Linear Model Regression on Time-series Data: Non-asymptotic Error Bounds and Applications

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Abstract—Data-driven methods for modeling dynamic systems have recently received considerable attention as they provide a mechanism for control synthesis directly from the observed time-series data. In the absence of prior assumptions on how the time-series had been generated, regression on the system model has been particularly popular. In the linear case, the resulting least squares setup for model regression, not only provides a computationally viable method to fit a model to the data, but also provides useful insights into the modal properties of the underlying dynamics. Although probabilistic estimates for this model regression have been reported, deterministic error bounds have not been examined in the literature, particularly as they pertain to the properties of the underlying system. In this paper, we provide deterministic non-asymptotic error bounds for fitting a linear model to observed time-series data, with a particular attention to the role of symmetry and eigenvalue multiplicity in the underlying system matrix.

Keywords: data-driven methods, linear regression, linear models, supervised learning

I. INTRODUCTION

Recent advances in measurement and sensing technologies have lead to the availability of an unprecedented amount of data generated by complex physical, social, and biological systems such as turbulent flow, opinion dynamics on social networks, transportation, financial trading, and drug discovery. This so-called big data revolution has resulted in the development of efficient computational tools that utilizes the data generated by a dynamic system to reason about reduce order representations of this data, subsequently utilized for classification or prediction on the underlying model. Such techniques have been particularly useful when the derivation of models from first principles is prohibitively complex or infeasible. In the meantime, utilizing data generated by the system directly for the purpose of control or estimation, poses a number of challenges, most notably for modelbased control design techniques such as \mathcal{H}_2 , \mathcal{H}_{∞} , and model predictive control (MPC). As such, it has become imperative to examine fundamental limits on fitting models to the timeseries data, that can subsequently be used for model-based control synthesis.

One caveat of such an approach for a wide range of complex systems is the absence of the ability to excite the system with desired (persistent) inputs for the purpose of system identification [1]. More recently, data-driven identification has also been examined in the context of machine learning as an extension of classification or prediction, with less attention given to the ability to excite the system with persistent inputs. In this direction, non-asymptotic bounds for finite sample complexity were obtained in [2]-[6] for the linear time-invariant systems. Maximum likelihood and subspace identification methods have been employed in [7] to learn linear systems with guaranteed stability. The problem was investigated in an online learning setup to find regret bounds on the average cost of linear quadratic (LQ) systems in [8]. In the context of data-driven analysis of dynamical systems, Koopman analysis has also been used for operatortheoretic identification of nonlinear systems and their spectral properties [9]-[11]. One of the key elements used in the aforementioned identification methods is a linear regression step in order to fit a model to data; linear regression is in fact one of the backbone of what is known as supervised learning. In its most basic form, linear regression is used to find the system parameters by solving a least-squares minimization problem constructed on the observed timeseries. Examples of such an approach can be found in [5], [12], [13].

The error analysis for fitting a linear dynamic system to data presented in this work is closely related to error estimates examined in [12] and [5] for linear quadratic synthesis. In both works, control synthesis involves an intermediate step of parameterizing the underlying system using the collected data; subsequently, probabilistic guarantees on the error between the true and the estimated models are presented. In [12], the underlying system is allowed to be excited by canonical inputs before time-series data is collected following each "episode". The same approach has been adopted in [5], where a Gaussian noise is used to excite the system. While meaningful upper bounds on the error of the estimate are examined in these works, the presented results are probabilistic in nature, with probabilistic bounds that are directly related to the number of data points. In this work, we provide non-asymptotic error bounds for adopting a regression approach to fit a linear model to data generated by the system, evolving from initial conditions and without a control input. Furthermore, this error bound is analyzed in an online (non-asymptotic) manner as more data becomes available. It is shown that the error guarantees are closely related to the system parameters, the rank of the collected

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¹Where a linear model is trained for labeling future instances of incoming data.

data, and not surprisingly, the initial conditions. We then focus on the case where the underlying linear model is symmetric and show that the modeling regression error depends on the spectral properties of the system.

The organization of the paper is as follows: In §II, we provide the necessary mathematical background. The problem setup is outlined in §III. §IV provides the error analysis and main results of the paper. We then examine the ramification of our results for fitting a linear model to network data in §V. The paper is concluded in §VI.

II. NOTATION AND PRELIMINARIES

The cardinality of a set S is denoted as |S|. A column vector with n real entries is denoted as $\mathbf{v} \in \mathbb{R}^n$, where \mathbf{v}_i represents its ith element. The matrix $M \in \mathbb{R}^{p \times q}$ contains p rows and q columns and $[M]_{ij}$ denotes its (real) entry at the ith row and ith column. The Moore-Penrose pseudoinverse of a full-rank matrix $M \in \mathbb{R}^{p \times q}$ is defined as $M^{\dagger} = (M^{\top}M)^{-1}M^{\top}$ if p > q and $M^{\dagger} = M^{\top}(MM^{\top})^{-1}$ otherwise; M^{\top} denotes the transpose of the matrix. The range and nullspace of matrix M are denoted by $\mathcal{R}(M)$ and $\mathcal{N}(M)$, respectively. The *basis* of the vector space \mathcal{V} is referred to as $\mathcal{B}(\mathcal{V})$; the span of a set of vectors, denoted by span, is the set of all linear combinations of these vectors. The unit vector e^i is the column vector with all zero entries except $[e^i]_i = 1$. The column vector of all ones is denoted by 1, with dimension implicit from the context. The $n \times n$ identity matrix is defined as $I_n = \mathbf{Diag}(1)$ for $1 \in \mathbb{R}^n$. The trace of $M \in \mathbb{R}^{n \times n}$ is designated as $\mathbf{Tr}(M) = \sum_{i=1}^{n} [M]_{ii} =$ $\sum_{i=1}^{n} \lambda_i$, where λ_i is the *i*th eigenvalue of \overline{M} . We write $M \succ 0$ when M is positive-definite (PD) and $M \succeq 0$ if M is positive-semidefinite (PSD). The *spectrum* of M (the set of its eigenvalues) is denoted by $\Lambda(M)$. The algebraic multiplicity of an eigenvalue λ is denoted by $m(\lambda)$, defined as the multiplicity of λ in $\Lambda(M)$. An eigenvalue λ is called simple if $m(\lambda) = 1$. The singular value decomposition (SVD) of a matrix $X \in \mathbb{R}^{n \times m}$ is the factorization X = $U\Sigma V^{\top}$, where the unitary matrices U and V consist of the left and right "singular" vectors of X, and Σ is the diagonal matrix of singular values. Economic SVD is the reduced order matrix obtained by truncating the factor matrices in the SVD to the first r columns of U and V, corresponding to the r non-zero singular values in Σ , where $r = \operatorname{rank}(X)$. From the SVD of a given matrix X, one can find its pseudoinverse as $X^{\dagger} = V \Sigma^{-1} U^{\top}$, resulting in $X X^{\dagger} = U U^{\top}$; when Σ has zero diagonals, the aforementioned inverse keeps these zeros untouched. The *Euclidean norm* of a vector $\mathbf{x} \in \mathbb{R}^n$ is defined as $\|\mathbf{x}\|_2 = (\mathbf{x}^{\top}\mathbf{x})^{1/2} = (\sum_{i=1}^n x_i^2)^{1/2}$. The spectral norm of matrix X is defined as $\|X\|_2 = \sup\{\|X\mathbf{u}\|_2 :$ $\|\mathbf{u}\|_2 = 1$. The Frobenius norm of a matrix $\|X\|_F$ is defined as $||X||_F = \sqrt{\text{Tr}(X^\top X)}$. Spectral and Frobenius norms satisfy the inequality $||X||_2 \le ||X||_F \le r||X||_2$, where $r = \operatorname{rank}(X)$.

III. PROBLEM SETUP

For a wide range of real-world systems, the underlying complex dynamics makes deriving the corresponding models from first principles difficult if not infeasible. This can be due to a range of factors from the unpredictable nature of the environment to perturbations and uncertainties in the complex system [5], [14]. However, with the availability of sensing technologies and high performance computing, time-series data can be collected from these systems. Hence, it becomes natural to consider to what extent the observed time-series can be used to reason about the underlying dynamic model. In the case when this data has been generated by simulations (a model, albeit complex, already exists), one might still be interested to reason about the dynamics using "simple" models. The adoption of this approach involves using prior knowledge of the underlying dynamics to choose a particular basis or library, and then postulating that the dynamic system is some combination of these basis elements. This problem then reduces to a parameter optimization problem -with respect to these basis or library- for their combination that best fits the given data, with respect to a suitable norm or metric. In the absence of any prior assumption on the dynamics, however, it is often desirable to explore simple models. This paper examines non-asymptotic error bounds for doing such a linear fit, for the case when the data had been generated by a linear system; generically, it is the case that if the data is rich enough and the system does not have degeneracies, exact model is obtained after the number of data snapshots is the dimension of the system.² In fact, we show that even in this most streamlined case, and even in the absence of noise or uncertainty in the collected data, understanding non-asymptotic behavior of the error requires some non-trivial analysis.

Consider the discrete linear time-invariant system described by the state equation,

$$\mathbf{x}_{t+1} = A\mathbf{x}_t, \qquad t = 0, 1, 2, \dots,$$
 (1)

where $A \in \mathbb{R}^{n \times n}$ is the unknown system matrix, $\mathbf{x}_t \in \mathbb{R}^n$ is the state of the system at time t, and the system has been initialized from \mathbf{x}_0 . The state snapshots collected up to (and including) time k can now be grouped as,

$$X_k = [\mathbf{x}_0 \quad \mathbf{x}_1 \quad \dots \quad \mathbf{x}_k] , Y_k = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \dots \quad \mathbf{x}_{k+1}].$$
 (2

This data collection approach is analogous to the first step of the so-called dynamic mode decomposition (DMD) algorithm [15], where this step is followed by the parameterization of the eigenvalues and eigenvectors of the underlying system model. Now to estimate the underlying system matrix at each k, we consider the the least-squares minimization, $\hat{A}_k = \operatorname{argmin}_A \|Y_k - AX_k\|_F$, whose solution is of the form $\hat{A}_k = Y_k X_k^{\dagger}$; see Fig. 1. Note that when $\operatorname{rank}(X_k) = n$, $\hat{A}_k = AX_k X_k^{\dagger} = AX_k X_k^{\top} (X_k X_k^{\top})^{-1} = A$.

 2 The model has n^2 unknown entries; as such, n^2 observations are generically needed for its exact recovery. One of course can get away with less data by invoking sparsity (say, using the ℓ_1 regularization) or structure on the model, e.g, assuming an underlying pattern for the system matrix.

³The main objective of DMD is however not "model" regression per se, as it is "modal" fitting, in order to provide useful insights into the underlying (possibly nonlinear) dynamics.

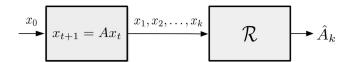


Fig. 1: Estimating the underlying dynamics A after k data snapshots using the model regression \mathcal{R}

The focus of this paper is on the analysis of the error $||A - \hat{A}_k||$, i.e., the non-asymptotic error between the original and estimated models, using linear regression when $rank(X_k) < n$. Our work considers an *online* estimation of the model A, where each new data snapshot is added to the previously collected set. At any time-step k, an estimate for A is found based on the received data up to k. The resulting data-driven recursive minimization is depicted in Fig. 1. Although not discussed further in this paper, we note that diagram in Fig. 1 can -in principle- be augmented with a model-based control or filtering scheme that utilizes \hat{A}_k after a suitable number of steps. In §IV, we first introduce an upper bound on the estimation error as a function of the system dynamics A, the iteration count k, dimension of the system n, and the initial conditions x_0 . In particular, we show that for each time-step, the left-singluar vectors of the SVD of the data matrix dictate the estimation error bound. Next, we focus on symmetric system matrices. In this case, it is shown that the model regression error can be characterized by the multiplicities in the spectrum of the underlying system.

IV. NON-ASYMPTOTIC ERROR ANALYSIS

In this section, we examine the error bound for the linear system regression in (1) based on the system characteristics and the observed data snapshots. We assume that k < n, i.e., the number of data snapshots is less than the size of the system. The next results, characterizes the regression estimation error as a function of the time step k.

Theorem 1. Consider the system in (1) and the corresponding data matrix. Let $X_k = U_k \Sigma_k V_k^{\top}$ be the SVD of X_k . Then the model estimate at time-step k is given by $\hat{A}_k = A(I - E_k)$ where,⁴

$$E_k = \left(I - \frac{S_k P_k}{\operatorname{Tr}(S_k P_k)}\right) S_k \,, \tag{3}$$

with

$$S_k = I - U_{k-1} U_{k-1}^{\mathsf{T}}, \quad P_k = \mathbf{x}_k \mathbf{x}_k^{\mathsf{T}} = A^k \mathbf{x}_0 \mathbf{x}_0^{\mathsf{T}} A^{k^{\mathsf{T}}}.$$
 (4)

Moreover,

$$||E_k||_2 \le \left| |I - \frac{S_k P_k}{\operatorname{Tr}(S_k P_k)} \right||_2. \tag{5}$$

Proof: From (2), $X_k = \begin{bmatrix} X_{k-1} & A^k \mathbf{x}_0 \end{bmatrix}$, and $Y_k = AX_k = A\begin{bmatrix} X_{k-1} & A^k \mathbf{x}_0 \end{bmatrix}$. Then the estimate of the system matrix after the k-th snapshot is given by $\hat{A}_k = Y_k X_k^{\dagger}$, where \hat{A}_k is the least-squares solution to $AX_k = Y_k$. Thus,

$$\hat{A}_k = Y_k X_k^{\dagger} = A X_k X_k^{\dagger} = A \begin{bmatrix} X_{k-1} & A^k \mathbf{x}_0 \end{bmatrix} X_k^{\dagger} . \tag{6}$$

Hence we need to characterize X_k^\dagger . To this end, we start from $X_k^\dagger = \left(X_k^\top X_k\right)^{-1} X_k^\top$. We first note that,

$$\begin{split} \boldsymbol{X}_{k}^{\top} \boldsymbol{X}_{k} &= \begin{bmatrix} \boldsymbol{X}_{k-1}^{\top} \\ \mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{k-1} & \boldsymbol{A}^{k} \mathbf{x}_{0} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{X}_{k-1}^{\top} \boldsymbol{X}_{k-1} & \boldsymbol{X}_{k-1}^{\top} \boldsymbol{A}^{k} \mathbf{x}_{0} \\ \mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \boldsymbol{X}_{k-1} & \mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \boldsymbol{A}^{k} \mathbf{x}_{0} \end{bmatrix}. \end{split}$$

Then

$$\left(X_k^\top X_k \right)^{-1} = \frac{1}{\zeta} \begin{bmatrix} \Phi & -X_{k-1}^\dagger A^k \mathbf{x}_0 \\ -\mathbf{x}_0^\top A^{k^\top} X_{k-1}^{\dagger^\top} & 1 \end{bmatrix},$$

where

$$\begin{split} & \Phi = \left(\boldsymbol{X}_{k-1}^{\top} \boldsymbol{X}_{k-1} \right)^{-1} \left[\zeta \boldsymbol{I} + \boldsymbol{X}_{k-1}^{\top} \boldsymbol{A}^{k} \mathbf{x}_{0} \mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \boldsymbol{X}_{k-1}^{\dagger^{\top}} \right], \\ & \zeta = \mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \left[\boldsymbol{I} - \boldsymbol{X}_{k-1} \left(\boldsymbol{X}_{k-1}^{\top} \boldsymbol{X}_{k-1} \right)^{-1} \boldsymbol{X}_{k-1}^{\top} \right] \boldsymbol{A}^{k} \mathbf{x}_{0} \\ & = -\mathbf{x}_{0}^{\top} \boldsymbol{A}^{k^{\top}} \left[\boldsymbol{X}_{k-1} \boldsymbol{X}_{k-1}^{\dagger} - \boldsymbol{I} \right] \boldsymbol{A}^{k} \mathbf{x}_{0} \ . \end{split}$$

In the meantime, $X_k^\dagger = \left(X_k^\top X_k\right)^{-1} X_k^\top \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$, with,

$$\Psi_{1} = X_{k-1}^{\dagger} + \frac{1}{\zeta} X_{k-1}^{\dagger} A^{k} \mathbf{x}_{0} \mathbf{x}_{0}^{\top} A^{k^{\top}} \left(U_{k-1} U_{k-1}^{\top} - I \right),$$

$$\Psi_{2} = -\frac{1}{\zeta} \mathbf{x}_{0}^{\top} A^{k^{\top}} \left(U_{k-1} U_{k-1}^{\top} - I \right),$$

where we have used the fact $X_{k-1}X_{k-1}^{\dagger} = U_{k-1}U_{k-1}^{\top}$. Thereby, we can expand \hat{A}_k from (6) as,

$$\begin{split} \hat{A}_k &= A X_k X_k^{\dagger} = A \left(U_{k-1} U_{k-1}^{\top} \right) \\ &+ A \left(\frac{1}{\zeta} \left(U_{k-1} U_{k-1}^{\top} - I \right) A^k \mathbf{x}_0 \mathbf{x}_0^{\top} A^{k^{\top}} \left(U_{k-1} U_{k-1}^{\top} - I \right) \right) \\ &= A \left(U_{k-1} U_{k-1}^{\top} \right) \\ &- A \left(\frac{\left(U_{k-1} U_{k-1}^{\top} - I \right) A^k \mathbf{x}_0 \mathbf{x}_0^{\top} A^{k^{\top}} \left(U_{k-1} U_{k-1}^{\top} - I \right)}{\mathbf{x}_0^{\top} A^{k^{\top}} \left(U_{k-1} U_{k-1}^{\top} - I \right) A^k \mathbf{x}_0} \right), \end{split}$$

and from (4), the estimated model \hat{A}_k at time-step k is given by $\hat{A}_k = A\left(I - E_k\right)$ with,

$$E_k = \frac{\left(\mathbf{Tr}(S_k P_k)I - S_k P_k\right) S_k}{\mathbf{Tr}(S_k P_k)} = \left(I - \frac{S_k P_k}{\mathbf{Tr}(S_k P_k)}\right) S_k.$$

The magnitude of this error simplifies for the case of the spectral norm as,

$$||E_k||_2 = \left\| \left(I - \frac{S_k P_k}{\operatorname{Tr}(S_k P_k)} \right) S_k \right\|_2 \le \left\| I - \frac{S_k P_k}{\operatorname{Tr}(S_k P_k)} \right\|_2$$

since $||S_k||_2 = 1$ for k < n. Lastly, we note that S_k is the projection onto the null space of X_{k-1} and P_k is the covariance matrix of the data at time-step k; as such both matrices are positive-semidefinite.

We note that the relation (3) captures -in a succinct waythe dependency of the model regression error on how new modes are revealed by the data stream over time.

⁴Note that we are quantifying the relative error in (3).

A. Non-asymptotic Error Analysis for Symmetric Systems

In this section, we consider linear systems with symmetric dynamics with the aim of characterizing fundamental bounds on the regression error in terms of the spectral properties of the system. This insight into the regression error is achieved through the spectral decomposition of the system matrix,

$$A = Q\Lambda Q^{\top} = \sum_{i=1}^{r} \lambda_i \mathbf{q}_i \mathbf{q}_i^{\top}, \tag{7}$$

where Q is the unitary matrix containing the eigenvectors corresponding to nonzero eigenvalues of A, Λ is the diagonal matrix of nonzero eigenvalues, and $r = \operatorname{rank}(A)$. Symmetric system matrices appear in a wide range of applications where interactions leading to the dynamics is bidirectional; such systems are of interest in biological networks [16], social interactions [17], robotic swarms [18], and networks security [19]. Using this spectral decomposition of symmetric systems, we show that the regression error is dependent on the multiplicity of eigenvalues in A. In particular, we show that if $m(\lambda) = 1, \forall \lambda \in \Lambda(A)$, then the upper bound (5) is a function of the largest and smallest singular values of the system matrix as well as its rank. Otherwise, the regression error is $\max_{i:m(\lambda_i)>1} |\lambda_i|$. We provide the details of the approach for each case.

1) Simple Eigenvalues: We first consider the case when the symmetric system matrix has simple eigenvalues and rank r. In reference to (7), consider the entire set of eigenvectors of A consisting of $Q = [\mathbf{q}_1 \cdots \mathbf{q}_r]$ and $\bar{Q} = [\bar{\mathbf{q}}_{r+1} \cdots \bar{\mathbf{q}}_n]$, where $\{\mathbf{q}_1, \cdots, \mathbf{q}_r, \bar{\mathbf{q}}_{r+1}, \cdots, \bar{\mathbf{q}}_n\}$ forms a basis for the entire \mathbb{R}^n . Then the nonzero random initial state \mathbf{x}_0 can be written as,

$$\mathbf{x}_0 = Q\nu + \bar{Q}\mu = \sum_{i=1}^r \nu_i \mathbf{q}_i + \sum_{i=r+1}^n \mu_i \bar{\mathbf{q}}_i \qquad \nu \neq 0.$$
 (8)

Lemma 1. For the symmetric linear system decomposed as (7), we have, $A - \hat{A}_k = A \left(I - \frac{S_k Q \Lambda^k \nu \nu^\top \Lambda^k Q^\top}{\|S_k Q \Lambda^k \nu\|^2} \right) S_k$.

Proof: From (8) and (4) we have, $P_k = A^k \mathbf{x}_0 \mathbf{x}_0^{\mathsf{T}} A^{k^{\mathsf{T}}}$

$$= (Q\Lambda^k Q^\top)(Q\nu + \bar{Q}\mu)(Q\nu + \bar{Q}\mu)^\top (Q\Lambda^k Q^\top)$$
$$= Q\Lambda^k \nu \nu^\top \Lambda^k Q^\top,$$

and, $\operatorname{Tr}(S_k P_k) = \operatorname{Tr}(S_k Q \Lambda^k \nu \nu^\top \Lambda^k Q^\top) = \|S_k Q \Lambda^k \nu\|^2$, where we have used that S_k is a symmetric projection. Substituting these in (3) completes the proof.

We now show that when k < n, the error depends on the largest and smallest eigenvalues of A.

Theorem 2. Consider the linear dynamical system with symmetric system matrix A as in (7) and the initial state x_0 as in (8). If k < n, then

$$||A - \hat{A}_k||_F^2 \le (n - \min\{k, |\nu| + \min\{|\mu|, 1\}\}) \lambda_1^2 - \lambda_n^2,$$

where $\lambda_n = \lambda_{\min}(A)$, $\lambda_1 = \lambda_{\max}(A)$, and $|\nu|$ and $|\mu|$ are the number of nonzero ν_i 's and μ_i 's, respectively.

Proof: From Lemma 1 we observe that,

$$\begin{split} &\|A - \hat{A}_k\|_F^2 = \mathbf{Tr}\Big((A - \hat{A}_k)^\top (A - \hat{A}_k)\Big) \\ &= \mathbf{Tr}(\Lambda^2 Q^\top S_k Q) - \frac{\nu^\top \Lambda^k Q^\top S_k Q \Lambda^2 Q^\top S_k Q \Lambda^k \nu}{\nu^\top \Lambda^k Q^\top S_k Q \Lambda^k \nu} \\ &= \|AS_k\|^2 - \frac{\|AS_k Q \Lambda^k \nu\|^2}{\|S_k Q \Lambda^k \nu\|^2} \,. \end{split}$$

In the meantime,

$$||AS_k||_F^2 \le \operatorname{rank}(S_k) ||AS_k||_2^2 \le \operatorname{rank}(S_k) ||A||_2^2 ||S_k||_2^2 = \operatorname{rank}(S_k) \lambda_1^2(A);$$

moreover, since $\lambda_n(A) = \inf_{y \neq 0} ||Ay||_2 / ||y||_2$, we have

$$\frac{\|AS_k Q\Lambda^k \nu\|^2}{\|S_k Q\Lambda^k \nu\|^2} \ge \lambda_n^2(A).$$

Since $A\mathbf{q}_i = \lambda_i \mathbf{q}_i$ and $A\bar{\mathbf{q}}_i = 0$, we have,

$$X_{k-1} = \begin{bmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^r \nu_i \mathbf{q}_i + \sum_{i=r+1}^n \mu_i \bar{\mathbf{q}}_i & \sum_{i=1}^r \lambda_i \nu_i \mathbf{q}_i & \cdots & \sum_{i=1}^r \lambda_i^{k-1} \nu_i \mathbf{q}_i \end{bmatrix}.$$

Thus

$$\mathbf{rank}(X_{k-1}) = \min\{k, |\nu| + \min\{|\mu|, 1\}\}, \qquad (9)$$

and,

 $\operatorname{rank}(S_k) = n - \operatorname{rank}(X_{k-1}) = n - \min\{k, |\nu| + \min\{|\mu|, 1\}\}.$ Hence

$$||A - \hat{A}_k||_F^2 \le \left(n - \min\{k, |\nu| + \min\{|\mu|, 1\}\}\right) \lambda_1^2 - \lambda_n^2,$$
 which completes the proof.

B. Effect of Eigenvalues Multiplicity on the Regression Error

In this section, we consider the symmetric systems whose eigenvalues are not necessary simple. We will see that for such systems, the regression error $\|E_k\|$ converges to the largest eigenvalue with multiplicity greater than one, i.e., $\|E_k\| = \max_{i:m(\lambda_i)>1} |\lambda_i|$ for $k \ge n$.

In order to show this, we will pursue the convention adopted in (2), (7), and (8). As in § IV-A, let $r = \operatorname{rank}(A)$ and define $\tilde{Q} = [Q \mid \bar{Q}] = [\mathbf{q}_1 \dots \mathbf{q}_r \quad \bar{\mathbf{q}}_{r+1} \dots \bar{\mathbf{q}}_n]$, where the columns of \tilde{Q} span the entire \mathbb{R}^n . Furthermore, let $\alpha = [\nu^\top \mu^\top]^\top$, where α and μ are from (8). The data matrix can now be re-written as,

$$X_k = [\mathbf{x}_0 \quad A\mathbf{x}_0 \quad A^2\mathbf{x}_0 \quad \dots \quad A^k\mathbf{x}_0]$$

= $\tilde{Q} \quad [\tilde{Q}^{\top}\mathbf{x}_0 \quad \Lambda \tilde{Q}^{\top}\mathbf{x}_0 \quad \dots \quad \Lambda^k \tilde{Q}^{\top}\mathbf{x}_0].$ (10)

From (8) and the fact that the columns of \tilde{Q} are orthonormal, we have $\tilde{Q}^{\top}\mathbf{x}_0 = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n]^{\top}$ and $\Lambda^j \tilde{Q}^{\top}\mathbf{x}_0 = [\alpha_1 \lambda_1^j \ \alpha_2 \lambda_2^j \ \dots \ \alpha_n \lambda_n^j]^{\top}$. Moreover, in light of (10) we can decompose the data matrix into $X_k = \tilde{Q}\Gamma V$ where,

$$\Gamma = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{bmatrix}, \quad V = \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^k \\ 1 & \lambda_2 & \dots & \lambda_2^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^k \end{bmatrix}.$$

$$(11)$$

The matrix $[V]_{ij} = \lambda_i^{j-1}$ is the Vandermonde matrix formed by the eigenvalues of A and $\Gamma = \mathbf{Diag}([\alpha_i]_{i=1}^n).^5$ Assume now that the system matrix A contains s distinct eigenvalues and let $\Lambda^*(A) = \{\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_s}\}$ be the set of these eigenvalues; as such, the other n-s eigenvalues are repetitions of the elements in $\Lambda^*(A)$.

For our subsequent error analysis, we will use the rank of X_k at each time-step. The next result characterizes the rank of X_k based on the number of the collected data.

Lemma 2. Given k < s, the $s \times k$ Vandermonde matrix defined by $[V_s]_{ij} = \lambda_i^{j-1}, \ i \in \{1, \dots, s\}, \ j \in \{1, \dots, k\}$, formed by the elements of $\Lambda^*(A)$, has full-rank.

Proof: Let v_i be the ith column of V_s and assume that $c_1v_1+c_2v_2+\cdots+c_kv_k=0$. Consider row p of the equation $c_1+c_2\lambda_p+\cdots+c_k\lambda_p^k=0$. Since $\lambda_i\neq\lambda_j$ for $i\neq j$, there exist s solutions to the k-degree polynomial,

$$P(x) = c_0 + c_1 x + \dots + c_k x^k = 0.$$

Hence $c_1 = c_2 = \cdots = c_k = 0$ and v_i 's are linearly independent and since k < s, V_s has full-rank.

Theorem 3. Let k be the number of collected data snapshots and $s = |\Lambda^*(A)|$. Then $\operatorname{rank}(X_k) = k$ when k < s and $\operatorname{rank}(X_k) = s$ when $k \ge s$.

Proof: For k < s, it is straightforward to show that from Lemma 2, $\operatorname{rank}(X_k) = \operatorname{rank}(V) = \operatorname{rank}(V_s) = k$. For $k \ge s$, we know that $\operatorname{rank}(V) = \operatorname{rank}(V_s)$, where $[V_s]_{ij} = \lambda_i^{j-1}$, $i \in \{1,\ldots,s\},\ j \in \{1,\ldots,k+1\}$. Since V_s has full-rank (this can be shown using the nonzero sub-matrix determinants for the first $s \times s$ block), we get $\operatorname{rank}(X_k) = \operatorname{rank}(V) = \operatorname{rank}(V_s) = s$.

Let $k \geq s$ and $X_k = U\Sigma V^{\top}$ be the SVD of X_k where,

$$X_{k} = \begin{bmatrix} U_{1} & U_{2} \end{bmatrix} \begin{bmatrix} \Sigma_{1} & \mathbf{0}_{s \times (k-s)} \\ \mathbf{0}_{(n-s) \times s} & \mathbf{0}_{(n-s) \times (k-s)} \end{bmatrix} \begin{bmatrix} V_{1} & V_{2} \end{bmatrix}^{\top},$$
(12)

with $U_1 \in \mathbb{R}^{n \times s}$, $U_2 \in \mathbb{R}^{n \times (n-s)}$, $\Sigma_1 \in \mathbb{R}^{s \times s}$, $V_1 \in \mathbb{R}^{k \times s}$, $V_2 \in \mathbb{R}^{k \times (k-s)}$. The existence of U_2 and V_2 is due to the fact that X_k is degenerate. Without loss of generality, we re-arrange the columns of \tilde{Q} and Λ such that,

$$\Lambda_R = \begin{bmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \Lambda_2 \end{bmatrix} \quad , \quad \tilde{Q}_R = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \end{bmatrix}, \quad (13)$$

where $\Lambda_1 = \mathbf{Diag}([\lambda_{t_1} \ \lambda_{t_2} \ \dots \ \lambda_{t_s}])$ contains the element of $\mathbf{\Lambda}^*(A)$ with the corresponding eigenvectors in $\tilde{Q}_1 \in \mathbb{R}^{n \times s}$. The remaining eigenvalues and the corresponding eigenvectors are stacked in $\Lambda_2 \in \mathbb{R}^{(n-s) \times (n-s)}$ and $\tilde{Q}_2 \in \mathbb{R}^{n \times (n-s)}$, respectively.

A crucial term in our analysis is $U_2^\top \tilde{Q}_R \in \mathbb{R}^{(n-s)\times n}$. This matrix multiplication combines a submatrix of U corresponding to the repeated eigenvalues, with the orthonormal eigenvectors of the system. In the next result, we show that this term has a specific row structure.

Lemma 3. Given the convention in (13), we have
$$U_2^{\top} \tilde{Q}_1 = 0$$
, i.e., $U_2^{\top} \tilde{Q}_R = U_2^{\top} [\tilde{Q}_1 \mid \tilde{Q}_2] = [\mathbf{0}_{(n-s)\times s} \mid U_2^{\top} \tilde{Q}_2]$.

⁵Note that it is assumed that for all i, $\mathbf{x}_0 \not\perp \mathbf{q}_i$; in this case, $\alpha_i \neq 0$ for all i and $\operatorname{rank}(X_k) = \operatorname{rank}(V)$.

Proof: From (12) we have $U_2^{\top} X_k = 0$. Then,

$$\begin{aligned} U_2^\top X_k &= U_2^\top \tilde{Q} \tilde{Q}^\top X_k = U_2^\top \tilde{Q} \tilde{Q}^\top [x_0 \quad Ax_0 \quad \dots \quad A^k x_0] \\ &= U_2^\top \tilde{Q} \big[\tilde{Q}^\top x_0 \quad \Lambda \tilde{Q}^\top x_0 \quad \dots \quad \Lambda^k \tilde{Q}^\top x_0 \big] \\ &= U_2^\top \tilde{Q} \Gamma V = 0, \end{aligned}$$

with Γ and V defined as in (11). Define $B = U_2^\top \tilde{Q}$ and let $\mathbf{b}_i = [\mathbf{b}_{i_1} \ \mathbf{b}_{i_2} \ \dots \ \mathbf{b}_{i_n}]$ and $\mathbf{v}_i = [1 \ \lambda_i \ \dots \ \lambda_i^k]$ be the i'th rows of B and V, respectively. Then for all $i \in \{1,2,\dots,n-s\}$ we have $B\Gamma V = 0$ implying that $\mathbf{b}_i\Gamma V = 0$; as such, $\sum_{j=1}^n \mathbf{b}_{i_j}\alpha_j\mathbf{v}_j = 0$ implies that $\sum_{t=1}^s \mathbf{c}_t\mathbf{v}_t^* = 0$, where \mathbf{v}_t^* 's are the rows of V corresponding to s distinct eigenvalues and \mathbf{c}_t 's are some combinations of $\mathbf{b}_{i_j}\alpha_j$'s. Since the vectors \mathbf{v}_t^* 's are linearly independent, we get $\mathbf{c}_i = 0$ for all $i = 1, 2, \dots, s$. Considering that $\mathbf{c}_i = \mathbf{b}_{i_j}$ for the elements with simple eigenvalues, the result implies that for any row \mathbf{v}_j corresponding to a simple eigenvalue λ_j the corresponding coefficient $\mathbf{b}_{i_j} = 0$. Hence, the structure $\begin{bmatrix} \mathbf{0}_{(n-s)\times s} \mid U_2^\top \tilde{Q}_1 \end{bmatrix}$ follows, implying that $U_2^\top \tilde{Q}_1 = 0$.

Definition 1. Define U_2' by permuting the columns of U_2 such that we obtain block diagonal matrix $U_2'^\top \tilde{Q}_2 = \mathbf{Diag}([P_1, P_2, \dots, P_\ell])$, where ℓ is the number of eigenvalues λ with $m(\lambda) > 1$ and $P_i = U_2^{i^\top} \tilde{Q}_2^i \in \mathbb{R}^{(m(\lambda_i)-1)\times m(\lambda_i)}$; as such $U_2'^i \in \mathbb{R}^{n\times (m(\lambda_i)-1)}$ is the matrix containing vectors in U_2' corresponding to λ_i and $\tilde{Q}_2^i \in \mathbb{R}^{n\times m(\lambda_i)}$ is the matrix of eigenvectors corresponding to λ_i .

Remark 1. To justify the existence of such a matrix U_2' , notice that the SVD factorization in terms of U and V are not unique and for any such factorization, $U_1 \perp U_2$.

We are now well positioned to prove the main theorem of this section.

Theorem 4. Consider the dynamics represented by (1) and (2). Let $s = |\Lambda^*(A)|$ be the number of distinct eigenvalues of A. Assume that $k \geq s$ and let $\lambda^* = \max_{i:m(\lambda_i) > 1} |\lambda_i|$ be the largest eigenvalue of A with multiplicity greater than one. Then $||E_k||_2 = ||A - \hat{A}_k||_2 = \lambda^*$.

Proof: We will show that $\Lambda(E_k) = \Lambda(A) \backslash \Lambda^*(A)$. The error can then be re-written as,

$$\begin{aligned} &\|E_k\|_2 \\ &= \|A - \hat{A}_k\|_2 = \|A - AX_k X_k^{\dagger}\|_2 = \|A(I - X_k X_k^{\dagger})\|_2 \\ &= \left\|A\left(I - U_1 \Sigma_1 V_1^{\top} V_1 \Sigma_1^{-1} U_1^{\top}\right)\right\|_2 = \|A(I - U_1 U_1^{\top})\|_2 \\ &= \|AU_2' U_2'^{\top}\|_2 = \|\tilde{Q}\Lambda \tilde{Q}^{\top} U_2' U_2'^{\top}\|_2 = \|\Lambda \tilde{Q}^{\top} U_2' U_2'^{\top} \tilde{Q}\|_2 \\ &= \|\Lambda (U_2'^{\top} \tilde{Q})^{\top} (U_2'^{\top} \tilde{Q})\|_2, \end{aligned}$$

where U_2' pertains to definition 1 and $I - U_1 U_1^{\top} = U_2' U_2'^{\top}$ is the projection matrix onto $\mathcal{N}(X_k^{\top})$. Note that since the columns of U_2' are linearly independent, we have $\operatorname{rank}(E_k) = \operatorname{rank}(AU_2'U_2'^{\top}) = n - s$. From Lemma 3,

$$\Lambda(U_2'^{\top}\tilde{Q})^{\top}(U_2'^{\top}\tilde{Q}) = \Lambda \begin{bmatrix} \mathbf{0} \\ (U_2'^{\top}\tilde{Q}_2)^{\top} \end{bmatrix} \begin{bmatrix} \mathbf{0} & U_2'^{\top}\tilde{Q}_2 \end{bmatrix} \\
= \begin{bmatrix} \mathbf{0}_{s \times s} & \mathbf{0}_{s \times (n-s)} \\ \mathbf{0}_{(n-s) \times s} & \Lambda_2(U_2'^{\top}\tilde{Q}_2)^{\top}(U_2'^{\top}\tilde{Q}_2) \end{bmatrix}.$$
(14)

Then from definition 1, $\Lambda_2(U_2^{\prime\top}\tilde{Q}_2)^{\top}(U_2^{\prime\top}\tilde{Q}_2) = \mathbf{Diag}([\lambda_1P_1^{\top}P_1 \ , \ \lambda_2P_2^{\top}P_2 \ , \ \dots \ , \ \lambda_\ell P_\ell^{\top}P_\ell])$. Consider the ith block $\lambda_i P_i^{\top}P_i$. Notice that from definition $\mathbf{rank}(P_i^{\top}P_i) = m(\lambda_i) - 1$, and since $U_2^{\prime i^{\top}}$ and \tilde{Q}_2^i are orthonormal, $\lambda_i P_i^{\top}P_i = \lambda_i \tilde{Q}_2^{i^{\top}} U_2^{\prime i} U_2^{\prime i^{\top}} \tilde{Q}_2^i$ has the spectrum $\Lambda(\lambda_i P_i^{\top}P_i) = \{0, \lambda_i, \dots, \lambda_i\}$. Then having ℓ of these blocks $\Lambda(E_k) = \Lambda(A) \backslash \Lambda^*(A)$, i.e., the spectrum of E_k contains all repeated eigenvalues of A.

Since both Λ and $(U_2'^\top \tilde{Q})^\top (U_2'^\top \tilde{Q})$ are symmetric square block diagonal matrices, the product $\Lambda(U_2'^\top \tilde{Q})^\top (U_2'^\top \tilde{Q})$ is symmetric and therefore $\|E_k\|_2 = \|\Lambda(U_2'^\top \tilde{Q})^\top (U_2'^\top \tilde{Q})\|_2 = \max_{i=1,\dots,\ell} \lambda_i = \lambda^*.^6$

V. MODEL REGRESSION ON NETWORKS

We now provide an example to demonstrate the applicability of the error bounds on networked systems. Consider the Petersen graph on 10 nodes and 15 edges as shown in Fig. 2. We use the weighted version of this specific structure to find error bounds on a system with simple eigenvalues. The dynamics of this system is defined using the graph Laplacian, defined as $\mathcal{L} = D - A$, where A is the adjacency matrix that defines the connections in the network and D is the degree matrix defined as $D_{ii} = \sum_{j} |W_{ij}|$; in this case W_{ij} is the weight of the edge between nodes i and j. Network symmetries typically induce eigenvalue multiplicities in the corresponding adjacency and Laplacian matrices. Hence to make the system more generic, we add weights $w_{1,6} = 1$, $w_{2,7} = 2$, $w_{3,8} = 3$, $w_{4,9} = 4$, and $w_{5,10} = 5$ and for all other weights we have $w_{i,j} = 1$. For each component i the dynamics depend on the adjacent nodes in the graph $\dot{x}_i = \sum_i |W_{ij}|(x_i - x_j)$. Then the overall dynamics can be written as $\dot{x} = -\mathcal{L}x$. The model regression algorithm discussed in this paper leads to the error shown in Fig. 2. For this simulation, the initial condition has been chosen as a (normalized) random vector $x_0 \in \mathbb{R}^{10}$. The upper subfigure shows the comparison of the bound for general case using the spectral norm and the lower subfigure demonstrates the same setup for IV-A. It can be seen that the error converges to zero after k = 10 steps, since the system matrix has simple eigenvalues.

VI. CONCLUSION

In this paper we consider the regression approach for learning linear time-invariant dynamic models from time-series data. In particular, we showed how the richness in the data as well as spectral properties of the model, dictate fundamental bounds on the error obtained from the streaming model regression. Our subsequent works will utilize these insights to provide an active learning mechanism that has the dual role of reducing the regression error in addition to achieving auxiliary control objectives.

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 $^6\mathrm{It}$ is straightforward to show an analogous result for the Frobenius norm $\|E_k\|_F^2=\|A-\hat{A}_k\|_F^2=\sum_{i=1}^\ell[m(\lambda_i)-1]\lambda_i^2.$

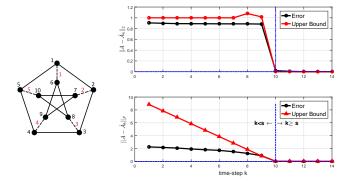


Fig. 2: (left) weighted Petersen graph; (right): model regression error the corresponding theoretical error bound

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