A Variable Sample-size Stochastic Quasi-Newton Method for Smooth and Nonsmooth Stochastic Convex Optimization

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Abstract

Classical theory for quasi-Newton schemes has focused on smooth deterministic unconstrained optimization while recent forays into stochastic convex optimization have largely resided in smooth, unconstrained, and strongly convex regimes. Naturally, there is a compelling need to address nonsmoothness, the lack of strong convexity, and the presence of constraints. Accordingly, this paper presents a quasi-Newton framework that can process merely convex and possibly nonsmooth (but smoothable) stochastic convex problems. We propose a framework that combines iterative smoothing and regularization with a variance-reduced scheme reliant on using an increasing sample-size of gradients. We make the following contributions. (i) We develop a regularized and smoothed variable sample-size BFGS update (rsL-BFGS) that generates a sequence of Hessian approximations and can accommodate nonsmooth convex objectives by utilizing iterative regularization and smoothing. (ii) In strongly convex regimes with statedependent noise, the proposed variable sample-size stochastic quasi-Newton (VS-SQN) scheme admits a non-asymptotic linear rate of convergence while the oracle complexity of computing an ϵ -solution is $\mathcal{O}(\kappa^{m+1}/\epsilon)$ where κ denotes the condition number and m > 1. In nonsmooth (but smoothable) regimes, using Moreau smoothing retains the linear convergence rate for the resulting smoothed VS-SQN (or sVS-SQN) scheme. Notably, the nonsmooth regime allows for accommodating convex constraints. To contend with the possible unavailability of Lipschitzian and strong convexity parameters, we also provide sublinear rates for diminishing steplength variants that do not rely on the knowledge of such parameters; (iii) In merely convex but smooth settings, the regularized VS-SQN scheme rVS-SQN displays a rate of $\mathcal{O}(1/k^{(1-\varepsilon)})$ with an oracle complexity of $\mathcal{O}(1/\epsilon^3)$. When the smoothness requirements are weakened, the rate for the regularized and smoothed VS-SQN scheme rsVS-SQN worsens to $\mathcal{O}(k^{-1/3})$. Such statements allow for a state-dependent noise assumption under a quadratic growth property on the objective. To the best of our knowledge, the rate results are amongst the first available rates for QN methods in nonsmooth regimes. Preliminary numerical evidence suggests that the schemes compare well with accelerated gradient counterparts on selected problems in stochastic optimization and machine learning with significant benefits in ill-conditioned regimes.

1 Introduction

We consider the stochastic convex optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \triangleq \mathbb{E}[F(x, \xi(\omega))], \tag{1}$$

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where $\xi: \Omega \to \mathbb{R}^o, F: \mathbb{R}^n \times \mathbb{R}^o \to \mathbb{R}$, and $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the associated probability space. Such problems have broad applicability in engineering, economics, statistics, and machine learning. Over the last two decades, two avenues for solving such problems have emerged via sample-average approximation (SAA) [23] and stochastic approximation (SA) [40]. In this paper, we focus on quasi-Newton variants of the latter. Traditionally, SA schemes have been afflicted by a key shortcoming in that such schemes display a markedly poorer convergence rate than their deterministic variants. For instance, in standard stochastic gradient schemes for strongly convex smooth problems with Lipschitz continuous gradients, the mean-squared error diminishes at a rate of $\mathcal{O}(1/k)$ while deterministic schemes display a geometric rate of convergence. This gap can be reduced by utilizing an increasing sample-size of gradients, an approach considered in [17, 9, 36, 13, 6], and subsequently refined for gradient-based methods for strongly convex [42, 22, 21], convex [20, 14, 22, 21], and nonsmooth convex regimes [21]. Variance-reduced techniques have also been considered for stochastic quasi-Newton (SQN) techniques [27, 51, 5] under twice differentiability and strong convexity requirements. To the best of our knowledge, the only available SQN scheme for merely convex but smooth problems is the regularized SQN scheme presented in our prior work [48] where an iterative regularization of the form $\frac{1}{2}\mu_k ||x_k - x_0||^2$ is employed to address the lack of strong convexity while μ_k is driven to zero at a suitable rate. Furthermore, a sequence of matrices $\{H_k\}$ is generated using a regularized L-BFGS update or (**rL-BFGS**) update. However, much of the extant schemes in this regime either have gaps in the rates (compared to deterministic counterparts) or cannot contend with nonsmoothness.

Quasi-Newton schemes for nonsmooth convex problems. There have been some attempts to apply (L-)BFGS directly to the deterministic nonsmooth convex problems. But the method may fail as shown in [28, 16, 24]; e.g. in [24], the authors consider minimizing $\frac{1}{2}||x||^2 + \max\{2|x_1| + x_2, 3x_2\}$ in \mathbb{R}^2 , BFGS takes a null step (steplength is zero) for different starting points and fails to converge to the optimal solution (0, -1) (except when initiated from (2, 2)) (See Fig. 1). Contending with nonsmoothness has been considered via a subgradient quasi-Newton method [49] for which global convergence can be recovered by identifying a descent direction and uti-

lizing a line search. An alternate approach [50] develops a



Figure 1: Lewis-Overton example

globally convergent trust region quasi-Newton method in which Moreau smoothing was employed. Yet, there appear to be neither non-asymptotic rate statements available nor considerations of stochasticity in nonsmooth regimes.

Gaps. Our research is motivated by several gaps. (i) First, can we develop smoothed generalizations of (**rL-BFGS**) that can contend with nonsmooth problems in a seamless fashion? (ii) Second, can one recover deterministic convergence rates (to the extent possible) by leveraging variance reduction techniques? (iii) Third, can one address nonsmoothness on stochastic convex optimization, which would allow for addressing more general problems as well as accounting for the presence of constraints? (iv) Finally, much of the prior results have stronger assumptions on the moment assumptions on the noise which require weakening to allow for wider applicability of the scheme.

1.1 A survey of literature

Before proceeding, we review some relevant prior research in stochastic quasi-Newton methods and variable sample-size schemes for stochastic optimization. In Table 1, we summarize the key advances in SQN methods where much of prior work focuses on strongly convex (with a few exceptions). Furthermore, from Table 2, it can be seen that an assumption of twice continuous differentiability and boundedness of eigenvalues on the true Hessian is often made. In addition, almost all results rely on having a uniform bound on the conditional second moment of stochastic gradient error.

	Convexity	Smooth	N _k	γ_k	Conver. rate	Iter. complex.	Oracle complex.
RES [29]	SC	1	N	1/k	$\mathcal{O}(1/k)$	-	-
Block BFGS [15] Stoch. L-BFGS [32]	SC	1	N (full grad periodically)	γ	$\mathcal{O}(ho^k)$	-	-
SQN [44]	NC	1	N	$k^{-0.5}$	$\mathcal{O}(1/\sqrt{k})$	$O(1/\epsilon^2)$	-
SdLBFGS-VR [44]	NC	1	N(full grad periodically $)$	γ	$\mathcal{O}(1/k)$	$\mathcal{O}(1/\epsilon)$	-
r-SQN [48]	С	1	1	$k^{-2/3+\varepsilon}$	$\mathcal{O}(1/k^{1/3-\varepsilon})$	-	-
SA-BFGS [51]	SC	1	N	γ_k	$\mathcal{O}(ho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon^2(\ln(1/\epsilon))^4)$
Progressive Batching [5]	NC	1	-	γ	$\mathcal{O}(1/k)$	-	-
Progressive Batching [5]	SC	1	-	γ	$\mathcal{O}(ho^k)$	-	-
(VS-SQN)	SC	1	$\lceil \rho^{-k} \rceil$	γ	$\mathcal{O}(ho^k)$	$\mathcal{O}(\kappa \ln(1/\epsilon))$	$\mathcal{O}(\kappa/\epsilon)$
(sVS-SQN)	SC	X	$\lceil \rho^{-k} \rceil$	γ	$\mathcal{O}(ho^k)$	$\mathcal{O}(\ln(1/\epsilon))$	$\mathcal{O}(1/\epsilon)$
$(\mathbf{rVS-SQN})$	С	1	$\lceil k^a \rceil$	$k^{-\varepsilon}$	$\mathcal{O}(1/k^{1-\varepsilon})$	$\mathcal{O}(1/\epsilon^{\frac{1}{1-\varepsilon}})$	$\mathcal{O}(1/\epsilon^{(3+\varepsilon)/(1-\varepsilon)})$
(rsVS-SQN)	C	X	$\lceil k^a \rceil$	$k^{-1/3+\varepsilon}$	$O(1/k^{1/3})$	$O(1/\epsilon^3)$	$\mathcal{O}\left(1/\epsilon^{(2+\varepsilon)/(1/3)}\right)$

Table 1: Comparing convergence rate of related schemes (note that a > 1)

	Convexity	Smooth	state-dep. noise	Assumptions
RES [29]	SC	1	×	$\underline{\lambda}\mathbf{I} \preceq H_k \preceq \overline{\lambda}\mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}, f \text{ is twice differentiable}$
Stoch. block BFGS [15]) $\mathbf{I} \neq \nabla^2 f(x) \neq \mathbf{\overline{J}} \mathbf{I} = 0 \neq \mathbf{\overline{J}} \neq \mathbf{\overline{J}} \mathbf{I}$ f is twice differentiable
Stoch. L-BFGS [32]	SC	1	×	$\underline{\lambda}\mathbf{I} \leq \mathbf{V} f(x) \leq \lambda\mathbf{I}, 0 \leq \underline{\lambda} \leq \lambda, f$ is twice differentiable
SQN for non convex [44]	NC	1	×	$\leq \nabla^2 f(x) \leq \overline{\lambda} \mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}, f \text{ is differentiable}$
SdLBFGS-VR [44]	NC	1	×	$\nabla^2 f(x) \preceq \overline{\lambda} \mathbf{I}, \overline{\lambda} \ge 0, f \text{ is twice differentiable}$
r-SQN [48]	C	1	×	$\underline{\lambda}\mathbf{I} \leq H_k \leq \overline{\lambda}\mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}, f \text{ is differentiable}$
SA-BEGS [51]	SC	1	×	$f_k(x)$ is standard self-concordant for every possible sam-
		•		pling, The Hessian is Lipschitz continuous,
				$\underline{\lambda}\mathbf{I} \preceq \nabla^2 f(x) \preceq \overline{\lambda}\mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}, \ f \text{ is } \mathbf{C}^2$
Progressive Batching [5]	NC	1	×	$\nabla^2 f(x) \preceq \overline{\lambda} \mathbf{I}, \overline{\lambda} \geq 0$, sample size is controlled by the
				exact inner product quasi-Newton test, f is C^2
Progressive Batching [5]	SC	1	×	$\underline{\lambda}\mathbf{I} \preceq \nabla^2 f(x) \preceq \overline{\lambda}\mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}, \text{ sample size controlled by}$
				exact inner product quasi-Newton test, f is C^2
(VS-SQN)	SC	1	1	$\underline{\lambda}\mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}_k, \ f \text{ is } \mathbf{C}^1$
(sVS-SQN)	SC	X	1	$\underline{\lambda}_k \mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}, 0 < \underline{\lambda}_k \leq \overline{\lambda}_k$
("VS SON)	C	1	1	$\underline{\lambda}\mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}_k, f \text{ is } \mathbf{C}^1, \text{ has quad. growth}$
(1 V 5-5Q1V)				prop.
			×	$\underline{\lambda}\mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}, f \text{ is } \mathbf{C}^1$
(reVS-SON)	C	¥	1	$\underline{\lambda}_k \mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}, 0 < \underline{\lambda} \leq \overline{\lambda}_k$, has quad. growth prop.
			×	$\underline{\lambda}_k \mathbf{I} \preceq H_k \preceq \overline{\lambda}_k \mathbf{I}$

Table 2: Comparing assumptions of related schemes

(i) Stochastic quasi-Newton (SQN) methods. QN schemes [26, 34] have proved enormously influential in solving nonlinear programs, motivating the use of stochastic Hessian information [6].

In 2014, Mokhtari and Riberio [29] introduced a regularized stochastic version of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method [12] by updating the matrix H_k using a modified BFGS update rule to ensure convergence while limited-memory variants [7, 30] and nonconvex generalizations [44] were subsequently introduced. In our prior work [48], an SQN method was presented for merely convex smooth problems, characterized by rates of $\mathcal{O}(1/k^{\frac{1}{3}-\varepsilon})$ and $\mathcal{O}(1/k^{1-\varepsilon})$ for the stochastic and deterministic case, respectively. In [47], via convolution-based smoothing to address nonsmoothness, we provide a.s. convergence guarantees and rate statements. (ii) Variance reduction schemes for stochastic optimization. Increasing sample-size schemes for finite-sum machine learning problems [13, 6] have provided the basis for a range of variance reduction schemes in machine learning [41, 45], amongst reduction others. By utilizing variable sample-size (VS) stochastic gradient schemes, linear convergence rates were obtained for strongly convex problems [42, 22] and these rates were subsequently improved (in a constant factor sense) through a VS-accelerated proximal method developed by Jalilzadeh et al. [21] (called (VS-APM)). In convex regimes, Ghadimi and Lan [14] developed an accelerated framework that admits the optimal rate of $\mathcal{O}(1/k^2)$ and the optimal oracle complexity (also see [22]), improving the rate statement presented in [20]. More recently, in [21], Jalilzadeh et al. present a smoothed accelerated scheme that admits the optimal rate of $\mathcal{O}(1/k)$ and optimal oracle complexity for nonsmooth problems, recovering the findings in [14] in the smooth regime. Finally, more intricate sampling rules are developed in [4, 37].

(iii) Variance reduced SQN schemes. Linear [27] and superlinear [51] convergence statements for variance reduced SQN schemes were provided in twice differentiable regimes under suitable assumptions on the Hessian. A (VS-SQN) scheme with L-BFGS [5] was presented in strongly convex regimes under suitable bounds on the Hessian.

1.2 Novelty and contributions

In this paper, we consider four variants of our proposed variable sample-size stochastic quasi-Newton method, distinguished by whether the function $F(x, \omega)$ is strongly convex/convex and smooth/nonsmooth. The vanilla scheme is given by

$$x_{k+1} := x_k - \gamma_k H_k \frac{\sum_{j=1}^{N_k} u_k(x_k, \omega_{j,k})}{N_k},$$
(2)

where H_k denotes an approximation of the inverse of the Hessian, $\omega_{j,k}$ denotes the j^{th} realization of ω at the k^{th} iteration, N_k denotes the sample-size at iteration k, and $u_k(x_k, \omega_{j,k})$ is given by one of the following: (i) (**VS-SQN**) where $F(.,\omega)$ is strongly convex and smooth, $u_k(x_k, \omega_{j,k}) \triangleq$ $\nabla_x F(x_k, \omega_{j,k})$; (ii) Smoothed (**VS-SQN**) or (**sVS-SQN**) where $F(.,\omega)$ is strongly convex and nonsmooth and $F_{\eta_k}(x,\omega)$ is a smooth approximation of $F(x,\omega)$, $u_k(x_k, \omega_{j,k}) \triangleq \nabla_x F_{\eta_k}(x_k, \omega_{j,k})$; (iii) Regularized (**VS-SQN**) or (**rVS-SQN**) where $F(.,\omega)$ is convex and smooth and $F_{\mu_k}(.,\omega)$ is a regularization of $F(.,\omega)$, $u_k(x_k, \omega_{j,k}) \triangleq \nabla_x F_{\mu_k}(x_k, \omega_{j,k})$; (iv) regularized and smoothed (**VS-SQN**) or (**rsVS-SQN**) where $F(.,\omega)$ is convex and possibly nonsmooth and $F_{\eta_k,\mu_k}(.,\omega)$ denotes a regularized smoothed approximation, $u_k(x_k, \omega_{j,k}) \triangleq \nabla_x F_{\eta_k,\mu_k}(x_k, \omega_{j,k})$. We recap these definitions in the relevant sections. We briefly discuss our contibutions and accentuate the novelty of our work.

(I) A regularized smoothed L-BFGS update. A regularized smoothed L-BFGS update (**rsL-BFGS**) is developed in Section 2.2, extending the realm of L-BFGS scheme to merely convex and possibly

nonsmooth regimes by integrating both regularization and smoothing. As a consequence, SQN techniques can now contend with merely convex and nonsmooth problems with convex constraints.

(II) Strongly convex problems. (II.i) (VS-SQN). In Section 3, we present a variable sample-size SQN scheme and prove that the convergence rate is $\mathcal{O}(\rho^k)$ (where $\rho < 1$) while the iteration and oracle complexity are proven to be $\mathcal{O}(\kappa^{m+1}\ln(1/\epsilon))$ and $\mathcal{O}(1/\epsilon)$, respectively. Notably, our findings are under a weaker assumption of either state-dependent noise (thereby extending the result from [5]) and do not necessitate assumptions of twice continuous differentiability [32, 15] or Lipschitz continuity of Hessian [51]. (II.ii). (sVS-SQN). By integrating a smoothing parameter, we extend (VS-SQN) to contend with nonsmooth but smoothable objectives. Via Moreau smoothing, we show that (sVS-SQN) retains the optimal rate and complexity statements of (VS-SQN). Additionally, in (II.i) and (II.ii), we derive rates that do not necessitate knowing either strong convexity or Lipschitzian parameters and rely on employing diminishing steplength sequences.

(III) Convex problems. (III.i) (**rVS-SQN**). A regularized (**VS-SQN**) scheme is presented in Section 4 based on the (**rL-BFGS**) update and admits a rate of $\mathcal{O}(1/k^{1-2\varepsilon})$ with an oracle complexity of $\mathcal{O}\left(\epsilon^{-\frac{3+\varepsilon}{1-\varepsilon}}\right)$, improving prior rate statements for SQN schemes for smooth convex problems and obviating prior inductive arguments. In addition, we show that (**rVS-SQN**) produces sequences that converge to the solution in an a.s. sense. Under a suitable growth property, these statements can be extended to the state-dependent noise regime. (III.ii) (**rsVS-SQN**). A regularized smoothed (**VS-SQN**) is presented that leverages the (**rsL-BFGS**) update and allows for developing rate $\mathcal{O}(k^{-\frac{1}{3}})$ amongst the first known rates for SQN schemes for nonsmooth convex programs. Again imposing a growth assumption allows for weakening the requirements to state-dependent noise.

(IV) Numerics. Finally, in Section 5, we apply the (**VS-SQN**) schemes on strongly convex/convex and smooth/nonsmooth stochastic optimization problems. In comparison with variable sample-size accelerated proximal gradient schemes, we observe that (**VS-SQN**) schemes compete well and outperform gradient schemes for ill-conditioned problems when the number of QN updates increases. In addition, SQN schemes do far better in computing sparse solutions, in contrast with standard subgradient and variance-reduced accelerated gradient techniques. Finally, via smoothing, (**VS-SQN**) schemes can be seen to resolve both nonsmooth and constrained problems.

Notation. $\mathbb{E}[\bullet]$ denotes the expectation with respect to the probability measure \mathbb{P} and we refer to $\nabla_x F(x, \xi(\omega))$ by $\nabla_x F(x, \omega)$. We denote the optimal objective value (or solution) of (1) by f^* (or x^*) and the set of the optimal solutions by X^* , which is assumed to be nonempty. For a vector $x \in \mathbb{R}^n$ and a nonempty set $X \subseteq \mathbb{R}^n$, the Euclidean distance of x from X is denoted by dist(x, X). Throughout the paper, unless specified otherwise, k denotes the iteration counter while K represents the total number of steps employed in the proposed methods.

2 Background and Assumptions

In Section 2.1, we provide some background on smoothing techniques and then proceed to define the regularized and smoothed L-BFGS method or (rsL-BFGS) update rule employed for generating the sequence of Hessian approximations H_k in Section 2.2. We conclude this section with a summary of the main assumptions in Section 2.3.

2.1 Smoothing of nonsmooth convex functions

We begin by defining of L-smoothness and (α, β) -smoothability [1].

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be L-smooth if it is differentiable and there exists an L > 0 such that $\|\nabla f(x) - \nabla f(y)\| \le L \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

Definition 2. $[(\alpha, \beta)$ -smoothable [1]] A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is (α, β) -smoothable if there exists a convex C^1 function $f_\eta : \mathbb{R}^n \to \mathbb{R}$ satisfying the following: (i) $f_\eta(x) \leq f(x) \leq f_\eta(x) + \eta\beta$ for all x; and (ii) $f_\eta(x)$ is α/η -smooth.

Some instances of (α, β) -smoothings [1] include the following: (i) If $f(x) \triangleq ||x||_2$ and $f_{\eta}(x) \triangleq \sqrt{||x||_2^2 + \eta^2} - \eta$, then f is (1, 1)-smoothable function; (ii) If $f(x) \triangleq \max\{x_1, x_2, \ldots, x_n\}$ and $f_{\eta}(x) \triangleq \eta \ln(\sum_{i=1}^n e^{x_i/\eta}) - \eta \ln(n)$, then f is $(1, \ln(n))$ -smoothable; (iii) If f is a proper, closed, and convex function and

$$f_{\eta}(x) \triangleq \min_{u} \left\{ f(u) + \frac{1}{2\eta} \|u - x\|^2 \right\},$$
 (3)

(referred to as Moreau proximal smoothing) [31], then f is $(1, B^2)$ -smoothable where B denotes a uniform bound on ||s|| where $s \in \partial f(x)$.

It may be recalled that Newton's method is the de-facto standard for computing a zero of a nonlinear equation [34] while variants such as semismooth Newton methods have been employed for addressing nonsmooth equations [10, 11]. More generally, in constrained regimes, such techniques take the form of interior point schemes which can be viewed as the application of Newton's method on the KKT system. Quasi-Newton variants of such techniques can then we applied when second derivatives are either unavailable or challenging to compute. However, in constrained stochastic regimes, there has been far less available via a direct application of quasi-Newton schemes. We consider a smoothing approach that leverages the unconstrained reformulation of a constrained convex program where X is a closed and convex set and $\mathbb{I}_X(x)$ is an indicator function:

$$\min_{x} f(x) + \mathbb{I}_X(x). \tag{P}$$

Then the smoothed problem can be represented as follows:

$$\min_{x} f(x) + \mathbb{I}_{X,\eta}(x), \tag{P}_{\eta}$$

where $\mathbb{I}_{X,\eta}(\cdot)$ denotes the Moreau-smoothed variant of $\mathbb{I}_X(\cdot)$ [31] defined as follows.

$$\mathbb{I}_{X,\eta}(x) \triangleq \min_{u \in \mathbb{R}^n} \left\{ \mathbb{I}_X(u) + \frac{1}{2\eta} \|x - u\|^2 \right\} = \frac{1}{2\eta} d_X^2(x), d_X(x) \triangleq (x - \operatorname{prox}_{\mathbb{I}_X}(x)) = (x - \Pi_X(x)), \quad (4)$$

 $\Pi_X(x) \triangleq \operatorname{argmin}_{y \in X}\{\|x - y\|^2\}$, and the second equality follows from [1, Ex. 6.53]. Note that $\mathbb{I}_{X,\eta}$ is continuously differentiable with gradient at x given by $\frac{1}{2\eta}\nabla_x(d_X^2(x)) = \frac{1}{\eta}(x - \operatorname{prox}_{\mathbb{I}_X}(x)) = \frac{1}{\eta}(x - \Pi_X(x))$. Our interest lies in reducing the smoothing parameter η after every iteration, a class of techniques (called *iterative smoothing schemes*) that have been applied for solving stochastic optimization [47, 21] and stochastic variational inequality problems [47]. Motivated by our recent work [21] in which a smoothed variable sample-size accelerated proximal gradient scheme is proposed for nonsmooth stochastic convex optimization, we consider a framework where at iteration k, an η_k -smoothed function f_{η_k} is utilized where the Lipschitz constant of $\nabla f_{\eta_k}(x)$ is $1/\eta_k$.

2.2 Regularized and Smoothed L-BFGS Update

When the function f is strongly convex but possibly nonsmooth, we adapt the standard L-BFGS scheme (by replacing the true gradient by a sample average) where the approximation of the inverse Hessian H_k is defined as follows using pairs (s_i, y_i) and η_i denotes a smoothing parameter:

$$s_{i} := x_{i} - x_{i-1},$$

$$y_{i} := \frac{\sum_{j=1}^{N_{i-1}} \nabla_{x} F_{\eta_{i}}(x_{i}, \omega_{j,i-1})}{N_{i-1}} - \frac{\sum_{j=1}^{N_{i-1}} \nabla_{x} F_{\eta_{i}}(x_{i-1}, \omega_{j,i-1})}{N_{i-1}},$$
(5)
$$(Strongly Convex (SC))$$

$$(SC)$$

$$(SC)$$

$$H_{k,j} := \left(\mathbf{I} - \frac{y_i s_i^T}{y_i^T s_i}\right)^T H_{k,j-1} \left(\mathbf{I} - \frac{y_i s_i^T}{y_i^T s_i}\right) + \frac{s_i s_i^T}{y_i^T s_i}, \quad i := k - 2(m-j), \ 1 \le j \le m, \ \forall i$$

where $H_{k,0} = \frac{s_k^T y_k}{y_k^T y_k} \mathbf{I}$. At iteration *i*, we generate $\nabla_x F_{\eta_i}(x_i, \omega_{j,i-1})$ and $\nabla_x F_{\eta_i}(x_{i-1}, \omega_{j,i-1})$, implying there are twice as many sampled gradients generated. Next, we discuss how the sequence of approximations H_k is generated when *f* is merely convex and not necessarily smooth. We overlay the regularized L-BFGS [29, 48] scheme with a smoothing, referring to the proposed scheme as the (**rsL-BFGS**) update. As in (**rL-BFGS**) [48], we update the regularization and smoothing parameters $\{\eta_k, \mu_k\}$ and matrix H_k at alternate iterations to keep the secant condition satisfied. We update the regularization parameter μ_k and smoothing parameter η_k as follows.

$$\begin{cases} \mu_k := \mu_{k-1}, & \eta_k := \eta_{k-1}, & \text{if } k \text{ is odd} \\ \mu_k < \mu_{k-1}, & \eta_k < \eta_{k-1}, & \text{otherwise.} \end{cases}$$
(6)

We construct the update in terms of s_i and y_i for convex problems,

$$s_i := x_i - x_{i-1},\tag{7}$$

$$y_{i} := \frac{\sum_{j=1}^{N_{i-1}} \nabla_{x} F_{\eta_{i}^{\delta}}(x_{i}, \omega_{j,i-1})}{N_{i-1}} - \frac{\sum_{j=1}^{N_{i-1}} \nabla_{x} F_{\eta_{i}^{\delta}}(x_{i-1}, \omega_{j,i-1})}{N_{i-1}} + \mu_{i}^{\bar{\delta}} s_{i}, \qquad (\text{Convex (C)})$$

where *i* is odd and $0 < \delta, \bar{\delta} \leq 1$ are scalars controlling the level of smoothing and regularization in updating matrix H_k , respectively. The update policy for H_k is given as follows:

$$H_k := \begin{cases} H_{k,m}, & \text{if } k \text{ is odd} \\ H_{k-1}, & \text{otherwise} \end{cases}$$

$$\tag{8}$$

where m < n (in large scale settings, $m \ll n$) is a fixed integer that determines the number of pairs (x_i, y_i) to be used to estimate H_k . The matrix $H_{k,m}$, for any $k \geq 2m - 1$, is updated using the following recursive formula:

$$H_{k,j} := \left(\mathbf{I} - \frac{y_i s_i^T}{y_i^T s_i}\right)^T H_{k,j-1} \left(\mathbf{I} - \frac{y_i s_i^T}{y_i^T s_i}\right) + \frac{s_i s_i^T}{y_i^T s_i}, \quad i := k - 2(m-j), \quad 1 \le j \le m, \quad \forall i, \qquad (9)$$

where $H_{k,0} = \frac{s_k^T y_k}{y_k^T y_k} \mathbf{I}$. It is important to note that our regularized method inherits the computational efficiency from (**L-BFGS**). Note that Assumption 3 holds for our choice of smoothing.

2.3 Main assumptions

A subset of our results require smoothness of $F(\cdot, \omega)$ as formalized by the next assumption.

Assumption 1. (a) The function $F(\cdot, \omega)$ is convex and continuously differentiable over \mathbb{R}^n for any $\omega \in \Omega$. (b) The function f is C^1 and L-smooth over \mathbb{R}^n .

We introduce the following assumptions of $F(\cdot, \omega)$, parts of which are imposed in a subset of results.

Assumption 2. (a) For every ω , $F(\cdot, \omega)$ is τ -strongly convex. (b) For every ω , $F(\cdot, \omega)$ is L-smooth. (c) $f(x) \triangleq g(x) + h(x)$, where $g(x) \triangleq \mathbb{E}[F(x, \omega)]$, $F(\cdot, \omega)$ is L-smooth and τ -strongly convex for every ω , and h is a closed, convex, and proper function.

In Sections 3.2 (II) and 4.2, we assume the following on the smoothed functions $F_{\eta}(\cdot, \omega)$.

Assumption 3. For any $\omega \in \Omega$, $F(\cdot, \omega)$ is $(1, \beta)$ smoothable, i.e. for any $\eta > 0$, there exists $F_{\eta}(\cdot, \omega)$ that is C^1 , convex, $\frac{1}{\eta}$ -smooth, and satisfies $F_{\eta}(z, \omega) \leq F(z, \omega) \leq F_{\eta}(z, \omega) + \eta\beta$ for any $z \in \mathbb{R}^n$ and any $\omega \in \Omega$.

Let $\mathcal{F}_k \triangleq \sigma\{x_0, \{\omega_{j,0}\}_{j=1}^{N_0}, \dots, \{\omega_{j,k}\}_{j=1}^{N_k}\}$. Now assume the following on the conditional second moment on the sampled gradient (in either the smooth or the smoothed regime) produced by the stochastic first-order oracle.

Assumption 4 (Moment requirements for state-dependent noise).

(Smooth) Suppose $\bar{w}_{k,N_k} \triangleq \nabla_x f(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F(x_k,\omega_{j,k})}{N_k}$.

(S-M) There exist
$$\nu_1, \nu_2 > 0$$
 such that $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k] \le \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k}$ a.s. for $k \ge 0$.

(S-B) For $k \geq 0$, $\mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] = 0$, a.s.

(Nonsmooth) Suppose
$$\bar{w}_{k,N_k} \triangleq \nabla f_{\eta_k}(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F_{\eta_k}(x_k, \omega_{j,k})}{N_k}, \ \eta_k > 0.$$

(NS-M) There exist $\nu_1, \nu_2 > 0$ such that $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k] \le \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k}$ a.s. for $k \ge 0$.

(NS-B) For $k \ge 0$, $\mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] = 0$, a.s.

(Structured smooth) Suppose
$$\bar{u}_k = \nabla_x g(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F(x_k, \omega_{j,k})}{N_k}$$
.

(SS-M) There exist ν_1, ν_2 such that $\mathbb{E}[\|\bar{u}_{k,N_k}\|^2 | \mathcal{F}_k] \leq \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k}$ a.s. for $k \geq 0$. (SS-B) For $k \geq 0$, $\mathbb{E}[\bar{u}_{k,N_k} | \mathcal{F}_k] = 0$, a.s.

Finally, we impose Assumption 1 on the sequence of Hessian approximations $\{H_k\}$. These properties follow when either the regularized update (**rL-BFGS**), the smoothed update (**sL-BFGS**), or the regularized smoothed update (**rsL-BFGS**) is employed (see Lemmas 1, 7, 8, and 10).

Property 1 (Properties of H_k). (i) H_k is \mathcal{F}_k -measurable; (ii) H_k is symmetric and positive definite and there exist $\underline{\lambda}_k, \overline{\lambda}_k > 0$ such that $\underline{\lambda}_k \mathbf{I} \leq H_k \leq \overline{\lambda}_k \mathbf{I}$ a.s. for all $k \geq 0$.

3 Smooth and nonsmooth strongly convex problems

In this section, we derive the rate and oracle complexity of the (rVS-SQN) scheme for smooth and nonsmooth strongly convex problems by considering the (VS-SQN) and (sVS-SQN) schemes.

3.1 Smooth strongly convex optimization

We begin by considering (1) when f is τ -strongly convex and L-smooth. Suppose κ is defined as $\kappa \triangleq L/\tau$. Throughout, we consider the (**VS-SQN**) scheme, defined next, where H_k is generated by the (**L-BFGS**) scheme.

$$x_{k+1} := x_k - \gamma_k H_k \frac{\sum_{j=1}^{N_k} \nabla_x F(x_k, \omega_{j,k})}{N_k}.$$
 (VS-SQN)

Next, we derive bounds on the eigenvalues of H_k under strong convexity (see Appendix for proof).

Lemma 1 (Properties of H_k produced by (L-BFGS)). Suppose Assumptions 1 and 2 (a,b) hold. Consider the (VS-SQN) method. Let s_i , y_i and H_k be given by (5), where $F_{\eta} = F$. Then H_k satisfies Property 1(S), with $\underline{\lambda}_k = \underline{\lambda} = \frac{1}{L(m+n)}$ and $\overline{\lambda}_k = \overline{\lambda} = \frac{((m+n)L)^{n+m-1}}{(n-1)!\tau^{n+m}}$ for all k.

Proposition 1 (Convergence in mean). Consider the iterates generated by the (VS-SQN) scheme. Suppose Assumptions 1, 2 (a,b), and 4 (S-M), (S-B) hold. In addition, suppose $\{N_k\}$ is an increasing sequence. Then the following inequality holds for all $k \ge 1$, where $N_0 \ge \frac{2\nu_1^2 \overline{\lambda}}{\tau^2 \underline{\lambda}}$ and $\gamma_k \triangleq \frac{1}{L\overline{\lambda}}$ for all k.

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*)\right] \le \left(1 - \frac{\tau \underline{\lambda}}{L\overline{\lambda}} + \frac{2\nu_1^2}{L\tau N_0}\right) \mathbb{E}\left[f(x_k) - f(x^*)\right] + \frac{2\nu_1^2 ||x^*||^2 + \nu_2^2}{2LN_k}.$$
Appendix.

Proof. See Appendix.

We now provide a result pivotal in deriving a rate and complexity statements under diminishing steplengths, an avenue that obviates knowing strong convexity and Lipschitzian parameters.

Lemma 2. [[39], Lemma 5] Suppose $\{u_k\}$ is a nonnegative sequence, where

$$u_{k+1} \le \left(1 - \frac{c}{k^s}\right)u_k + \frac{d}{k^t}, \quad k \ge 0$$

$$\tag{10}$$

where 0 < s < 1, s < t, and c, d > 0. Then for $k \ge K$,

$$u_k \le \frac{d}{ck^{t-s}} + o\left(\frac{1}{k^{t-s}}\right).$$

Theorem 1 (Optimal rate and oracle complexity). Consider the iterates generated by the (VS-SQN) scheme. Suppose Assumptions 1, 2 (a,b) and 4 (S-M), (S-B) hold. In addition, suppose $\gamma_k = \frac{1}{L\lambda}$ for all k.

(i) If $a \triangleq \left(1 - \frac{\tau \lambda}{L\overline{\lambda}} + \frac{2\nu_1^2}{L\tau N_0}\right)$, $N_k \triangleq \lceil N_0 \rho^{-k} \rceil$ where $\rho < 1$ and $N_0 \ge \frac{2\nu_1^2 \overline{\lambda}}{\tau^2 \underline{\lambda}}$. Then for every $k \ge 1$ and some scalar C, the following holds: $\mathbb{E}\left[f(x_{K+1}) - f(x^*)\right] \le C(\max\{a, \rho\})^K$.

(ii) Suppose x_{K+1} is an ϵ -solution such that $\mathbb{E}[f(x_{K+1}) - f^*] \leq \epsilon$. Then the iteration and oracle complexity of (**VS-SQN**) are $\mathcal{O}(\kappa^{m+1}\ln(1/\epsilon))$ and $\mathcal{O}(\frac{\kappa^{m+1}}{\epsilon})$, respectively implying that $\sum_{k=1}^{K} N_k \leq \mathcal{O}\left(\frac{\kappa^{m+n+1}}{\epsilon}\right)$.

(iii) Suppose $\gamma_k = k^{-s}$ and $N_k = \lceil k^{p-s} \rceil$ for every k where 0 < s < 1 and s < p. In addition, suppose $c \triangleq \frac{\lambda \tau}{2}$ and $d \triangleq \frac{\overline{\lambda}^2 L(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2}$. Then for K sufficiently large, we have that

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*)\right] \le \left(\frac{d}{ck^p}\right) + o\left(\frac{1}{k^p}\right), \quad k \ge K.$$

Proof. See Appendix.

We prove a.s. convergence of iterates by using the super-martingale convergence lemma from [39].

Lemma 3 (super-martingale convergence theorem). Let $\{v_k\}$ be a sequence of nonnegative random variables, where $\mathbb{E}[v_0] < \infty$ and let $\{\chi_k\}$ and $\{\beta_k\}$ be deterministic scalar sequences such that $0 \leq \chi_k \leq 1$ and $\beta_k \geq 0$ for all $k \geq 0$, $\sum_{k=0}^{\infty} \chi_k = \infty$, $\sum_{k=0}^{\infty} \beta_k < \infty$, and $\lim_{k\to\infty} \frac{\beta_k}{\chi_k} = 0$, and $\mathbb{E}[v_{k+1} \mid \mathcal{F}_k] \leq (1-\chi_k)v_k + \beta_k$ a.s. for all $k \geq 0$. Then, $v_k \to 0$ almost surely as $k \to \infty$.

Theorem 2 (a.s. convergence under strong convexity). Consider the iterates generated by the (VS-SQN) scheme. Suppose Assumptions 1, 2 (a,b), and 4 (S-M), (S-B) hold. In addition, suppose $\gamma_k = \frac{1}{L\lambda}$ for all $k \ge 0$. Let $\{N_k\}_{k\ge 0}$ be an increasing sequence such that $\sum_{k=0}^{\infty} \frac{1}{N_k} < \infty$ and $N_0 > \frac{2\nu_1^2\lambda}{\tau^2\lambda}$. Then $\lim_{k\to\infty} f(x_k) = f(x^*)$ almost surely.

Proof. From Assumption 2 (a,b), f is τ -strongly convex and L-smooth. Recall that in (31), we derived the following for $k \ge 0$.

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*) \mid \mathcal{F}_k\right] \le \left(1 - \tau \gamma_k \underline{\lambda} + \frac{2\nu_1^2}{L\tau N_k}\right) \left(f(x_k) - f(x^*)\right) + \frac{2\nu_1^2 ||x^*||^2 + \nu_2^2}{2LN_k}.$$

If $v_k \triangleq f(x_k) - f(x^*)$, $\chi_k \triangleq \tau \gamma_k \underline{\lambda} - \frac{2\nu_1^2}{L\tau N_k}$, $\beta_k \triangleq \frac{2\nu_1^2 \|x^*\|^2 + \nu_2^2}{2LN_k}$, $\gamma_k = \frac{1}{L\overline{\lambda}}$, and $\{N_k\}_{k\geq 0}$ be an increasing sequence such that $\sum_{k=0}^{\infty} \frac{1}{N_k} < \infty$ where $N_0 > \frac{2\nu_1^2\overline{\lambda}}{\tau^2\underline{\lambda}}$, (e.g. $N_k \ge \lceil N_0 k^{1+\epsilon} \rceil$) the requirements of Lemma 3 are seen to be satisfied. Hence, $f(x_k) - f(x^*) \to 0$ a.s. as $k \to \infty$ and by strong convexity of f, it follows that $\|x_k - x^*\|^2 \to 0$ a.s.

Having presented the variable sample-size SQN method, we now consider the special case where $N_k = 1$. Similar to Proposition 1, the following inequality holds for $N_k = 1$:

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*)\right] \leq f(x_k) - f(x^*) - \gamma_k \left(1 - \frac{L}{2} \gamma_k \overline{\lambda}\right) \|H_k^{1/2} \nabla f(x_k)\|^2 + \frac{\gamma_k^2 \overline{\lambda}^2 L(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{2} \\
\leq \left(1 - 2\gamma_k \frac{L^2}{\tau} \overline{\lambda} (1 - \frac{L}{2} \gamma_k \overline{\lambda})\right) (f(x_k) - f(x^*)) + \frac{\gamma_k^2 \overline{\lambda}^2 L(\nu_1^2 \|x_k - x^* + x^*\|^2 + \nu_2^2)}{2} \\
\leq \left(1 - 2\gamma_k \overline{\lambda} \frac{L^2}{\tau} + \gamma_k^2 \overline{\lambda}^2 \frac{L^3}{\tau} + \frac{2\nu_1^2 \gamma_k^2 \overline{\lambda}^2 L}{\tau}\right) (f(x_k) - f(x^*)) + \frac{\gamma_k^2 \overline{\lambda}^2 L(2\nu_1^2 \|x^*\|^2 + \nu_2^2)}{2}, \quad (11)$$

where the second inequality is obtained by using Lipschitz continuity of $\nabla f(x)$ and the strong convexity of f(x). Next, to obtain the convergence rate of SQN, we use the following lemma [46].

Lemma 4. Suppose $e_{k+1} \leq (1 - 2a\gamma_k + \gamma_k^2 b)e_k + \gamma_k^2 c$ for all $k \geq 1$. Let $\gamma_k = \gamma/k, \ \gamma > 1/(2a), K \triangleq \lceil \frac{\gamma^2 b}{2a\gamma - 1} \rceil + 1$ and $Q(\gamma, K) \triangleq \max\left\{ \frac{\gamma^2 c}{2a\gamma - \gamma^2 b/K - 1}, Ke_K \right\}$. Then $\forall k \geq K, \ e_k \leq \frac{Q(\gamma, K)}{k}$.

Now from inequality (11) and Lemma 4, the following proposition follows.

Proposition 2 (Rate of convergence of SQN with $N_k = 1$). Consider the iterates generated by the (VS-SQN) scheme. Suppose Assumptions 1–2, and 4 (S-M),(S-B) hold. Let $a = \frac{L^2 \overline{\lambda}}{\tau}$, $b = \frac{\overline{\lambda}^2 L^3 + 2\nu_1^2 \overline{\lambda}^2 L}{\tau}$ and $c = \frac{\overline{\lambda}^2 L(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2}$. Then, $\gamma_k = \frac{\gamma}{k}$, $\gamma > \frac{1}{L\overline{\lambda}}$ and $N_k = 1$ the following holds: $\mathbb{E}[f(x_{k+1}) - f(x^*)] \leq \frac{Q(\gamma, K)}{k}$, where $Q(\gamma, K) \triangleq \max\left\{\frac{\gamma^2 c}{2a\gamma - \gamma^2 b/K - 1}, K(f(x_K) - f(x^*))\right\}$ and $K \triangleq \lceil \frac{\gamma^2 b}{2a\gamma - 1} \rceil + 1$.

Remark 1. It is worth emphasizing that the proof techniques, while aligned with avenues adopted in [6, 13, 5], extend results in [5] to the regime of state-dependent noise [13]. We also observe that in the analysis of deterministic/stochastic first-order methods, any non-asymptotic rate statements rely on utilizing problem parameters (e.g. the strong convexity modulus, Lipschitz constants, etc.). Similarly, in the context of QN methods, obtaining non-asymptotic bounds also requires $\underline{\lambda}$ and $\overline{\lambda}$ (cf. [5, Theorem 3.1], [3, Theorem 3.4], and [44, Lemma 2.2]) since the impact of H_k needs to be addressed. One avenue for weakening the dependence on such parameters lies in using line search schemes. However when the problem is expectation-valued, the steplength arising from a line search leads to a dependence between the steplength (which is now random) and the direction. Consequently, standard analysis fails and one has to appeal to more refined analysis (cf. [18, 8, 35, 43]). This remains the focus of future work.

Remark 2. Note that Assumption 2, $F(\cdot, \omega)$ is L-smooth and τ -strongly convex for every ω and is commonly employed in stochastic quasi-Newton schemes; cf. [3, 7, 30]. This is necessitated by the need to provide bounds on the eigenvalues of H_k . While deriving such Lipschitz constants is challenging, there are instances when this is possible.

(i) Consider the following problem.

$$\min_{x \in X} f(x) \triangleq \mathbb{E}\left[\frac{1}{2}x^T Q(\omega)x + c^T x\right].$$

If one has access to the form of $Q(\omega)$, then by leveraging Jensen's inequality and under suitable integrability requirements, one may be able to prove that f is L-smooth. More generally, if $\nabla_x f(x) = \mathbb{E}[\nabla_x f(x,\omega)]$ and $\nabla_x f(\cdot,\omega)$ is $L(\omega)$ -Lipschitz where $L(\omega)$ has finite mean given by L, then one may conclude that f is L-smooth. We may either derive L (if we have access to the structure of $L(\omega)$) or postulate the existence of L if $L(\omega)$ has finite expectation.

(ii) Suppose we consider the setting when $F(\cdot,\xi)$ is the ℓ_2 -squared loss function; i.e. $F(x,\xi_i) \triangleq \frac{1}{2}(a_i^T x - b_i)^2$ where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ denote the *i*th pair of the input and output data, respectively, and $\xi_i \triangleq (a_i; b_i) \in \mathbb{R}^{n+1}$. We obtain that $\nabla F(x,\xi_i) = (a_i^T x - b_i)a_i = (a_i a_i^T)x - b_i a_i$. This implies that $\nabla F(\cdot,\xi_i)$ is a Lipschitz continuous mapping with the parameter $L_{\xi_i} := ||a_i a_i^T||_2$. If ξ is assumed to have finite support based on an empirical distribution, we have that

$$\nabla f(x) = \mathbb{E}\left[\nabla F(x,\xi)\right] = \frac{1}{N} \sum_{i=1}^{N} \nabla F(x,\xi_i).$$

From the preceding relation and that every sample path function is $L(\xi)$ -smooth, for any $x, y \in \mathbb{R}^n$:

$$\begin{aligned} \|\nabla f(x) - \nabla f(y)\|_{2} &= \|\mathbb{E}\left[\nabla F(x,\xi_{i}) - \nabla F(y,\xi_{i})\right]\|_{2} \leq \frac{\sum_{i=1}^{N} \|\nabla F(x,\xi_{i}) - \nabla F(y,\xi_{i})\|_{2}}{N} \\ &\leq \frac{\sum_{i=1}^{N} \|a_{i}a_{i}^{T}\|_{2} \|x - y\|_{2}}{N}. \end{aligned}$$

This implies that $f(x) \triangleq \mathbb{E}[F(x,\xi)]$ has Lipschitz gradients with the parameter $L \triangleq \frac{1}{N} \sum_{i=1}^{N} ||a_i a_i^T||_2$. We, however, note that the computation of L may become costly in cases where either N or n are massive. This can be addressed to some extent by deriving an upper bound on L as follows:

$$L \triangleq \frac{\sum_{i=1}^{N} \|a_i a_i^T\|_2}{N} \le \frac{\sum_{i=1}^{N} \|a_i\|_{\infty} \|a_i\|_2}{N} \le \frac{\max_i \|a_i\|_{\infty}}{N} \sum_{i=1}^{N} \|a_i\|_2 \le \frac{\|A\|_F \max_i \|a_i\|_{\infty}}{\sqrt{N}}$$

where $A \in \mathbb{R}^{N \times n}$ is defined as $A = [a_1^T; \ldots; a_N^T]$ and $||A||_F$ denotes the Frobenius norm of A.

3.2 Nonsmooth strongly convex optimization

Consider (1) where f is a strongly convex but nonsmooth function. In this section, we focus on the case where $f(x) \triangleq h(x) + g(x)$, h is a deterministic, closed, convex, and proper function, g is L-smooth and strongly convex, $F(\cdot, \omega)$ is a convex function for every ω , where $g(x) \triangleq \mathbb{E}[F(x, \omega)]$. We begin by noting that the Moreau envelope of f, denoted by f_{η} and defined as (3), retains both the minimizers of f as well as its strong convexity as captured by the following result based on [38, Lemma 2.19].

Lemma 5. Consider a convex, closed, and proper function f and its Moreau envelope f_{η} . Then the following hold: (i) x^* is a minimizer of f over \mathbb{R}^n if and only if x^* is a minimizer of $f_{\eta}(x)$; (ii) f is σ -strongly convex on \mathbb{R}^n if and only if f_{η} is $\frac{\sigma}{\eta\sigma+1}$ -strongly convex on \mathbb{R}^n .

Consequently, it suffices to minimize the (smooth) Moreau envelope with a fixed smoothing parameter η , as shown in the next result. For notational simplicity, we choose m = 1 but the rate results hold for m > 1 and define $f_{N_k}(x) \triangleq h(x) + \frac{1}{N_k} \sum_{j=1}^{N_k} F(x, \omega_{j,k})$. Throughout this subsection, we consider the smoothed variant of (**VS-SQN**), referred to the (**sVS-SQN**) scheme, defined next, where H_k is generated by the (**sL-BFGS**) update rule, $\nabla_x f_{\eta_k}(x_k)$ denotes the gradient of the Moreau-smoothed function, given by $\frac{1}{\eta_k}(x_k - \operatorname{prox}_{\eta_k,f}(x_k))$, while $\nabla_x f_{\eta_k,N_k}(x_k)$, the gradient of the Moreau-smoothed and sample-average function $f_{N_k}(x)$, is defined as $\frac{1}{\eta_k}(x_k - \operatorname{prox}_{\eta_k,f_{N_k}}(x_k))$ and $\bar{w}_k \triangleq \nabla_x f_{\eta_k,N_k}(x_k) - \nabla_x f_{\eta_k}(x_k)$. Consequently the update rule for x_k becomes the following.

$$x_{k+1} := x_k - \gamma_k H_k(\nabla_x f_{\eta_k}(x_k) + \bar{w}_k), \qquad (sVS-SQN)$$

and the update rule of L-BFGS that we use in this section is as follows:

$$s_{i} := x_{i} - x_{i-1}, \quad y_{i} := \nabla_{x} f_{\eta_{i}, N_{i-1}}(x_{i}) - \nabla_{x} f_{\eta_{i}, N_{i-1}}(x_{i-1}),$$
$$H_{k,j} := \left(\mathbf{I} - \frac{y_{i} s_{i}^{T}}{y_{i}^{T} s_{i}}\right)^{T} H_{k,j-1} \left(\mathbf{I} - \frac{y_{i} s_{i}^{T}}{y_{i}^{T} s_{i}}\right) + \frac{s_{i} s_{i}^{T}}{y_{i}^{T} s_{i}}, \quad i := k - 2(m-j), \ 1 \le j \le m, \ \forall i.$$

At each iteration of (**sVS-SQN**), the error in the gradient is captured by \bar{w}_k . We show that \bar{w}_k satisfies Assumption 4 (NS) by utilizing the following assumption on the gradient of function.

Assumption 5. Suppose there exists $\nu_1, \nu_2 > 0$ such that for all $i \ge 1$, $\mathbb{E}[\|\bar{u}_k\|^2 | \mathcal{F}_k] \le \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k}$ holds almost surely, where $\bar{u}_k = \nabla_x g(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F(x_k, \omega_{j,k})}{N_k}$.

Lemma 6. Suppose Assumptions 1, 2 (a,c), and 4 hold. Let f_{η} denote the Moreau smoothed approximation of f and $\eta < 2/L$. Then, $\mathbb{E}[\|\bar{w}_k\|^2 \mid \mathcal{F}_k] \leq \frac{4(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{\tau^2 \eta^2 N_k}$ for all $k \geq 0$.

Proof. We begin by noting that $f_{N_k}(x)$ is convex. Consider the two problems:

$$\operatorname{prox}_{\eta,f}(x_k) \triangleq \arg\min_{u} \left[f(u) + \frac{1}{2\eta} \|x_k - u\|^2 \right],$$
(12)

$$\operatorname{prox}_{\eta, f_{N_k}}(x_k) \triangleq \arg\min_{u} \left[f_{N_k}(u) + \frac{1}{2\eta} \|x_k - u\|^2 \right].$$
(13)

Suppose x_k^* and $x_{N_k}^*$ denote the optimal unique solutions of (12) and (13), respectively. From the definition of Moreau smoothing, it follows that

$$\bar{w}_{k} = \nabla_{x} f_{\eta, N_{k}}(x_{k}) - \nabla_{x} f_{\eta}(x_{k}) = \frac{1}{\eta} (x_{k} - \operatorname{prox}_{\eta, f_{N_{k}}}(x_{k})) - \frac{1}{\eta} (x_{k} - \operatorname{prox}_{\eta, f}(x_{k}))$$
$$= \operatorname{prox}_{\eta, f_{N_{k}}}(x_{k}) - \operatorname{prox}_{\eta, f}(x_{k}) = \frac{1}{\eta} (x_{N_{k}}^{*} - x_{k}^{*}),$$

which implies $\mathbb{E}[\|\bar{w}_k\|^2 | \mathcal{F}_k] = \frac{1}{\eta^2} \mathbb{E}[\|x_k^* - x_{N_k}^*\|^2 | \mathcal{F}_k]$. The following inequalities are a consequence of invoking strong convexity of f, convexity of f_{N_k} and the optimality conditions of (12) and (13):

$$f(x_{N_k}^*) + \frac{1}{2\eta} \|x_{N_k}^* - x_k\|^2 \ge f(x_k^*) + \frac{1}{2\eta} \|x_k^* - x_k\|^2 + \frac{1}{2} \left(\tau + \frac{1}{\eta}\right) \|x_k^* - x_{N_k}^*\|^2,$$

$$f_{N_k}(x_k^*) + \frac{1}{2\eta} \|x_k^* - x_k\|^2 \ge f_{N_k}(x_{N_k}^*) + \frac{1}{2\eta} \|x_{N_k}^* - x_k\|^2 + \frac{1}{2\eta} \|x_{N_k}^* - x_k^*\|^2.$$

Adding the above inequalities, we have that

$$f(x_{N_k}^*) - f_{N_k}(x_{N_k}^*) + f_{N_k}(x_k^*) - f(x_k^*) \ge \left(\frac{\tau}{2} + \frac{1}{\eta}\right) \|x_{N_k}^* - x_k^*\|^2.$$

From the definition of $f_{N_k}(x_k)$ and $\beta \triangleq \frac{\tau}{2} + \frac{1}{\eta}$, and by the convexity of $F(\cdot, \omega)$ in x for a.e. ω and and L-smoothness of function g, we may prove the following.

$$\begin{split} \beta \|x_k^* - x_{N_k}^*\|^2 &\leq f(x_{N_k}^*) - f_{N_k}(x_{N_k}^*) + f_{N_k}(x_k^*) - f(x_k^*) \\ &= \frac{\sum_{j=1}^{N_k} (g(x_{N_k}^*) - F(x_{N_k}^*, \omega_{j,k}))}{N_k} + \frac{\sum_{j=1}^{N_k} (F(x_k^*, \omega_{j,k}) - g(x_k^*))}{N_k} \\ &\leq \frac{\sum_{j=1}^{N_k} \left(g(x_k^*) + \nabla_x g(x_k^*)^T(x_{N_k}^* - x_k^*) + \frac{L}{2} \|x^* - x_{N_k}^*\|^2 - F(x_k^*, \omega_{j,k}) - \nabla_x F(x_k^*, \omega_{j,k})^T(x_{N_k}^* - x_k^*) \right)}{N_k} \\ &+ \frac{\sum_{j=1}^{N_k} (F(x_k^*, \omega_{j,k}) - g(x_k^*))}{N_k} = \frac{\sum_{j=1}^{N_k} (\nabla_x g(x_k^*) - \nabla_x F(x_k^*, \omega_{j,k}))^T(x_{N_k}^* - x_k^*)}{N_k} + \frac{L}{2} \|x^* - x_{N_k}^*\|^2 \\ &= \bar{u}_k^T (x_{N_k}^* - x_k^*) + \frac{L}{2} \|x_k^* - x_{N_k}^*\|^2. \end{split}$$

Consequently, by taking conditional expectations and using Assumption 5, we have the following.

$$\begin{split} \mathbb{E}[\beta \|x_k^* - x_{N_k}^*\|^2 \mid \mathcal{F}_k] &= \mathbb{E}[\bar{u}_k^T (x_{N_k}^* - x_k^*) \mid \mathcal{F}_K] + \frac{L}{2} \mathbb{E}[\|x_k^* - x_{N_k}^*\|^2 \mid \mathcal{F}_k] \\ &\leq \frac{1}{\tau} \mathbb{E}[\|\bar{u}_k\|^2 \mid \mathcal{F}_k] + (\frac{\tau}{4} + \frac{L}{2}) \mathbb{E}[\|x_k^* - x_{N_k}^*\|^2 \mid \mathcal{F}_k] \\ \implies \mathbb{E}[\|x_k^* - x_{N_k}^*\|^2 \mid \mathcal{F}_k] &\leq \frac{4}{\tau^2} \mathbb{E}[\|\bar{u}_k\|^2 \mid \mathcal{F}_k] \leq \frac{4}{\tau^2} \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k}, \text{ if } \eta < 2/L. \end{split}$$

We may then conclude that $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k] \leq \frac{4(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{\eta^2 \tau^2 N_k}.$

Next, we derive bounds on the eigenvalues of H_k under strong convexity (similar to Lemma 1).

Lemma 7 (Properties of H_k produced by (sL-BFGS)). Suppose Assumptions 1 and 2 (a,c) hold. Let s_i , y_i and H_k be given by (5). Then H_k satisfies Property 1 with $\underline{\lambda}_k = \frac{\eta_k}{(m+n)}$ and $\overline{\lambda}_k = \left(\frac{n+m}{\eta_k}\right)^{m+n-1} \left(\frac{1}{(n-1)!\tau^{n+m}}\right)$.

We now show that under Moreau smoothing, a linear rate of convergence is retained.

Theorem 3. Consider the iterates generated by the (**sVS-SQN**) scheme where $\eta_k = \eta$ for all k. Suppose Assumptions 1, 2(a,c), 4(SS-M, SS-B) and 5 hold. In addition, suppose m = 1, $\eta \leq \min\{2/L, (8(n+1)^2/\tau^2)^{1/3}\}, d \triangleq 1 - \frac{\tau^2 \eta^3}{8(n+1)^2(1+\eta\tau)}, N_k \triangleq \lceil N_0 q^{-k} \rceil$ for all $k \geq 1$, $N_0 \geq \frac{5(1+\eta\tau)^2}{\tau^4} \frac{(\nu_1^2)}{\eta\gamma \lambda} \gamma \triangleq \frac{\tau\eta^2}{4(1+n)}, c_1 \triangleq \max\{q, d\}, and c_2 \triangleq \min\{q, d\}.$ (i) (a) Suppose $c_1 > c_2$. Then $\mathbb{E}[\|x_{k+1} - x^*\|^2] \leq Dc_1^{k+1}$ for all k where where

$$D \triangleq \left(\frac{2\mathbb{E}[f_{\eta}(x_0) - f_{\eta}(x^*)](1 + \eta\tau)}{\tau}\right) + \left(\frac{10(1 + \eta\tau)(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{4\tau^3 \eta N_0(c_1 - c_2)}\right).$$

(b) Suppose $c_1 = c_2$. Then $\mathbb{E}[\|x_{K+1} - x^*\|^2] \le D\tilde{d}^{K+1}$ where $\tilde{d} \in (d, 1), \ \tilde{D} > \frac{1}{\ln(\tilde{d}/d)^e}$, and

$$D \triangleq \left(\frac{2\mathbb{E}[f_{\eta}(x_0) - f_{\eta}(x^*)](1+\eta\tau)}{\tau}\right) + \left(\frac{10(1+\eta\tau)(2\nu_1^2 \|x^*\|^2 + \nu_2^2)\tilde{D}}{4\tau^3\eta N_0 d}\right)$$

(ii) Suppose x_{K+1} is an ϵ -solution such that $\mathbb{E}[f(x_{K+1}) - f^*] \leq \epsilon$. Then, the iteration and oracle complexity of computing x_{K+1} are $\mathcal{O}(\ln(1/\epsilon))$ steps and $\mathcal{O}(1/\epsilon)$, respectively.

(iii) Suppose $\gamma_k = k^{-s}$ and $N_k = \lceil k^p \rceil$ where 0 < s < 1 and s < p. Suppose $c \triangleq \frac{\lambda \tau}{2(1+\eta\tau)}$, $d \triangleq \left(\frac{\overline{\lambda}^2}{\eta} + \frac{\eta}{4}\right) \frac{(8\nu_1^2 \|x^*\|^2 + 4\nu_2^2)}{\tau^2 \eta^2}$, and $\tilde{d} \triangleq \frac{(1+\eta\tau)d}{\tau}$. Then for K sufficiently large, we have that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{\tilde{d}}{ck^p} + o\left(\frac{1}{k^p}\right), \text{ for } k \ge K.$$

Proof. (i) From Lipschitz continuity of ∇f_{η} and update (**sVS-SQN**), we have the following:

$$\begin{split} f_{\eta}(x_{k+1}) &\leq f_{\eta}(x_{k}) + \nabla f_{\eta}(x_{k})^{T}(x_{k+1} - x_{k}) + \frac{1}{2\eta} \|x_{k+1} - x_{k}\|^{2} \\ &= f_{\eta}(x_{k}) + \nabla f_{\eta}(x_{k})^{T} \left(-\gamma H_{k}(\nabla f_{\eta}(x_{k}) + \bar{w}_{k,N_{k}})\right) + \frac{1}{2\eta}\gamma^{2} \|H_{k}(\nabla f_{\eta}(x_{k}) + \bar{w}_{k,N_{k}})\|^{2} \\ &= f_{\eta}(x_{k}) - \gamma \nabla f_{\eta}(x_{k})^{T} H_{k} \nabla f_{\eta}(x_{k}) - \gamma \nabla f_{\eta}(x_{k})^{T} H_{k} \bar{w}_{k,N_{k}} + \frac{\gamma^{2}}{2\eta} \|H_{k} \nabla f_{\eta}(x_{k})\|^{2} \\ &+ \frac{\gamma^{2}}{2\eta} \|H_{k} \bar{w}_{k,N_{k}}\|^{2} + \frac{\gamma^{2}}{\eta} H_{k} \nabla f_{\eta}(x_{k})^{T} H_{k} \bar{w}_{k,N_{k}} \\ &\leq f_{\eta}(x_{k}) - \gamma \nabla f_{\eta}(x_{k})^{T} H_{k} \nabla f_{\eta}(x_{k}) + \frac{\eta}{4} \|\bar{w}_{k,N_{k}}\|^{2} + \frac{\gamma^{2}}{\eta} \|\nabla f_{\eta}(x_{k})^{T} H_{k}\|^{2} + \frac{\gamma^{2}}{2\eta} \|H_{k} \nabla f_{\eta}(x_{k})\|^{2} \\ &+ \frac{\overline{\lambda}^{2} \gamma^{2}}{2\eta} \|\bar{w}_{k,N_{k}}\|^{2} + \frac{\gamma^{2}}{2\eta} \|H_{k} \nabla f_{\eta}(x_{k})\|^{2} + \frac{\overline{\lambda}^{2} \gamma^{2}}{2\eta} \|\bar{w}_{k,N_{k}}\|^{2}, \end{split}$$

where in the last inequality, we use the fact that $2a^Tb \leq \frac{\eta}{2\gamma} ||a||^2 + \frac{2\gamma}{\eta} ||b||^2$. From Lemma 6, $\mathbb{E}[||\bar{w}_k||^2 | \mathcal{F}_k] \leq \frac{4(\nu_1^2 ||x_k||^2 + \nu_2^2)}{\eta^2 \tau^2 N_k}$. Now by taking conditional expectations with respect to \mathcal{F}_k , using Lemma 7, we obtain the following.

$$\mathbb{E}\left[f_{\eta}(x_{k+1}) - f_{\eta}(x_{k}) \mid \mathcal{F}_{k}\right] \\
\leq -\gamma \nabla f_{\eta}(x_{k})^{T} H_{k} \nabla f_{\eta}(x_{k}) + \frac{2\gamma^{2}}{\eta} \|H_{k} \nabla f_{\eta}(x_{k})\|^{2} + \left(\frac{\overline{\lambda}^{2} \gamma^{2}}{\eta} + \frac{\eta}{4}\right) \frac{4(\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2})}{\tau^{2} \eta^{2} N_{k}} \qquad (14) \\
= \gamma \nabla f_{\eta}(x_{k})^{T} H_{k}^{1/2} \left(-I + \frac{2\gamma}{\eta} H_{k}^{T}\right) H_{k}^{1/2} \nabla f_{\eta}(x_{k}) + \left(\frac{\overline{\lambda}^{2} \gamma^{2}}{\eta} + \frac{\eta}{4}\right) \frac{4(\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2})}{\tau^{2} \eta^{2} N_{k}} \\
\leq -\gamma \left(1 - \frac{2\gamma}{\eta} \overline{\lambda}\right) \|H_{k}^{1/2} \nabla f_{\eta}(x_{k})\|^{2} + \left(\frac{\overline{\lambda}^{2} \gamma^{2}}{\eta} + \frac{\eta}{4}\right) \frac{8(\nu_{1}^{2} \|x_{k} - x^{*}\|^{2})}{\tau^{2} \eta^{2} N_{k}} \\
+ \left(\frac{\overline{\lambda}^{2} \gamma^{2}}{\eta} + \frac{\eta}{4}\right) \frac{(8\nu_{1}^{2} \|x^{*}\|^{2} + 4\nu_{2}^{2})}{\tau^{2} \eta^{2} N_{k}} \\
= \frac{-\gamma}{2} \|H_{k}^{1/2} \nabla f_{\eta}(x_{k})\|^{2} + \left(\frac{5\eta}{16}\right) \frac{8(\nu_{1}^{2} \|x_{k} - x^{*}\|^{2})}{\tau^{2} \eta^{2} N_{k}} + \left(\frac{5\eta}{16}\right) \frac{(2\nu_{1}^{2} \|x^{*}\|^{2} + 4\nu_{2}^{2})}{\tau^{2} \eta^{2} N_{k}},$$

where the second equality follows from $\gamma = \frac{\eta}{4\overline{\lambda}}$. Since f_{η} is $\tau/(1+\eta\tau)$ -strongly convex (Lemma 5), $\|\nabla f_{\eta}(x_k)\|^2 \ge 2\tau/(1+\eta\tau) \left(f_{\eta}(x_k) - f_{\eta}(x^*)\right)$ and $f_{\eta}(x_k) - f_{\eta}(x^*) \ge \frac{\tau}{(1+\eta\tau)} \|x_k - x^*\|^2$. Consequently, by subtracting $f_{\eta}(x^*)$ from both sides by invoking Lemma 7, we obtain

$$\mathbb{E}\left[f_{\eta}(x_{k+1}) - f_{\eta}(x^{*}) \mid \mathcal{F}_{k}\right] \\
\leq f_{\eta}(x_{k}) - f_{\eta}(x^{*}) - \frac{\gamma \underline{\lambda}}{2} \|\nabla f_{\eta}(x_{k})\|^{2} + \frac{5(\nu_{1}^{2} \|x_{k} - x^{*}\|^{2})}{2\tau^{2} \eta N_{k}} + \frac{5(2\nu_{1}^{2} \|x^{*}\|^{2} + \nu_{2}^{2})}{4\tau^{2} \eta N_{k}} \\
\leq \left(1 - \frac{\tau}{1 + \eta\tau} \gamma \underline{\lambda} + \frac{(1 + \eta\tau)}{\tau} \frac{(5\nu_{1}^{2})}{2\tau^{2} \eta N_{k}}\right) (f_{\eta}(x_{k}) - f_{\eta}(x^{*})) + \frac{5(2\nu_{1}^{2} \|x^{*}\|^{2} + \nu_{2}^{2})}{4\tau^{2} \eta N_{k}} \\
\leq \left(1 - \frac{\tau}{1 + \eta\tau} \gamma \underline{\lambda} + \frac{5\nu_{1}^{2}(1 + \eta\tau)}{2\tau^{3} \eta N_{0}}\right) (f_{\eta}(x_{k}) - f_{\eta}(x^{*})) + \frac{5(2\nu_{1}^{2} \|x^{*}\|^{2} + \nu_{2}^{2})}{4\tau^{2} \eta N_{k}}.$$
(15)

By observing the following relations,

$$-\frac{\tau}{1+\eta\tau}\gamma\underline{\lambda} + \frac{5\nu_1^2(1+\eta\tau)}{2\tau^3\eta N_0} \le -\frac{1}{2}\frac{\tau}{1+\eta\tau}\gamma\underline{\lambda} \iff \frac{5\nu_1^2(1+\eta\tau)}{2\tau^3\eta N_0} \le \frac{1}{2}\frac{\tau}{1+\eta\tau}\gamma\underline{\lambda}$$
$$\iff N_0 \ge \frac{5\nu_1^2(1+\eta\tau)^2}{\tau^4\eta\gamma\underline{\lambda}},$$

we may then conclude that if $N_0 \geq \frac{5(1+\eta\tau)^2}{\tau^4} \frac{(\nu_1^2)}{\eta\gamma\underline{\lambda}}$, we have that

$$\mathbb{E}\left[f_{\eta}(x_{k+1}) - f_{\eta}(x^{*}) \mid \mathcal{F}_{k}\right] \leq \left(1 - \frac{\tau}{2(1+\eta\tau)}\gamma\underline{\lambda}\right)\left(f_{\eta}(x_{k}) - f_{\eta}(x^{*})\right) + \frac{4(2\nu_{1}^{2}||x^{*}||^{2} + \nu_{2}^{2})}{4\tau^{2}\eta N_{k}}.$$

Then by taking unconditional expectations, we obtain the following sequence of inequalities:

$$\mathbb{E}\left[f_{\eta}(x_{k+1}) - f_{\eta}(x^{*})\right] \leq \left(1 - \frac{\tau}{2(1+\eta\tau)}\gamma\underline{\lambda}\right) \mathbb{E}\left[f_{\eta}(x_{k}) - f_{\eta}(x^{*})\right] + \frac{5(2\nu_{1}^{2}\|x^{*}\|^{2} + \nu_{2}^{2})}{4\tau^{2}\eta N_{k}} \\
= \left(1 - \frac{(\tau\eta)^{n+2}(n-1)!}{8(n+1)^{n+1}(1+\eta\tau)}\right) \mathbb{E}\left[f_{\eta}(x_{k}) - f_{\eta}(x^{*})\right] + \frac{5(2\nu_{1}^{2}\|x^{*}\|^{2} + \nu_{2}^{2})}{4\tau^{2}\eta N_{k}}, \quad (16)$$

where the last equality arises from choosing $\underline{\lambda} = \frac{\eta}{1+n}$, $\overline{\lambda} = \frac{1+n}{\tau\eta}$ (by Lemma 7 for m = 1), $\gamma = \frac{\eta}{4\overline{\lambda}} = \frac{\tau\eta^2}{4(1+n)}$ and using the fact that $N_k \ge N_0$ for all k > 0. Let $d \triangleq \left(1 - \frac{(\tau\eta)^{n+2}(n-1)!}{8(n+1)^{n+1}(1+\eta\tau)}\right)$ and $b_k \triangleq \frac{5(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{4\tau^2 \eta N_k}$. Then for $\eta < \frac{1}{\tau} \left(\frac{8(n+1)^{n+1}}{(n-1)!}\right)^{1/(n+2)}$, we have d < 1. Furthermore, by recalling that $N_k = \lceil N_0 q^{-k} \rceil$, it follows that $b_k \leq \frac{5(2\nu_1^2 ||x^*||^2 + \nu_2^2)q^k}{4\tau^2 \eta N_0}$, we obtain the following bound from (16).

$$\mathbb{E}\left[f_{\eta}(x_{K+1}) - f_{\eta}(x^{*})\right] \leq d^{K+1}\mathbb{E}\left[f_{\eta}(x_{0}) - f_{\eta}(x^{*})\right] + \sum_{i=0}^{K} d^{K-i}b_{i}$$
$$\leq d^{K+1}\mathbb{E}\left[f_{\eta}(x_{0}) - f_{\eta}(x^{*})\right] + \frac{5(2\nu_{1}^{2}||x^{*}||^{2} + \nu_{2}^{2})}{4\tau^{2}\eta N_{0}}\sum_{i=0}^{K} d^{K-i}q^{i}.$$

We now consider three cases.

Case (1) q < d. If q < d, then $\sum_{i=0}^{K} d^{K-i}q^i = d^K \sum_{i=0}^{K} (q/d)^i \leq d^K \left(\frac{1}{1-q/d}\right)$. Since f_η retains the minimizers of f, $\frac{\tau}{2(1+\eta\tau)} \|x_k - x^*\|^2 \leq f_\eta(x_k) - f_\eta(x^*)$) by strong convexity of f_η , implying the following following.

$$\frac{\tau}{2(1+\eta\tau)}\mathbb{E}[\|x_{K+1}-x^*\|^2] \le d^{K+1}\mathbb{E}[f_\eta(x_0)-f_\eta(x^*)] + d^K\frac{5(2\nu_1^2\|x^*\|^2+\nu_2^2)}{4\tau^2\eta N_0\left(1-q/d\right)}$$

Dividing both sides by $\frac{\tau}{2(1+\eta\tau)}$, the desired result is obtained.

$$\mathbb{E}[\|x_{K+1} - x^*\|^2] \le d^{K+1} \left(\frac{2\mathbb{E}[f_\eta(x_0) - f_\eta(x^*)](1+\eta\tau)}{\tau}\right) + d^K \frac{10(1+\eta\tau)(2\nu_1^2\|x^*\|^2 + \nu_2^2)}{4\tau^3\eta N_0 (1-q/d)} = Dd^{K+1},$$

where $D \triangleq \left(\frac{2\mathbb{E}[f_\eta(x_0) - f_\eta(x^*)](1+\eta\tau)}{\tau}\right) + \left(\frac{10(1+\eta\tau)(2\nu_1^2\|x^*\|^2 + \nu_2^2)}{4\tau^3\eta N_0 (d-q)}\right).$

Case (2) q > d. Similarly, if d < q, $\mathbb{E}[||x_{K+1} - x^*||^2] \le Dq^{K+1}$ where

$$D \triangleq \left(\frac{2\mathbb{E}[f_{\eta}(x_0) - f_{\eta}(x^*)](1 + \eta\tau)}{\tau}\right) + \left(\frac{10(1 + \eta\tau)(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{4\tau^3 \eta N_0(q - d)}\right).$$

Case (3) q = d. Then, we have that

$$\sum_{i=0}^{K} d^{K-i} q^{i} = (K+1)d^{K} \le \frac{1}{d}(K+1)d^{k+1} \le \frac{1}{d}\tilde{D}\tilde{d}^{K+1},$$

where $\tilde{d} \in (d, 1)$ and $\tilde{D} > \frac{1}{\ln(\tilde{d}/d)^e}$. Consequently, we have that $\mathbb{E}[\|x_{K+1} - x^*\|^2] \leq D\tilde{d}^{K+1}$ where

$$D \triangleq \left(\frac{2\mathbb{E}[f_{\eta}(x_0) - f_{\eta}(x^*)](1 + \eta\tau)}{\tau}\right) + \left(\frac{10(1 + \eta\tau)(2\nu_1^2 ||x^*||^2 + \nu_2^2)\tilde{D}}{4\tau^3\eta N_0 d}\right).$$

(ii) To find an x_{K+1} such that $\mathbb{E}[||x_{K+1} - x^*||^2] \leq \epsilon$, suppose d < q with no loss of generality. Then for some C > 0, $Cq^K \leq \epsilon$, implying that $K = \lceil \log_{1/q}(C/\epsilon) \rceil$. It follows that

$$\sum_{k=0}^{K} N_k \le \sum_{k=0}^{1+\log_{1/q}\left(\frac{C}{\epsilon}\right)} N_0 q^{-k} = N_0 \frac{\left(\left(\frac{1}{q}\right)\left(\frac{1}{q}\right)^{\log 1/q\left(\frac{C}{\epsilon}\right)} - 1\right)}{(1/q-1)} \le N_0 \frac{\left(\frac{C}{\epsilon}\right)}{1-q} = \mathcal{O}(1/\epsilon).$$

(iii) Similar to (14) and (15) we can obtain:

$$\mathbb{E}\left[f_{\eta}(x_{k+1}) - f_{\eta}(x_{k}) \mid \mathcal{F}_{k}\right] \leq -\gamma_{k} \left(1 - \frac{2\gamma_{k}}{\eta}\overline{\lambda}\right) \|H_{k}^{1/2}\nabla f_{\eta}(x_{k})\|^{2} + \left(\frac{\overline{\lambda}^{2}\gamma_{k}^{2}}{\eta} + \frac{\eta}{4}\right) \frac{8(\nu_{1}^{2}\|x_{k} - x^{*}\|^{2})}{\tau^{2}\eta^{2}N_{k}} + \left(\frac{\overline{\lambda}^{2}\gamma_{k}^{2}}{\eta} + \frac{\eta}{4}\right) \frac{(8\nu_{1}^{2}\|x^{*}\|^{2} + 4\nu_{2}^{2})}{\tau^{2}\eta^{2}N_{k}}$$

$$\implies E\left[f_{\eta}(x_{k+1}) - f_{\eta}(x^{*}) \mid \mathcal{F}_{k}\right] \leq \left(1 - \gamma_{k}\left(1 - \frac{2\gamma_{k}}{\eta}\overline{\lambda}\right)\frac{\lambda\tau}{1+\eta\tau} + \frac{1+\tau\eta}{\tau}\left(\frac{\overline{\lambda}^{2}\gamma_{k}^{2}}{\eta} + \frac{\eta}{4}\right)\frac{8\nu_{1}^{2}}{\eta^{2}\tau^{2}N_{k}}\right)\left(f_{\eta}(x_{k}) - f_{\eta}(x^{*})\right) \\ + \left(\frac{\overline{\lambda}^{2}\gamma_{k}^{2}}{\eta} + \frac{\eta}{4}\right)\frac{(8\nu_{1}^{2}\|x^{*}\|^{2} + 4\nu_{2}^{2})}{\tau^{2}\eta^{2}N_{k}}.$$

Since γ_k is a diminishing sequence and N_k is an increasing sequence, for sufficiently large K we have that $\frac{2\gamma_k^2 \overline{\lambda} \underline{\lambda} \tau}{\eta(1+\eta\tau)} + \frac{1+\tau\eta}{\tau} \left(\frac{\overline{\lambda}^2 \gamma_k^2}{\eta} + \frac{\eta}{4}\right) \frac{8\nu_1^2}{\eta^2 \tau^2 N_k} \leq \frac{1}{2} \left(\frac{\gamma_k \underline{\lambda} \tau}{1+\eta\tau}\right)$. Therefore, we obtain:

$$E[f_{\eta}(x_{k+1}) - f_{\eta}(x^*) \mid \mathcal{F}_k] \le (1 - c\gamma_k) (f(x_k) - f(x^*)) + \frac{d}{N_k}, \text{ for } k \ge K,$$

where we use the fact that $\gamma_k \leq 1$ and we set $c = \frac{\underline{\lambda}\tau}{2(1+\eta\tau)}$ and $d = \left(\frac{\overline{\lambda}^2}{\eta} + \frac{\eta}{4}\right) \frac{(8\nu_1^2 ||x^*||^2 + 4\nu_2^2)}{\tau^2 \eta^2}$. Since $\gamma_k = \frac{1}{k^s}$ and $N_k = \lceil k^{p+s} \rceil$ where 0 < s < 1 and p > 0, by taking unconditional expectations,

$$\mathbb{E}\left[f_{\eta}(x_k) - f_{\eta}(x^*)\right] \le \frac{d}{ck^p} + o\left(\frac{1}{k^p}\right), \text{ for } k \ge K.$$

By leveraging the $\frac{\tau}{1+\eta\tau}$ -strong convexity of f_{η} , we may claim that

$$\mathbb{E}\left[\|x_k - x^*\|^2\right] \le \frac{(1+\eta\tau)d}{c\tau k^p} + o\left(\frac{1}{k^p}\right), \text{ for } k \ge K.$$

4 Smooth and nonsmooth convex optimization

In this section, we weaken the strong convexity requirement and analyze the rate and oracle complexity of $(\mathbf{rVS}-\mathbf{SQN})$ and $(\mathbf{rsVS}-\mathbf{SQN})$ in smooth and nonsmooth regimes, respectively.

4.1 Smooth convex optimization

Consider the setting when f is an L-smooth convex function. In such an instance, a regularization of f and its gradient can be defined as follows.

Definition 3 (Regularized function and gradient map). Given a sequence $\{\mu_k\}$ of positive scalars, the function f_{μ_k} and its gradient $\nabla f_k(x)$ are defined as follows for any $x_0 \in \mathbb{R}^n$:

$$f_{\mu_k}(x) \triangleq f(x) + \frac{\mu_k}{2} ||x - x_0||^2$$
, for any $k \ge 0$, $\nabla f_{\mu_k}(x) \triangleq \nabla f(x) + \mu_k(x - x_0)$, for any $k \ge 0$.

Then, f_{μ_k} and ∇f_{μ_k} satisfy the following: (i) f_{μ_k} is μ_k -strongly convex; (ii) f_{μ_k} has Lipschitzian gradients with parameter $L+\mu_k$; (iii) f_{μ_k} has a unique minimizer over \mathbb{R}^n , denoted by x_k^* . Moreover, for any $x \in \mathbb{R}^n$ [39, sec. 1.3.2],

$$2\mu_k(f_{\mu_k}(x) - f_{\mu_k}(x_k^*)) \le \|\nabla f_{\mu_k}(x)\|^2 \le 2(L + \mu_k) \left(f_{\mu_k}(x) - f_{\mu_k}(x_k^*)\right).$$

We consider the following update rule (**rVS-SQN**), where H_k is generated by **rL-BFGS** scheme.

$$x_{k+1} := x_k - \gamma_k H_k \frac{\sum_{j=1}^{N_k} \nabla_x F_{\mu_k}(x_k, \omega_{j,k})}{N_k}.$$
 (rVS-SQN)

For a subset of the results, we assume quadratic growth property.

Assumption 6. (Quadratic growth) Suppose that the function f has a nonempty set X^* of minimizers. There exists $\alpha > 0$ such that $f(x) \ge f(x^*) + \frac{\alpha}{2} \operatorname{dist}^2(x, X^*)$ holds for all $x \in \mathbb{R}^n$:

In the next lemma the bound for eigenvalues of H_k is derived (see Lemma 6 in [48]).

Lemma 8 (Properties of Hessian approximations produced by (rL-BFGS)). Consider the (rVS-SQN) method. Let H_k be given by the update rule (8)-(9) with $\eta_k = 0$ for all k, and s_i and y_i are defined in (7). Suppose μ_k is updated according to the procedure (6). Let Assumption. 1(a,b) hold. Then the following hold.

- (a) For any odd k > 2m, $s_k^T y_k > 0$; (b) For any odd k > 2m, $H_k y_k = s_k$;
- (c) For any k > 2m, H_k satisfies Assumption 1(S), $\underline{\lambda} = \frac{1}{(m+n)(L+\mu_0^{\overline{\delta}})}$, $\lambda = \frac{(m+n)^{n+m-1}(L+\mu_0^{\overline{\delta}})^{n+m-1}}{(n-1)!}$ and $\overline{\lambda}_k = \lambda \mu_k^{-\overline{\delta}(n+m)}$, for scalars $\delta, \overline{\delta} > 0$. Then for all k, we have that $H_k = H_k^T$ and $\mathbb{E}[H_k \mid \mathcal{F}_k] = H_k$ and $\underline{\lambda}\mathbf{I} \leq H_k \leq \overline{\lambda}_k\mathbf{I}$ both hold in an a.s. fashion.

Lemma 9 (An error bound). Consider the (VS-SQN) method and suppose Assumptions 1, 4(S-M), 4(S-B), 1(S) and 6 hold. Suppose { μ_k } is a non-increasing sequence, and γ_k satisfies

$$\gamma_k \le \frac{\underline{\lambda}}{\overline{\lambda}_k^2 (L+\mu_0)}, \quad for \ all \ k \ge 0.$$
 (17)

Then, the following inequality holds for all k:

$$\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) \mid \mathcal{F}_k] - f^* \leq (1 - \underline{\lambda}\mu_k\gamma_k)(f_{\mu_k}(x_k) - f^*) + \frac{\underline{\lambda}dist^2(x_0, X^*)}{2}\mu_k^2\gamma_k + \frac{(L + \mu_k)\overline{\lambda}_k^2(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{2N_k}\gamma_k^2.$$
(18)

Proof. By the Lipschitzian property of ∇f_{μ_k} , update rule (**rVS-SQN**) and Def. 3, we obtain

$$f_{k}(x_{k+1}) \leq f_{\mu_{k}}(x_{k}) + \nabla f_{\mu_{k}}(x_{k})^{T}(x_{k+1} - x_{k}) + \frac{(L + \mu_{k})}{2} ||x_{k+1} - x_{k}||^{2}$$

$$\leq f_{\mu_{k}}(x_{k}) - \gamma_{k} \underbrace{\nabla f_{\mu_{k}}(x_{k})^{T} H_{k}(\nabla f_{\mu_{k}}(x_{k}) + \bar{w}_{k,N_{k}})}_{\text{Term 1}} + \frac{(L + \mu_{k})}{2} \gamma_{k}^{2} \underbrace{||H_{k}(\nabla f_{\mu_{k}}(x_{k}) + \bar{w}_{k,N_{k}})||^{2}}_{\text{Term 2}}, \quad (19)$$

where $\bar{w}_{k,N_k} \triangleq \frac{\sum_{j=1}^{N_k} (\nabla_x F_{\mu_k}(x_k, \omega(\omega_{j,k})) - \nabla f_{\mu_k}(x_k))}{N_k}$. Next, we estimate the conditional expectation of Terms 1 and 2. From Assumption 1, we have

Term
$$1 = \nabla f_{\mu_k}(x_k)^T H_k \nabla f_{\mu_k}(x_k) + \nabla f_{\mu_k}(x_k)^T H_k \bar{w}_{k,N_k} \ge \underline{\lambda} \|\nabla f_{\mu_k}(x_k)\|^2 + \nabla f_{\mu_k}(x_k)^T H_k \bar{w}_{k,N_k}.$$

Thus, taking conditional expectations, from (19), we obtain

$$\mathbb{E}[\text{Term 1} \mid \mathcal{F}_k] \geq \underline{\lambda} \|\nabla f_{\mu_k}(x_k)\|^2 + \mathbb{E}[\nabla f_{\mu_k}(x_k)^T H_k \bar{w}_{k,N_k} \mid \mathcal{F}_k] \\ = \underline{\lambda} \|\nabla f_{\mu_k}(x_k)\|^2 + \nabla f_{\mu_k}(x_k)^T H_k \mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] = \underline{\lambda} \|\nabla f_{\mu_k}(x_k)\|^2,$$
(20)

where $\mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] = 0$ and $\mathbb{E}[H_k \mid \mathcal{F}_k] = H_k$ a.s. Similarly, invoking Assumption 1(S), we may bound Term 2.

Term 2 =
$$(\nabla f_{\mu_k}(x_k) + \bar{w}_{k,N_k})^T H_k^2 (\nabla f_{\mu_k}(x_k) + \bar{w}_{k,N_k}) \le \overline{\lambda_k}^2 ||\nabla f_{\mu_k}(x_k) + \bar{w}_{k,N_k}||^2$$

= $\overline{\lambda_k}^2 (||\nabla f_{\mu_k}(x_k)||^2 + ||\bar{w}_{k,N_k}||^2 + 2\nabla f_{\mu_k}(x_k)^T \bar{w}_{k,N_k}).$

Taking conditional expectations in the preceding inequality and using Assumption 4 (S-M), 4 (S-B), we obtain

$$\mathbb{E}[\text{Term } 2 \mid \mathcal{F}_k] \leq \overline{\lambda}_k^2 \Big(\|\nabla f_{\mu_k}(x_k)\|^2 + \mathbb{E}[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k] \\ + 2\nabla f_{\mu_k}(x_k)^T \mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] \Big) \leq \overline{\lambda}_k^2 \left(\|\nabla f_{\mu_k}(x_k)\|^2 + \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{N_k} \right).$$
(21)

By taking conditional expectations in (19), and by (20)-(21),

$$\mathbb{E}[f_{\mu_{k}}(x_{k+1}) \mid \mathcal{F}_{k}] \leq f_{\mu_{k}}(x_{k}) - \gamma_{k}\underline{\lambda} \|\nabla\mu_{k}(x_{k})\|^{2} + \overline{\lambda}_{k}^{2} \frac{(L+\mu_{k})}{2} \gamma_{k}^{2} \left(\|\nabla f_{\mu_{k}}(x_{k})\|^{2} + \frac{\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2}}{N_{k}} \right) \\ \leq f_{\mu_{k}}(x_{k}) - \frac{\gamma_{k}\underline{\lambda}}{2} \|\nabla f_{\mu_{k}}(x_{k})\|^{2} \left(2 - \frac{\overline{\lambda}_{k}^{2} \gamma_{k}(L+\mu_{k})}{\underline{\lambda}} \right) + \overline{\lambda}_{k}^{2} \frac{(L+\mu_{k})}{2} \frac{\gamma_{k}^{2}(\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2})}{N_{k}}.$$

From (17), $\gamma_k \leq \frac{\lambda}{\overline{\lambda}_k^2(L+\mu_0)}$ for any $k \geq 0$. Since $\{\mu_k\}$ is a non-increasing sequence, it follows that

$$\gamma_k \leq \frac{\underline{\lambda}}{\overline{\lambda}_k^2(L+\mu_k)} \implies 2 - \frac{\overline{\lambda}_k^2 \gamma_k(L+\mu_k)}{\underline{\lambda}} \geq 1.$$

Hence, the following holds.

$$\mathbb{E}[f_{\mu_{k}}(x_{k+1}) \mid \mathcal{F}_{k}] \leq f_{\mu_{k}}(x_{k}) - \frac{\gamma_{k}\underline{\lambda}}{2} \|\nabla f_{\mu_{k}}(x_{k})\|^{2} + \overline{\lambda}_{k}^{2} \frac{(L+\mu_{k})}{2} \frac{\gamma_{k}^{2}(\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2})}{N_{k}}$$

$$\stackrel{\text{(iii) in Def. 3}}{\leq} f_{\mu_{k}}(x_{k}) - \underline{\lambda}\mu_{k}\gamma_{k}(f_{\mu_{k}}(x_{k}) - f_{\mu_{k}}(x_{k}^{*})) + \overline{\lambda}_{k}^{2} \frac{(L+\mu_{k})}{2} \frac{\gamma_{k}^{2}(\nu_{1}^{2} \|x_{k}\|^{2} + \nu_{2}^{2})}{N_{k}}.$$

By using Definition 3 and non-increasing property of $\{\mu_k\}$,

$$\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) \mid \mathcal{F}_k] \leq \mathbb{E}[f_{\mu_k}(x_{k+1}) \mid \mathcal{F}_k] \Longrightarrow \\ \mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) \mid \mathcal{F}_k] \leq f_{\mu_k}(x_k) - \underline{\lambda}\mu_k \gamma_k (\overbrace{f_{\mu_k}(x_k) - f_{\mu_k}(x_k^*)}^{\text{Term 3}}) + \overline{\lambda}_k^2 \frac{(L+\mu_k)}{2} \frac{\gamma_k^2 (\nu_1^2 ||x_k||^2 + \nu_2^2)}{N_k}.$$
(22)

Next, we derive a lower bound for Term 3. Since x_k^* is the unique minimizer of f_{μ_k} , we have $f_{\mu_k}(x_k^*) \leq f_{\mu_k}(x^*)$. Therefore, invoking Definition 3, for an arbitrary optimal solution $x^* \in X^*$,

$$f_{\mu_k}(x_k) - f_{\mu_k}(x_k^*) \ge f_{\mu_k}(x_k) - f_{\mu_k}(x^*) = f_{\mu_k}(x_k) - f^* - \frac{\mu_k}{2} ||x^* - x_0||^2$$

From the preceding relation and (22), we have

$$\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) \mid \mathcal{F}_k] \le f_{\mu_k}(x_k) - \underline{\lambda}\mu_k\gamma_k\mathbb{E}[f_{\mu_k}(x_k) - f^*] + \frac{\underline{\lambda}\|x^* - x_0\|^2\mu_k^2\gamma_k}{2} + \frac{(L+\mu_k)\overline{\lambda}_k^2(\nu_1^2\|x_k\|^2 + \nu_2^2)\gamma_k^2}{2N_k}.$$

By subtracting f^* from both sides and by noting that this inequality holds for all $x^* \in X^*$ where X^* denotes the solution set, the desired result is obtained.

We now derive the rate for sequences produced by (**rVS-SQN**) under the following assumption.

Assumption 7. Let the positive sequences $\{N_k, \gamma_k, \mu_k, t_k\}$ satisfy the following conditions:

(a) $\{\mu_k\}, \{\gamma_k\}$ are non-increasing sequences such that $\mu_k, \gamma_k \to 0$; $\{t_k\}$ is an increasing sequence;

(b)
$$\left(1 - \underline{\lambda}\mu_k\gamma_k + \frac{2(L+\mu_0)\overline{\lambda}_k^2\nu_1^2\gamma_k^2}{N_k\alpha}\right)t_{k+1} \le t_k, \ \forall k \ge \tilde{K} \ for \ some \ \tilde{K} \ge 1;$$

(c) $\sum_{k=0}^{\infty}\mu_k^2\gamma_k t_{k+1} = \bar{c}_0 < \infty; \ (d) \ \sum_{k=0}^{\infty}\frac{\mu_k^{-2\bar{\delta}(n+m)}\gamma_k^2}{N_k}t_{k+1} = \bar{c}_1 < \infty;$

Theorem 4 (Convergence of (rVS-SQN) in mean). Consider the (rVS-SQN) scheme and suppose Assumptions 1, 4(S-M), 4(S-B), 1(S), 6 and 7 hold. There exists $\tilde{K} \ge 1$ and scalars \bar{c}_0, \bar{c}_1 (defined in Assumption 7) such that the following inequality holds for all $K \ge \tilde{K} + 1$:

$$\mathbb{E}[f(x_K) - f^*] \le \frac{t_{\tilde{K}}}{t_K} \mathbb{E}[f_{\mu_{\tilde{K}}}(x_{\tilde{K}}) - f^*] + \frac{\bar{c}_0 + \bar{c}_1}{t_K}.$$
(23)

Proof. We begin by noting that Assumption 7(a,b) implies that (18) holds for $k \ge \tilde{K}$, where \tilde{K} is defined in Assumption 7(b). Since the conditions of Lemma 9 are met, taking expectations on both sides of (18):

$$\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) - f^*] \le (1 - \underline{\lambda}\mu_k\gamma_k) \mathbb{E}[f_{\mu_k}(x_k) - f^*] + \frac{\underline{\lambda}\operatorname{dist}^2(x_0, X^*)}{2} \mu_k^2 \gamma_k + \frac{(L + \mu_0)\overline{\lambda}_k^2(\nu_1^2 \| x_k - x^* + x^* \|^2 + \nu_2^2)}{2N_k} \gamma_k^2 \quad \forall k \ge \tilde{K}.$$

Now by using the quadratic growth property i.e. $||x_k - x^*||^2 \leq \frac{2}{\alpha} (f(x) - f(x^*))$ and the fact that $||x_k - x^* + x^*||^2 \leq 2||x_k - x^*||^2 + 2||x^*||^2$, we obtain the following relationship

$$\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) - f^*] \le \left(1 - \underline{\lambda}\mu_k \gamma_k + \frac{2(L+\mu_0)\overline{\lambda}_k^2 \nu_1^2 \gamma_k^2}{N_k \alpha}\right) \mathbb{E}[f_{\mu_k}(x_k) - f^*] + \frac{\underline{\lambda} \text{dist}^2(x_0, X^*)}{2} \mu_k^2 \gamma_k + \frac{(L+\mu_0)\overline{\lambda}_k^2 (2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2N_k} \gamma_k^2.$$

By multiplying both sides by t_{k+1} , using Assumption 7(b) and $\overline{\lambda}_k = \lambda \mu_k^{-\overline{\delta}(n+m)}$, we obtain

$$t_{k+1}\mathbb{E}[f_{\mu_{k+1}}(x_{k+1}) - f^*] \le t_k\mathbb{E}[f_{\mu_k}(x_k) - f^*] + A_1\mu_k^2\gamma_k t_{k+1} + \frac{A_2\mu_k^{-2\delta(n+m)}}{N_k}\gamma_k^2 t_{k+1}, \qquad (24)$$

where $A_1 \triangleq \frac{\lambda \operatorname{dist}^2(x_0, X^*)}{2}$ and $A_2 \triangleq \frac{(L+\mu_0)\lambda^2(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2}$. By summing (24) from $k = \tilde{K}$ to K-1, for $K \ge \tilde{K} + 1$, and dividing both sides by t_K , we obtain

$$\mathbb{E}[f_{\mu_{K}}(x_{K}) - f^{*}] \leq \frac{t_{\tilde{K}}}{t_{K}} \mathbb{E}[f_{\mu_{\tilde{K}}}(x_{\tilde{K}}) - f^{*}] + \frac{\sum_{k=\tilde{K}}^{K-1} A_{1} \mu_{k}^{2} \gamma_{k} t_{k+1}}{t_{K}} + \frac{\sum_{k=\tilde{K}}^{K-1} A_{2} \mu_{k}^{-2\delta(n+m)} \gamma_{k}^{2} t_{k+1} N_{k}^{-1}}{t_{K}}.$$

From Assumption 7(c,d), $\sum_{k=\tilde{K}}^{K-1} \left(A_1 \mu_k^2 \gamma_k t_{k+1} + A_2 \mu_k^{-2\bar{\delta}(n+m)} \gamma_k^2 \frac{t_{k+1}}{N_k} \right) \leq A_1 \bar{c}_0 + A_2 \bar{c}_1.$ Therefore, by using the fact that $f(x_K) \leq f_{\mu_K}(x_K)$, we obtain $\mathbb{E}[f(x_K) - f^*] \leq \frac{t_{\tilde{K}}}{t_K} \mathbb{E}[f_{\mu_{\tilde{K}}}(x_{\tilde{K}}) - f^*] + \frac{\bar{c}_0 + \bar{c}_1}{t_K}.$

We now show that the requirements of Assumption 7 are satisfied under suitable assumptions. **Corollary 1.** Let $N_k \triangleq \lceil N_0 k^a \rceil$, $\gamma_k \triangleq \gamma_0 k^{-b}$, $\mu_k \triangleq \mu_0 k^{-c}$ and $t_k \triangleq t_0 (k-1)^h$ for some a, b, c, h > 0. Let $2\bar{\delta}(m+n) = \varepsilon$ for $\varepsilon > 0$. Then Assumption 7 holds if $a + 2b - c\varepsilon \ge b + c$, $N_0 \ge \frac{(L+\mu_0)\lambda^2 \nu_1^2 \gamma_0}{\alpha \underline{\lambda} \mu_0}$, b + c < 1, $h \le 1$, b + 2c - h > 1 and $a + 2b - h - c\varepsilon > 1$.

Proof. From $N_k = \lceil N_0 k^a \rceil \ge N_0 k^a$, $\gamma_k = \gamma_0 k^{-b}$ and $\mu_k = \mu_0 k^{-c}$, the requirements to satisfy Assumption 7 are as follows:

- (a) $\lim_{k\to\infty} \gamma_0 k^{-b} = 0$, $\lim_{k\to\infty} \mu_0 k^{-c} = 0 \Leftrightarrow b, c > 0$;
- (b) $\left(1 \underline{\lambda}\mu_k\gamma_k + \frac{2(L+\mu_0)\overline{\lambda}_k^2\nu_1^2\gamma_k^2}{N_k\alpha}\right) \leq \frac{t_k}{t_{k+1}} \Leftrightarrow \left(1 \frac{1}{k^{b+c}} + \frac{1}{k^{a+2b-c\varepsilon}}\right) \leq (1 1/k)^h$. From the Taylor expansion of right hand side and assuming $h \leq 1$, we get $\left(1 \frac{1}{k^{b+c}} + \frac{1}{k^{a+2b-c\varepsilon}}\right) \leq 1 M/k$ for some M > 0 and $\forall k \geq \tilde{K}$ which means $\left(1 \underline{\lambda}\mu_k\gamma_k + \frac{2(L+\mu_0)\overline{\lambda}_k^2\nu_1^2\gamma_k^2}{N_k\alpha}\right) \leq \frac{t_k}{t_{k+1}} \Leftrightarrow h \leq 1, b+c < 1, a+2b-c\varepsilon \geq b+c$ and $N_0 = \frac{(L+\mu_0)\lambda^2\nu_1^2\gamma_0}{\alpha\underline{\lambda}\mu_0}$;

(c)
$$\sum_{k=0}^{\infty} \mu_k^2 \gamma_k t_{k+1} < \infty \Leftarrow \sum_{k=0}^{\infty} \frac{1}{k^{b+2c-h}} < \infty \Leftrightarrow b+2c-h > 1;$$

(d)
$$\sum_{k=0}^{\infty} \frac{\mu_k^{-2\bar{\delta}(n+m)} \gamma_k^2}{N_k} t_{k+1} < \infty \Leftarrow \sum_{k=0}^{\infty} \frac{1}{k^{a+2b-h-c\varepsilon}} < \infty \Leftrightarrow a+2b-h-c\varepsilon > 1;$$

One can easily verify that $a = 2 + \varepsilon$, $b = \varepsilon$ and $c = 1 - \frac{2}{3}\varepsilon$ and $h = 1 - \varepsilon$ satisfy these conditions. We derive complexity statements for (**rVS-SQN**) for a specific choice of parameter sequences.

Theorem 5 (Rate statement and Oracle complexity). Consider the (rVS-SQN) scheme and suppose Assumptions 1, 4(S-M), 4(S-B), 1(S), 6 and 7 hold. Suppose $\gamma_k \triangleq \gamma_0 k^{-b}$, $\mu_k \triangleq \mu_0 k^{-c}$, $\triangleq t_k = t_0(k-1)^h$ and $N_k \triangleq \lceil N_0 k^a \rceil$ where $N_0 = \frac{(L+\mu_0)\lambda^2 \nu_1^2 \gamma_0}{\alpha \Delta \mu_0}$, $a = 2 + \varepsilon$, $b = \varepsilon$ and $c = 1 - \frac{2}{3}\varepsilon$ and $h = 1 - \varepsilon$.

(i) Then the following holds for $K \geq \tilde{K}$ where $\tilde{K} \geq 1$ and $\tilde{C} \triangleq f_{\mu_{\tilde{K}}}(x_{\tilde{K}}) - f^*$.

$$\mathbb{E}[f(x_K) - f^*] \le \frac{\tilde{C} + \bar{c}_0 + \bar{c}_1}{K^{1-\varepsilon}}.$$
(25)

(ii) Let $\epsilon > 0$ and $K \ge \tilde{K} + 1$ such that $\mathbb{E}[f(x_K)] - f^* \le \epsilon$. Then, $\sum_{k=0}^K N_k \le \mathcal{O}\left(\epsilon^{-\frac{3+\varepsilon}{1-\varepsilon}}\right)$.

Proof. (i) By choosing the sequence parameters as specified, the result follows immediately from Theorem 4. (ii) To find an x_K such that $\mathbb{E}[f(x_K)] - f^* \leq \epsilon$ we have $\frac{\tilde{C} + \bar{c}_0 + \bar{c}_1}{\tilde{K}^{1-\varepsilon}} \leq \epsilon$ which implies that $K = \left\lceil \left(\frac{\tilde{C} + \bar{c}_0 + \bar{c}_1}{\epsilon}\right)^{\frac{1}{1-\varepsilon}} \right\rceil$. Hence, the following holds.

$$\sum_{k=0}^{K} N_k \le \sum_{k=0}^{1+(C/\epsilon)^{\frac{1}{1-\varepsilon}}} 2N_0 k^{2+\varepsilon} \le 2N_0 \int_0^{1+(C/\epsilon)^{\frac{1}{1-\varepsilon}}} x^{2+\varepsilon} \, dx = \frac{2N_0 \left(1+(C/\epsilon)^{\frac{1}{1-\varepsilon}}\right)^{3+\varepsilon}}{3+\varepsilon} \le \mathcal{O}\left(\epsilon^{-\frac{3+\varepsilon}{1-\varepsilon}}\right).$$

One may instead consider the following requirement on the conditional second moment on the sampled gradient instead of state-dependent noise (Assumption 4).

Assumption 8. Let $\bar{w}_{k,N_k} \triangleq \nabla_x f(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F(x_k, \omega_{j,k})}{N_k}$. Then there exists $\nu > 0$ such that $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k] \leq \frac{\nu^2}{N_k}$ and $\mathbb{E}[\bar{w}_{k,N_k} \mid \mathcal{F}_k] = 0$ hold almost surely for all k, where $\mathcal{F}_k \triangleq \sigma\{x_0, x_1, \dots, x_{k-1}\}$.

By invoking Assumption 8, we can derive the rate result without requiring a quadratic growth property of objective function.

Corollary 2 (Rate statement and Oracle complexity). Consider (**rVS-SQN**) and suppose Assumptions 1, 1(S), 7 and 8 hold. Suppose $\gamma_k = \gamma_0 k^{-b}$, $\mu_k = \mu_0 k^{-c}$, $t_k = t_0 (k-1)^h$ and $N_k = \lceil k^a \rceil$ where $a = 2 + \varepsilon$, $b = \varepsilon$ and $c = 1 - \frac{4}{3}\varepsilon$ and $h = 1 - \varepsilon$. (i) Then for $K \ge \tilde{K}$ where $\tilde{K} \ge 1$ and $\tilde{C} \triangleq f_{\mu_{\tilde{K}}}(x_{\tilde{K}}) - f^*$, $\mathbb{E}[f(x_K) - f^*] \le \frac{\tilde{C} + \bar{c}_0 + \bar{c}_1}{K^{1-\varepsilon}}$. (ii) Let $\epsilon > 0$ and $K \ge \tilde{K} + 1$ such that $\mathbb{E}[f(x_K)] - f^* \le \epsilon$. Then, $\sum_{k=0}^K N_k \le \mathcal{O}\left(\epsilon^{-\frac{3+\varepsilon}{1-\varepsilon}}\right)$.

Remark 3. Although the oracle complexity of (**rVS-SQN**) is poorer than the canonical $O(1/\epsilon^2)$, there are several reasons to consider using the SQN schemes when faced with a choice between gradient-based counterparts. (a) Sparsity. In many machine learning problems, the sparsity properties of the estimator are of relevance. However, averaging schemes tend to have a detrimental impact on the sparsity properties while non-averaging schemes do a far better job in preserving such properties. Both accelerated and unaccelerated gradient schemes for smooth stochastic convex optimization rely on averaging and this significantly impacts the sparsity of the estimators. (See Table 6 in Section 5). (b) Ill-conditioning. As is relatively well known, quasi-Newton schemes do a far better job of contending with ill-conditioning in practice, in comparison with gradient-based techniques. (See Tables 9 and 10 in Section 5.)

4.2 Nonsmooth convex optimization

We now consider problem (1) when f is nonsmooth but (α, β) -smoothable and consider the (**rsVS-SQN**) scheme, defined as follows, where H_k is generated by **rsL-BFGS** scheme.

$$x_{k+1} := x_k - \gamma_k H_k \frac{\sum_{j=1}^{N_k} \nabla_x F_{\eta_k,\mu_k}(x_k,\omega_{j,k})}{N_k}.$$
 (rsVS-SQN)

Note that in this section, we set m = 1 for the sake of simplicity but the analysis can be extended to m > 1. Next, we generalize Lemma 8 to show that Assumption 1 is satisfied and both the secant condition (**SC**) and the secant equation (**SE**). (See Appendix for Proof.)

Lemma 10 (Properties of Hessian approximation produced by (rsL-BFGS)). Consider the (rsVS-SQN) method, where H_k is updated by (8)-(9), s_i and y_i are defined in (7) and η_k and μ_k are updated according to procedure (6). Let Assumption 3 holds. Then the following hold.

- (a) For any odd k > 2m, (SC) holds, i.e., $s_k^T y_k > 0$;
- (b) For any odd k > 2m, (SE) holds, i.e., $H_k y_k = s_k$.

(c) For any k > 2m, H_k satisfies Assumption 1(NS) with $\underline{\lambda}_k = \frac{1}{(m+n)(1/\eta_k^{\delta} + \mu_0^{\overline{\delta}})}$ and $\overline{\lambda}_k = \frac{(m+n)^{n+m-1}(1/\eta_k^{\delta} + \mu_0^{\overline{\delta}})^{n+m-1}}{(n-1)!\mu_k^{(n+m)\overline{\delta}}}$, for scalars $\delta, \overline{\delta} > 0$. Then for all k, we have that $H_k = H_k^T$ and $\mathbb{E}[H_k \mid \mathcal{F}_k] = H_k$ and $\underline{\lambda}_k \mathbf{I} \leq H_k \leq \overline{\lambda}_k \mathbf{I}$ both hold in an a.s. fashion.

We now derive a rate statement for the mean sub-optimality.

Theorem 6 (Convergence in mean). Consider the (rsVS-SQN) scheme. Suppose Assumptions 3, 4 (NS-M), 4 (NS-B), 1 (NS), and 6 hold. Let $\gamma_k = \gamma$, $\mu_k = \mu$, and $\eta_k = \eta$ be chosen such that (17) holds (where $L = 1/\eta$). If $\bar{x}_K \triangleq \frac{\sum_{k=0}^{K-1} x_k (\Delta \mu \gamma - C/N_k)}{\sum_{k=0}^{K-1} (\Delta \mu \gamma - C/N_k)}$, then (26) holds for $K \ge 1$ and $C = \frac{2(1+\mu\eta)\bar{\lambda}^2 \nu_1^2 \gamma^2}{\alpha \eta}$.

$$\left(K\underline{\lambda}\mu\gamma - \sum_{k=0}^{K-1} \frac{C}{N_k}\right) \mathbb{E}[f_{\eta,\mu}(\bar{x}_K) - f^*] \leq \mathbb{E}[f_{\eta,\mu}(x_0) - f^*] + \eta B^2 + \frac{\underline{\lambda}dist^2(x_0, X^*)}{2}\mu^2\gamma K + \sum_{k=0}^{K-1} \frac{(1+\mu\eta)\overline{\lambda}^2(2\nu_1^2||x^*||^2 + \nu_2^2)\gamma^2}{2N_k\eta}.$$
(26)

Proof. Since Lemma 9 may be invoked, by taking expectations on both sides of (18), for any $k \geq 0$ letting $\bar{w}_{k,N_k} \triangleq \frac{\sum_{j=1}^{N_k} (\nabla_x F_{\eta_k,\mu_k}(x_k,\omega_{j,k}) - \nabla f_{\eta_k,\mu_k}(x_k))}{N_k}$, and by letting $\underline{\lambda} \triangleq \frac{1}{(m+n)(1/\eta^{\delta} + \mu^{\overline{\delta}})}, \overline{\lambda} \triangleq \frac{(m+n)^{n+m-1}(1/\eta^{\delta} + \mu^{\overline{\delta}})^{n+m-1}}{(n-1)!\mu^{(n+m)\overline{\delta}}}$, using the quadratic growth property i.e. $\|x_k - x^*\|^2 \leq \frac{2}{\alpha} (f(x) - f(x^*))$ and the fact that $\|x_k - x^* + x^*\|^2 \leq 2\|x_k - x^*\|^2 + 2\|x^*\|^2$, we obtain the following

$$\mathbb{E}[f_{\eta,\mu}(x_{k+1}) - f^*] \leq \left(1 - \underline{\lambda}\mu\gamma + \frac{2(1+\mu\eta)\overline{\lambda}^2\nu_1^2\gamma^2}{\alpha N_k\eta}\right) \mathbb{E}[f_{\eta,\mu}(x_k) - f^*] + \frac{\underline{\lambda}\operatorname{dist}^2(x_0, X^*)}{2}\mu^2\gamma + \frac{(1+\mu\eta)\overline{\lambda}^2(2\nu_1^2\|x^*\|^2 + \nu_2^2)\gamma^2}{2N_k\eta}$$

$$\Longrightarrow \left(\underline{\lambda}\mu\gamma - \frac{2(1+\mu\eta)\overline{\lambda}^2\nu_1^2\gamma^2}{\alpha N_k\eta}\right) \mathbb{E}[f_{\eta,\mu}(x_k) - f^*] \le \mathbb{E}[f_{\eta,\mu}(x_k) - f^*] - \mathbb{E}[f_{\eta,\mu}(x_{k+1}) - f^*] \\ + \frac{\underline{\lambda}\text{dist}^2(x_0, X^*)\mu^2\gamma}{2} + \frac{(1+\mu\eta)\overline{\lambda}^2(2\nu_1^2\|x^*\|^2 + \nu_2^2)\gamma^2}{2N_k\eta}.$$

Summing from k = 0 to K - 1 and by invoking Jensen's inequality, we obtain the following

$$\left(K\underline{\lambda}\mu\gamma - \sum_{k=0}^{K-1} \frac{C}{N_k}\right) \mathbb{E}[f_{\eta,\mu}(\bar{x}_K) - f^*] \leq \mathbb{E}[f_{\eta,\mu}(x_0) - f^*] - \mathbb{E}[f_{\eta,\mu}(x_K) - f^*] \\
+ \frac{\lambda \text{dist}^2(x_0, X^*)}{2} \mu^2 \gamma K + \sum_{k=0}^{K-1} \frac{(1+\mu\eta)\overline{\lambda}^2 (2\nu_1^2 ||x^*||^2 + \nu_2^2) \gamma^2}{2N_k \eta},$$

where $C = \frac{2(1+\mu\eta)\overline{\lambda}^2\nu_1^2\gamma^2}{\alpha\eta}$ and $\bar{x}_K \triangleq \frac{\sum_{k=0}^{K-1} x_k(\underline{\lambda}\mu\gamma - C/N_k)}{\sum_{k=0}^{K-1} (\underline{\lambda}\mu\gamma - C/N_k)}$. Since $\mathbb{E}[f(x)] \leq \mathbb{E}[f_\eta(x)] + \eta_k B^2$ and $f_\mu(x) = f(x) + \frac{\mu}{2} ||x - x_0||^2$, we have that $-\mathbb{E}[f_{\eta,\mu}(x_K) - f^*] \leq -\mathbb{E}[f_\mu(x_K) - f^*] + \eta B^2 \leq \eta B^2$. Therefore, we obtain the following:

$$\left(K\underline{\lambda}\mu\gamma - \sum_{k=0}^{K-1} \frac{C}{N_k}\right) \mathbb{E}[f_{\eta,\mu}(\bar{x}_K) - f^*] \le \mathbb{E}[f_{\eta,\mu}(x_0) - f^*] + \eta B^2 + \frac{\lambda \text{dist}^2(x_0, X^*)}{2} \mu^2 \gamma K + \sum_{k=0}^{K-1} \frac{(1+\mu\eta)\overline{\lambda}^2(2\nu_1^2 ||x^*||^2 + \nu_2^2)\gamma^2}{2N_k \eta}.$$

We refine this result for a set of parameter sequences.

Theorem 7 (Rate statement and oracle complexity). Consider (rsVS-SQN) and suppose Assumptions 3, 4 (NS-M), 4 (NS-B), 1 (NS), and 6 hold, $\gamma \triangleq c_{\gamma} K^{-1/3+\bar{\varepsilon}}$, $\mu \triangleq K^{-1/3}$, $\eta \triangleq K^{-1/3}$ and $N_k \triangleq \lceil N_0(k+1)^a \rceil$, where $\bar{\varepsilon} \triangleq \frac{5\varepsilon}{3}$, $\varepsilon > 0$, $N_0 > \frac{C}{\Delta\mu\gamma}$, $C = \frac{2(1+\mu\eta)\bar{\lambda}^2\nu_1^2\gamma^2}{\alpha\eta}$ and a > 1. Let $\delta = \frac{\varepsilon}{n+m-1}$ and $\bar{\delta} = \frac{\varepsilon}{n+m}$. (i) For any $K \ge 1$, $\mathbb{E}[f(\bar{x}_K)] - f^* \le \mathcal{O}(K^{-1/3})$. (ii) Let $\epsilon > 0$, $a = (1+\epsilon)$, and $K \ge 1$ such that $\mathbb{E}[f(\bar{x}_K)] - f^* \le \epsilon$. Then, $\sum_{k=0}^{K} N_k \le \mathcal{O}\left(\epsilon^{-\frac{(2+\varepsilon)}{1/3}}\right)$.

Proof. (i) First, note that for a > 1 and $N_0 > \frac{C}{\lambda\mu\gamma}$ we have $\sum_{k=0}^{K-1} \frac{C}{N_k} < \infty$. Therefore we can let $C_4 \triangleq \sum_{k=0}^{K-1} \frac{C}{N_k}$. Dividing both sides of (26) by $K \underline{\lambda} \mu \gamma - C_4$ and by recalling that $f_{\eta}(x) \leq f(x) \leq f_{\eta}(x) + \eta B^2$ and $f(x) \leq f_{\mu}(x)$, we obtain

$$\mathbb{E}[f(\bar{x}_{K}) - f^{*}] \leq \frac{\mathbb{E}[f_{\mu}(x_{0}) - f^{*}]}{K\underline{\lambda}\mu\gamma - C_{4}} + \frac{\eta B^{2}}{K\underline{\lambda}\mu\gamma - C_{4}} + \frac{\frac{\lambda \operatorname{dist}^{2}(x_{0}, X^{*})}{2}\mu^{2}\gamma K}{K\underline{\lambda}\mu\gamma - C_{4}} + \frac{\sum_{k=0}^{K-1} \frac{(1+\mu\eta)\overline{\lambda}^{2}(2\nu_{1}^{2}||x^{*}||^{2}+\nu_{2}^{2})\gamma^{2}}{2N_{k}\eta}}{K\underline{\lambda}\mu\gamma - C_{4}} + \eta B^{2}.$$

Note that by choosing $\gamma = c_{\gamma} K^{-1/3+\bar{\varepsilon}}$, $\mu = K^{-1/3}$ and $\eta = K^{-1/3}$, where $\bar{\varepsilon} = 5/3\varepsilon$, inequality (17) is satisfied for sufficiently small c_{γ} . By choosing $N_k = \lceil N_0(k+1)^a \rceil \ge N_0(k+2)^a$ for any a > 1 and $N_0 > \frac{C}{\lambda\mu\gamma}$, we have that

$$\sum_{k=0}^{K-1} \frac{1}{(k+1)^a} \le 1 + \int_0^{K-1} (x+1)^{-a} dx \le 1 + \frac{K^{1-a}}{1-a}$$
$$\implies \mathbb{E}[f(\bar{x}_K) - f^*] \le \frac{C_1}{K\underline{\lambda}\mu\gamma - C_4} + \frac{\eta B^2}{K\underline{\lambda}\mu\gamma - C_4} + \frac{C_2\underline{\lambda}\mu^2\gamma K}{K\underline{\lambda}\mu\gamma - C_4} + \frac{C_3(1+\mu\eta)\overline{\lambda}^2\gamma^2}{\eta N_0(K\mu\gamma - C_4)}(1+K^{1-a}) + \eta B^2,$$

where $C_1 = \mathbb{E}[f_{\mu}(x_0) - f^*]$, $C_2 = \frac{\operatorname{dist}^2(x_0, X^*)}{2}$ and $C_3 = \frac{2\nu_1^2 \|x^*\|^2 + \nu_2^2}{2(1-a)}$. Choosing the parameters γ, μ and η as stated and noting that $\underline{\lambda} = \frac{1}{(m+n)(1/\eta^{\delta} + \mu^{\overline{\delta}})} = \mathcal{O}(\eta^{\delta}) = \mathcal{O}(K^{-\delta/3})$ and $\overline{\lambda} = \frac{(m+n)^{n+m-1}(1/\eta^{\delta} + \mu^{\overline{\delta}})^{n+m-1}}{(n-1)!\mu^{(n+m)\overline{\delta}}} = \mathcal{O}(\eta^{-\delta(n+m-1)/\mu^{\overline{\delta}(n+m)}}) = \mathcal{O}(K^{2\varepsilon/3})$, where we used the assumption

that $\delta = \frac{\varepsilon}{n+m-1}$ and $\bar{\delta} = \frac{\varepsilon}{n+m}$. Therefore, we obtain $\mathbb{E}[f(\bar{x}_K) - f^*] \leq \mathcal{O}(K^{-1/3 - 5\varepsilon/3} + \delta/3) + \mathcal{O}(K^{-2/3 - 5\varepsilon/3 + \delta/3}) + \mathcal{O}(K^{-1/3}) + \mathcal{O}(K^{-2/3 + 3\varepsilon}) + \mathcal{O}(K^{-1/3}) = \mathcal{O}(K^{-1/3}).$ (ii) The proof is similar to part (ii) of Theorem 5.

Remark 4. Note that in Theorem 7 we choose steplength, regularization, and smoothing parameters as constant parameters in accordance with the length of the simulation trajectory K, i.e. γ, μ, η are constants. This is akin to the avenue chosen by Nemirovski et al. [33] where the steplength is chosen in accordance with the length of the simulation trajectory K.

Next, we relax Assumption 6 (quadratic growth property) and impose a stronger bound on the conditional second moment of the sampled gradient.

Assumption 9. Let $\bar{w}_{k,N_k} \triangleq \nabla_x f_{\eta_k}(x_k) - \frac{\sum_{j=1}^{N_k} \nabla_x F_{\eta_k}(x_k, \omega_{j,k})}{N_k}$. Then there exists $\nu > 0$ such that $\mathbb{E}[\|\bar{w}_{k,N_k}\|^2 | \mathcal{F}_k] \leq \frac{\nu^2}{N_k}$ and $\mathbb{E}[\bar{w}_{k,N_k} | \mathcal{F}_k] = 0$ hold almost surely for all k and $\eta_k > 0$, where $\mathcal{F}_k \triangleq \sigma\{x_0, x_1, \dots, x_{k-1}\}.$

Corollary 3 (**Rate statement and Oracle complexity**). Consider the (**rsVS-SQN**) scheme. Suppose Assumptions 3, 1 (NS) and 9 hold and $\gamma \triangleq c_{\gamma} K^{-1/3 + \bar{\varepsilon}}$, $\mu \triangleq K^{-1/3}$, $\eta \triangleq K^{-1/3}$ and $N_k \triangleq \lceil (k+1)^a \rceil$, where $\bar{\varepsilon} \triangleq \frac{5\varepsilon}{3}$, $\varepsilon > 0$ and a > 1. (*i*) For any $K \ge 1$, $\mathbb{E}[f(\bar{x}_K)] - f^* \le \mathcal{O}(K^{-1/3})$.

(ii) Let $\epsilon > 0$, $a = (1+\epsilon)$, and $K \ge 1$ such that $\mathbb{E}[f(\bar{x}_K)] - f^* \le \epsilon$. Then, $\sum_{k=0}^K N_k \le \mathcal{O}\left(\epsilon^{-\frac{(2+\epsilon)}{1/3}}\right)$.

Remark 5. It is worth emphasizing that the unavailability of problem parameters such as the Lipschitz constant and the strong convexity modulus may render some methods unimplementable. In fact, this is a prime motivation for utilizing smoothing, regularization, and diminishing steplengths in this work; smoothing and regularization address the unavailability of Lipschitz and strong convexity constants, respectively while diminishing steplengths obviate the need for knowing both sets of parameters. Remark 2 cover some instances where such constants may indeed be available. We briefly summarize avenues for contending with the absence of such parameters.

- Absence/unavailability of Lipschitz constant in strongly-convex regimes (sVS-SQN). In the absence of a Lipschitz constant but in the presence of strong convexity, we propose two distinct avenues. We can employ Moreau-smoothing with fixed parameter η or iterative smoothing which relies on a diminishing sequence {η_k}. This framework relies on a modified L-BFGS update, referred to as smoothed L-BFGS and denoted by (sL-BFGS). This case is discussed in section 3.2 and the main results are provided by Theorem 3.
- 2. Absence/unavailability of strong convexity constants in smooth regimes (**rVS-SQN**). If the problem is either merely convex or has an unknown strong convexity modulus but satisfies L-smoothness, then we present a regularization scheme, extending our prior work in this area to the variance-reduced arena. The resulting scheme, referred to as the regularized VS-SQN and denoted by (**rVS-SQN**), is reliant on the analogous L-BFGS update. This case is discussed in section 4.1 and the main results are provided by Theorems 4 and 5.
- 3. Absence/unavailability of Lipschitz constants and strong convexity constants. If both the Lipschitz and the strong convexity constants are unavailable, then we may overlay smoothing

and regularization. This requires a regularized and smoothed L-BFGS update, referred to as (**rsL-BFGS**), and the resulting scheme is referred to as the regularized smoothed VS-SQN and denoted by (**rsVS-SQN**). This case is discussed in section 4.2 and the main results are provided by Theorems 6 and 7.

- 4. Addressing unavailability of Lipschitz and strong convexity constants via diminishing steplength schemes. One may also obviate the need for knowing the constants by utilizing diminishing steplength sequences. This allows for deriving rate statements for a shifted recursion via some well known results on deterministic recursions. This has now been added in Theorem 1 (iii) ((VS-SQN) in the smooth and strongly convex regime) and Theorem 3 (iii) ((sVS-SQN) in the nonsmooth and strongly convex regime).
- 5. Stochastic line-search techniques. One avenue to contend with the absence of Lipschitzian guarantees lies in leveraging line-search techniques in stochastic regimes [22, 8, 35, 43]. This remains a goal of future work.

5 Numerical results

In this section, we compare the behavior of the proposed VS-SQN schemes with their accelerated/unaccelerated gradient counterparts on a class of strongly convex/convex and smooth/nonsmooth stochastic optimization problems with the intent of examining empirical error and sparsity of estimators (in machine learning problems) as well as the ability to contend with ill-conditioning. **Example 1.** First, we consider the logistic regression problem, defined as follows:

$$\min_{x \in \mathbb{R}^n} f(x) \triangleq \frac{1}{N} \sum_{i=1}^N \ln\left(1 + \exp\left(-u_i^T x v_i\right)\right),$$
(LRM)

where $u_i \in \mathbb{R}^n$ is the input binary vector associated with article *i* and $v_i \in \{-1, 1\}$ represents the class of the *i*th article. A μ -regularized variant of such a problem is defined as follows.

$$\min_{x \in \mathbb{R}^n} f(x) \triangleq \frac{1}{N} \sum_{i=1}^N \ln\left(1 + \exp\left(-u_i^T x v_i\right)\right) + \frac{\mu}{2} \|x\|^2.$$
 (reg-LRM)

We consider the SIDOO dataset [25] where N = 12678 and n = 4932.

(1.1) Strongly convex and smooth problems: To apply (VS-SQN), we consider (Reg-LRM) where the problem is strongly convex and $\mu = 0.1$. We compare the behavior of the scheme with an accelerated gradient scheme [21] and set the overall sampling buget equal to 1*e*4. We observe that (VS-SQN) competes well with (VS-APM). (see Table 3 and Fig. 2 (a)).

	SC, s	mooth	SC, nonsmooth (Moreau smoothing)		
	VS-SQN VS-APM		sVS-SQN	sVS-APM	
sample size: N_k	ρ^{-k}	ρ^{-k}	$\lfloor q^{-k} \rfloor$	$\lfloor q^{-k} \rfloor$	
steplength: γ_k	0.1	0.1	η_k^2	η_k^2	
smoothing: η_k	-	-	0.1	0.1	
$f(x_k)$	5.015e-1	5.015e-1	8.905e-1	1.497e + 0	

Table 3: sido0: SC, smooth and nonsmooth

		SC, smooth		SC, nonsmooth (Moreau smoothing)			
	VS-SQN	VS-SQN	VS-SQN	sVS-SQN	sVS-SQN	sVS-SQN	
sample size: N_k	ρ^{-k}	k	k^2	$\lfloor q^{-k} \rfloor$	k	k^2	
steplength: γ_k	0.1	$1/\sqrt{k}$	$1/\sqrt{k}$	η_k^2	$1/\sqrt{k}$	$1/\sqrt{k}$	
smoothing: η_k	-	-	-	0.1	0.1	0.1	
$f(x_k)$	5.015e-1	5.019e-1	5.098e-1	8.905e-1	1.358e+0	8.989e-1	

Table 4: sido0: SC, smooth and nonsmooth

Next, in Table 4, we compare the behavior of (VS-SQN) in constant versus diminishing steplength regimes and observe that the distinctions are modest for these problem instances.

(1.2) Strongly convex and nonsmooth: We consider a nonsmooth variant where an ℓ_1 regularization is added with $\lambda = \mu = 0.1$:

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{N} \sum_{i=1}^N \ln\left(1 + \mathbb{E}\left(-u_i^T x v_i\right)\right) + \frac{\mu}{2} \|x\|^2 + \lambda \|x\|_1.$$
(27)

,

From [2], a smooth approximation of $||x||_1$ is given by the following

$$\sum_{i=1}^{n} H_{\eta}(x_i) = \begin{cases} x_i^2/2\eta, & \text{if } |x_i| \le \eta \\ |x_i| - \eta/2, & \text{o.w.} \end{cases}$$

where η is a smoothing parameter. The perfomance of (**sVS-SQN**) is shown in Figure 2 (b) while parameter choices are provided in Table 3 and the total sampling budget is 1*e*5. We see that empirical behavior of (**VS-SQN**) and (**sVS-SQN**) is similar to (**VS-APM**) [21] and (**rsVS-APM**) [21], respectively. Note that while in the strongly convex regimes, both schemes display similar (linear) rates, we do not have a rate statement for smoothed (**sVS-APM**) [21].



Figure 2: Left to right: (a) SC smooth, (b) SC nonsmooth, (c) C smooth, (d) C nonsmooth

(1.3) Convex and smooth: We implement (**rVS-SQN**) on the (LRM) problem and compare the result with VS-APM [21] and r-SQN [48]. We again consider the SIDOO dataset with a total budget of 1e5 while the parameters are tuned to ensure good performance. In Figure 2 (c) we compare three different methods while the choices of steplength and sample size can be seen in Table 5. We note that (VS-APM) produces slightly better solutions, which is not surprising since it enjoys a rate of $\mathcal{O}(1/k^2)$ with an optimal oracle complexity. However, (**rVS-SQN**) is competitive and appears to be better than (r-SQN) by a significant margin in terms of the function value. In addition, the last three columns of Table 4 capture the distinctions in performance between employing constant versus diminishing steplengths.

	con	vex, smoot	convex, no	convex, nonsmooth		
	rVS-SQN r-SQN		VS-APM	rsVS-SQN	sVS-APM	
sample size: N_k	$k^{2+\varepsilon}$	1	$k^{2+\varepsilon}$	$(k+1)^{1+\varepsilon}$	$(k+1)^{1+\varepsilon}$	
steplength: γ_k	$k^{-\varepsilon}$	$k^{-2/3}$	1/(2L)	$K^{-1/3+\varepsilon}$	1/(2k)	
regularizer: μ_k	$k^{2/3\varepsilon-1}$	$k^{-1/3}$	-	$K^{-1/3}$	-	
smoothing: η_k	-	-	-	$K^{-1/3}$	1/k	
$f(x_k)$	1.38e-1	2.29e-1	9.26e-2	6.99e-1	7.56e-1	

Table 5: sido0: C, smooth and nonsmooth

(1.4.) Convex and nonsmooth: Now we consider the nonsmooth problem in which $\lambda = 0.1$.

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{N} \sum_{i=1}^N \ln\left(1 + \exp\left(-u_i^T x v_i\right)\right) + \lambda \|x\|_1.$$
(28)

We implement **rsVS-SQN** scheme with a total budget of 1e4. (see Table 5 and Fig. 2 (d)) observe that it competes well with (sVS-APM) [21], which has a superior convergence rate of $\mathcal{O}(1/k)$. (1.5.) Sparsity We now compare the sparsity of the estimators obtained via (**rVS-SQN**) scheme

with averaging-based stochastic gradient schemes. Consider the following example where we consider the smooth approximation of $\|.\|_1$, leading to a convex and smooth problem.

$$\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{N} \sum_{i=1}^N \ln\left(1 + \exp\left(-u_i^T x v_i\right)\right) + \lambda \sum_{i=1}^n \sqrt{x_i^2 + \lambda_2},$$

where we set $\lambda = 1e-4$. We chose the parameters according to Table 5, total budget is 1e5 and $||x_K||_0$ denotes the number of entries in x_K that are greater than 1e-4. Consequently, $n_0 \triangleq n - ||x_K||_0$ denotes the number of "zeros" in the vector. As it can be seen in Table 6, the solution obtained by (**rVS-SQN**) is significantly sparser than that obtained by (**vS-APM**) and standard stochastic gradient. In fact, SGD produces nearly dense vectors while (**rVS-SQN**) produces vectors, 10% of which are sparse for $\lambda_2 = 1e-6$.

	rVS-SQN	(VS-APM)	SGD
N_k	$k^{2+\epsilon}$	$k^{2+\epsilon}$	1
# of iter.	66	66	1e5
n_0 for $\lambda_2 = 1e-5$	144	31	0
n_0 for $\lambda_2 = 1e-6$	497	57	2

Table 6: sido0: Convex, smooth

Example 2. Impact of size and ill-conditioning. In Example 1, we observed that (**rVS-SQN**) competes well with VS-APM for a subclass of machine learning problems. We now consider a stochastic quadratic program over a general probability space and observe similarly competitive behavior. In fact, (**rVS-SQN**) often outperforms (**VS-APM**) [21] (see Tables 7 and 8). We consider the following problem.

$$\min_{x \in \mathbb{R}^n} \mathbb{E}\left[\frac{1}{2}x^T Q(\omega)x + c(\omega)^T x\right],$$

where $Q(\omega) \in \mathbb{R}^{n \times n}$ is a random symmetric matrix such that the eigenvalues are chosen uniformly at random and the minimum eigenvalue is one and zero for strongly convex and convex problem,

	(VS-SQN)	(VS-APM)
n	$\mathbb{E}[f(x_k) - f(x^*)]$	$\mathbb{E}[f(x_k) - f(x^*)]$
20	3.28e-6	5.06e-6
60	9.54e-6	1.57e-5
100	1.80e-5	2.92e-5

Table 7: Strongly convex: (VS-SQN) vs (VS-APM)

	$(\mathbf{rVS-SQN})$	(VS-APM)
n	$\mathbb{E}[f(x_k) - f(x^*)]$	$\mathbb{E}[f(x_k) - f(x^*)]$
20	9.14e-5	1.89e-4
60	2.67e-4	4.35e-4
100	5.41e-4	8.29e-4

Table 8: Convex: (rVS-SQN) vs (VS-APM)

respectively. Furthermore, $c_{\omega} = -Q(\omega)x^0$, where $x^0 \in \mathbb{R}^{n \times 1}$ is a vector whose elements are chosen randomly from the standard Gaussian distribution. In Tables 9 and 10, we compare the behavior of (**rVS-SQN**) and (**VS-APM**) when the problem is ill-conditioned in strongly convex and convex regimes, respectively. In strongly convex regimes, we set the total budget equal to 2*e*8 and maintain the steplength as equal for both schemes. The sample size sequence is chosen to be $N_k = \lceil 0.99^{-k} \rceil$, leading to 1443 steps for both methods. We observe that as *m* grows, the relative quality of the solution compared to (**VS-APM**) improves even further. These findings are reinforced in Table 10, where for merely convex problems, although the convergence rate for (**VS-APM**) is $\mathcal{O}(1/k^2)$ (superior to $\mathcal{O}(1/k)$ for (**rVS-SQN**), (**rVS-SQN**) outperforms (**VS-APM**) in terms of empirical error. Note that parameters are chosen similar to Table 5.

	$\mathbb{E}[f(x_k) - f(x^*)]$				$\mathbb{E}[f(x_k) - f(x^*)]$		
κ	(VS-SQN), m = 1	(VS-SQN), m = 10	(VS-APM)	L	$(\mathbf{rVS-SQN}), m = 1$	(rVS-SQN), m = 10	(VS-APM)
1e5	9.25e-4	2.656e-4	2.600e-3	1e3	4.978e-4	1.268e-4	1.942e-4
1e6	9.938e-5	4.182e-5	4.895e-4	1e4	3.288e-3	2.570e-4	3.612e-2
1e7	1.915e-5	1.478e-5	1.079e-4	1e5	8.571e-2	3.075e-3	2.794e+0
1e8	1.688e-5	6.304e-6	4.135e-5	1e6	3.367e-1	3.203e-1	4.293e+0

Table 9: Strongly convex:

Table 10: Convex:

Performance vs Condition number (as m changes) Performance vs Condition number (as m changes)

Example 3. Constrained Problems. We consider the isotonic constrained LASSO problem.

$$\min_{x=[x_i]_{i=1}^n \in \mathbb{R}^n} \left\{ \frac{1}{2} \sum_{i=1}^p \|A_i x - b_i\|^2 \mid x_1 \le x_2 \le \dots \le x_n \right\},\tag{29}$$

where $A = [A_i]_{i=1}^p \in \mathbb{R}^{n \times p}$ is a matrix whose elements are chosen randomly from standard Gaussian distribution such that the $A^{\top}A \succeq 0$ and $b = [b_i]_{i=1}^p \in \mathbb{R}^p$ such that $b = A(x_0 + \sigma)$ where $x_0 \in \mathbb{R}^n$ is chosen such that the first and last $\frac{n}{4}$ of its elements are chosen from U([-10,0]) and U([0,10]) in ascending order, respectively, while the other elements are set to zero. Further, $\sigma \in \mathbb{R}^n$ is a random vector whose elements are independent normally distributed random variables with mean zero and standard deviation 0.01. Let $C \in \mathbb{R}^{n-1 \times n}$ be a matrix that captures the constraint, i.e., C(i,i) = 1and C(i,i+1) = -1 for $1 \le i \le n-1$ and its other components are zero and let $X \triangleq \{x : Cx \le 0\}$. Hence, we can rewrite the problem (29) as $\min_{x \in \mathbb{R}^n} f(x) := \frac{1}{2} \sum_{i=1}^p ||A_ix - b_i||^2 + \mathcal{I}_X(x)$. We know that the smooth approximation of the indicator function is $\mathcal{I}_{X,\eta} = \frac{1}{2\eta} d_X^2(x)$. Therefore, we apply (**rsVS-SQN**) on the following problem

$$\min_{x \in \mathbb{R}^n} f(x) \triangleq \frac{1}{2} \sum_{i=1}^p \|A_i x - b_i\|^2 + \frac{1}{2\eta} d_X^2(x).$$
(30)

Parameter choices are similar to those in Table 5 and we note from Fig. 3 (Left) that empirical behavior appears to be favorable.



Figure 3: Left: (sVS-SQN) Right: (sVS-SQN) vs. BFGS

Example 4. Comparison of (s-QN) with BFGS In [24], the authors show that a nonsmooth BFGS scheme may take null steps and fails to converge to the optimal solution (See Fig. 1) and consider the following problem.

$$\min_{x \in \mathbb{R}^2} \qquad \frac{1}{2} \|x\|^2 + \max\{2|x_1| + x_2, 3x_2\}$$

In this problem, BFGS takes a null step after two iterations (steplength is zero); however (s-QN) (the deterministic version of (sVS-SQN)) converges to the optimal solution. Note that the optimal solution is (0, -1) and (s-QN) reaches (0, -1.0006) in just 0.095 seconds (see Fig. 3 (Right)).

6 Conclusions

Most SQN schemes can process smooth and strongly convex stochastic optimization problems and there appears be a gap in the asymptotics and rate statements in addressing merely convex and possibly nonsmooth settings. Furthermore, a clear difference exists between deterministic rates and their stochastic counterparts, paving the way for developing variance-reduced schemes. In addition, much of the available statements rely on a somewhat stronger assumption of uniform boundedness of the conditional second moment of the noise, which is often difficult to satisfy in unconstrained regimes. Accordingly, the present paper makes three sets of contributions. First, a regularized smoothed L-BFGS update is proposed that combines regularization and smoothing, providing a foundation for addressing nonsmoothness and a lack of strong convexity. Second, we develop a variable sample-size SQN scheme (VS-SQN) for strongly convex problems and its Moreau smoothed variant (sVS-SQN) for nonsmooth (but smoothable) variants, both of which attain a linear rate of convergence and an optimal oracle complexity. To contend with the possible unavailability of strong convexity and Lipschitzian parameters, we also derive sublinear rates of convergence for diminishing steplength variants. Third, in merely convex regimes, we develop a regularized VS-SQN (**rVS-SQN**) and its smoothed variant (**rsVS-SQN**) for smooth and nonsmooth problems respectively. The former achieves a rate of $\mathcal{O}(1/K^{1-\epsilon})$ while the rate degenerates to $\mathcal{O}(1/K^{1/3-\epsilon})$ in the case of the latter. Finally, numerics suggest that the SQN schemes compare well with their variable sample-size accelerated gradient counterparts and perform particularly well in comparison when the problem is afflicted by ill-conditioning.

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7 Appendix

Proof of Proposition 1. From Assumption 2 (a,b), f is τ -strongly convex and L-smooth. From Lipschitz continuity of $\nabla f(x)$ and update rule (**VS-SQN**), we have the following:

$$f(x_{k+1}) \leq f(x_k) + \nabla f(x_k)^T (x_{k+1} - x_k) + \frac{L}{2} ||x_{k+1} - x_k||^2$$

= $f(x_k) + \nabla f(x_k)^T (-\gamma_k H_k (\nabla f(x_k) + \bar{w}_{k,N_k})) + \frac{L}{2} \gamma_k^2 ||H_k (\nabla f(x_k) + \bar{w}_{k,N_k})||^2,$

where $\bar{w}_{k,N_k} \triangleq \frac{\sum_{j=1}^{N_k} (\nabla_x F(x_k, \omega_{j,k}) - \nabla f(x_k))}{N_k}$. By taking expectations conditioned on \mathcal{F}_k , using Lemma 1, and Assumption 4 (S-M) and (S-B), we obtain the following.

$$\mathbb{E}\left[f(x_{k+1}) - f(x_k) \mid \mathcal{F}_k\right] \leq -\gamma_k \nabla f(x_k)^T H_k \nabla f(x_k) + \frac{L}{2} \gamma_k^2 \|H_k \nabla f(x_k)\|^2 + \frac{\gamma_k^2 \overline{\lambda}^2 L}{2} \mathbb{E}\left[\|\bar{w}_{k,N_k}\|^2 \mid \mathcal{F}_k\right] \\ = \gamma_k \nabla f(x_k)^T H_k^{1/2} \left(-I + \frac{L}{2} \gamma_k H_k\right) H_k^{1/2} \nabla f(x_k) + \frac{\gamma_k^2 \overline{\lambda}^2 L(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{2N_k} \\ \leq -\gamma_k \left(1 - \frac{L}{2} \gamma_k \overline{\lambda}\right) \|H_k^{1/2} \nabla f(x_k)\|^2 + \frac{\gamma_k^2 \overline{\lambda}^2 L(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{2N_k} = \frac{-\gamma_k}{2} \|H_k^{1/2} \nabla f(x_k)\|^2 + \frac{\nu_1^2 \|x_k\|^2 + \nu_2^2}{2LN_k}$$

where the last equality follows from $\gamma_k = \frac{1}{L\overline{\lambda}}$ for all k. Since f is strongly convex with modulus τ , $\|\nabla f(x_k)\|^2 \ge 2\tau (f(x_k) - f(x^*))$. Therefore by subtracting $f(x^*)$ from both sides, we obtain:

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*) \mid \mathcal{F}_k\right] \le f(x_k) - f(x^*) - \frac{\gamma_k \underline{\lambda}}{2} \|\nabla f(x_k)\|^2 + \frac{\nu_1^2 \|x_k - x^* + x^*\|^2 + \nu_2^2}{2LN_k} \le \left(1 - \tau \gamma_k \underline{\lambda} + \frac{2\nu_1^2}{L\tau N_k}\right) (f(x_k) - f(x^*)) + \frac{2\nu_1^2 \|x^*\|^2 + \nu_2^2}{2LN_k}, \quad (31)$$

where the last inequality is a consequence of $f(x_k) \ge f(x^*) + \frac{\tau}{2} ||x_k - x^*||^2$. Taking unconditional expectations on both sides of (31), choosing $\gamma_k = \frac{1}{L\overline{\lambda}}$ for all k and invoking the assumption that $\{N_k\}$ is an increasing sequence, we obtain the following.

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*)\right] \le \left(1 - \frac{\tau \underline{\lambda}}{L\overline{\lambda}} + \frac{2\nu_1^2}{L\tau N_0}\right) \mathbb{E}\left[f(x_k) - f(x^*)\right] + \frac{2\nu_1^2 ||x^*||^2 + \nu_2^2}{2LN_k}.$$

Proof of Theorem 1. From Assumption 2 (a,b), f is τ -strongly convex and L-smooth. (i) Let $a \triangleq \left(1 - \frac{\tau \lambda}{L\lambda} + \frac{2\nu_1^2}{L\tau N_0}\right)$, $b_k \triangleq \frac{2\nu_1^2 \|x^*\|^2 + \nu_2^2}{2LN_k}$, and $N_k \triangleq \lceil N_0 \rho^{-k} \rceil \ge N_0 \rho^{-k}$. Note that, choosing $N_0 \ge \frac{2\nu_1^2 \overline{\lambda}}{\tau^2 \underline{\lambda}}$ leads to a < 1. Consider $C \triangleq \mathbb{E}[f(x_0) - f(x^*)] + \left(\frac{2\nu_1^2 \|x^*\|^2 + \nu_2^2}{2N_0 L}\right) \frac{1}{1 - (\min\{a, \rho\}/\max\{a, \rho\})}$. Then by Prop. 1, we obtain the following for every $k \ge 1$.

$$\mathbb{E}\left[f(x_{K+1}) - f(x^*)\right] \le a^{K+1} \mathbb{E}\left[f(x_0) - f(x^*)\right] + \sum_{i=0}^{K} a^{K-i} b_i$$

$$\le a^{K+1} \mathbb{E}\left[f(x_0) - f(x^*)\right] + \frac{(\max\{a,\rho\})^K (2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2N_0 L} \sum_{i=0}^{K} \left(\frac{\min\{a,\rho\}}{\max\{a,\rho\}}\right)^{K-i}$$

$$\le a^{K+1} \mathbb{E}\left[f(x_0) - f(x^*)\right] + \left(\frac{(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2N_0 L}\right) \frac{(\max\{a,\rho\})^K}{1 - (\min\{a,\rho\}/\max\{a,\rho\})} \le C(\max\{a,\rho\})^K.$$

Furthermore, we may derive the number of steps K to obtain an ϵ -solution. Without loss of generality, suppose $\max\{a, \rho\} = a$. Choose $N_0 \geq \frac{4\nu_1^2 \lambda}{\tau^2 \overline{\lambda}}$, then $a = \left(1 - \left(\frac{\tau \lambda}{2L\overline{\lambda}}\right)\right) = 1 - \frac{1}{\alpha \kappa}$, where $\alpha = \frac{2\overline{\lambda}}{\overline{\lambda}}$. Therefore, since $\frac{1}{a} = \frac{1}{(1 - \frac{1}{\alpha \kappa})}$, by using the definition of $\underline{\lambda}$ and $\overline{\lambda}$ in Lemma 1 to get $\alpha = \frac{2\overline{\lambda}}{\overline{\lambda}} = \mathcal{O}(\kappa^{m+n})$, we obtain that

$$\left(\frac{\ln(C) - \ln(\epsilon)}{\ln(1/a)}\right) = \left(\frac{\ln(C/\epsilon)}{\ln(1/(1 - \frac{1}{\alpha\kappa}))}\right) = \left(\frac{\ln(C/\epsilon)}{-\ln((1 - \frac{1}{\alpha\kappa}))}\right) \le \left(\frac{\ln(C/\epsilon)}{\frac{1}{\alpha\kappa}}\right) = \mathcal{O}(\kappa^{m+n+1}\ln(\tilde{C}/\epsilon)),$$

where the bound holds when $\alpha \kappa > 1$. It follows that the iteration complexity of computing an ϵ -solution is $\mathcal{O}(\kappa^{m+1}\ln(\frac{C}{\epsilon}))$. (ii) To compute a vector x_{K+1} satisfying $\mathbb{E}[f(x_{K+1}) - f^*] \leq \epsilon$, we consider the case where $a > \rho$ while the other case follows similarly. Then we have that $Ca^K \leq \epsilon$, implying that $K = \lceil \ln_{(1/a)}(C/\epsilon) \rceil$. To obtain the optimal oracle complexity, we require $\sum_{k=1}^{K} N_k$ gradients. If $N_k = \lceil N_0 a^{-k} \rceil \leq 2N_0 a^{-k}$, we obtain the following since $(1-a) = 1/(\alpha \kappa)$.

$$\sum_{k=1}^{n_{(1/a)}(C/\epsilon)+1} 2N_0 a^{-k} \le \frac{2N_0}{\left(\frac{1}{a}-1\right)} \left(\frac{1}{a}\right)^{3+\ln_{(1/a)}(C/\epsilon)} \le \left(\frac{C}{\epsilon}\right) \frac{2N_0}{a^2(1-a)} = \frac{2N_0\alpha\kappa C}{a^2\epsilon}.$$

Note that $a = 1 - \frac{1}{\alpha \kappa}$ and $\alpha = \mathcal{O}(\kappa^{m+n})$, implying that

$$a^{2} = 1 - 2/(\alpha\kappa) + 1/(\alpha^{2}\kappa^{2}) \ge \frac{\alpha^{2}\kappa^{2} - 2\alpha\kappa^{2} + 1}{\alpha^{2}\kappa^{2}} \ge \frac{\alpha^{2}\kappa^{2} - 2\alpha\kappa^{2}}{\alpha^{2}\kappa^{2}} = \frac{(\alpha^{2} - 2\alpha)}{\alpha^{2}}$$
$$\implies \frac{\kappa}{a^{2}} \le \frac{\alpha^{2}\kappa}{(\alpha^{2} - 2\alpha)} = \left(\frac{\alpha}{\alpha - 2}\right)\kappa \implies \sum_{k=1}^{\ln_{(1/a)}(C/\epsilon) + 1} a^{-k} \le \frac{2N_{0}\alpha^{2}\kappa C}{(\alpha - 2)\epsilon} = \mathcal{O}\left(\frac{\kappa^{m+n+1}}{\epsilon}\right)$$

(iii) Similar to (31), using strong convexity of f and choosing $N_k = \lceil k^{p-s} \rceil$ and $\gamma_k = k^{-s}$, we can obtain the following.

$$\mathbb{E}\left[f(x_{k+1}) - f(x_k) \mid \mathcal{F}_k\right] \le -\gamma_k \left(1 - \frac{L}{2} \gamma_k \overline{\lambda}\right) \|H_k^{1/2} \nabla f(x_k)\|^2 + \frac{\gamma_k^2 \overline{\lambda}^2 L(\nu_1^2 \|x_k\|^2 + \nu_2^2)}{2N_k} \\ \implies \mathbb{E}\left[f(x_{k+1}) - f(x^*) \mid \mathcal{F}_k\right] \le \left(1 - \gamma_k (1 - \frac{L\overline{\lambda}\gamma_k}{2}) \underline{\lambda}\tau + \frac{\gamma_k^2 \overline{\lambda}^2 L\nu_1^2}{\tau N_k}\right) (f(x_k) - f(x^*)) + \frac{\gamma_k^2 \overline{\lambda}^2 L(2\nu_1^2 \|x^*\|^2 + \nu_2^2)}{2N_k}$$

Since γ_k is a diminishing sequence and N_k is an increasing sequence, for sufficiently large K we have that $\frac{L\bar{\lambda}\gamma_k^2\underline{\lambda}\tau}{2} + \frac{\gamma_k^2\bar{\lambda}^2L\nu_1^2}{\tau N_k} \leq \frac{1}{2}\gamma_k\underline{\lambda}\tau$. Therefore, we obtain:

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*) \mid \mathcal{F}_k\right] \le \left(1 - \frac{\gamma_k \underline{\lambda} \tau}{2}\right) \left(f(x_k) - f(x^*)\right) + \frac{\gamma_k^2 \overline{\lambda}^2 L(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2N_k} \\ = \left(1 - \frac{c}{k^s}\right) \left(f(x_k) - f(x^*)\right) + \frac{d}{k^{p+s}}, \text{ for } k \ge K,$$

where $c \triangleq \frac{\lambda \tau}{2}$ and $d \triangleq \frac{\overline{\lambda}^2 L(2\nu_1^2 ||x^*||^2 + \nu_2^2)}{2}$. Then by taking unconditional expectations and recalling that 0 < s < 1, s < p, we may invoke Lemma 2 to claim that there exists K such that

$$\mathbb{E}\left[f(x_{k+1}) - f(x^*)\right] \le \left(\frac{d}{ck^p}\right) + o\left(\frac{1}{k^p}\right), \quad k \ge K.$$

Proof of Lemma 1 and Lemma 10 : First we prove Lemma 10 and then we show that how the result in Lemma 1 can be proved similarly. Recall that $\underline{\lambda}_k$ and $\overline{\lambda}_k$ denote the minimum and maximum eigenvalues of H_k , respectively. Also, we denote the inverse of matrix H_k by B_k .

Lemma 11. [48] Let $0 < a_1 \le a_2 \le \ldots \le a_n$, P and S be positive scalars such that $\sum_{i=1}^n a_i \le S$ and $\prod_{i=1}^n a_i \ge P$. Then, we have $a_1 \ge \frac{(n-1)!P}{S^{n-1}}$.

Proof of Lemma 10: It can be seen, by induction on k, that H_k is symmetric and \mathcal{F}_k measurable, assuming that all matrices are well-defined. We use induction on odd values of k > 2m to show that the statements of part (a), (b) and (c) hold and that the matrices are well-defined. Suppose k > 2m is odd and for any odd value of t < k, we have $s_t^T y_t > 0$, $H_t y_t = s_t$, and part (c) holds for t. We show that these statements also hold for k. First, we prove that the secant condition holds.

$$s_{k}^{T}y_{k} = (x_{k} - x_{k-1})^{T} \left(\frac{\sum_{j=1}^{N_{k-1}} \left(\nabla F_{\eta_{k}^{\delta}}(x_{k}, \omega_{j,k-1}) - \nabla F_{\eta_{k}^{\delta}}(x_{k-1}, \omega_{j,k-1}) \right)}{N_{k-1}} + \mu_{k}^{\delta}(x_{k} - x_{k-1}) \right) \\ = \frac{\sum_{j=1}^{N_{k-1}} \left[(x_{k} - x_{k-1})^{T} (\nabla F_{\eta_{k}^{\delta}}(x_{k}, \omega_{j,k-1}) - \nabla F_{\eta_{k}^{\delta}}(x_{k-1}, \omega_{j,k-1})) \right]}{N_{k-1}} + \mu_{k}^{\delta} \|x_{k} - x_{k-1}\|^{2} \ge \mu_{k}^{\delta} \|x_{k} - x_{k-1}\|^{2}$$

where the inequality follows from the monotonicity of the gradient map $\nabla F(\cdot, \omega)$. From the induction hypothesis, H_{k-2} is positive definite, since k-2 is odd. Furthermore, since k-2 is odd, we have $H_{k-1} = H_{k-2}$ by the update rule (8). Therefore, H_{k-1} is positive definite. Note that since k-2 is odd, the choice of μ_{k-1} is such that $\frac{1}{N_{k-1}} \sum_{j=1}^{N_{k-1}} \nabla F_{\eta_k^{\delta}}(x_{k-1}, \omega_{j,k-1}) + \mu_{k-1}x_{k-1} \neq 0$ (see the discussion following (6)). Since H_{k-1} is positive definite, we have $H_{k-1}\left(\frac{1}{N_{k-1}}\sum_{j=1}^{N_{k-1}} \nabla F_{\eta_k^{\delta}}(x_{k-1}, \omega_{j,k-1}) + \mu_{k-1}x_{k-1}\right) \neq 0$

0, implying that $x_k \neq x_{k-1}$. Hence $s_k^T y_k \geq \mu_k^{\delta} ||x_k - x_{k-1}||^2 > 0$, where the second inequality is a consequence of $\mu_k > 0$. Thus, the secant condition holds. Next, we show that part (c) holds for k. Let $\underline{\lambda}_k$ and $\overline{\lambda}_k$ denote the minimum and maximum eigenvalues of H_k , respectively. Denote the inverse of matrix H_k in (9) by B_k . It is well-known that using the Sherman-Morrison-Woodbury formula, B_k is equal to $B_{k,m}$ given by

$$B_{k,j} = B_{k,j-1} - \frac{B_{k,j-1}s_i s_i^T B_{k,j-1}}{s_i^T B_{k,j-1} s_i} + \frac{y_i y_i^T}{y_i^T s_i}, \quad i := k - 2(m-j) \quad 1 \le j \le m,$$
(32)

where s_i and y_i are defined by (7) and $B_{k,0} = \frac{y_k^T y_k}{s_k^T y_k} \mathbf{I}$. First, we show that for any i,

$$\mu_{k}^{\delta} \leq \frac{\|y_{i}\|^{2}}{y_{i}^{T} s_{i}} \leq 1/\eta_{k}^{\delta} + \mu_{k}^{\delta},$$
(33)

Let us consider the function $h(x) := \frac{1}{N_{i-1}} \sum_{j=1}^{N_{i-1}} F_{\eta_k^{\delta}}(x, \omega_{j,i-1}) + \frac{\mu_k^{\delta}}{2} ||x||^2$ for fixed *i* and *k*. Note that this function is strongly convex and has a gradient mapping of the form $\frac{1}{N_{i-1}} \sum_{j=1}^{N_{i-1}} \nabla F_{\eta_k^{\delta}}(x_{i-1}, \omega_{j,i-1}) + \mu_k^{\delta} \mathbf{I}$ that is Lipschitz with parameter $\frac{1}{\eta_k^{\delta}} + \mu_k^{\delta}$. For a convex function *h* with Lipschitz gradient with parameter $1/\eta_k^{\delta} + \mu_k^{\delta}$, the following inequality, referred to as co-coercivity property, holds for any $x_1, x_2 \in \mathbb{R}^n$ (see [39], Lemma 2): $\|\nabla h(x_2) - \nabla h(x_1)\|^2 \leq (1/\eta_k^{\delta} + \mu_k^{\delta})(x_2 - x_1)^T (\nabla h(x_2) - \nabla h(x_1))$. Substituting x_2 by x_i, x_1 by x_{i-1} , and recalling (7), the preceding inequality yields

$$\|y_i\|^2 \le (1/\eta_k^{\delta} + \mu_k^{\delta}) s_i^T y_i.$$
(34)

Note that function h is strongly convex with parameter μ_k^{δ} . Applying the Cauchy-Schwarz inequality, we can write

$$\frac{\|y_i\|^2}{s_i^T y_i} \ge \frac{\|y_i\|^2}{\|s_i\| \|y_i\|} = \frac{\|y_i\|}{\|s_i\|} \ge \frac{\|y_i\| \|s_i\|}{\|s_i\|^2} \ge \frac{y_i^T s_i}{\|s_i\|^2} \ge \mu_k^{\delta}.$$

Combining this relation with (34), we obtain (33). Next, we show that the maximum eigenvalue of B_k is bounded. Let $Trace(\cdot)$ denote the trace of a matrix. Taking trace from both sides of (32) and summing up over index j, we obtain

$$Trace(B_{k,m}) = Trace(B_{k,0}) - \sum_{j=1}^{m} Trace\left(\frac{B_{k,j-1}s_i s_i^T B_{k,j-1}}{s_i^T B_{k,j-1}s_i}\right) + \sum_{j=1}^{m} Trace\left(\frac{y_i y_i^T}{y_i^T s_i}\right)$$
(35)

$$= Trace\left(\frac{\|y_i\|^2}{y_i^T s_i}\mathbf{I}\right) - \sum_{j=1}^m \frac{\|B_{k,j-1}s_i\|^2}{s_i^T B_{k,j-1}s_i} + \sum_{j=1}^m \frac{\|y_i\|^2}{y_i^T s_i} \le n\frac{\|y_i\|^2}{y_i^T s_i} + \sum_{j=1}^m (1/\eta_k^\delta + \mu_k^\delta) = (m+n)(1/\eta_k^\delta + \mu_k^\delta),$$

where the third relation is obtained by positive-definiteness of B_k (this can be seen by induction on k, and using (32) and $B_{k,0} \succ 0$). Since $B_k = B_{k,m}$, the maximum eigenvalue of the matrix B_k is bounded. As a result,

$$\underline{\lambda}_k \ge \frac{1}{(m+n)(1/\eta_k^{\delta} + \mu_k^{\delta})}.$$
(36)

In the next part of the proof, we establish the bound for $\overline{\lambda}_k$. From Lemma 3 in [30], we have $det(B_{k,m}) = det(B_{k,0}) \prod_{j=1}^m \frac{s_i^T y_i}{s_i^T B_{k,j-1} s_i}$. Multiplying and dividing by $s_i^T s_i$, using the strong convexity of the function h, and invoking (33) and the result of (35), we obtain

$$det(B_{k}) = det\left(\frac{y_{k}^{T}y_{k}}{s_{k}^{T}y_{k}}\mathbf{I}\right)\prod_{j=1}^{m}\left(\frac{s_{i}^{T}y_{i}}{s_{i}^{T}s_{i}}\right)\left(\frac{s_{i}^{T}s_{i}}{s_{i}^{T}B_{k,j-1}s_{i}}\right) \ge \left(\frac{y_{k}^{T}y_{k}}{s_{k}^{T}y_{k}}\right)^{n}\prod_{j=1}^{m}\mu_{k}^{\delta}\left(\frac{s_{i}^{T}s_{i}}{s_{i}^{T}B_{k,j-1}s_{i}}\right) \ge (\mu_{k})^{(n+m)\delta}\prod_{j=1}^{m}\frac{1}{(m+n)(1/\eta_{k}^{\delta}+\mu_{k}^{\delta})} = \frac{\mu_{k}^{(n+m)\delta}}{(m+n)^{m}(1/\eta_{k}^{\delta}+\mu_{k}^{\delta})^{m}}.$$
(37)

Let $\alpha_{k,1} \leq \alpha_{k,2} \leq \ldots \leq \alpha_{k,n}$ be the eigenvalues of B_k sorted non-decreasingly. Note that since $B_k \succ 0$, all the eigenvalues are positive. Also, from (35), we know that $\alpha_{k,\ell} \leq (m+n)(L+\mu_0^{\delta})$. Taking (34) and (37) into account, and employing Lemma 11, we obtain

$$\alpha_{k,1} \ge \frac{(n-1)! \mu_k^{(n+m)\delta}}{(m+n)^{n+m-1} (1/\eta_k^{\delta} + \mu_k^{\delta})^{n+m-1}}$$

This relation and that $\alpha_{k,1} = \overline{\lambda}_k^{-1}$ imply that

$$\overline{\lambda}_k \le \frac{(m+n)^{n+m-1} (1/\eta_k^{\delta} + \mu_k^{\delta})^{n+m-1}}{(n-1)! \mu_k^{(n+m)\delta}}.$$
(38)

Therefore, from (36) and (38) and that μ_k is non-increasing, we conclude that part (c) holds for k as well. Next, we show that $H_k y_k = s_k$. From (32), for j = m we obtain

$$B_{k,m} = B_{k,m-1} - \frac{B_{k,m-1}s_k s_k^T B_{k,m-1}}{s_k^T B_{k,m-1} s_k} + \frac{y_k y_k^T}{y_k^T s_k},$$

where we used i = k - 2(m - m) = k. Multiplying both sides of the preceding equation by s_k , and using $B_k = B_{k,m}$, we have $B_k s_k = B_{k,m-1} s_k - B_{k,m-1} s_k + y_k = y_k$. Multiplying both sides of the preceding relation by H_k and invoking $H_k = B_k^{-1}$, we conclude that $H_k y_k = s_k$. Therefore, we showed that the statements of (a), (b), and (c) hold for k, assuming that they hold for any odd 2m < t < k. In a similar fashion to this analysis, it can be seen that the statements hold for t = 2m + 1. Thus, by induction, we conclude that the statements hold for any odd k > 2m. To complete the proof, it is enough to show that part (c) holds for any even value of k > 2m. Let t = k - 1. Since t > 2m is odd, relation part (c) holds. Writing it for k - 1, and taking into account that $H_k = H_{k-1}$, and $\mu_k < \mu_{k-1}$, we can conclude that part (c) holds for any even value of k > 2mand this completes the proof.

Proof of Lemma 1: We observe that Lemma 1 is the special case of Lemma 10, where the objective f function is L-smooth and τ -strongly convex. Similar to (36), by noting that $\mu_k = 0$ and $\frac{1}{\eta_k} = L$ for all k, we may show that $\underline{\lambda}_k = \underline{\lambda} = \frac{1}{L(m+n)}$ for all k. Furthermore from the τ -strongly convex nature of f, akin to (38), we obtain that $\overline{\lambda} = \frac{((m+n)L)^{n+m-1}}{(n-1)!\tau^{n+m}}$.