterra series model with a short memory, and a fixed (long memory) LTI, as long as the memory depth  $t_{max}$  of  $\mathbf{F}$  is short, relative to the sampling time T.

In most applications, with an appropriate scaling and time delay, the system S to be inverted can be viewed as a small perturbation of identity, i.e.  $S = I + \Delta$ . When  $\Delta$  is "small" in an appropriate sense (e.g., has small incremental L2 gain  $||\Delta|| \ll 1$ ), the inverse of S can be well approximated by  $S^{-1} \approx I - \Delta = 2I - S$ . Hence the result of this paper suggests a specific structure of the compensator (pre-distorter)  $C \approx I - \Delta = 2I - S$ . In other words, a plain Volterra monomials structure is, in general, not good enough for C, as it lacks the capacity to implement the long-memory LTI post-filter L. Instead, C should be sought in the form  $C = I - L_0 XV$ , where V is the system generating all Volterra series monomials of a limited depth and limited degree,  $L_0$  is a fixed LTI system with a very long time constant, and X is a matrix of coefficients to be optimized to fit the data available.

#### 3 Main Result

Given a sequence  $\tau = (\tau_1, \dots, \tau_d)$  of d non-negative real numbers  $\tau_i$  let  $\mathbf{F}_{\tau} : \mathcal{L} \to \mathcal{L}$  be the CT system mapping inputs  $x \in \mathcal{L}$  to the outputs  $y \in \mathcal{L}$  defined by

$$y(t) = x(t - \tau_1)x(t - \tau_2)\dots x(t - \tau_d).$$

Given d-tuple  $m = (m_1, ..., m_d) \in [1:4]^d$  and  $k \in [1:4]$  let  $S_k(m) = \{i \in [1:d]: m_i = k\}$ , and define

$$N_1(m) = |S_1(m) \cup S_2(m)|, \ N_2(m) = |S_3(m) \cup S_4(m)|.$$

Let  $(\cdot,\cdot):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$  be a map defined by  $(x,y)=\sum_{i=1}^d x_i\cdot y_i$  (i.e. the standard scalar product in  $\mathbb{R}^d$ ), and let maps  $\bar{\sigma},\sigma:\mathbb{R}^d\to\mathbb{R}$  be defined by  $\bar{\sigma}(x)=\sum_{i=1}^d x_i$  and  $\sigma(x)=\bar{\sigma}(x)-1$ . Also for a given  $m\in[1:4]^d$ , we define map  $\pi_m:\mathbb{R}^d\to\mathbb{R}$  by

$$\pi_m(x) = \prod_{i \in S_3(m) \cup S_4(m)} x_i,$$

and projection operators  $\mathcal{P}_m^i: \mathbb{R}^d \to \mathbb{R}^{N_i(m)}, i=1,2$  by

$$\mathcal{P}_{m}^{i} x = \begin{bmatrix} x_{n_{1}} & \dots & x_{n_{N_{i}(m)}} \end{bmatrix}^{T},$$
  
$$\{n_{1}, \dots, n_{N_{i}(m)}\} = S_{i}(m) \cup S_{i+1}(m), n_{1} < \dots < n_{N_{i}(m)}.$$

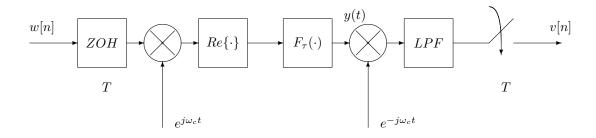


Figure 3:

Given a vector  $\tau \in \mathbb{R}^d_+$  let k be a vector in  $(\mathbb{N} \cup \{0\})^d$ , such that  $\tau = kT + \tau'$ , with  $\tau' \in [0, T)^d$ . It is obvious that for a given  $\tau$  vector k is uniquely defined.

Given a positive real number T, let us denote by  $p_{ZOH}(t)$ , impulse response of the zero-order hold (ZOH) system. We have  $p_{ZOH}(t) = \frac{1}{T}(u(t) - u(t-T))$ , where u(t) is the Heaviside step function. Moreover for a given  $m \in [1:4]^d$  and  $\tau' \in [0,T)^d$ , we define

$$\tau_{min} = \begin{cases} \min_{i \in S_2(m) \cup S_4(m)} \tau_i', & |S_2(m) \cup S_4(m)| > 0 \\ 0, & o/w \end{cases},$$

and

$$\tau_{max} = \begin{cases} \max_{i \in S_1(m) \cup S_3(m)} \tau'_i, & |S_1(m) \cup S_3(m)| > 0 \\ T, & o/w \end{cases}.$$

Now let  $p_{m,\tau}: \mathbb{R} \to \mathbb{R}$  be the continuous time signal defined by

$$p_{m,\tau}(t) = \begin{cases} \frac{1}{T} (u(t - \tau_{min}) - u(t - \tau_{max})), & \tau_{min} < \tau_{max} \\ 0, & o/w \end{cases}.$$

We denote its Fourier transform by  $P_{m,\tau}(j\omega)$ .

From (2) we can see that general CT Volterra model is a linear combination of subsistems of form  $F_{\tau}$ , so in order to find system decomposition  $\mathbf{S} = \mathbf{L}\mathbf{V}$  it is clearly sufficient to find what happens with one particular element  $F_{\tau}$ , i.e. to find map  $\mathbf{D}\mathbf{H}\mathbf{F}_{\tau}\mathbf{M}$ . The following theorem gives answer to that question.

**Theorem 2.1.** A DT system  $\mathbf{DHF}_{\tau}\mathbf{M}: \ell(\mathbb{C}) \to \ell(\mathbb{C})$ , mapping  $w[n] = i[n] + j \cdot q[n]$  to v[n], is given by

$$v[n] = \sum_{m \in \{1,2,3,4\}^d} f_{m,k}[n] * h_{m,\tau}[n],$$

where

$$f_{m,k}[n] = \prod_{i \in S_1(m)} i[n - k_i - 1] \cdot \prod_{i \in S_2(m)} i[n - k_i] \cdot \prod_{i \in S_3(m)} q[n - k_i - 1] \cdot \prod_{i \in S_4(m)} q[n - k_i],$$

and Fourier transform of a unit sample response  $h_{m,\tau}[n]$  is given by

$$H_{m,\tau}(e^{j\Omega}) = \frac{(-j)^{N_2(m)}}{2^d} \sum_{r \in \{-1,1\}^d} (-1)^{\pi_m(r)} P_{m,\tau} \left( j\frac{\Omega}{T} - j\omega_c \sigma(r) \right) e^{-j\omega_c(r,\tau)}.$$

*Proof.* We first state and prove the following Lemma, which is very similar to Theorem 2.1 but considers somewhat simpler case when  $\tau \in [0,T)^d$ , i.e. k=0. The proof of Theorem 2.1 then immediately follows from this Lemma.

**Lemma 2.2.** Suppose that  $\tau \in [0,T)^d$ . A DT system  $\mathbf{DHF}_{\tau}\mathbf{M}: \ell(\mathbb{C}) \to \ell(\mathbb{C})$ , mapping  $w[n] = i[n] + j \cdot q[n]$  to v[n], is given by

$$v[n] = \sum_{m \in \{1,2,3,4\}^d} f_m[n] * h_{m,\tau}[n],$$

where

$$f_m[n] = i[n-1]^{|S_1(m)|} \cdot i[n]^{|S_2(m)|} \cdot q[n-1]^{|S_3(m)|} \cdot q[n]^{|S_4(m)|},$$

and Fourier transform of a unit sample response  $h_{m,\tau}[n]$  is given by

$$H_{m,\tau}(e^{j\Omega}) = \frac{(-j)^{N_2(m)}}{2^d} \sum_{r \in \{-1,1\}^d} (-1)^{\pi_m(r)} P_{m,\tau} \left( j\frac{\Omega}{T} - j\omega_c \sigma(r) \right) e^{-j\omega_c(r,\tau)}.$$

*Proof.* Let us first analyze what happens in the case when d=1, i.e. system  $\mathbf{F}_{\tau}$  is just a delay by  $\tau \in [0,T)$ . Output y(t) of  $F_{\tau}$  becomes

$$y(t) = i(t - \tau)\cos(t - \tau) - q(t - \tau)\sin(t - \tau).$$

We observe that  $\mathbf{F}_{\tau}$  commutes with the modulation subsystem M, following an appropriate splitting of the ZOH impulse response, thus allowing us to move  $\mathbf{F}_{\tau}$  out of the Mod/Demod part of the system. Now system  $\mathbf{D}\mathbf{H}\mathbf{F}_{\tau}\mathbf{M}$  is equivalent to the one shown in Fig. 4, where the impulse responses  $p_1(t)$  and  $p_2(t)$  are given by

$$p_1(t) = \frac{1}{T}(u(t) - u(t - \tau)),$$
  
$$p_2(t) = \frac{1}{T}(u(t - \tau) - u(t - T)).$$

It is clear that  $p_1(t)$  and  $p_2(t)$  form the above mentioned splitting of the  $p_{ZOH}(t)$ , in the sense that the ZOH impulse response satisfies  $p_{ZOH}(t) = p_1(t) + p_2(t)$ . Thus subsystem  $\mathbf{F}_{\tau}\mathbf{M}$ , mapping w[n] to y(t), can be represented as a parallel connection of four LTI systems whose inputs are current and previous values of in-phase and quadrature components of the input signal w[n]. Hence output y(t) can be written as

$$y(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t),$$

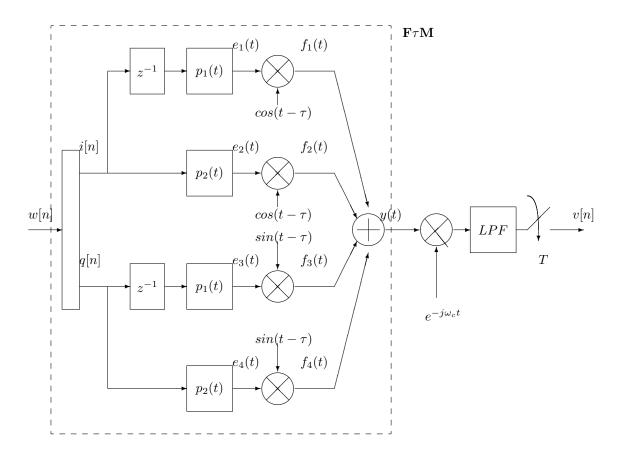


Figure 4:

where definition of signals  $f_i(t)$  is obvious from Fig. 4.

Now suppose that order d of  $\mathbf{F}_{\tau}$  is an arbitrary positive integer. From analysis in the case when d=1, it immediately follows that block structure shown in Fig. 5 is an equivalent representation of the system  $\mathbf{DHF}_{\tau}\mathbf{M}$ . Hence, by using the same notation as in Figs 4 and 5, signal y(t) can be written as

$$y(t) = \prod_{i=1}^{d} f^{i}(t) = \prod_{i=1}^{d} (f_{1}^{i}(t) + f_{2}^{i}(t) + f_{3}^{i}(t) + f_{4}^{i}(t)) = \sum_{m \in \{1,2,3,4\}^{d}} f_{m_{1}}^{1}(t) \cdot \dots \cdot f_{m_{d}}^{d}(t).$$
 (3)

Now it is clear that product in the last sum in (3) can be written as

$$f_{m_1}^1(t) \cdot \ldots \cdot f_{m_d}^d(t) = \prod_{i=1}^d e_{m_i}^i(t) \cdot \prod_{k \in S_1(m) \cup S_2(m)} \cos(t - \tau_k) \cdot \prod_{l \in S_3(m) \cup S_4(m)} \sin(t - \tau_l), \quad (4)$$

where  $e^i_{m_i}(t)$  equals  $f^i_{m_i}(t)/\cos(t-\tau_i)$  for  $m_i=1,2,$  or  $f^i_{m_i}(t)/\sin(t-\tau_i)$  otherwise. Since our goal is to find a transfer function from w[n] to v[n], it is more convenient to express the above products of

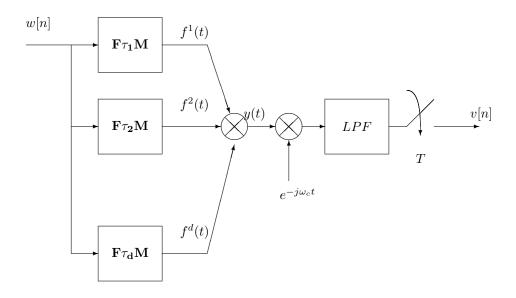


Figure 5:

cosines and sines as sums of complex exponentials, i.e.

$$\prod_{k \in S_1(m) \cup S_2(m)} \cos(t - \tau_k) = \frac{1}{2^{N_1(m)}} \sum_{r \in \{-1,1\}^{N_1(m)}} e^{j\omega_c \bar{\sigma}(r)t} \cdot e^{j\omega_c(r,\mathcal{P}_m^1 \tau)},$$

$$\prod_{l \in S_3(m) \cup S_4(m)} \sin(t - \tau_l) = \frac{1}{(2j)^{N_2(m)}} \sum_{r \in \{-1,1\}^{N_2(m)}} (-1)^{\prod_{i=1}^{N_2(m)} r_i} \cdot e^{j\omega_c \bar{\sigma}(r)t} \cdot e^{j\omega_c(r,\mathcal{P}_m^2 \tau)}.$$

Signals  $e_{m_i}^i(t)$  are obtained by applying pulse amplitude modulation with  $p_1(t)$  or  $p_2(t)$  on in-phase or quadrature components of the input signal (or their delayed counterparts). Now their product can be written as

$$\prod_{i=1}^{d} e_{m_i}^i(t) = \sum_{n=-\infty}^{\infty} i[n]^{|S_1(m)|} i[n-1]^{|S_2(m)|} \cdot \cdot q[n]^{|S_3(m)|} q[n-1]^{|S_4(m)|} p_m(t-nT).$$
 (5)

If we denote this product by  $e_m(t)$ , we can write (4) as

$$f_{m_1}^1(t) \cdot \dots \cdot f_{m_d}^d(t) = e_m(t) \cdot \frac{(-j)^{N_2(m)}}{2^d} \cdot \sum_{r \in \{-1,1\}^d} (-1)^{\pi_m(r)} \cdot e^{j\omega\bar{\sigma}(r)t} \cdot e^{-j\omega_c(r,\tau)}.$$
 (6)

Finally from (3),(5) and (6) it follows that the output v[n] is equal to

$$v[n] = \sum_{m \in \{1,2,3,4\}^d} f_m[n] * h_{m,\tau}[n],$$

where

$$f_m[n] = i[n-1]^{|S_1(m)|} \cdot i[n]^{|S_2(m)|} \cdot q[n-1]^{|S_3(m)|} \cdot q[n]^{|S_4(m)|},$$

and Fourier transforms of impulse responses  $h_{m,\tau}[n]$  are given by

$$H_{m,\tau}(e^{j\Omega}) = \sum_{m \in \{1,2,3,4\}^d} \frac{(-j)^{N_2(m)}}{2^d} \sum_{r \in \{-1,1\}^d} (-1)^{\pi_m(r)} \cdot P_m\left(j\frac{\Omega}{T} - j\omega_c\sigma(r)\right) \cdot e^{-j\omega_c(r,\tau)}.$$

This concludes the proof of Lemma 2.2.

In Lemma 2.2 we assumed that  $\tau_i \in [0,T)$ ,  $\forall i \in [1:d]$ , but in general  $\tau_i$  can take any positive real value depending on the depth of (2), i.e. vector k associated to  $\tau$  is not necessarily zero vector. Now assume that  $\tau = kT + \bar{\tau}$ , where  $\bar{\tau} \in [0,T)^d$ . The input/output relation for system  $\mathbf{DHF}_{\tau}\mathbf{M}$  readily follows from Lemma 2.2, and we have

$$v[n] = \sum_{m \in \{1,2,3,4\}^d} f_m[n] * h_m[n],$$

where signals  $f_m[n]$  are given by

$$f_m[n] = \prod_{i \in S_1(m)} i[n - k_i - 1] \cdot \prod_{i \in S_2(m)} i[n - k_i] \cdot \prod_{i \in S_3(m)} q[n - k_i - 1] \cdot \prod_{i \in S_4(m)} q[n - k_i],$$

and unit sample responses  $h_{m,\tau}[n]$  have the following Fourier transforms

$$H_{m,\tau}(e^{j\Omega}) = \frac{(-j)^{N_2(m)}}{2^d} \sum_{r \in \{-1,1\}^d} (-1)^{\pi_m(r)} \cdot P_m \left( j \frac{\Omega}{T} - j \omega_c \sigma(r) \right) \cdot e^{-j\omega_c(r,\tau)}.$$

### 4 Simulation Results

In this section, through MATLAB simulations, we illustrate performance of the proposed compensator structure. We compare this structure with some standard compensator structures, together with ideal compensator, and show that it closely resembles dynamics of ideal compensator, thus achieving very good compensation performance.

The underlying system S is given in Figure 1, with the distortion subsystem F given by

$$(Fx)(t) = x(t) - \delta \cdot x(t - \tau_1)x(t - \tau_2)x(t - \tau_3), \tag{7}$$

where  $0 \le \tau_1 \le \tau_2 \le \tau_3 \le T$ , with T sampling time, and  $\delta > 0$  parameter specifying magnitude of distortion  $\Delta$  in  $\mathbf{S} = \mathbf{I} + \Delta$ . We assume that parameter  $\delta$  is relatively small, in particular  $\delta \in (0, 0.2)$ ,

so that the inverse  $S^{-1}$  of S can be well approximated by 2I - S. Then our goal is to build compensator  $C = S^{-1}$  with different structures, and compare their performance, which is measured as output Error Vector Magnitude (EVM) [3]. EVM, for an input u and output  $\hat{u}$ , is defined as

EVM(dB) = 
$$20 \log_{10} \left( \frac{||u - \hat{u}||_2}{||u||_2} \right)$$
.

Analytical results from the previous section suggest that the compensator structure should be of the form depicted in Figure 2. It is easy to see from the proof of Theorem 1.1, that transfer functions in  $\mathbf{L}$ , from each nonlinear component  $g_k[n]$  of g[n], to the output v[n], are smooth functions, hence can be well approximated by low order polynomials in  $\Omega$ . In this example we choose second order polynomial approximation of components of  $\mathbf{L}$ . This observation, together with the true structure of  $\mathbf{S}$ , suggests that compensator  $\mathbf{C}$  should be fit within a family of models with structure shown on the block diagram in Fig 6.

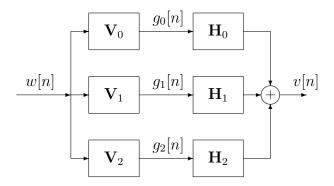


Figure 6: Proposed compensator structure

Subsystems  $\mathbf{H}_i$ , i = 1, 2, 3, are LTI systems, with transfer functions  $H_i$  given by

$$H_0(e^{j\Omega}) = 1, H_1(e^{j\Omega}) = j\Omega, H_2(e^{j\Omega}) = \Omega^2, \forall \Omega \in [-\pi, \pi].$$

Nonlinear subsystems  $V_i$  are modeled as third order Volterra series, with memory m=1, i.e.

$$(\mathbf{V}_{j}w)[n] = \sum_{(\alpha(k),\beta(k))} c_{k}^{j} \prod_{l=0}^{1} i[n-l]^{\alpha_{l}(k)} \prod_{l=0}^{1} q[n-l]^{\beta_{l}(k)},$$
  
$$\alpha_{l}(k), \beta_{l}(k) \in \mathbb{Z}_{+}, \qquad \sum_{l=0}^{1} \alpha_{l}(k) + \sum_{l=0}^{1} \beta_{l}(k) \leq 3,$$

where i[n] = Re w[n] and q[n] = Im w[n], and  $(\alpha(k), \beta(k)) = (\alpha_0(k), \alpha_1(k), \beta_0(k), \beta_1(k))$ .

We compare performance of this compensator with the widely used one obtained by utilizing simple Volterra series structure [3]:

$$(\mathbf{C}w)[n] = \sum_{(\alpha(k),\beta(k))} c_k \prod_{l=-m_1}^{m_2} i[n-l]^{\alpha_l(k)} \prod_{l=-m_1}^{m_2} q[n-l]^{\beta_l(k)},$$

$$\alpha_l(k), \beta_l(k) \in \mathbb{Z}_+, \qquad \sum_{l=-m_1}^{m_2} \alpha_l(k) + \sum_{l=-m_1}^{m_2} \beta_l(k) \le d.$$

Parameters which could be varied in this case are forward and backward memory depth  $m_1$  and  $m_2$ , respectively, and degree d of this model. We consider three cases for different sets of parameter values:

- Case 1:  $m_1 = 0$ ,  $m_2 = 2$ , d = 5
- Case 2:  $m_1 = 0$ ,  $m_2 = 4$ , d = 5
- Case 3:  $m_1 = 2$ ,  $m_2 = 2$ , d = 5

Table 1: Number of coefficients  $c_k$  being optimized for different compensator models

Model	$\#$ of $c_k$	# of significant $c_k$
New structure	210	141
Volterra 1	924	177
Volterra 2	6006	2058
Volterra 3	6006	1935

After fixing compensator structure, coefficients  $c_k$  are obtained by applying straightforward least squares optimization.

We should emphasize here that fitting has to be done for both real and imaginary part of v[n], thus the actual compensator structure is twice that depicted in Figure 6.

Simulation parameters for system **S** are as follows: symbol rate  $f_{symb} = 2$ MHz, carrier frequency  $f_c = 20$ MHz, with 64QAM input symbol sequence. Nonlinear distortion subsystem **F** of **S**, used in simulation, is defined in (7), where the delays  $\tau_1, \tau_2, \tau_3$  are given by the vector  $\tau = [0.2T \quad 0.3T \quad 0.4T]$ , with  $T = 1/f_{symb}$ . Digital simulation of the continuous part of **S** was done by representing continuous signals by their discrete counterparts, obtained by sampling with high sampling rate  $f_s = 1000 \cdot f_{symb}$ . As input to **S**, we assume periodic 64QAM symbol sequence, with period  $N_{symb} = 4096$ . This period length is used for generating input/output data for fitting coefficients  $c_k$ , as well as generating input/output data for performance validation.

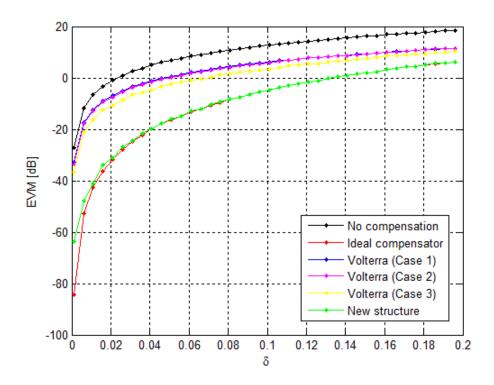


Figure 7: Output EVM for different compensator structures

In Figure 7 we present EVM obtained for different compensator structures, as well as output EVM with no compensation, and case with ideal compensator  $C = S^{-1} \approx 2I - S$ . As can be seen from Figure 7, compensator fitted using the proposed structure in Figure 6 outperforms other compensators, and gives output EVM almost identical to the ideal compensator. This result was to be expected, since model in Figure 6 approximates the original system S very closely, and thus is capable of approximating system 2I - S closely as well. This is not the case for compensators modeled with simple Volterra series, due to inherently long (or more precisely infinite) memory introduced by the LTI part of S. Even if we use noncausal Volterra series model (i.e.  $m_1 \neq 0$ ), which is expected to capture true dynamics better, we are still unable to get good fitting of the system S, and consequently of the compensator  $C \approx 2I - S$ .

Advantage of the proposed compensator structure is not only in better compensation performance, but also in that it achieves better performance with much more efficient structure. That is, we need far less coefficients in order to represent nonlinear part of the compensator, in both least squares optimization and actual implementation (Table 1). In Table 1 we can see a comparison in the number of coefficients between different compensator structures, for nonlinear subsystem parameter value  $\delta = 0.02$ . Data in the first column is number of coefficients (i.e. basis elements) needed for general Volterra model, i.e. coefficients which are optimized by least squares. The second column shows actual number of coefficients used to build compensator. Least squares optimization yields many nonzero coefficients, but only subset of those are considered significant and thus used in actual com-

pensator implementation. Coefficient is considered significant if its value falls above a certain treshold t, where t is chosen such that increas in EVM after zeroing nonsignificant coefficients is not larger than 1% of the best achievable EVM (i.e. when all basis elements are used for building compensator). From Table 1 we can see that for case 3 Volterra structure, 10 times more coefficients are needed in order to implement compensator, than in the case of our proposed structure. And even when such a large number of coefficients is used, its performance is still below the one achieved by this new compensator model.

#### 5 Discussion

The potential significance of the result presented in this paper lies in revealing a special structure of a digital pre-distortion compensator which appears to be both necessary and sufficient to match the discrete time dynamics resulting from combining modulation and demodulation with a dynamic non-linearity in continuous time. The "necessity" somewhat relies on the input signal u having "full" spectrum. While, theoretically, the baseband signal u is supposed to be shaped so that only a lower DT frequency spectrum of it remains significant, a practical implementation of amplitude-phase modulation will frequently employ the a signal component separation approach, such as LINC [7], where the low-pass signal u is decomposed into two components of constant amplitude,  $u = u_1 + u_2$ ,  $|u_1[n]| \equiv |u_2[n]| = \text{const}$ , after which the components  $u_i$  are fed into two separate modulators, to produce continuous time outputs  $y_1, y_2$ , to be combined into a single output  $y = y_1 + y_2$ . Even when u is band-limited, the resulting components  $u_1, u_2$  are not, and the full range of modulator's nonlinearity is likely to be engaged when producing  $y_1$  and  $y_2$ .

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