Linearly Convergent Variable Sample-Size Schemes for Stochastic Nash Games: Best-Response Schemes and Distributed Gradient-Response Schemes

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Abstract— This paper considers an N-player stochastic Nash game in which the *i*th player minimizes a composite objective $f_i(x) + r_i(x_i)$, where f_i is expectation-valued and r_i has an efficient prox-evaluation. In this context, we make the following contributions. (i) Under a strong monotonicity assumption on the concatenated gradient map, we derive (optimal) rate statements and oracle complexity bounds for the proposed variable sample-size proximal stochastic gradient-response (VS-PGR) scheme; (ii) We overlay (VS-PGR) with a consensus phase with a view towards developing distributed protocols for aggregative stochastic Nash games. Notably, when the samplesize and the number of consensus steps at each iteration grow at a suitable rate, a linear rate of convergence can be achieved; (iii) Finally, under a suitable contractive property associated with the proximal best-response (BR) map, we design a variable sample-size proximal BR (VS-PBR) scheme, where the proximal BR is computed by solving a sample-average problem. If the batch-size for computing the sample-average is raised at a suitable rate, we show that the resulting iterates converge at a linear rate and derive the oracle complexity.

I. INTRODUCTION

Noncooperative game theory [1], [2] is a branch of game theory that considers the resolution of conflicts among selfish players, each of which tries to optimize its payoff function, given the rivals' strategies. Nash games represent an important subclass of noncooperative games, originating from the seminal work by [3]. Such models have seen wide applicability in a breadth of engineered systems, such as power grids, communication networks, and sensor networks. In this paper, we consider the Nash equilibrium problem (NEP) with a finite set of N players indexed by i where $i \in \mathcal{N} \triangleq \{1, \dots, N\}$. For any $i \in \mathcal{N}$, the *i*th player is characterized by a strategy $x_i \in \mathbb{R}^{n_i}$ and a payoff function $F_i(x_i, x_{-i})$ dependent on its strategy x_i and parametrized by rivals' strategies $x_{-i} \triangleq \{x_j\}_{j \neq i}$. If $n \triangleq \sum_{i=1}^N n_i$ and xdenotes the strategy profile, defined as $x \triangleq (x_1, \cdots, x_N) \in$ \mathbb{R}^n . We consider a stochastic Nash game \mathcal{P} where the objective of player *i*, given rivals' strategies x_{-i} , is to solve the following stochastic composite optimization problem:

$$\min_{x_i \in \mathbb{R}^{n_i}} F_i(x_i, x_{-i}) \triangleq f_i(x_i, x_{-i}) + r_i(x_i) \qquad (\mathcal{P}_i(x_{-i}))$$

where $f_i(x) \triangleq \mathbb{E}[\psi_i(x;\xi(\omega))]$, the random variable ξ : $\Omega \to \mathbb{R}^d$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\psi_i : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is a scalar-valued function, and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure \mathbb{P} . We restrict our attention to nonsmooth convex Nash games where $f_i(x_i, x_{-i})$ is assumed to be smooth and convex in x_i for any x_{-i} while $r_i(x_i)$ is assumed to be convex but a possibly nonsmooth function with an efficient prox-evaluation. A Nash equilibrium (NE) of the stochastic Nash game in which the *i*th player solves the parametrized problem $(\mathcal{P}_i(x_{-i}))$ is a tuple $x^* = \{x_i^*\}_{i=1}^N \in \mathbb{R}^n$ such that the following holds for each player $i \in \mathcal{N}$:

$$F_i(x_i^*, x_{-i}^*) \le F_i(x_i, x_{-i}^*) \quad \forall x_i \in \mathbb{R}^{n_i}.$$

In other words, x^* is an NE if no player can improve the payoff by unilaterally deviating from the strategy x_i^* .

Our focus is two-fold: (i) Development of variable samplesize stochastic proximal gradient-response (PGR) and proximal best-response (PBR) schemes with **optimal (deterministic)** rates of convergence; (ii) Extension of PGR schemes to distributed (consensus-based) regimes, allowing for resolving aggregative games with a prescribed communication graph, where **linear** rates of convergence are achieved by combining increasing number of consensus steps with a growing sample-sizes of sampled gradients.

Prior research. We discuss some relevant research on continuous-strategy Nash games and variance reduction schemes for stochastic optimization.

(i) *Deterministic Nash games.* Early work considered convex Nash games (where players solve convex programs) where the concatenated gradient map is either strongly monotone [4] or merely monotone maps [5], [6]. While the aforementioned schemes utilized gradient-response techniques, best-response schemes reliant on the contractive nature of the best-response map were examined in [7].

(ii) *Stochastic Nash games.* Regularized stochastic approximation schemes were presented for monotone stochastic Nash games [8] while extensions have been developed to contend with misspecification [9] and the lack of Lipschitzian properties [10]. More recently, sampled bestresponse schemes have been developed in [11] while rate statements and iteration complexity bounds have been provided for a class of inexact stochastic best-response schemes in [12]–[14]. In fact, we draw inspiration from our work in [14] to develop superior rate statements and extensions to distributed regimes. Finally, a.s. and mean convergence of sequences produced by BR schemes was proven in [13], [15] for stochastic and misspecified potential games.

(iii) Consensus-based distributed schemes for Nash games. Inspired by the advances in consensus-based protocols for re-

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solving distributed optimization problems, Koshal et al. [16] developed two sets of distributed algorithms for monotone aggregative Nash games on graphs. More recently, in [17], [18], the authors combine gradient-based schemes with consensus protocols to address generalized Nash games.

(iv) Variance reduction schemes for stochastic optimization. There has been an effort to utilize increasing batchsizes of sampled gradients in stochastic gradient schemes, leading to improved rates of convergence, as seen in strongly convex [19]–[21] and convex regimes [20]–[23].

Novelty and Contributions.

(i). *VS-PGR*. In Section II, under a strong monotonicity assumption, we prove that a variable sample-size proximal gradient response (VS-PGR) scheme is characterized by a linear rate of convergence in mean-suqared error (Th. 1) while in Th. 2, we establish that the iteration complexity (in terms of proximal evaluations) and oracle complexity to achieve an ϵ -NE denoted by x where x satisfies $\mathbb{E}[||x - x^*||^2] \le \epsilon$ are $\mathcal{O}(\ln(1/\epsilon))$ and $\mathcal{O}((1/\epsilon)^{1+\delta})$, respectively, where $\delta \ge 0$ and $\delta = 0$ under a suitable selection of parameters. Furthermore, it is shown in Corollary 1 that with some specific algorithmic parameters, the iteration and oracle complexity to obtain an ϵ -NE are bounded by $\mathcal{O}(\kappa^2 \ln(1/\epsilon))$ and by $\mathcal{O}(\kappa^2/\epsilon)$, where κ denotes the condition number.

(ii). Distributed VS-PGR. In Section III, addressing an open question in stochastic Nash game, we design a distributed VS-PGR scheme to compute an equilibrium of an aggregative stochastic Nash game over a communication graph. By increasing the number of consensus steps and sample-size at each iteration, this scheme is characterized by a linear rate of convergence (Th. 3). In Th. 4, we show that the iteration, oracle, and communication complexity to compute an ϵ -Nash equilibrium are $\mathcal{O}(\ln(1/\epsilon))$, $\mathcal{O}((1/\epsilon)^{1+\delta})$, and $\mathcal{O}(\ln^2(1/\epsilon))$ respectively, where $\delta \geq 0$ and ϵ -NE₂ denotes an x satisfying $\mathbb{E}[||x - x^*||] \leq \epsilon$.

(iii). *VS-PBR*. In Section IV, we develop a variable sample-size proximal BR (VS-PBR) scheme (see Alg. 2) to solve a class of stochastic Nash games with contractive proximal BR maps, where each player solves a sample-average best-response problem per step. We show in Th. 5 that the generated iterates converge to the NE in mean at a linear rate under suitable number of scenarios, and also establish that the iteration and oracle complexity to achieve an ϵ -NE₂ are $\mathcal{O}((\ln(1/\epsilon))$ and $\mathcal{O}((1/\epsilon)^{2(1+\delta)})$ with $\delta \geq 0$.

Notation: When referring to a vector x, it is assumed to be a column vector while x^T denotes its transpose. Generally, ||x|| denotes the Euclidean vector norm, i.e., $||x|| = \sqrt{x^T x}$. We write *a.s.* as the abbreviation for "almost surely". For a real number x, we define by $\lceil x \rceil$ the smallest integer greater than x. For a closed convex function $r(\cdot)$, the proximal operator is defined in the following for any $\alpha > 0$:

$$\operatorname{prox}_{\alpha r}(x) \triangleq \operatorname{argmin}_{y} \left(r(y) + \frac{1}{2\alpha} \|y - x\|^2 \right).$$
(1)

For simplicity, ξ denotes $\xi(\omega)$ and in a slight abuse of notation, N denotes the number of players while N_k denotes the batch-size of sampled gradients at iteration k.

II. VARIABLE SAMPLE-SIZE GRADIENT RESPONSE

This section considers the development of a variable sample-size stochastic gradient response scheme for a class of strongly monotone Nash games associated with a strongly monotone concatenated gradient map. We proceed to show that this scheme produces a sequence of iterates that converges to the Nash equilibrium at a linear rate and establish the oracle complexity to achieve an ϵ -Nash equilibrium.

A. Variable sample-size proximal GR (VS-PGR)

We impose the following assumptions on \mathcal{P} .

Assumption 1: Let the following hold.

(a) The function r_i is lower semicontinuous and convex with effective domain denoted by $\mathcal{R}_i \triangleq \operatorname{dom}(r_i)$. Suppose $\mathcal{R} \triangleq \prod_{j=1}^N \mathcal{R}_i$ and $\mathcal{R}_{-i} = \prod_{j \neq i} \mathcal{R}_j$.

(b) $f_i(x_i, x_{-i})$ is C¹ and convex in x_i over on an open set containing \mathcal{R}_i for every fixed $x_{-i} \in \mathcal{R}_{-i}$;

(c) For all $x_{-i} \in \mathcal{R}_{-i}$ and any $\xi \in \mathbb{R}^d$, $\psi_i(x_i, x_{-i}; \xi)$ is differentiable in x_i over an open set containing \mathcal{R}_i such that $\nabla_{x_i} f_i(x_i, x_{-i}) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)].$

 $\nabla_{x_i} f_i(x_i, x_{-i}) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)].$ If $G(x; \xi) \triangleq (\nabla_{x_i} \psi_i(x; \xi))_{i=1}^N$ and $G(x) \triangleq \mathbb{E}[G(x; \xi)],$ then $G(x) = (\nabla_{x_i} f_i(x))_{i=1}^N$ by Assumption 1(iii). The following lemma establishes a tuple x^* is an NE of \mathcal{P} if and only if it is a fixed point of a suitable map.

Lemma 1 (Equivalence between NE and fixed point): Given the stochastic Nash game \mathcal{P} , suppose Assumption 1 holds for each player $i \in \mathcal{N}$. Define $r(x) \triangleq (r_i(x_i))_{i=1}^N$. Then $x^* \in X$ is an NE if and only if x^* is a fixed point of $\operatorname{prox}_{\alpha r}(x - \alpha G(x))$, i.e.,

$$x^* = \operatorname{prox}_{\alpha r}(x^* - \alpha G(x^*)), \quad \forall \alpha > 0.$$
(2)

Suppose the iteration index is denoted by k and player i's strategy at time k is denoted by $x_{i,k} \in \mathbb{R}^{n_i}$, which is an estimate of its equilibrium strategy x_i^* . We consider a variable sample-size generalization of the standard proximal stochastic gradient method, in which N_k sampled gradients are utilized at iteration k. Given a sample $\xi_k^1, \dots, \xi_k^{N_k}$ of N_k realizations of the random vector ξ , for any $i \in \mathcal{N}$, given $x_{i,0} \in \mathcal{R}_i$, player i updates $x_{i,k+1}$ as follows:

$$x_{i,k+1} = \operatorname{prox}_{\alpha r_i} \left[x_{i,k} - \alpha \frac{\sum_{p=1}^{N_k} \nabla_{x_i} \psi_i(x_k; \xi_k^p)}{N_k} \right],$$

where $\alpha > 0$ is the constant step size, $\nabla_{x_i}\psi_i(x_k;\xi_k^p)$, $p = 1, \cdots, N_k$ denote the sampled gradients. Define $w_k^p \triangleq G(x_k;\xi_k^p) - G(x_k)$, and $\bar{w}_{k,N_k} \triangleq \frac{1}{N_k}\sum_{p=1}^{N_k} w_k^p$. Then the aforementioned scheme can be expressed as

$$x_{k+1} = \operatorname{prox}_{\alpha r} \left[x_k - \alpha \left(G(x_k) + \bar{w}_{k,N_k} \right) \right]. \quad (\text{VS-PGR})$$

We make the following assumptions on the gradient map and the noise.

Assumption 2: (i) The mapping G(x) is Lipschitz continuous over the set \mathcal{R} with a constant L, namely,

$$||G(x) - G(y)|| \le L||x - y|| \quad \forall x, y \in \mathcal{R}.$$

(ii) G(x) is strongly monotone with parameter η , i.e.,

$$(G(x) - G(y))^T (x - y) \ge \eta ||x - y||^2 \quad \forall x, y \in \mathcal{R}.$$

(iii) There exists a constant ν such that the following holds for any $k \ge 0$: $\mathbb{E}[\bar{w}_{k,N_k} | \mathcal{F}_k] = 0$, $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 | \mathcal{F}_k] \le \nu^2/N_k$ a.s., where $\mathcal{F}_k \triangleq \sigma\{x_0, x_1, \cdots, x_k\}$.

B. Rate analysis

We begin with a simple recursion for the conditional mean squared error in terms of sample size N_k , step size α , and the problem parameters.

Lemma 2: Consider (VS-PGR) and let Assumptions 1 and 2 hold. Define $q \triangleq 1 - 2\alpha\eta + \alpha^2 L^2$. Then for all $k \ge 0$,

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 | \mathcal{F}_k] \le q \|x_k - x^*\|^2 + \alpha^2 \nu^2 / N_k, \quad a.s.$$

Using Lemma 2, we are able to show the linear convergence rate of algorithm (VS-PGR).

Theorem 1 (Linear convergence rate of VS-PGR): Let (VS-PGR) be applied to \mathcal{P} , where $N_k = \lceil \rho^{-(k+1)} \rceil$ for some $\rho \in (0, 1)$, and $\mathbb{E}[||x_0 - x^*||^2] \leq C$ for some constant C > 0. Suppose Assumptions 1 and 2 hold, and $\alpha < 2\eta/L^2$. Define $q \triangleq 1 - 2\alpha\eta + \alpha^2 L^2$. Then the following holds for any $k \geq 0$.

(i) If $\rho \neq q$, then $\mathbb{E}[||x_k - x^*||^2] \leq C(\rho, q) \max\{\rho, q\}^k$, where $C(\rho, q) \triangleq C + \frac{\alpha^2 \nu^2}{1 - \min\{\rho/q, q/\rho\}}$. (ii) If $\rho = q$, then for any $\tilde{\rho} \in (\rho, 1)$. $\mathbb{E}[||x_k - x^*||^2] \leq \widetilde{D} \tilde{\rho} k$.

(ii) If $\rho = q$, then for any $\tilde{\rho} \in (\rho, 1)$, $\mathbb{E}[||x_k - x^*||^2] \leq \widetilde{D}\tilde{\rho}^k$, where $\widetilde{D} \triangleq \left(C + \frac{\alpha^2 \nu^2}{\ln((\tilde{\rho}/\rho)^e)}\right)$.

C. Iteration and Oracle Complexity

Next, we examine the iteration (in terms of proximal evaluations) and oracle complexity of this scheme to compute an ϵ -Nash equilibrium, defined next. Recall that a random strategy profile $x: \Omega \to \mathbb{R}^n$ is an ϵ -NE if $\mathbb{E}[||x-x^*||^2] \leq \epsilon$.

Theorem 2 (Iteration and Oracle Complexity): Let (VS-PGR) be applied to \mathcal{P} , where $N_k = \lceil \rho^{-(k+1)} \rceil$ for some $\rho \in (0,1)$, and $\mathbb{E}[||x_0 - x^*||^2] \leq C$. Suppose Assumptions 1 and 2 hold. Define $q \triangleq 1 - 2\alpha\eta + \alpha^2 L^2$. Let $\alpha < 2\eta/L^2$, $\tilde{\rho} \in (\rho, 1)$, $C(\rho, q)$ and \tilde{D} be defined in Theorem 1. Then the number of proximal evaluations needed to obtain an ϵ -NE is bounded by $K(\epsilon)$, defined as

$$K(\epsilon) \triangleq \begin{cases} \frac{1}{\ln(1/q)} \ln\left(\frac{C(\rho,q)}{\epsilon}\right) & \text{if } \rho < q < 1, \\ \frac{1}{\ln(1/\tilde{\rho})} \ln\left(\frac{\tilde{D}}{\epsilon}\right) & \text{if } q = \rho, \\ \frac{1}{\ln(1/\rho)} \ln\left(\frac{C(\rho,q)}{\epsilon}\right) & \text{if } q < \rho < 1, \end{cases}$$
(3)

and the number of sampled gradients required is bounded by $M(\epsilon)$, defined as

$$M(\epsilon) \triangleq \begin{cases} \frac{1}{\rho \ln(1/\rho)} \left(\frac{C(\rho,q)}{\epsilon}\right)^{\frac{\ln(1/\rho)}{\ln(1/q)}} + K(\epsilon) \text{ if } \rho < q < 1, \\ \frac{1}{\rho \ln(1/\rho)} \left(\frac{\tilde{D}}{\epsilon}\right)^{\frac{\ln(1/\rho)}{\ln(1/\rho)}} + K(\epsilon) \text{ if } q = \rho \\ \frac{1}{\rho \ln(1/\rho)} \left(\frac{C(\rho,q)}{\epsilon}\right) + K(\epsilon) \text{ if } q < \rho < 1. \end{cases}$$

$$(4)$$

Proof. We first consider the case $\rho \neq q$. By Theorem 1(i), the following holds:

$$\mathbb{E}[\|x_k - x^*\|^2] \le \epsilon \implies k \ge K_1(\epsilon) \triangleq \frac{\ln\left(C(\rho, q)/\epsilon\right)}{\ln\left(1/\max\{\rho, q\}\right)}.$$

Then we achieve the bound given in equation (3) for cases $\rho < q < 1$ and $q < \rho < 1$. Note that for $\lambda > 1$ and any positive integer K, the following holds:

$$\sum_{k=0}^{K} \lambda^k \le \int_0^{K+1} \lambda^x dx \le \frac{\lambda^{K+1}}{\ln(\lambda)}.$$
 (5)

Then we may obtain the following bound:

$$\sum_{k=0}^{K_1(\epsilon)-1} N_k \le \sum_{k=0}^{K_1(\epsilon)-1} \rho^{-(k+1)} + K_1(\epsilon) \le \frac{\rho^{-K_1(\epsilon)}}{\rho \ln(1/\rho)} + K_1(\epsilon)$$

Note that for any $0 < \epsilon, p < 1, c_1 > 0$, the following holds:

$$\rho^{-\frac{\ln(c_1/\epsilon)}{\ln(1/p)}} = \left(e^{\ln(\rho^{-1})}\right)^{\frac{\ln(c_1/\epsilon)}{\ln(1/p)}} = e^{\ln(c_1/\epsilon))^{\frac{\ln(1/\rho)}{\ln(1/p)}}} = (c_1/\epsilon)^{\frac{\ln(1/\rho)}{\ln(1/p)}}.$$
(6)

Thus, the number of sampled gradients required to obtain an ϵ -NE is bounded by

$$\frac{1}{\rho \ln(1/\rho)} \left(\frac{C(\rho,q)}{\epsilon}\right)^{\frac{\ln(1/\rho)}{\ln(1/\max\{\rho,q\})}} + K_1(\epsilon).$$

Thus, we achieve the bound given in equation (4) for cases $\rho < q < 1$ and $q < \rho < 1$. The resultd for the case $\rho = q$ can be similarly proved.

Remark 1: (i) The above theorem establishes that the iteration and oracle complexity to achieve an ϵ -NE are $\mathcal{O}(\ln(1/\epsilon))$ and $\mathcal{O}((1/\epsilon)^{1+\delta})$, where $\delta = 0$ when $\rho \in (p, 1)$, $\delta = \frac{\ln(q/\rho)}{\ln(1/q)}$ when $\rho < q < 1$, and $\delta = \frac{\ln(\bar{\rho}/\rho)}{\ln(1/\bar{\rho})}$ when $q = \rho$. (ii) Suppose we use an alternative metric to describe the ϵ -Nash equilibrium: $x : \Omega \to \mathbb{R}^n$ is an ϵ -NE₂ if $\mathbb{E}[\|x - x^*\|] \leq \epsilon$. Then by Jensen's inequality if follows that an ϵ^2 -NE is also an ϵ -NE₂, and hence by Theorem 2 we obtain that the iteration and oracle complexity to achieve an ϵ -NE₂ are $\mathcal{O}(\ln(1/\epsilon))$ and $\mathcal{O}((1/\epsilon)^{2(1+\delta)})$, respectively.

Corollary 1: Let (VS-PGR) be applied to \mathcal{P} , where $\mathbb{E}[||x_0 - x^*||^2] \leq C$ for some constant C > 0. Suppose Assumptions 1 and 2 hold. Define the condition number $\kappa \triangleq \frac{1}{\eta}$. Set $\alpha = \frac{\eta}{L^2}$ and $N_k = \lceil \rho^{-(k+1)} \rceil$ with $\rho = 1 - \frac{1}{2\kappa^2}$. Then the number of proximal evaluations and samples required to obtain an ϵ -NE are bounded by $\mathcal{O}(\kappa^2 \ln(1/\epsilon))$ and by $\mathcal{O}(\kappa^2/\epsilon)$, respectively.

III. DISTRIBUTED VS-PGR FOR AGGREGATIVE GAMES

Next, we consider an aggregative game \mathcal{P}^{agg} , where the *i*th player solves the following parametrized problem:

$$\min_{c_i \in \mathbb{R}^{n_i}} F_i^{\text{agg}}(x_i, x_{-i}), \qquad (\mathcal{P}_i^{\text{agg}}(x_{-i}))$$

where $F_i^{\text{agg}}(x_i, x_{-i}) \triangleq f_i(x_i, x_i + \bar{x}_{-i}) + r_i(x_i), \bar{x} \triangleq \sum_{i=1}^N x_i$ denotes the aggregate of all players' decisions, $\bar{x}_{-i} = \sum_{j=1, j \neq i}^N x_j$ denotes the aggregate of all players' decisions except player *i*, and $f_i(x_i, x_i + \bar{x}_{-i}) \triangleq \mathbb{E} [\psi_i(x_i, x_i + \bar{x}_{-i}; \xi)]$ is expectation valued. We impose the following assumptions on the stochastic aggregative game. Assumption 3: Let the following hold.

(a) The function r_i is lower semicontinuous and convex with

effective domain denoted by \mathcal{R}_i required to be compact. (b) For any $y \in \overline{\mathcal{R}} \triangleq \sum_{i=1}^N \mathcal{R}_i$, $f_i(x_i, y)$ is C^1 and convex in x_i over an open set containing \mathcal{R}_i .

(c) For all $y \in \overline{\mathcal{R}}$ and any $\xi \in \mathbb{R}^d$, $\psi_i(x_i, y; \xi)$ is differentiable in x_i over an open set containing \mathcal{R}_i s.t. $\nabla_{x_i} f_i(x_i, y) = \mathbb{E}[\nabla_{x_i} \psi_i(x_i, y; \xi)].$

A. Algorithm Design

We aim to design a distributed algorithm to compute an NE of \mathcal{P}^{agg} , where each player may exchange information with its local neighbors, and subsequently update its estimate of the equilibrium strategy and the aggregate. The communication among players is defined by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} \triangleq \{1, \ldots, N\}$ is the set of players and \mathcal{E} is the set of undirected edges between players. The set of neighbors of player *i*, denoted \mathcal{N}_i , is defined as $\mathcal{N}_i \triangleq \{j \in \mathcal{N} : (i, j) \in \mathcal{E}\}$. Define the adjacency matrix $A = [a_{ij}]_{i,j=1}^N$, where $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$, otherwise. A path in \mathcal{G} with length p from v_1 to v_{p+1} is a sequence of distinct nodes, $v_1v_2 \ldots v_{p+1}$, such that $(v_m, v_{m+1}) \in \mathcal{E}$, for all $m = 1, \ldots, p$. The graph \mathcal{G} is termed connected if there is a path between any two distinct players.

Though each player does not have access to all players' decisions, it may estimate the aggregate \bar{x} by communicating with its neighbors. Player *i* at time *k* holds an estimate $x_{i,k}$ for its equilibrium strategy and an estimate $v_{i,k}$ for the average of the aggregate. To overcome the fact that the communication network is sparse, we assume that to compute $v_{i,k+1}$, players communicate not once but τ_k rounds at major iteration k + 1. The strategy of each player is updated by a variable sample-size proximal stochastic gradient scheme that depends on parameters α and N_k , similar to (VS-PGR) developed in Section II. We now specify the scheme in Algorithm 1.

Algorithm 1 Distrib. VS-PGR for Agg. Stoch. Nash Games *Initialize:* Set k = 0, and $v_{i,0} = x_{i,0} \in \mathcal{R}_i$ for any $i \in \mathcal{N}$. Let $\alpha > 0$ and $\{\tau_k, N_k\}$ be deterministic sequences. *Iterate until convergence*

Consensus. $\hat{v}_{i,k} := v_{i,k} \ \forall i \in \mathcal{N}$ and repeat τ_k times

$$\hat{v}_{i,k} := \sum_{j \in \mathcal{N}_i} a_{ij} \hat{v}_{j,k} \quad \forall i \in \mathcal{N}.$$

Strategy Update. for any $i \in \mathcal{N}$

$$\begin{aligned} x_{i,k+1} &:= \operatorname{prox}_{\alpha r_i} \left[x_{i,k} - \alpha \left(\nabla_{x_i} f_i(x_{i,k}, N \hat{v}_{i,k}) + e_{i,k} \right) \right], \quad (7) \\ v_{i,k+1} &:= v_{i,k} + x_{i,k+1} - x_{i,k}. \end{aligned}$$

$$v_{i,k+1} := v_{i,k} + x_{i,k+1} - x_{i,k},$$
(8)

where $e_{i,k} \triangleq \frac{\sum_{p=1}^{k} \sum_{i,k \in \mathcal{I}} \sum_{i,k \in \mathcal{I}} \sum_{i,k \in \mathcal{I}} \sum_{i,k \in \mathcal{I}} -\nabla_{x_i} f_i(x_{i,k}, N\hat{v}_{i,k}).$

We impose the following conditions on the communication graph, gradient mapping, and observation noise.

Assumption 4: (i) The undirected graph \mathcal{G} is connected and the associated adjacency matrix A is symmetric with row sums equal to one.

(ii) $\phi(x) \triangleq (\nabla_{x_i} f_i(x_i, \bar{x}))_{i=1}^N$ is strongly monotone over \mathcal{R} with parameter η , i.e., $(\phi(x) - \phi(y))^T (x - y) \ge \eta ||x - y||^2 \quad \forall x, y \in \mathcal{R}.$ (iii) The mapping $\phi(x)$ is Lipschitz continuous over \mathcal{R} with a constant L, i.e., $\|\phi(x) - \phi(y)\| \le L \|x - y\| \quad \forall x, y \in \mathcal{R}$. (iv) For any $i \in \mathcal{N}, \nabla_{x_i} f_i(x_i, y)$ is Lipschitz continuous in y over the set $\overline{\mathcal{R}}$ for every fixed $x_i \in \mathcal{R}_i$, i.e., there exists some constant L_i such that for any $x_i \in \mathcal{R}_i$,

 $\begin{aligned} \|\nabla_{x_i} f_i(x_i, y_1) - \nabla_{x_i} f_i(x_i, y_2)\| &\leq L_i \|y_1 - y_2\| \quad \forall y_1, y_2 \in \bar{\mathcal{R}}. \end{aligned}$ (v) If $\mathcal{F}_k \triangleq \sigma\{x_0, x_1, \cdots, x_k\}$, for any $i \in \mathcal{N}$, there exists a constant ν_i such that the following holds for any $k \geq 0$:

$$\mathbb{E}[e_{i,k}|\mathcal{F}_k] = 0 \quad \text{and} \quad \mathbb{E}[\|e_{i,k}\|^2|\mathcal{F}_k] \le \nu_i^2/N_k \quad a.s.$$

B. Convergence Analysis

Define $A(k) \triangleq A^{\tau_k}$. Then by Assumption 4(a), A(k) is also symmetric with the sum of each row equaling one. Note from the consensus step in Algorithm 1 that $\hat{v}_{i,k} = \sum_{j=1}^{N} [A(k)]_{ij} v_{j,k}$. We now recall a prior result.

Lemma 3: By Assumption 4 and [24, Proposition 1], there exists a constant $\theta > 0$ and $\beta \in (0, 1)$ such that

$$\left| \left[A^k \right]_{ij} - \frac{1}{N} \right| \le \theta \beta^k \quad \forall i, j \in \mathcal{N}.$$
(9)

We introduce the transition matrices $\Phi(k, s)$ from time s to time $k \ge s$ as follows:

$$\Phi(k,s) = A(k)A(k-1)\cdots A(s), \forall \ 0 \le s < k,$$

where $\Phi(k,k) = A(k)$. We then obtain the following recursion on the mean-squared error.

Proposition 1: Consider Algorithm 1, where $\tau_k = k + 1$ and $N_k = \lceil \beta^{-(k+1)/2} \rceil$. Define $M \triangleq \sum_{j=1}^N \max_{x_j \in \mathcal{R}_j} ||x_j||$, $C_1 \triangleq M\theta \left(1 + 2e\sqrt{1/\ln(\beta^{-1/2})} \right)$, $C_2 \triangleq \frac{4M\theta}{\ln(1/\beta)}$, and $\varrho \triangleq \left(1 - 2\alpha\eta + 2\alpha^2 L^2 \right)$, where θ and β are defined in Lemma 3. Let Assumptions 3 and 4 hold. Then for any $k \ge 0$,

$$\mathbb{E}[\|x_{k+1} - x^*\|^2] \le \varrho \mathbb{E}[\|x_k - x^*\|^2] + C_3 \beta^{(k+1)/2}, \quad (10)$$

where C_3 is defined as

$$C_{3} \triangleq \alpha^{2} \sum_{i=1}^{N} \nu_{i}^{2} + 4\alpha MN \left(C_{1}\beta^{1/2} + C_{2} \right) \sum_{i=1}^{N} L_{i} + 4\alpha^{2} N^{2} \left(C_{1}^{2}\beta^{3/2} + C_{2}^{2}\beta^{1/2} \right) \sum_{i=1}^{N} L_{i}^{2}.$$
(11)

Proof. For purposes of brevity, we merely outline the proof. Firstly, we give a recursion on the conditional mean-squared error as follows: N

$$\mathbb{E}[\|x_{k+1} - x^*\|^2 |\mathcal{F}_k] \le \varrho \|x_k - x^*\|^2 + \alpha^2 \sum_{i=1}^N \nu_i^2 / N_k + 4\alpha M N \sum_{i=1}^N L_i \|\hat{v}_{i,k} - y_k\| + 2\alpha^2 N^2 \sum_{i=1}^N L_i^2 \|\hat{v}_{i,k} - y_k\|^2.$$

We then establish an upper bound on the consensus error:

$$\|y_k - \hat{v}_{i,k}\| \le M\theta\beta^{\sum_{p=0}^k \tau_p} + 2M\theta\sum_{s=1}^n \beta^{\sum_{p=s}^k \tau_p}.$$

Finally, by getting an upper bound on $\sum_{s=1}^{k} \beta^{\sum_{p=s}^{k} \tau_p} \leq e\sqrt{1/\ln(\beta^{-1/2})}\beta^{(k+1)(k+2)/2} + \frac{2\beta^{(k+1)/2}}{(k+1)\ln(1/\beta)}$ and taking the unconditional expectation, we prove the result. \Box

Based on Prop. 1, we can show the linear rate of convergence of Algorithm 1.

Theorem 3 (Linear convergence rate of Algorithm 1): Suppose Assumptions 3 and 4 hold. Consider Algorithm 1, where $\tau_k = k+1$, $N_k = \lceil \beta^{-(k+1)/2} \rceil$ and $\mathbb{E}[||x_0 - x^*||^2] \leq C$ for some C > 0. Let $\alpha \in (0, \eta/L^2)$ and define $\varrho \triangleq (1 - 2\alpha\eta + 2\alpha^2L^2)$. Then we have the following assertions for any $k \geq 0$. (i) If $\beta \neq \varrho^2$, then $\mathbb{E}[||x_k - x^*||^2] \leq \widetilde{C}(\varrho, \beta) \max\{\varrho, \sqrt{\beta}\}^k$, where $\widetilde{C}(\varrho, \beta) \triangleq C + \frac{C_3}{1 - \min\{\varrho/\sqrt{\beta}, \sqrt{\beta}/\varrho\}}$ with C_3 defined in Proposition 1. (ii) If $\beta = \varrho^2$, then for any $\widetilde{\varrho} \in (\varrho, 1)$, $\mathbb{E}[||x_k - x^*||^2] \leq \widetilde{D}(\varrho)\widetilde{\varrho}^k$, where $\widetilde{D}(\varrho) \triangleq \left(C + \frac{C_3}{\ln((\widetilde{\varrho}/\varrho)^e)}\right)$. Similar to Theorem 2, we may derive bounds on the

Similar to Theorem 2, we may derive bounds on the iteration and oracle complexity as well as the communication complexity to compute an ϵ -Nash equilibrium.

Theorem 4: Suppose Assumptions 3 and 4 hold. Consider Algorithm 1, where $\tau_k = k + 1$, $N_k = \lceil \beta^{-(k+1)/2} \rceil$ and $\mathbb{E}[\|x_0 - x^*\|^2] \leq C$ for some constant C > 0. Let $\alpha \in (0, \eta/L^2)$ and define $\varrho \triangleq (1 - 2\alpha\eta + 2\alpha^2L^2)$. Let $\tilde{\varrho} \in (\varrho, 1), \ \tilde{C}(\varrho, \beta)$ and $\tilde{D}(\varrho)$ be defined in Theorem 3. Then the number of proximal evaluations needed to obtain an ϵ -NE is bounded as follows:

$$K(\epsilon) \triangleq \begin{cases} \frac{1}{\ln(1/\varrho)} \ln\left(\frac{\widetilde{C}(\varrho,\beta)}{\epsilon}\right) & \text{if } \beta < \varrho^2 < 1, \\ \frac{1}{\ln(1/\widetilde{\varrho})} \ln\left(\frac{\widetilde{D}(\varrho)}{\epsilon}\right) & \text{if } \beta = \varrho^2, \\ \frac{1}{\ln(1/\beta^{1/2})} \ln\left(\frac{\widetilde{C}(\varrho,\beta)}{\epsilon}\right) & \text{if } \varrho < \beta^{1/2} < 1, \end{cases}$$

and the round of communications is $\frac{(K(\epsilon)+1)(K(\epsilon)+2)}{2}$, and the number of sampled gradients required is bounded by

$$M(\epsilon) \triangleq \begin{cases} \frac{\left(\frac{\tilde{C}(\varrho,\beta)}{\epsilon}\right)^{\frac{\ln(1/\beta^{1/2})}{\ln(1/\varrho)}}}{\beta^{1/2}\ln((1/\beta^{1/2}))} + K(\epsilon) & \text{if } \beta < \varrho^2 < 1, \\ \frac{\left(\frac{\tilde{D}(\varrho)}{\epsilon}\right)^{\frac{\ln(1/\varrho)}{\ell}}}{\beta^{1/2}\ln((1/\beta^{1/2}))} + K(\epsilon) & \text{if } \beta = \varrho^2, \\ \frac{\left(\frac{\tilde{C}(\varrho,\beta)}{\epsilon}\right)}{\beta^{1/2}\ln((1/\beta^{1/2}))} + K(\epsilon) & \text{if } \varrho < \beta^{1/2} < 1 \end{cases}$$

Remark 2: Recall that in [25], a fast distributed gradient algorithm based on Nesterov's accelerated gradient algorithm is employed to solve a distributed convex optimization problem, where at each step, $\mathcal{O}(\ln(k))$ consensus steps are taken. In [25], the authors show that in merely convex settings, the rate is $\mathcal{O}(1/k^2)$ (optimal) and total number of communications rounds is $\mathcal{O}(k \ln(k))$ up to time k. Our scheme (Algorithm 1) requires $\mathcal{O}(k^2)$ rounds of communications to recover the optimal linear rate of convergence but does so in a stochastic game-theoretic regime. In future work, we intend to investigate how the number of consensus steps may be chosen to maintain geometric convergence while reducing communication overhead.

IV. VARIABLE SAMPLE-SIZE PROX. BEST RESPONSE

In this section, we consider the class of stochastic Nash games in which the proximal BR map is contractive [26]. We propose a variable sample size proximal BR scheme for computing an equilibrium, and derive rate statements and establish iteration and oracle complexity bounds.

A. Background on proximal best-reponse maps

For any $i \in \mathcal{N}$ and any tuple $y \in \mathbb{R}^n$, define the proximal BR map $\hat{x}_i(y)$ as $\hat{x}_i(y) \triangleq \operatorname{argmin}_{x_i \in \mathbb{R}^{n_i}} \left[\mathbb{E} \left[\psi_i(x_i, y_{-i}; \xi) \right] + r_i(x_i) + \frac{\mu}{2} \|x_i - y_i\|^2 \right].$ We impose the following assumption on problem $(\mathcal{P}_i(x_{-i}))$.

Assumption 5: (i) Assumption 3(i). (ii) For every fixed $x_{-i} \in \mathcal{R}_{-i}$, $f_i(x_i, x_{-i})$ is C² and convex in x_i over on an open set containing \mathcal{R}_i . Moreover, $\nabla_{x_i} f_i(x_i, x_{-i})$ is assumed to be Lipschitz continuous in x_i uniformly in x_{-i} with constant L_i , i.e., for any $x_i, x'_i \in \mathcal{R}_i$,

$$\|\nabla_{x_i} f_i(x_i, x_{-i}) - \nabla_{x_i} f_i(x'_i, x_{-i})\| \le L_i \|x_i - x'_i\|.$$

(iii) For all $x_{-i} \in \mathcal{R}_{-i}$ and any $\xi \in \mathbb{R}^d$, $\psi_i(x_i, x_{-i}; \xi)$ is differentiable in x_i over an open set containing \mathcal{R}_i . Moreover, for any $i \in \mathcal{N}$ and all $x \in \mathcal{R}$, there exists $M_i > 0$ such that $\mathbb{E}[\|\nabla_{x_i} f_i(x) - \nabla_{x_i} \psi_i(x_i, x_{-i}; \xi)\|^2] \leq M_i^2$.

By Assumption 5, the second derivatives of the functions $f_i, \forall i \in \mathcal{N}$ on \mathcal{R} are bounded. Analogous to the avenue adopted in [26], we may define

$$\Gamma \triangleq \begin{pmatrix} \frac{\mu}{\mu+\zeta_{1,\min}} & \frac{\zeta_{12,\max}}{\mu+\zeta_{1,\min}} & \cdots & \frac{\zeta_{1N,\max}}{\mu+\zeta_{1,\min}} \\ \frac{\zeta_{21,\max}}{\mu+\zeta_{2,\min}} & \frac{\mu}{\mu+\zeta_{2,\min}} & \cdots & \frac{\zeta_{2N,\max}}{\mu+\zeta_{2,\min}} \\ \vdots & & \ddots & \\ \frac{\zeta_{N1,\max}}{\mu+\zeta_{N,\min}} & \frac{\zeta_{N2,\max}}{\mu+\zeta_{N,\min}} & \cdots & \frac{\mu}{\mu+\zeta_{N,\min}} \end{pmatrix}$$
(12)

where $\zeta_{i,\min} \triangleq \inf_{x \in X} \lambda_{\min} \left(\nabla_{x_i}^2 f_i(x) \right)$ and $\zeta_{ij,\max} \triangleq \sup_{x \in X} \| \nabla_{x_i x_j}^2 f_i(x) \| \quad \forall j \neq i$. Then by [11, Theorem 4] we obtain that

$$\begin{pmatrix} \|\widehat{x}_{1}(y') - \widehat{x}_{1}(y)\| \\ \vdots \\ \|\widehat{x}_{N}(y') - \widehat{x}_{N}(y)\| \end{pmatrix} \leq \Gamma \begin{pmatrix} \|y'_{1} - y_{1}\| \\ \vdots \\ \|y'_{N} - y_{N}\| \end{pmatrix}.$$
(13)

If the spectral radius $\rho(\Gamma) < 1$, then the proximal bestresponse map is contractive w.r.t. some monotonic norm. These sufficient conditions for the contractive property of the BR map $\hat{x}(\bullet)$ can be found in [11], [26].

B. Variable sample-size proximal BR scheme

Suppose at iteration k, we have N_k samples $\xi_k^1, \dots, \xi_k^{N_k}$ of the random vector ξ . For any $x_i \in X_i$, we approximate $f_i(x_i, y_{-i,k})$ by $\frac{1}{N_k} \sum_{p=1}^{N_k} \psi_i(x_i, y_{-i,k}; \xi_k^p)$ and solve the sample-average best-response problem (14). We then get the variable-size proximal BR scheme (Algorithm 2).

Algorithm 2 Variable-size proximal BR scheme

Set k := 0. Let $y_{i,0} = x_{i,0} \in X_i$, and $\{\gamma_{i,k}\}_{k\geq 0}$ be a given deterministic sequence for i = 1, ..., N.

(1) For i = 1, ..., N, player *i* updates estimate $x_{i,k+1}$ as

$$x_{i,k+1} = \underset{x_{i} \in \mathbb{R}^{n_{i}}}{\operatorname{argmin}} \left[\frac{1}{N_{k}} \sum_{p=1}^{N_{k}} \psi_{i}(x_{i}, y_{-i,k}; \xi_{k}^{p}) + r_{i}(x_{i}) + \frac{\mu}{2} \|x_{i} - y_{i,k}\|^{2} \right].$$
(14)

(2) For i = 1, ..., N, $y_{i,k+1} := x_{i,k+1}$; (3) Set k := k + 1 and return to (1).

C. Oracle and iteration complexity

Define $\varepsilon_{i,k+1} \triangleq x_{i,k+1} - \widehat{x}_i(y_k)$. We may obtain an bound on $\mathbb{E}[\|\varepsilon_{i,k+1}\|^2]$ in the following lemma.

Lemma 4: Suppose Assumption 5 holds. Consider Algorithm 2. Then $\mathbb{E}[\|\varepsilon_{i,k+1}\|^2] \leq \frac{M_i^2 C_r^2}{N_k}$, where $C_r \triangleq \frac{\mu}{\mu^2 + L^2} (1 - L/\sqrt{\mu^2 + L^2})^{-1}$ with $L \triangleq \max_i L_i$. **Proof.** Define $\bar{w}_{i,k}(x_i) \triangleq \frac{1}{N_k} \sum_{p=1}^{N_k} \nabla_{x_i} \psi_i(x_i, y_{-i,k}; \xi_k^p) - \nabla_{x_i} f_i(x_i, y_{-i,k})$. By the optimality condition, $x_{i,k+1}$ and $\hat{x}_i(y_k)$ are respectively a fixed point of the map $\operatorname{prox}_{\alpha r_i} [x_i - \alpha \tilde{f}_i(x_i, y_k) + \bar{w}_{i,k}(x_i))]$ and $\operatorname{prox}_{\alpha r_i} [x_i - \alpha \tilde{f}_i(x_i, y_k)]$ for any $\alpha > 0$. Then by the nonexpansive property of the proximal operator, the following holds for any $\alpha > 0$:

$$\begin{aligned} \|\varepsilon_{i,k+1}\| &\leq \alpha \|\bar{w}_{i,k}(x_{i,k+1})\| \\ &+ \left\| x_{i,k+1} - \hat{x}_i(y_k) - \alpha \left(\tilde{f}_i(x_{i,k+1}, y_k) - \tilde{f}_i(\hat{x}_i(y_k), y_k) \right) \right\| \\ &\leq \sqrt{(1 - \alpha \mu)^2 + \alpha^2 L^2} \|\varepsilon_{i,k+1}\| + \alpha \|\bar{w}_{i,k}(x_{i,k+1})\|. \end{aligned}$$

In the above inequality, by setting $\alpha = \frac{\mu}{\mu^2 + L^2}$, we obtain that $\|\varepsilon_{i,k+1}\| \leq C_r \|\bar{w}_{i,k}(x_{i,k+1})\|$. Then by using Assumption 5(iii), the lemma is proved.

Similar to [14, Prp. 4], we can prove the linear rate of convergence. We then establish the iteration and oracle complexity to obtain an ϵ -NE₂, which is random strategy profile $x : \Omega \to \mathbb{R}^n$ satisfies $\mathbb{E}[||x - x^*||] \le \epsilon$.

Theorem 5: Suppose Ass. 5 holds and $a \triangleq ||\Gamma|| < 1$. Let Algorithm 2 be applied to the stochastic Nash game $(\mathcal{P}_i(x_{-i}))$, where $\mathbb{E}[||x_{i,0} - x_i^*||] \leq C$ and $N_k = \left\lceil \frac{\max_i M_i^2 C_r^2}{\eta^{2k}} \right\rceil$ for some $\eta \in (0,1)$. Define $c \triangleq \max\{a,\eta\}$, let $\tilde{\eta} \in (c,1)$, and $D = 1/\ln((\tilde{\eta}/c)^e)$. Then the number of the deterministic optimization problems solved and samples required by player *i* to obtain an ϵ -NE₂ are $\mathcal{O}(\ln(\sqrt{N}/\epsilon))$ and $\mathcal{O}\left(\left(\sqrt{N}/\epsilon\right)^{\frac{2\ln(1/\eta)}{\ln(1/\tilde{\eta})}}\right)$, respectively.

V. CONCLUDING REMARKS

We consider a class of stochastic Nash games where each player-specific objective is a sum of an expectation-valued smooth function and a convex nonsmooth function. We consider three schemes: (i) Variable sample-size proximal gradient response (VS-PGR) for strongly monotone stochastic Nash games; (ii) Distributed VS-PGR for strongly monotone aggregative Nash games; and (iii) VS proximal best-response (VS-PBR) for stochastic Nash games with contractive best response maps. Under suitable assumptions, we show that all schemes generate sequences that converge at the (**optimal**) linear rate and derive bounds on the computational, oracle, and communication complexity.

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