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Compositional abstraction of interconnected control systems under dynamic interconnection topology

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***Abstract*— In this work, we derive conditions under which compositional abstractions of networks of control systems, interconnected via some dynamic interconnection topology, can be constructed using the dynamic interconnection and joint dissipativity-type properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process. Moreover, we derive conditions for the construction of abstractions for a class of control systems involving nonlinearities satisfying an incremental quadratic inequality. We provide an example to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning by reducing a 150-dimensional nonlinear system to a 3-dimensional one.**

I. Introduction

For large interconnected control systems, controller design to achieve some complex specifications in a reliable and cost effective way is a challenging task. One direction which has been explored to overcome this challenge is to use a simpler (e.g. lower dimension) (in)finite approximation of the given system as a replacement in the controller design process. This allows for a design of a controller for the approximation, which can be refined to the one for the original complex system. The error between the output behaviour of the concrete system and the one of its approximation can be quantified in this detour controller synthesis approach.

Rather than treating the interconnected system in a monolithic manner, an approach which severely restricts the capability of existing techniques to deal with many number of subsystems, compositional schemes provide network-level certifications from main structural properties of the subsystems and their interconnections. Recently, there have been several results on the compositional construction of (in)finite abstractions of deterministic control systems including [1], [2], [3], [4], and probabilistic control systems [5]. These results use a small-gain type condition to enable the compositional construction of abstractions. However, as shown in [6], this type of condition is a function of the size of the network and can be violated as the number of subsystems grows.

Recently in [7], a compositional framework for the construction of abstractions of networks of control systems has been proposed using dissipativity theory. In this result a notion of storage function is proposed which describes joint dissipativity properties of control systems and their abstractions. This notion is used to derive compositional conditions under which a network of abstractions approximate a network of the concrete subsystems. Those conditions can be independent of the number of the subsystems under some properties on the interconnection topologies.

In this work, we extend the results in [7] to networks of control systems in which the interconnection topology is dynamic [8]. In such interconnected systems, the additional dynamics introduced due to the interconnection/interaction system has to be taken into account in the compositional reasoning. We derive conditions under which we can form compositional abstractions of a given interconnected control system under dynamic interconnection.

In addition, for a class of control systems which takes into account a more general class of nonlinearities than the one considered in [7], we derive a set of linear matrix (in)equalities facilitating the construction of their abstractions together with the corresponding storage functions. We illustrate the effectiveness of the proposed results by deriving a compositional abstraction for a network of control systems of this class in which compositionality conditions are satisfied independent of the number or gains of the subsystems.

II. Control Systems

A. Notation

The sets of non-negative integer and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Those symbols are footnoted with subscripts to restrict them in the usual way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. The symbol $\mathbb{R}^{n\times m}$ denotes the vector space of real matrices with n rows and m columns. The symbols $\vec{1}_n, \vec{0}_n, I_n, 0_{n\times m}$ denote the vector in $\mathbb{R}^n$ with all its elements to be one, the zero vector, identity, and zero matrices in $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times m}$, respectively. For $a, b \in \mathbb{R}$ with $a \le b$, the closed, open, and half-open intervals in $\mathbb{R}$ are denoted by $[a, b],]a, b[, [a, b[$, and $]a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \le b$, we use $[a; b],]a; b[, [a; b[$, and $]a; b]$ to denote the corresponding intervals in $\mathbb{N}$. Given $N \in \mathbb{N}_{\ge 1}$, vectors $x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}_{\ge 1}$ and $i \in [1; N]$, we use $x = [x_1; \ldots; x_N]$ to denote the concatenated vector in $\mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. Similarly, we use $X = [X_1; \ldots; X_N]$ to denote the matrix in $\mathbb{R}^{n\times m}$ with $n = \sum_{i=1}^N n_i$, given $N \in \mathbb{N}_{\ge 1}$, matrices $X_i \in \mathbb{R}^{n_i\times m}, n_i \in \mathbb{N}_{\ge 1}$, and $i \in [1; N]$. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the Euclidean norm of x. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the Euclidean norm of M. Given matrices $M_1, \ldots, M_n$, the notation $\mathsf{diag}(M_1, \ldots, M_n)$ represents a block diagonal

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matrix with diagonal matrix entries $M_1, \ldots, M_n$. Given a symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of A, respectively. Given a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|,\ t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the map $\beta(r, t)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed nonzero r, the map $\beta(r, t)$ is decreasing with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$.

B. Control Systems

Here, we define the class of control systems being investigated in this paper.

Definition 2.1: The class of control systems studied in this paper is a tuple

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2, h_{u1}, h_{u2}),$$

where

- $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$, $\mathbb{R}^{q_1}$, and $\mathbb{R}^{q_2}$ are the state, external input, internal input, external output, and internal output spaces, respectively;
- $\mathcal{U}$ and $\mathcal{W}$ are subsets of sets of all measurable functions of time taking values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively;
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuous map satisfying the following Lipshitz assumption: for every compact set $\mathsf{D} \subset \mathbb{R}^n$, there exists a constant $Z \in \mathbb{R}_{>0}$ such that $\|f(x, u, w) - f(y, u, w)\| \leq Z\|x - y\|$ for all $x, y \in \mathsf{D}$, all $u \in \mathbb{R}^m$, and all $w \in \mathbb{R}^p$;
- $h_1 : \mathbb{R}^n \to \mathbb{R}^{q_1}$ is the external output map;
- $h_2 : \mathbb{R}^n \to \mathbb{R}^{q_2}$ is the internal output map.
- $h_{u1} : \mathbb{R}^m \to \mathbb{R}^{q_1}$ is the external feedforward map;
- $h_{u2} : \mathbb{R}^m \to \mathbb{R}^{q_2}$ is the internal feedforward map.

A control system Σ satisfies

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)) + h_{u1}(\upsilon(t)), \\ \zeta_2(t) = h_2(\xi(t)) + h_{u2}(\upsilon(t)), \end{cases} \tag{II.1}$$

for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where a locally absolutely continous curve $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called a *state trajectory* of Σ, $\zeta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_1}$ is called an external output trajectory of Σ, and $\zeta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_2}$ is called an internal output trajectory of Σ. We call the tuple $(\xi, \zeta_1, \zeta_2, \upsilon, \omega)$ a *trajectory* of Σ, consisting of a state trajectory ξ, output trajectories ζ_1 and ζ_2, and input trajectories υ and ω, satisfying (II.1). We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the state trajectory at time $t \in \mathbb{R}_{\geq 0}$ under the input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$, where $a \in \mathbb{R}^n$. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to the state trajectory $\xi_{a\upsilon\omega}$.

Remark 2.2: If the control system Σ does not have internal and external feedforward maps, the description of the system defined in Definition 2.1 reduces to the tuple

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2).$$

Correspondingly, equation (II.1) describing the evolution of state and output trajectories reduces to:

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)). \end{cases} \tag{II.2}$$

We use the notion of control system in (II.2) later to refer to control subsystems in an interconnected system.

Remark 2.3: If the control system Σ does not have internal inputs and outputs, the description of the control system in Definition 2.1 reduces to the tuple

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h, h_u).$$

Correspondingly, the equation (II.1) describing the evolution of state and output trajectories reduces to:

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)) + h_u(\upsilon(t)). \end{cases} \tag{II.3}$$

We use the notion of control system in (II.3) later to refer to a dynamical interconnection topology in an interconnected system.

Remark 2.4: If the control system Σ does not have internal inputs and outputs, and external feedforward map, the definition of the control system in Definition 2.1 reduces to the tuple

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h).$$

Correspondingly, the equation (II.1) describing the state and output trajectories reduces to:

$$\Sigma : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{II.4}$$

We use the notion of control system in (II.4) later to refer to an overall interconnected control system.

III. Storage Function

In this section, we recall the notion of so-called storage function introduced in [7] with some modifications to accommodate for dynamic interconnection topology.

Definition 3.1: Let

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

be two control subsystems with the same external output space dimension. A continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ is called a storage function from $\hat{\Sigma}$ to Σ, if there exist functions $\alpha, \eta \in \mathcal{K}_\infty$, $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, some matrices $W, \hat{W}$, and H of appropriate dimensions, and some symmetric matrix X of appropriate dimension with conformal block partitions $X^{ij}, i, j \in [1; 2]$, such that for any $x \in \mathbb{R}^n$ and any $\hat{x} \in \mathbb{R}^{\hat{n}}$, one has

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|) \leq V(x, \hat{x}), \tag{III.1}$$

and $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$, $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$, $\exists u \in \mathbb{R}^m$, such that $\forall \hat{w} \in \mathbb{R}^{\hat{p}}$ $\forall w \in \mathbb{R}^p$, one obtains

$$\nabla V(x,\hat{x})^T \begin{bmatrix} f(x,u,w) \\ \hat{f}(\hat{x},\hat{u},\hat{w}) \end{bmatrix} \leq -\eta(V(x,\hat{x})) + \psi_{\text{ext}}(\|\hat{u}\|) + z^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} z, \quad \text{(III.2)}$$

where

$$z = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$

We now recall the notion of simulation functions introduced in [9] with some modification.

Definition 3.2: Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$, be two interconnected control systems. A continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ is called a *simulation function* from $\hat{\Sigma}$ to Σ if there exist $\alpha, \eta \in \mathcal{K}_\infty$ and $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ such that for any $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, one has

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x,\hat{x}), \quad \text{(III.3)}$$

and $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$, $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$, $\exists u \in \mathbb{R}^m$ such that

$$\nabla V(x,\hat{x})^T \begin{bmatrix} f(x,u) \\ \hat{f}(\hat{x},\hat{u}) \end{bmatrix} \leq -\eta(V(x,\hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|). \quad \text{(III.4)}$$

The next theorem, borrowed from [7], shows the importance of the existence of a simulation function by quantifying the error between the output behaviours of Σ and the ones of its abstraction $\hat{\Sigma}$.

Theorem 3.3: Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$, and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$. Suppose V is a simulation function from $\hat{\Sigma}$ to Σ. Then, there exists a $\mathcal{KL}$ function ϑ such that for any $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, $\hat{\upsilon} \in \hat{\mathcal{U}}$, there exists $\upsilon \in \mathcal{U}$ such that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\|\zeta_{x\upsilon}(t) - \hat{\zeta}_{\hat{x}\hat{\upsilon}}(t)\| \leq \alpha^{-1}(2\vartheta(V(x,\hat{x}),t)) + \alpha^{-1}(2\eta^{-1}(2\rho_{\text{ext}}(\|\hat{\upsilon}\|_\infty))). \quad \text{(III.5)}$$

Remark 3.4: If functions α and η in Definition 3.2 satisfy the triangle inequality, then one can drop coefficients 2 in inequality (III.5) to get a less conservative upper bound.

IV. Interconnected Systems

Here, we define interconnected control systems under dynamic interconnection topology.

Definition 4.1: Consider $N \in \mathbb{N}_{\geq 1}$ control subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1;N]$, and a so-called interconnection system

$$\Sigma_o = (\mathbb{R}^{n_o}, \mathbb{R}^{m_o}, \mathcal{U}_o, f_o, \mathbb{R}^{q_o}, h_o, h_{ou}), \quad \text{(IV.1)}$$

where, for any $x_o \in \mathbb{R}^{n_o}, u_o \in \mathbb{R}^{m_o}$,

$$\begin{aligned} f_o(x_o,u_o) &:= A_o x_o + B_o u_o, \\ h_o(x_o) &:= C_o x_o, \\ h_{ou}(u_o) &:= D_o u_o, \end{aligned}$$

for some matrices A_o, B_o, C_o, and D_o of appropriate dimensions, $q_o = \sum_{i=1}^N p_i$, and $m_o = \sum_{i=1}^N q_{2i}$. The interconnected control system

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \tilde{f}, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}_{\Sigma_o}(\Sigma_1, \ldots, \Sigma_N)$, follows by $n = \sum_{i=1}^N n_i + n_o, m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{1i}$, and the functions

$$\begin{aligned} \tilde{f}(x,x_o,u) &= \begin{bmatrix} [f_1(x_1,u_1,w_1);\ldots;f_N(x_N,u_N,w_N)] \\ f_o(x_o,u_o) \end{bmatrix}, \\ h(x) &= [h_{11}(x);\ldots;h_{1N}(x_N)], \end{aligned}$$

where $u = [u_1;\ldots;u_N]$, $x = [x_1;\ldots;x_N]$, and with the internal inputs equal to the output of Σ_o, i.e. $[w_1;\ldots;w_N] = h_o(x_o) + h_{ou}(u_o)$, and the input of Σ_o equal to the internal outputs, i.e. $u_o = [h_{21}(x_1);\ldots;h_{2N}(x_N)]$.

The next theorem provides a compositional approach on the construction of abstractions of dynamically interconnected networks of control systems.

Theorem 4.2: Consider an interconnected control system $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$, induced by $N \in \mathbb{N}$ control subsystems Σ_i as in (II.2), and the interconnection system Σ_o as in (IV.1). Suppose each subsystem admits an abstraction $\hat{\Sigma}_i$ with the corresponding storage function V_i. If there exist $\mu_i > 0$, $i \in [1;N]$, and positive constant κ_o such that the inequality[1] (IV.2) and the following equalities

$$\begin{aligned} C_o \Pi \hat{A}_o &= C_o A_o \Pi, \\ C_o \Pi \hat{B}_o &= C_o B_o H, \\ \hat{W}\hat{D}_o &= W D_o H, \\ \hat{W}\hat{C}_o &= W C_o \Pi, \end{aligned} \quad \text{(IV.3)}$$

hold for some matrices $\hat{A}_o, \hat{B}_o, \hat{C}_o, \hat{D}_o$, and Π of appropriate dimensions, where

$$W = \mathsf{diag}(W_1,\ldots,W_N), \hat{W} = \mathsf{diag}(\hat{W}_1,\ldots,\hat{W}_N), \\ H = \mathsf{diag}(H_1,\ldots,H_N),$$

$$X(\mu_1 X_1,\ldots,\mu_N X_N) = \begin{bmatrix} \mu_1 X_1^{11} & & & \mu_1 X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{11} & & & \mu_N^{12} X_N^{12} \\ \mu_1 X_1^{21} & & & \mu_1 X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{21} & & & \mu_N^{22} X_N^{22} \end{bmatrix},$$

then

$$V(x,x_o,\hat{x},\hat{x}_o) = \bar{V}(x,\hat{x}) + V_o(x_o,\hat{x}_o),$$

is a simulation function from $\hat{\Sigma} = \mathcal{I}_{\hat{\Sigma}_o}(\hat{\Sigma}_1,\ldots,\hat{\Sigma}_N)$ to $\Sigma = \mathcal{I}_{\Sigma_o}(\Sigma_1,\ldots,\Sigma_N)$, where $\bar{V}(x,\hat{x}) = \sum_{i=1}^N \mu_i V_i(x_i,\hat{x}_i)$, $V_o(x_o,\hat{x}_o) = (x_o - \Pi\hat{x}_o)^T C_o^T C_o (x_o - \Pi\hat{x}_o)$ and

$$\hat{\Sigma}_o = (\mathbb{R}^{\hat{n}_o}, \mathbb{R}^{\hat{m}_o}, \hat{\mathcal{U}}_o, \hat{f}_o, \mathbb{R}^{\hat{q}_o}, \hat{h}_o, \hat{h}_{ou}),$$

[1]Matrices 0_* and I_* in (IV.2) represent zero and identity matrices of appropriate dimensions, respectively.

$$\begin{bmatrix} C_o^TC_oA_o+A_o^TC_o^TC_o & C_o^TC_oB_o \\ B_o^TC_o^TC_o & 0_* \end{bmatrix} + \begin{bmatrix} WC_o & WD_o \\ 0_* & I_* \end{bmatrix}^T X(\mu_1X_1,\ldots,\mu_NX_N) \begin{bmatrix} WC_o & WD_o \\ 0_* & I_* \end{bmatrix} \preceq \begin{bmatrix} -\kappa_oC_o^TC_o & 0_* \\ 0_* & 0_* \end{bmatrix} \quad \text{(IV.2)}$$

where, for any $\hat{x}_o \in \mathbb{R}^{\hat{n}_o}$ and any $\hat{u}_o \in \mathbb{R}^{\hat{m}_o}$,

$$\begin{aligned} \hat{f}(\hat{x}_o,\hat{u}_o) &:= \hat{A}_o\hat{x}_o + \hat{B}_o\hat{u}_o, \\ \hat{h}_o(\hat{x}_o) &:= \hat{C}_o\hat{x}_o, \\ \hat{h}_{ou}(\hat{u}_o) &:= \hat{D}_o\hat{u}_o. \end{aligned}$$

Proof: The proof is inspired by that of Theorem 4.2 in [7]. First we show that inequality (III.3) holds for some $\mathcal{K}_\infty$ function α. For any $x = [x_1;\ldots;x_N] \in \mathbb{R}^n$, and $\hat{x} = [\hat{x}_1;\ldots;\hat{x}_N] \in \mathbb{R}^{\hat{n}}$, one gets:

$$\begin{aligned} \|h(x)-\hat{h}(\hat{x})\|^2 &\le \sum_{i=1}^N \|h_{1i}(x_i)-\hat{h}_{1i}(\hat{x}_i)\|^2 \\ &\le \sum_{i=1}^N \alpha_i^{-1}(V_i(x_i,\hat{x}_i)) \le \underline{\alpha}(\bar{V}(x,\hat{x})), \end{aligned}$$

where $\underline{\alpha}$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}(s) := \begin{cases} \max\limits_{\vec{s}\ge 0} & \sum_{i=1}^N \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu^T\vec{s} = s, \end{cases}$$

where[2] $\vec{s} = [s_1;\ldots;s_N] \in \mathbb{R}^N$ and $\mu = [\mu_1;\ldots;\mu_N]$. By defining $\mathcal{K}_\infty$ function $\alpha(s) = \underline{\alpha}^{-1}(s), \forall s \in \mathbb{R}_{\ge 0}$, one obtains $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$, $\forall x_o \in \mathbb{R}^{n_o}$, $\forall \hat{x}_o \in \mathbb{R}^{\hat{n}_o}$,

$$\begin{aligned} \alpha(\|h(x)-\hat{h}(\hat{x})\|^2) \le \bar{V}(x,\hat{x}) &\le \bar{V}(x,\hat{x}) + V_o(x_o,\hat{x}_o) \\ &= V(x,x_o,\hat{x},\hat{x}_o), \end{aligned}$$

hence satisfying inequality (III.3). Now we prove inequality (III.4). Consider any $x = [x_1;\ldots;x_N] \in \mathbb{R}^n, \hat{x} = [\hat{x}_1;\ldots;\hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $\hat{u} = [\hat{u}_1;\ldots;\hat{u}_N] \in \mathbb{R}^{\hat{m}}$. For any $i \in [1;N]$, there exists $u_i \in \mathbb{R}^{m_i}$, consequently, a vector $u = [u_1;\ldots;u_N] \in \mathbb{R}^m$, satisfying (III.2) for each pair of Σ_i and $\hat{\Sigma}_i$, with the internal inputs given by the outputs of the interconnection systems Σ_o and $\hat{\Sigma}_o$, respectively, i.e.

$$[w_1;\ldots;w_N] = h_o(x_o) + h_{ou}(u_o),$$

$$[\hat{w}_1;\ldots;\hat{w}_N] = \hat{h}_o(\hat{x}_o) + \hat{h}_{ou}(\hat{u}_o),$$

where the inputs to Σ_o and $\hat{\Sigma}_o$ are the internal outputs of the subsystems Σ_i and $\hat{\Sigma}_i$, respectively, i.e.

$$u_o = [h_{21}(x_1);\ldots;h_{2N}(x_N)],$$

$$\hat{u}_o = [\hat{h}_{21}(\hat{x}_1);\ldots;\hat{h}_{2N}(\hat{x}_N)].$$

We employ the conditions in (IV.2) and (IV.3), which results in the chain of inequalities in (IV.4), where the functions $\eta \in \mathcal{K}_\infty$, and $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ are defined as

$$\eta(s) := \begin{cases} \min\limits_{[\vec{s};s_o]\ge 0} & \sum_{i=1}^N \mu_i\eta_i(s_i) + \kappa_o s_o \\ \text{s.t.} & \mu^T\vec{s} + s_o = s, \end{cases}$$

$$\psi_{\text{ext}}(s) := \begin{cases} \max\limits_{\vec{s}\ge 0} & \sum_{i=1}^N \mu_i\psi_{i\text{ext}}(s_i) \\ \text{s.t.} & \mu^T\vec{s} = s. \end{cases}$$

Hence we conclude that V is a simulation function from $\hat{\Sigma}$ to Σ. ∎

Remark 4.3: Note that the case of static interconnection and its associated conditions presented in [7] can readily be recovered by the results here if C_o is equal to the zero matrix and D_o is equal to the static interconnection matrix (values of A_o and B_o become irrelevant since x_o does not affect the internal input to Σ).

In the following section, we consider a specific class of control subsystems Σ, and a specific candidate storage function V. We derive conditions under which candidate V is a storage function from an abstraction $\hat{\Sigma}$ to Σ. Those conditions facilitate the construction of $\hat{\Sigma}$.

V. A Class of Nonlinear Control Systems

We consider a specific class of control subsystems given by

$$\begin{aligned} \dot{\xi}(t) &= A\xi(t) + B\upsilon(t) + E\varphi(t,F\xi) + D\omega(t), \\ \zeta_1(t) &= C_1\xi(t), \\ \zeta_2(t) &= C_2\xi(t), \end{aligned} \quad \text{(V.1)}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times p}, E \in \mathbb{R}^{n\times l_k}, F \in \mathbb{R}^{l_k\times n}, C_1 \in \mathbb{R}^{q_1\times n}$, and $C_2 \in \mathbb{R}^{q_2\times n}$. The time-varying non-linearity is the one considered in [10] satisfying an incremental quadratic inequality. Similar to [10], we assume that $\varphi(t,Fx) \in \mathbb{R}^{l_k}$ is defined by an implicit relation:

$$\varphi(t,Fx) = \phi(t,k),$$

where $k = Fx + D_k\varphi$, ϕ is a continuous function, $k \in \mathbb{R}^{l_k}$, and $D_k \in \mathbb{R}^{l_k\times l_k}$. For all $\tilde{M} \in \mathcal{M}$, where $\mathcal{M}$ is the set of symmetric matrices referred to as incremental multiplier matrices, the following incremental quadratic constraint holds for all $t \in \mathbb{R}_{\ge 0}$, and all $k_1,k_2 \in \mathbb{R}^{l_k}$:

$$\begin{bmatrix} k_2-k_1 \\ \phi(t,k_2)-\phi(t,k_1) \end{bmatrix}^T \tilde{M} \begin{bmatrix} k_2-k_1 \\ \phi(t,k_2)-\phi(t,k_1) \end{bmatrix} \ge 0.$$

We recall the assumption given in [10] which will be used in the rest of the paper.

Assumption 1: There is a continuous function ς such that, for all $t \in \mathbb{R}_{\ge 0}$ and $z_k = Fx$,

$$\varphi = \varsigma(t,z_k)$$

[2]We interpret inequality $\vec{s} \ge 0$ component-wise.

$$
\begin{aligned}
&\dot{V}(x,x_o,\hat{x},\hat{x}_o) = \sum_{i=1}^{N}\mu_i\dot{V}_i(x_i,\hat{x}_i) + 2(x_o-\Pi\hat{x}_o)^T C_o^T C_o(\dot{x}_o - \Pi\dot{\hat{x}}_o)\\
&\le \sum_{i=1}^{N}\mu_i\left(-\eta_i(V_i(x_i,\hat{x}_i)) + \psi_{i\mathrm{ext}}(\|\hat{u}_i\|) + \begin{bmatrix} W_i w_i - \hat{W}_i\hat{w}_i \\ h_{2i}(x_i) - H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}^T \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22}\end{bmatrix}\begin{bmatrix} W_i w_i - \hat{W}_i\hat{w}_i \\ h_{2i}(x_i) - H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}\right)\\
&\quad + 2(x_o-\Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - \Pi\hat{A}_o\hat{x}_o - \Pi\hat{B}_o\hat{u}_o)\\
&= \sum_{i=1}^{N}-\mu_i\eta_i(V_i(x_i,\hat{x}_i)) + \sum_{i=1}^{N}\mu_i\psi_{i\mathrm{ext}}(\|\hat{u}_i\|)\\
&\quad + \begin{bmatrix} W\begin{bmatrix} w_1\\ \vdots \\ w_N\end{bmatrix} - \hat{W}\begin{bmatrix} \hat{w}_1\\ \vdots \\ \hat{w}_N\end{bmatrix} \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}^T X(\mu_1X_1,\ldots,\mu_NX_N)\begin{bmatrix} W\begin{bmatrix} w_1\\ \vdots \\ w_N\end{bmatrix} - \hat{W}\begin{bmatrix} \hat{w}_1\\ \vdots \\ \hat{w}_N\end{bmatrix} \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}\\
&\quad + 2(x_o-\Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - A_o\Pi\hat{x}_o - B_oH\hat{u}_o)\\
&= \sum_{i=1}^{N}-\mu_i\eta_i(V_i(x_i,\hat{x}_i)) + \sum_{i=1}^{N}\mu_i\psi_{i\mathrm{ext}}(\|\hat{u}_i\|)\\
&\quad + \begin{bmatrix} WC_ox_o + WD_ou_o - \hat{W}\hat{C}_o\hat{x}_o - \hat{W}\hat{D}_o\hat{u}_o \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}^T X(\mu_1X_1,\ldots,\mu_NX_N)\begin{bmatrix} WC_ox_o + WD_ou_o - \hat{W}\hat{C}_o\hat{x}_o - \hat{W}\hat{D}_o\hat{u}_o \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}\\
&\quad + 2(x_o-\Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - A_o\Pi\hat{x}_o - B_oH\hat{u}_o)\\
&\le -\sum_{i=1}^{N}\mu_i\eta_i(V_i(x_i,\hat{x}_i)) + \psi_{\mathrm{ext}}(\|\hat{u}\|)\\
&\quad + \begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}^T \begin{bmatrix} WC_o & WD_o \\ 0_* & I_*\end{bmatrix}^T X(\mu_1X_1,\ldots,\mu_NX_N)\begin{bmatrix} WC_o & WD_o \\ 0_* & I_*\end{bmatrix}\begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}\\
&\quad + \begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}^T \begin{bmatrix} C_o^TC_oA_o + A_o^TC_o^TC_o & C_o^TC_oB_o \\ B_o^TC_o^TC_o & 0_*\end{bmatrix}\begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\end{bmatrix}\\
&\le -\sum_{i=1}^{N}\mu_i\eta_i(V_i(x_i,\hat{x}_i)) - \kappa_o(x_o-\Pi\hat{x}_o)^T C_o^T C_o(x_o-\Pi\hat{x}_o) + \psi_{\mathrm{ext}}(\|\hat{u}\|)\\
&= -\sum_{i=1}^{N}\mu_i\eta_i(V_i(x_i,\hat{x}_i)) - \kappa_o V_o(x_o,\hat{x}_o) + \psi_{\mathrm{ext}}(\|\hat{u}\|)\\
&\le -\eta(V(x,x_o,\hat{x},\hat{x}_o)) + \psi_{\mathrm{ext}}(\|\hat{u}\|)
\end{aligned} \tag{IV.4}
$$

uniquely solves the implicit relationship $\varphi = \phi(t, z_k + D_k\varphi)$. To facilitate subsequent analysis, we write the incremental multiplier matrix $\tilde{M}$ in the following conformal partitioned form

$$\tilde{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22}\end{bmatrix}.$$

We use the tuple

$$\Sigma = (A, B, C_1, C_2, D, E, F, \varphi),$$

to refer to the class of system of the form (V.1). We now consider a candidate function and derive conditions under which it is a storage function from a $\hat{\Sigma}$ to Σ.

A. Storage function

Here, we consider a candidate storage function of the form

$$V(x,\hat{x}) = (x - P\hat{x})^T\widehat{M}(x - P\hat{x}), \tag{V.2}$$

where P and $\widehat{M} \succ 0$, are matrices of appropriate dimensions. In order to show that $V(x, \hat{x})$ in (V.2) is a storage function from an abstraction $\hat{\Sigma}$ to the concrete system Σ, we require the following assumptions on the concrete system Σ.

Assumption 2: Let $\Sigma = (A, B, C_1, C_2, D, E, F, \varphi)$. There exist matrices $\widehat{M} \succ 0, K, W, X^{11}, X^{12}, X^{21}, X^{22}$, L_n, L_1, and Z of appropriate dimensions, and positive constant $\hat{\kappa}$, such that the following (in)equalities hold:

$$D = ZW,$$

$$\begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1+E) \\ Z^T\widehat{M} & 0 & 0 \\ (BL_1+E)^T\widehat{M} & 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^TX^{22}C_2 - \varPhi_{11} & C_2^TX^{21} & -\varPhi_{12} \\ X^{12}C_2 & X^{11} & 0 \\ -\varPhi_{12}^T & 0 & -\varPhi_{22} \end{bmatrix},$$

where

$$\Delta := (A+BK)^T\widehat{M} + \widehat{M}(A+BK),$$

$$\varPhi_{11} := (F+L_n)^TM_{11}(F+L_n), \tag{V.3}$$

$$\varPhi_{12} := (F+L_n)^TM_{11}D_k + (F+L_n)^TM_{12}, \tag{V.4}$$

$$\varPhi_{22} := D_k^TM_{11}D_k + M_{12}D_k + D_k^TM_{12} + M_{22}. \tag{V.5}$$

The next theorem provides the main result of this section.

Theorem 5.1: Let $\Sigma = (A, B, C_1, C_2, D, E, F, \varphi)$, and $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \varphi)$ with the same external output dimension. Suppose Assumptions 1 and 2 hold and there exist matrices Q, H, $\hat{W}$, L_2, and L_3 of appropriate dimensions such that:

$$AP = P\hat{A} - BQ \tag{V.6a}$$

$$C_1P = \hat{C}_1 \tag{V.6b}$$

$$C_2P = H\hat{C}_2 \tag{V.6c}$$

$$FP = \hat{F} \tag{V.6d}$$

$$E = B(L_2 - L_1) \tag{V.6e}$$

$$BL_3 = P\hat{E} \tag{V.6f}$$

$$P\hat{D} = Z\hat{W}. \tag{V.6g}$$

Then function V defined in (V.2) is a storage function from $\hat{\Sigma}$ to Σ, where X in Definition 2.1 is given by:

$$X = \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix}.$$

Proof: Note that from (V.6b), and for all $x \in \mathbb{R}^n$, and $\hat{x} \in \mathbb{R}^{\hat{n}}$, we have $\|C_1x - \hat{C}_1\hat{x}\|^2 = (x - P\hat{x})^TC_1^TC_1(x - P\hat{x})$. It can be readily verified that $\frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^TC_1)}\|C_1x - \hat{C}_1\hat{x}\|^2 \leq V(x, \hat{x})$ implying that inequality (III.1) holds with $\alpha(r) = \frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^TC_1)}r^2$ for any $r \in \mathbb{R}_{\geq 0}$, which is a $\mathcal{K}_\infty$ function. We proceed to prove inequality (III.2). By the definition of V, one has

$$\nabla V^T = \left[2(x - P\hat{x})^T\widehat{M} \quad -2(x - P\hat{x})^T\widehat{M}P\right].$$

For any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}$, one obtains:

$$\begin{aligned}\dot{V}(x, \hat{x}) &= 2(x - P\hat{x})^T\widehat{M}(Ax + E\varphi(t, Fx) + Bu + Dw) \\ &\quad - 2(x - P\hat{x})^T\widehat{M}P(\hat{A}\hat{x} + \hat{E}\varphi(t, \hat{F}\hat{x}) + \hat{B}\hat{u} + \hat{D}\hat{w}).\end{aligned}$$

Given any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}$, and $\hat{u} \in \mathbb{R}^{\hat{m}}$, we use the following *interface* function providing $u \in \mathbb{R}^m$:

$$\begin{aligned}u = K(x - P\hat{x}) + Q\hat{x} + \tilde{R}\hat{u} + L_1\varphi(t, Fx) - L_2\hat{\varphi} \\ + L_3\varphi(t, \hat{F}\hat{x}),\end{aligned} \tag{V.7}$$

where

$$\hat{\varphi} = \varsigma(t, \hat{F}\hat{x} + L_n(x - P\hat{x})),$$

and $\tilde{R}$ is a matrix of appropriate dimension. Using the interface function in (V.7), and the conditions (V.6a), (V.6d), (V.6e), (V.6f), and (V.6g), one obtains:

$$\begin{aligned}\dot{V}(x, \hat{x}) = 2(x - P\hat{x})^T\widehat{M}\Big(A(x - P\hat{x}) + BK(x - P\hat{x}) \\ + ZWw - Z\hat{W}\hat{w} + (B\tilde{R} - P\hat{B})\hat{u} + (BL_1 + E)\delta\varphi\Big),\end{aligned}$$

where $\delta\varphi = \varsigma(t, Fx) - \varsigma(t, Fx + (F + L_n)(x - P\hat{x}))$. Using Young's inequality [11], Cauchy-Schwarz inequality, (V.3), (V.4), (V.5), and (V.6c), one obtains the upper bound for $\dot{V}(x, \hat{x})$ in (V.8) for any positive $\pi < \hat{\kappa}$. Remark that we used the following inequality [10] for any $x \in \mathbb{R}^n$ and $\hat{x} \in \mathbb{R}^{\hat{n}}$ to show inequality (V.8):

$$\begin{bmatrix} x - P\hat{x} \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} F + L_n & D_k \\ 0_{l_k} & I_{l_k} \end{bmatrix}^T \tilde{M} \begin{bmatrix} F + L_n & D_k \\ 0_{l_k} & I_{l_k} \end{bmatrix} \begin{bmatrix} x - P\hat{x} \\ \delta\varphi \end{bmatrix} \geq 0.$$

Using the upper bound in (V.8), the inequality (III.2) is readily satisfied with $\eta(s) = (\hat{\kappa} - \pi)s$, and $\psi_{\text{ext}}(s) = \frac{\|\sqrt{\widehat{M}}(B\tilde{R} - P\hat{B})\|^2}{\pi}s^2, \forall s \in \mathbb{R}_{\geq 0}$, and matrix $X = \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix}$. ■

In the next section, we provide a practical example for compositional abstraction of an interconnected system with dynamic interconnection topology.

VI. Example

Consider n first order resistance-capacitance (R-C) circuits, interconnected via resistance-inductance (R-L) series branches. The i-th R-C circuit has the dynamics given by:

$$\dot{v}_{c_i} = -\frac{1}{R_iC_i}v_{c_i} + \frac{1}{R_iC_i}v_{s_i} + \frac{1}{C_i}\tilde{w}_i, \tag{VI.1}$$

where $i \in [1; n]$, $v_{s_i} \in \mathbb{R}$ represents the input source voltage (external input), $v_{c_i} \in \mathbb{R}$ the voltage across capacitor, C_i the capacitance, R_i the resistance, and $\tilde{w}_i \in \mathbb{R}$ the total current inflow from other R-L branches connected to the R-C circuit. Assuming identical values of the resistance and inductance in all R-L branches, one can write the dynamics of the total current inflow for the i-th R-C circuit as:

$$\dot{\tilde{w}}_i = a_{o_i}\tilde{w}_i + b_{o_i}v,$$

$$
\begin{aligned}
\dot V(x,\hat x) &= \begin{bmatrix} x-P\hat x \\ Ww-\hat W\hat w \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1+E) \\ Z^T\widehat{M} & 0 & 0 \\ (BL_1+E)^T\widehat{M} & 0 & 0 \end{bmatrix} \begin{bmatrix} x-P\hat x \\ Ww-\hat W\hat w \\ \delta\varphi \end{bmatrix}^T + 2(x-P\hat x)^T\widehat{M}(B\tilde R-P\hat B)\hat u \\
&\le \begin{bmatrix} x-P\hat x \\ Ww-\hat W\hat w \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} -\hat\kappa\widehat{M}+C_2^TX^{22}C_2-\Phi_{11} & C_2^TX^{21} & -\Phi_{12} \\ X^{12}C_2 & X^{11} & 0 \\ -\Phi_{12}^T & 0 & -\Phi_{22} \end{bmatrix} \begin{bmatrix} x-P\hat x \\ Ww-\hat W\hat w \\ \delta\varphi \end{bmatrix} + 2(x-P\hat x)^T\widehat{M}(B\tilde R-P\hat B)\hat u \\
&\le -(\hat\kappa-\pi)(x-P\hat x)^T\widehat{M}(x-P\hat x) + \frac{\|\sqrt{\widehat{M}}(B\tilde R-P\hat B)\|^2}{\pi}\|\hat u\|^2 \\
&\quad - \begin{bmatrix} x-P\hat x \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} F+L_n & D_k \\ 0_{l_k} & I_{l_k} \end{bmatrix}^T \tilde M \begin{bmatrix} F+L_n & D_k \\ 0_{l_k} & I_{l_k} \end{bmatrix} \begin{bmatrix} x-P\hat x \\ \delta\varphi \end{bmatrix} + \underbrace{\begin{bmatrix} Ww-\hat W\hat w \\ C_2x-H\hat C_2\hat x \end{bmatrix}}_{z:=}^T X \begin{bmatrix} Ww-\hat W\hat w \\ C_2x-H\hat C_2\hat x \end{bmatrix} \\
&\le -(\hat\kappa-\pi)(x-P\hat x)^T\widehat{M}(x-P\hat x) + \frac{\|\sqrt{\widehat{M}}(B\tilde R-P\hat B)\|^2}{\pi}\|\hat u\|^2 + z^TXz \\
&\le -\eta(V(x,\hat x)) + \psi_{\mathrm{ext}}(\|\hat u\|) + z^TXz. \qquad \text{(V.8)}
\end{aligned}
$$

where $v = [v_{c_1};\ldots;v_{c_n}]$, and $a_{o_i} \in \mathbb{R}$, and $b_{o_i} \in \mathbb{R}^{1\times n}$ represent the parameters of dynamics of the R-L series branch(es) connected to the i-th R-C circuit. We consider the above interconnected system as an interconnection of N concrete subsystems Σ_i, $i \in [1;N]$, wherein each subsystem Σ_i is formed by clustering n_i R-C circuits ($n_i \le n$). Each subsystem, $\Sigma_i = (A_i, B_i, C_{1i}, I_{ni}, I_{ni}, \vec{1}_{ni}, \vec{1}_{ni}^T, \varphi)$, generates a scalar (external) output. We also add a nonlinearity belonging to the class of nonlinearities presented in this paper. We have:

$$
\Sigma_i : \begin{cases} \dot\xi_i = A_i\xi_i + B_iu_i + D_iw_i + \vec{1}_{ni}\varphi(\vec{1}_{ni}^T\xi_i), \\ \zeta_{1i} = C_{1i}\xi_i, \\ \zeta_{2i} = \xi_i, \end{cases}
$$

where $\xi_i = \mathsf{L}_i v$, $\mathsf{L}_i := [e_{i1};\ldots;e_{in_i}]$, $e_{ij} \in \mathbb{R}^{1\times n}$ is a row vector whose k-th element is defined as

$$
e_{ij}^{(k)} = \begin{cases} 1 \text{ if } k\text{-th R-C circuit is part of the } i\text{-th cluster} \\ 0 \text{ otherwise}, \end{cases}
$$

$A_i, B_i, D_i \in \mathbb{R}^{n_i\times n_i}$ are readily obtained from (VI.1), $C_{1i} \in \mathbb{R}^{1\times n_i}$, $u_i = \mathsf{L}_i v_s$, $v_s = [v_{s_1};\ldots;v_{s_n}]$, $w_i = \mathsf{L}_i\tilde w$, $\tilde w = [\tilde w_1;\ldots;\tilde w_n]$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as

$$
\varphi(x) = \sin(x).
$$

The dynamic of the interconnection topology Σ_o is given by

$$
\Sigma_o : \begin{cases} \dot x_o = A_ox_o + B_ov \\ y_o = C_ox_o + D_ov, \end{cases}
$$

where $A_o = \mathsf{diag}(a_{o_1},\ldots,a_{o_n})$, $B_o = [b_{o_1};\ldots;b_{o_n}]$, $C_o = I_n$, and $D_o = 0_n$. We aggregate each Σ_i into a scalar abstraction $\hat\Sigma_i = (\hat A_i, \hat B_i, \hat C_{1i}, 1, 1, 1, 1, \varphi)$ given by the following dynamics

$$
\hat\Sigma_i : \begin{cases} \dot{\hat\xi}_i = \hat A_i\hat\xi_i + \hat B_i\hat u_i + \hat w_i + \varphi(\hat\xi_i), \\ \hat\zeta_{1i} = \hat C_{1i}\hat\xi_i, \\ \hat\zeta_{2i} = \hat\xi_i, \end{cases}
$$

where $\hat A_i$ satisfies $A_i\vec{1}_{n_i} = \vec{1}_{n_i}\hat A_i$, $\hat B_i$ is chosen arbitrarily (in this example we choose $\hat B_i = 1$), $\hat C_{1i} = C_{1i}\vec{1}_{n_i}$. For $R_j = C_j = 1$ in (VI.1), $\forall j \in [1;n]$, the function $V_i(x_i,\hat x_i) = (x_i - \vec{1}_{n_i}\hat x_i)^T(x_i - \vec{1}_{n_i}\hat x_i)$ (i.e. $\widehat{M}_i = I_{n_i}, P_i = \vec{1}_{n_i}$) is a storage function from $\hat\Sigma_i$ to Σ_i, with the following parameters

$$
\begin{aligned}
&K_i = -2I_{n_i}, Z_i = I_{n_i}, W_i = I_{n_i}, X_i^{11} = 0_{n_i}, \\
&X_i^{22} = -2I_{n_i}, X_i^{12} = X_i^{21} = I_{n_i}, \hat\kappa_i = 2, Q_i = 0_{n_i}, \\
&H_i = \hat W_i = \vec{1}_{n_i}, L_{ni} = 0,
\end{aligned}
$$

and with $\alpha_i(r) = \frac{1}{\lambda_{\max}(C_{1i}^TC_{1i})}r^2$, $\eta_i(r) = 2r$, $\psi_{i\mathrm{ext}}(r) = 0, \forall r \in \mathbb{R}_{\ge 0}$. The interface function in (V.7) is given here by

$$
u_i = -2(x_i - \vec{1}_{n_i}\hat x_i) + \vec{1}_{n_i}\hat u_i - \vec{1}_{n_i}\varphi(\vec{1}_{n_i}^Tx_i) + \vec{1}_{n_i}\varphi(\hat x_i).
$$

By selecting $\mu_1 = \cdots = \mu_N = 1$, the function $V(x, x_o, \hat x, \hat x_o) = \sum_{i=1}^N \mu_iV_i(x_i,\hat x_i) + (x_o - \Pi\hat x_o)^T(x_o - \Pi\hat x_o)$, where $\Pi = \mathsf{diag}(\vec{1}_{n_1},\ldots,\vec{1}_{n_N})$, is a simulation function from $\hat\Sigma$ to Σ, where $\hat\Sigma$ is the interconnection of the abstract subsystems with the dynamic interconnection topology $\hat\Sigma_o$ satisfying conditions (IV.2) and (IV.3). For this example, we choose $C_{1i} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$, and the dynamic interconnection system as follows

$$
\begin{aligned}
A_o &= -3I_n \\
B_o &= \begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 1 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \ldots & 1 & -2 \end{bmatrix}.
\end{aligned}
$$

For this dynamic interconnection, there always exists $\hat\Sigma_o$ satisfying conditions (IV.2) and (IV.3) for any even n. Using

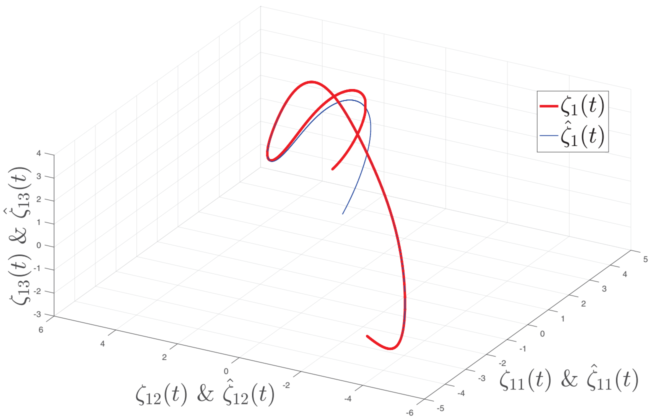

Fig. 1: Evolution of the external outputs of the concrete and abstract interconnected systems.

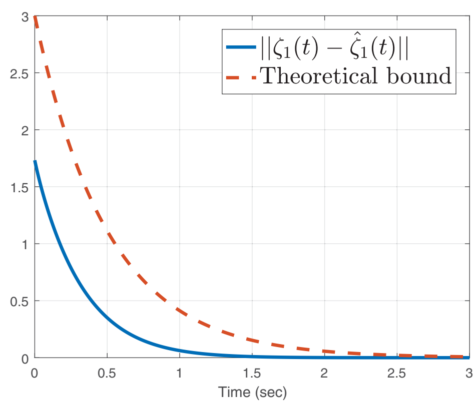

Fig. 2: The evolution of the norm of error, i.e. $\|\zeta_1(t) - \hat{\zeta}_1(t))\|$, along with the theoretical bound given in (III.5).

(IV.3), the abstract interconnection system is given by

$$\hat{A}_o = -3I_N, \hat{B}_o = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \hat{C}_o = I_N, \hat{D}_o = 0_N.$$

For the sake of simulation, we choose $N = 3, n = 150$, and $n_i = 50, \forall i \in [1; N]$. The simulation results are shown in Figures 1 and 2. The initial condition of ξ_i is chosen as $\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, while $\hat{\xi}_i$, x_o, and $\hat{x}_o$ are initialized from zero. The input to the abstract interconnected system $\hat{\Sigma}$ is chosen as $\hat{v}(t) = \begin{bmatrix} 10\cos(t) & 10\sin(2t) & 10\cos(4t) \end{bmatrix}^T$.

VII. Conclusion

In this paper, we derived conditions under which compositional abstractions of networks of control systems under dynamic interconnection topologies can be constructed using abstractions of components. The approximation errors are quantified using the notion of simulation function. For a class of nonlinear control systems, we derived a set of linear matrix (in)equalities facilitating the construction of their abstractions. Finally, we showed the effectiveness of

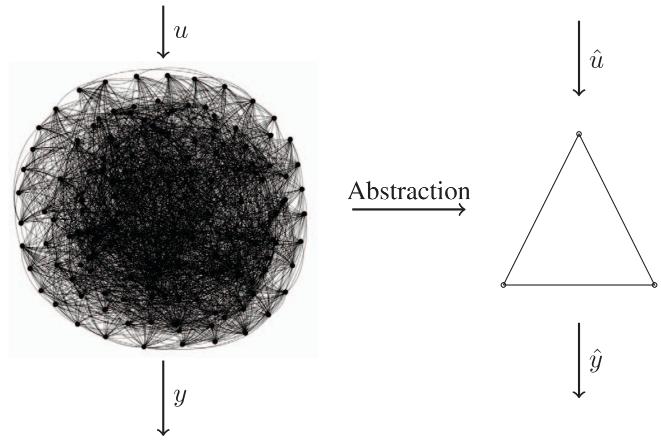

Fig. 3: 150-dimensional interconnected system is reduced to a 3-dimensional interconnected system.

the results on a 150-dimensional network of R-C circuits by reducing it to a 3-dimensional abstraction.

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