Unsafe Point Avoidance in Linear State Feedback

Philipp Braun[1,2], Christopher M. Kellett[1] and Luca Zaccarian[3]

***Abstract*— We propose a hybrid solution for the stabilization of the origin of a linear time-invariant stabilizable system with the property that a suitable neighborhood of a pre-defined unsafe point in the state space is avoided by the closed-loop solutions. Hybrid tools are motivated by the fact that the task at hand cannot be solved with continuous feedback, whereas the proposed hybrid solution induces nominal and robust asymptotic stability of the origin. More specifically, we formulate a semiglobal version of the problem at hand and describe a fully constructive approach under the assumption that the unsafe point to be avoided does not belong to the equilibrium subspace induced by the control input on the linear dynamics. The approach is illustrated on a numerical example.**

I. Introduction

Lyapunov functions [5] provide a well established tool to analyze and characterize stability properties of general dynamical systems and are an important mechanism in the control literature to construct stabilizing feedback laws. While global asymptotic stability/stabilization (GAS) of unconstrained dynamical systems is well understood, stability/stabilization of dynamical systems subject to bounded state constraints, e.g., obstacle avoidance for mobile robots or collision avoidance in the coordination of drones, has yet to be addressed rigorously for general classes of dynamical systems. While in the context of unconstrained stabilization, discontinuous control laws only need to be considered for the class of systems that are asymptotically controllable but not Lipschitz continuous feedback stabilizable (e.g., the nonholonomic integrator [3]), discontinuous feedback laws are necessary in the presence of bounded constraints, independent of the system dynamics (see [2] for an illustrative proof). A similar need for discontinuous feedback laws is discussed in [6] in terms of topological obstructions on manifolds.

When using control Lyapunov functions, the need for discontinuous feedback laws precludes the use of Sontag's universal formula [13], for example, since it leads to a continuous feedback law. Thus, approaches extending classical results on control Lyapunov functions by control barrier functions [17] to include constraints in the state space, are limited to constraints defining unbounded sets. In particular, this impacts approaches in [7], [14], [1], [10], since they rely on the existence of continuous feedback laws.

Additionally, note that the model predicitive control literature does not provide a general framework for obstacle avoidance and global stabilization. Even though it is simple to define an optimization problem to iteratively compute a feedback law, proving GAS of the closed loop and recursive feasibility is nontrivial.

One way to define discontinuous feedback laws, and which we will follow in this paper, is to unite local and global controllers. This approach traces back to [15] and was further investigated and established using the formalism of hybrid dynamical systems in [8], [16], [9], [11], [12]. While the results in these works are promising and motivating, the papers address particular applications and do not provide a general tool for controller design subject to bounded state contsraints.

In contrast to the approaches discussed above, we propose a constructive method to design a hybrid control law for a controllable linear system that simultaneously guarantees GAS of the origin and avoidance of a neighborhood around a given obstacle described by a single point. While we address the case of a single unsafe point, our approach easily extends to the case of multiple points.

The paper is structured as follows. In Section II the mathematical setting and the problem under consideration are formalized. In Section III the "wipeout" property is introduced, ensuring that solutions getting close to the obstacle are guaranteed to leave a neighborhood around the obstacle in finite time. This result is used in Section IV to define a local obstacle avoidance controller. Section V combines the results to obtain a global hybrid control law. Here, the main result providing GAS while avoiding the obstacle is stated. The results of the hybrid controller are illustrated on a numerical example in Section VI before the paper concludes in Section VII.

Throughout the paper the following notation is used. For $x \in \mathbb{R}^n$ we use the vector norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. Similarly, the distance to a point $y \in \mathbb{R}^n$ is denoted by $|x|_y = |x - y|$. For a closed set $\mathcal{A} \in \mathbb{R}^n$ and $r > 0$ we define $\mathcal{B}_r(\mathcal{A}) = \{x \in \mathbb{R}^n \,|\, \min_{y \in \mathcal{A}} |x - y| \leq r\}$. The closure, the boundary and the interior of a set are denoted by $\overline{\mathcal{A}}$, $\partial \mathcal{A}$ and $\operatorname{int}(\mathcal{A})$, respectively. The identiy matrix of appropriate dimension is denoted by I.

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[1] P. Braun and C. M. Kellett are with the School of Electrical Engineering and Computing at the University of Newcastle, Callaghan, New South Wales 2308, Australia, {chris.kellett, philipp.braun}@newcastle.edu.au.

[2] P. Braun is with the Mathematical Institute, Universität Bayreuth, 95440 Bayreuth, Germany.

[3] L. Zaccarian is with the Laboratoire d'Analyse et d'Architecture des Systèmes-Centre Nationnal de la Recherche Scientifique, Université de Toulouse, CNRS, UPS, 31400 Toulouse, France, and also with the Dipartimento diIngegneria Industriale, University of Trento, 38122 Trento, Italy, luca.zaccarian@laas.fr.

II. Setting & problem formulation

In this paper we consider linear dynamical systems

$$\dot{x} = Ax + Bu, \qquad x_0 = x(0) \in \mathbb{R}^n \tag{1}$$

with state $x \in \mathbb{R}^n$, one dimensional input $u \in \mathbb{R}$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$. An extension to the multidimensional input case $u \in \mathbb{R}^m$, $m \in \mathbb{N}$, is straightforward. As motivated in the introduction, the paper addresses the following general problem and provides a solution under some simplifying assumptions described below.

Problem 1: (Semiglobal x_a-avoidance augmentation with GAS) Given an "unsafe" point $x_\mathrm{a} \in \mathbb{R}^n$ that must be avoided by the controller, and a stabilizing state feedback $u_\mathrm{s} = K_\mathrm{s} x$, for each $\delta > 0$, design a feedback selection of u that guarantees

(i) (GAS) uniform global asymptotic stability of the origin;

(ii) (semiglobal preservation) the feedback matches the original stabilizer $u(x) = K_\mathrm{s} x$ in $\mathbb{R}^n \setminus \mathcal{B}_\delta(x_\mathrm{a})$; and

(iii) (semiglobal x_a-avoidance) all solutions starting outside the ball $\mathcal{B}_\delta(x_\mathrm{a})$ never enter a suitable "safety" neighborhood of x_a. ⌟

Problem 1 entails the desirable property that the modifications enforced by the avoidance augmentation are minimally invasive, because semiglobal preservation ensures that the pre-defined stabilizer u_s is unchanged in an arbitrarily large subset of the state space. Note that this goal is similar but not quite the same as the one of safety region avoidance, addressed in [10] and [1].

In contrast to those works, we do not assume the existence of a control Lyapunov function and a control barrier function avoiding an a priori fixed neighborhood around x_a, characterized in item (iii). Instead we provide a constructive method to design the control and provide a corresponding bound on the size of the neighborhood that can be avoided. To keep the discussion simple, we only address one point x_a but our construction carries over trivially to the case of multiple unsafe points, always providing a constructive solution to the corresponding semiglobal avoidance design. We emphasize again that Problem 1 cannot be solved by a continuous feedback, as motivated in the introduction, and thus we provide a hybrid solution to the problem here. For our construction, we will enforce the following assumption on the system data.

Assumption 1: Basic assumptions:

(a) Matrix $A_\mathrm{s} := A + BK_\mathrm{s}$ is Hurwitz.

(b) Vectors Ax_a and B are linearly independent.

(c) Vector B is a unit vector (namely $|B| = 1$).

(d) The norm $x \mapsto |x|^2$ is contractive under the stabilizer $u_\mathrm{s} = K_\mathrm{s} x$ (equivalently, $A_\mathrm{s} + A_\mathrm{s}^T < 0$). ⌟

We note that Assumption 1(a) is a necessary condition for Problem 1(i,ii), whereas Assumptions 1(c) and 1(d) are simplifying assumptions that can be easily removed by suitable input and state transformations, respectively. In particular, Assumption 1(c) can be achieved through the definition $\dot{x} = Ax + B_\circ u_\circ$, where $u_\circ := |B|u$ and $B_\circ := B/|B|$. With respect to Assumption 1(d), if $V(x) = x^T S x$ is a Lyapunov function for the closed-loop system $\dot{x} = A_\mathrm{s} x$, then $V(\tilde{x}) = |\tilde{x}|^2$ is a Lyapunov function in the coordinates $\tilde{x} = S_F x$, where $S_F^T S_F = S$ denotes the Cholesky factorization of S.

Assumption 1(b) is the only substantial restriction that we make in this paper and will be addressed in future work. Even under this simplifying assumption it appears that Problem 1 requires a sufficient amount of sophistication. Assumption 1(b) enables us to exploit the convenient property that solutions transit through any small enough neighborhood of x_a independently of the input u. This property, that we call the "wipeout" property, is characterized in Section III hereafter, and is one of the two main ingredients of our solution. The other ingredient corresponds to a suitable repulsive control design, characterized in Section IV, ensuring that solutions that approach x_a are suitably modified to avoid entering a peculiar "shell" corresponding to the above characterized "safety neighborhood" of x_a. We emphasize that the two above mentioned ingredients, developed in Sections III and IV below, are independent of each other, which establishes a desirable modularity in our design, prone for future developments of this research direction.

III. η-neighborhood and wipeout property

In this section we provide a thorough characterization of the implications of Assumption 1(b) to ensure that local equilibria around x_a cannot be created by whatever feedback solution u we may design to solve Problem 1. We first provide a few equivalences.

Lemma 1: The following items are equivalent:

(i) Assumption 1(b) holds.

(ii) The point x_a cannot be an induced equilibrium of the linear dynamics, namely

$$x_\mathrm{a} \notin \mathcal{E} := \{y \in \mathbb{R}^n : \exists u^*, Ay + Bu^* = 0\}. \tag{2}$$

(iii) It holds that

$$A_B x_\mathrm{a} := (I - BB^T)Ax_\mathrm{a} \neq 0. \tag{3}$$

⌟

Proof: The equivalence between (i) and (ii) is a trivial consequence of the definition of linear independence.
"(ii) ⇒ (iii):" If $(I - BB^T)Ax_\mathrm{a} = 0$, then selecting $u^* = -B^T Ax_\mathrm{a}$ leads to $Ax_\mathrm{a} + Bu^* = 0$.
"(iii) ⇒ (ii):" If $\exists u^*$ such that $Ax_\mathrm{a} + Bu^* = 0$, then, using $B^T B = 1$ and $Ax_\mathrm{a} = -Bu^*$ implies $(I - BB^T)Ax_\mathrm{a} = Ax_\mathrm{a} + BB^T Bu^*$. ■

In light of the property in (2), an important parameter in the control design proposed here is the (positive) distance between x_a and the subspace $\mathcal{E}$, defined as

$$\eta^2 := \min_{y\in\mathcal{E}} |x_\mathrm{a} - y|^2. \tag{4}$$

The parameter η is a positive scalar under Assumption 1 (by virtue of Lemma 1) and its positivity is essential for establishing that there exists a linear function of the state that monotonically increases in the interior of $\mathcal{B}_\eta(x_\mathrm{a})$, regardless of the choice of the input u. This property, called "wipeout" henceforth, is useful to establish that any solution flowing in $\mathcal{B}_\eta(x_\mathrm{a})$ must approach its boundary and leave any compact subset of its interior, in finite time. This wipeout feature helps in the analysis of the evolution of solutions within $\mathcal{B}_\eta(x_\mathrm{a})$, because solutions naturally drift away from small enough neighborhoods of x_a, regardless of the input u.

Proposition 1: (*Wipeout Property*). Let Assumption 1 hold. Consider the function $H(x) := x_\mathrm{a}^T A_B^T x$, where A_B is defined in (3), and the scalar $\eta > 0$ is defined in (4). For each $x \in \mathcal{B}_\eta(x_\mathrm{a})$ we have $\langle \nabla H(x), Ax + Bu \rangle \geq 0$ for all $u \in \mathbb{R}$. Moreover, for each $\bar{\eta} < \eta$, there exists $\underline{h} > 0$ such that

$$\langle \nabla H(x), Ax + Bu \rangle \geq \underline{h}, \quad \forall u \in \mathbb{R}, \forall x \in \mathcal{B}_{\bar{\eta}}(x_\mathrm{a}). \tag{5}$$

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Proof: Consider the identities, where we use the fact that the projection $\Pi_B := (I - BB^T)$ satisfies $\Pi_B^2 = \Pi_B$:

$$\begin{aligned} \dot{H}(x) &= x_\mathrm{a}^T A_B^T \dot{x} = x_\mathrm{a}^T A^T (I - BB^T)(Ax + Bu) \\ &= x_\mathrm{a}^T A^T (I - BB^T) Ax = (A_B x_\mathrm{a})^T (A_B x). \end{aligned} \tag{6}$$

By definition of η, and the left expression in (6), we know that $\dot{H}(x) \neq 0$ in $\mathcal{B}_\eta(x_\mathrm{a})$. By the right expression in (6), we know that $\dot{H}(x_\mathrm{a}) > 0$ and from continuity we obtain $\dot{H}(x) > 0$ for all $x \in \mathrm{int}(\mathcal{B}_\eta(x_\mathrm{a}))$. Then (5) follows from $\mathcal{B}_{\bar{\eta}}(x_\mathrm{a}) \subset \mathrm{int}(\mathcal{B}_\eta(x_\mathrm{a}))$, for all $\bar{\eta} < \eta$. Finally, $\langle \nabla H(x), Ax + Bu \rangle \geq 0$ in $\mathcal{B}_\eta(x_\mathrm{a})$ follows from continuity of $\dot{H}(\cdot)$. ■

Remark 1: Observe that due to the linearity of $H(x)$, $\nabla H(x)$ is independent of x and defines a direction

$$w_{x_\mathrm{a}} = \tfrac{\nabla H(x)}{|\nabla H(x)|} = \tfrac{A_B x_\mathrm{a}}{\sqrt{x_\mathrm{a}^T A_B^T A_B x_\mathrm{a}}},$$

which is well-defined through (3). This implies that (5) provides a lower bound on the speed the solution $x(t)$ moves in direction w_{x_a} for all $\bar{\eta} \leq \eta$, i.e., $\langle w_{x_\mathrm{a}}, \dot{x} \rangle \geq h_{\bar{\eta}}$ for $h_{\bar{\eta}} = \underline{h}/|\nabla H(x)|$. In particular, a solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_{\bar{\eta}}(x_\mathrm{a})$ for all $t \in [0, T]$, satisfies

$$\langle w_{x_\mathrm{a}}, x(T) - x(0) \rangle \geq T h_{\bar{\eta}}. \tag{7}$$

Moreover, a solution $x(\cdot)$ such that $x(t) \in \mathcal{B}_\eta(x_\mathrm{a})$ for all $t \in [0, T]$, satisfies

$$\langle w_{x_\mathrm{a}}, x(t_2) - x(t_1) \rangle \geq 0 \qquad \text{for all } 0 \leq t_1 \leq t_2 \leq T. \tag{8}$$

IV. Unsafe shell and avoidance controller

A. The eye-shaped shell $\mathcal{S}$

A second ingredient used in this paper, whose construction is parallel to, and independent of the wipeout function H introduced in the previous section, is the safety or avoidance controller u_a, acting in a neighborhood of the unsafe point x_a. The neighborhood is a nonsmooth compact set, having the shape of an eye, as visualized in Figure 1, defined based on two geometrical parameters:

1) the size $\delta \in \mathbb{R}_{>0}$ of the shell;
2) the aspect ratio $\mu \in (0, 2)$ of the shell.

Based on these two parameters, the shell $\mathcal{S}$ is the following intersection between two balls centered at some shifted versions of the unsafe point x_a:

$$\delta_\mu := \delta \left(\tfrac{1}{\mu} - \tfrac{\mu}{4} \right), \tag{9a}$$

$$\mathcal{O}_q := \mathcal{B}_{\left(\frac{\mu\delta}{2} + \delta_\mu\right)}(x_\mathrm{a} - q\delta_\mu B), \quad q \in \{1, -1\}, \tag{9b}$$

$$\mathcal{S}(\delta) := \mathcal{O}_1 \bigcap \mathcal{O}_{-1}. \tag{9c}$$

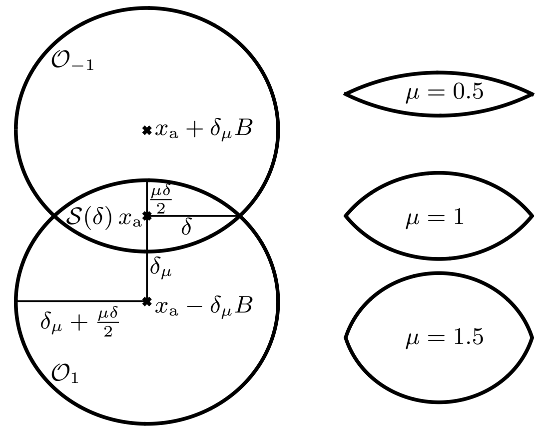

Fig. 1. The construction of the eye-shaped shell $\mathcal{S}(\delta)$ around the unsafe point x_a, based on the size $\delta \in \mathbb{R}_{>0}$ and the aspect ratio $\mu \in (0, 2)$.

Figure 1 represents a few possible shapes of these sets together with the distances that go with them. Note that $\mu \in (0, 2)$ fixes the aspect ratio of the shell, whose height corresponds to $\mu\delta$, which resembles an eye that is increasingly closed as μ approaches its lower limit 0. Conversely, as μ approaches its upper limit 2, the eye is increasingly open and converges to a circle. In our construction, we will assume that a certain desired aspect ratio μ is fixed a priori, and we will establish suitable results by exploiting the fact that the shell $\mathcal{S}(\delta)$ can be made arbitrarily large and arbitrarily small, by adjusting the positive parameter δ. In particular, the following fact will be used throughout our constructions.

Lemma 2: Given an aspect ratio $\mu \in (0, 2)$, for each $\delta > 0$, the following inclusions hold for the shell $\mathcal{S}(\delta)$ defined in (9):

$$\mathcal{B}_{\frac{\mu\delta}{2}}(x_\mathrm{a}) \subset \mathcal{S}(\delta) \subset \mathcal{B}_\delta(x_\mathrm{a}). \tag{10}$$

Proof: Let $x \in \mathcal{B}_{\frac{\mu\delta}{2}}(x_\mathrm{a})$, i.e., $|x - x_\mathrm{a}| \leq \frac{\mu\delta}{2}$. Then for each $q \in \{-1, 1\}$ the triangle inequality leads to the estimate

$$\begin{aligned} |x - x_\mathrm{a} + q\delta_\mu B| &\leq |x - x_\mathrm{a}| + |q\delta_\mu B| \\ &\leq \tfrac{\mu\delta}{2} + \delta_\mu = \delta \left(\tfrac{1}{\mu} + \tfrac{\mu}{4} \right), \end{aligned}$$

which implies that $x \in \mathcal{O}_q$ for all $q \in \{-1, 1\}$, and thus $x \in \mathcal{S}(\delta)$. Hence $\mathcal{B}_{\frac{\mu\delta}{2}}(x_\mathrm{a}) \subset \mathcal{S}(\delta)$ satisfied.

We define the set

$$\mathcal{S}_{\max} = \left\{ x \in \mathcal{S}(\delta) : |x|_{x_\mathrm{a}} \geq \max\nolimits_{y \in \mathcal{S}(\delta)} |y|_{x_\mathrm{a}} \right\}.$$

It is clear that for all $x \in \mathcal{S}_{\max}$ either $x \in \partial\mathcal{O}_1$ and/or $x \in \partial\mathcal{O}_{-1}$ is satisfied since otherwise the condition $|x|_{x_\mathrm{a}} \geq \max_{y \in \mathcal{S}(\delta)} |y|_{x_\mathrm{a}}$ cannot hold. Similarly if $x \in \mathcal{S}_{\max}$, $x \in \partial\mathcal{O}_q$ and $x \in \mathrm{int}(\mathcal{O}_{-q})$, $q \in \{-1, 1\}$, for all $\varepsilon > 0$ there needs to exist $\tilde{x} \in \mathcal{B}_\varepsilon(x) \cap \partial\mathcal{O}_q \cap \mathcal{O}_{-q}$ such that $|\tilde{x}|_{x_\mathrm{a}} > |x|_{x_\mathrm{a}}$. Thus, the set $\mathcal{S}_{\max}$ satisfies $\mathcal{S}_{\max} \subset \partial\mathcal{O}_1 \cap \partial\mathcal{O}_{-1}$. Using the definitions of $\mathcal{O}_1$ and $\mathcal{O}_{-1}$, and Pythagoras theorem for pairwise orthogonal vectors provides the identities

$$\begin{aligned} |x - x_\mathrm{a}|^2 &= \left(\delta_\mu + \tfrac{\mu\delta}{2} \right)^2 - \delta_\mu^2 \\ &= \delta^2 \left(\tfrac{1}{\mu} + \tfrac{\mu}{4} \right)^2 - \left(\tfrac{1}{\mu} - \tfrac{\mu}{4} \right)^2 = \delta^2 \end{aligned}$$

for all $x \in \partial\mathcal{O}_1 \cap \partial\mathcal{O}_{-1}$ (visualized in Figure 1). This particularly implies that $\mathcal{S}(\delta) \subset \mathcal{B}_\delta(x_\mathrm{a})$. ■

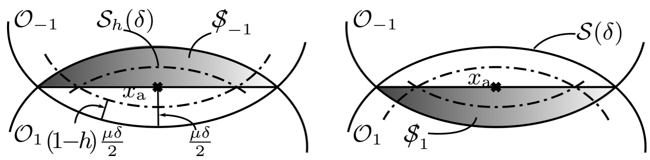

Fig. 2. The shrunken shell $\mathcal{S}_h(\delta)$ and the half shells $\$_1$ and $\$_{-1}$ considered in Proposition 2.

B. Avoidance Controller

The shell $\mathcal{S}(\delta)$ introduced in the previous section is intrinsically composed of two separate boundaries, thereby simplifying the design of a hybrid-based avoidance controller that depends on a logical state $q \in \{1, -1\}$. The value of q indicates whether the avoidance controller should cause sliding of the solution "under" the shell (so to speak, based on the "up" direction of the unit vector B) if $q = -1$, or over the shell if $q = 1$.

We define such a "binary" avoidance controller as a parametric state feedback defined for $q \in \{1, -1\}$ as

$$u_{\mathrm{a}}(x, q) := -B^T Ax - \frac{\langle x - (x_{\mathrm{a}} - q\delta_\mu B), (I - BB^T)Ax\rangle}{\langle x - (x_{\mathrm{a}} - q\delta_\mu B), B\rangle}. \tag{11}$$

The avoidance control law (11) is activated by some hybrid logic in the solution proposed in Section V, wherein a suitable h-hysteresis switching is enforced, based on a region $\mathcal{S}_h(\delta)$ obtained by shrinking $\mathcal{S}(\delta)$ by a factor $h \in (0, 1)$ as follows, and according to the pictorial representation in Figure 2:

$$\mathcal{O}_{h,q} := \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(x_{\mathrm{a}} - q\delta_\mu B), \quad q \in \{1, -1\}, \tag{12}$$

$$\mathcal{S}_h(\delta) := \mathcal{O}_{h,1} \bigcap \mathcal{O}_{h,-1}. \tag{13}$$

It is clear that for each $q \in \{1, -1\}$ the set $\mathcal{O}_{h,q}$ is a ball sharing the same center as $\mathcal{O}_q$ but having a smaller radius that approaches δ_μ as h approaches 0. As a consequence, $\mathcal{S}_h(\delta)$ is a smaller eye-shaped set, with the same aspect ratio as $\mathcal{S}(\delta)$ (see Figure 2).

The desirable features of the avoidance controller (11) is that it enforces sliding of the solution above or below the shell $\mathcal{S}_h(\delta)$ because it does enforce a constant distance from the upper and the lower balls $\mathcal{O}_{h,1}$, $\mathcal{O}_{h,-1}$ involved in the definition of $\mathcal{S}_h(\delta)$. Such a desirable sliding mechanism is well understood in terms of the following closed half shells

$$\$_q := \mathcal{S}(\delta) \cap \{x \in \mathbb{R}^n : \, qB^T(x - x_{\mathrm{a}}) \geq 0\}, \quad q \in \{-1, 1\}, \tag{14}$$

represented in Figure 2.

The following lemma ensures that whenever using the avoidance controller (11) with a suitable value of q, the ensuing solution does not enter the shell $\mathcal{S}_h(\delta)$ and actually remains at a constant distance from the corresponding ball containing $\mathcal{S}_h(\delta)$.

Proposition 2: Let $\mu \in (0, 2/\sqrt{3})$, $\delta > 0$ and $h \in (0, 1)$ be given. For each $q \in \{-1, 1\}$ and any point $x_0 \in \mathcal{S}_h(\delta) \subset \mathcal{S}(\delta)$, the local controller $u = u_{\mathrm{a}}(x_0, q)$, in (11), is well defined. Moreover, the solution to (1) with $u = u_{\mathrm{a}}(x, q)$ starting at x_0 remains at a constant (non-negative) distance from the center $x_{\mathrm{a}} - q\delta_\mu B$ of the ball $\mathcal{O}_{h,q}$ until it remains in $\mathcal{S}(\delta)$. ⌟

Proof: We show the assertion of the lemma for all $x_0 \in \mathcal{S}(\delta)$, which includes the results for $x_0 \in \mathcal{S}_h(\delta)$ due to the set inclusion $\mathcal{S}_h(\delta) \subset \mathcal{S}(\delta)$ for all $h \in (0, 1)$. To simplify the notation we define the points

$$p_q = x_{\mathrm{a}} - q\delta_\mu B, \qquad q \in \{-1, 1\},$$

as the centers of $\mathcal{O}_q$. As a first step, we show that the local control law (11) is well defined under the condition $\mu \in (0, 2/\sqrt{3})$, i.e., we show that $\langle x - p_q, B\rangle \neq 0$ for all $x \in \mathcal{S}(\delta)$, $q \in \{-1, 1\}$. Due to the definition of δ_μ in (9a) and μ satisfying $0 < \mu < \mu^* := 2/\sqrt{3}$, it holds that

$$\delta_\mu = \tfrac{\delta}{\mu}\left(1 - \tfrac{\mu^2}{4}\right) > \tfrac{\delta}{\mu^*}\left(1 - \tfrac{(\mu^*)^2}{4}\right) = \tfrac{\delta}{\sqrt{3}} = \tfrac{\delta\mu^*}{2} > \tfrac{\delta\mu}{2},$$

which particularly implies that $p_q \notin \mathcal{O}_{-q}$, $q \in \{-1, 1\}$. Thus, every $x_0 \in \mathcal{S}(\delta)$ can be represented as $x_0 = p_q + q\alpha B + \beta B^\perp$, where $\alpha > 0$, $\beta \in \mathbb{R}$ and $B^\perp \in \mathbb{R}^n$ satisfies $\langle B, B^\perp\rangle = 0$. Due to this definition, it holds that

$$\begin{aligned}\langle x_0 - p_q, B\rangle &= \langle p_q + q\alpha B + \beta B^\perp - p_q, B\rangle \\ &= q\alpha|B| = q\alpha \neq 0\end{aligned}$$

and the local controller (11) is well defined for all $x_0 \in \mathcal{S}(\delta)$.

To show the second statement of the lemma, which means $|x(t)|_{p_q}$ is constant for the closed loop system using the feedback law $u_{\mathrm{a}}(x, q)$, we show that $\frac{d}{dt}|x(t) - p_q|^2 = 0$ is satisfied. Hence, we show that

$$\langle x - p_q, Ax + Bu_{\mathrm{a}}(x, q)\rangle = 0$$

holds for all $x \in \mathcal{S}(\delta)$. To shorten the expressions we use the notation $x_{p_q} = x - p_q$ and $A_B = (I - BB^T)A$. With the definition of $u_{\mathrm{a}}(x, q)$ and using the fact that $\langle x - p_q, B\rangle \neq 0$ for all $x \in \mathcal{S}(\delta)$ it holds that

$$\begin{aligned}&\langle x_{p_q}, B\rangle \cdot \langle x_{p_q}, Ax - BB^T Ax - B\frac{\langle x_{p_q}, A_B x\rangle}{\langle x_{p_q}, B\rangle}\rangle \\ &\quad = \langle x_{p_q}, B\rangle \cdot \langle x_{p_q}, A_B x\rangle - \langle x_{p_q}, A_B x\rangle \cdot \langle x_{p_q} B\rangle = 0.\end{aligned}$$

which completes the proof. ∎

The avoidance controller u_{a} provides a tool to ensure that the inner shell $\mathcal{S}_h(\delta)$ is not entered by the closed loop-solution. In the next section we show how the avoidance controller can be combined with the stabilizing controller u_{s} to ensure asymptotic stability of the origin.

Remark 2: Note that the non-smoothness of the boundary of the shell $\mathcal{S}(\delta)$ in $\partial\mathcal{O}_{-1} \cap \partial\mathcal{O}_1$ is an essential property for the avoidance controller (11). The idea of sliding along the boundary $\mathcal{S}(\delta)$ cannot be replaced by sliding along the boundary of a set with a smooth boundary, e.g., a ball centered around x_{a}. In the case of a smooth boundary there always exists at least one point x on the boundary such that $\langle x - (x_{\mathrm{a}} + \alpha B), B\rangle = 0$ for some $\alpha \in \mathbb{R}$, i.e., the tangent of the boundary in x is aligned with the direction of the vector B. Thus, a finite input u does not keep the closed-loop solution on the boundary. ⌟

V. A HYBRID CONTROL SOLUTION

A. Hybrid dynamics selection

To ensure global asymptotic stability of the origin for the closed loop, we need to patch the two feedback laws

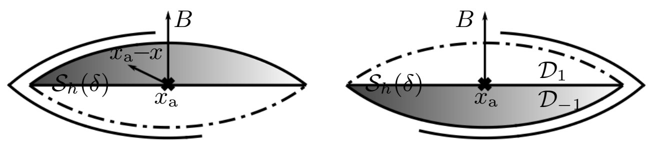

Fig. 3. The upper and lower half-shells associated to $\mathcal{D}_1$ and $\mathcal{D}_{-1}$, respectively, in (16).

$u_{\mathrm{s}}(x)$ (the stabilizing controller), and $u_{\mathrm{a}}(x,q)$ (the avoidance controller). Such a patching operation is done here using a hybrid switching strategy exploiting the h-hysteresis margin between $\mathcal{S}_h(\delta)$) and $\mathcal{S}(\delta)$. Hybrid feedback is a natural choice in light of the discussion above that no continuous feedback can simultaneously ensure GAS of the origin and avoidance of x_{a}.

To suitably orchestrate the choice of the active controller, we define an augmented state $\xi = (x,q)$ for the hybrid dynamics, comprising the plant state x and the quantity $q \in \{1,0,-1\}$ already discussed in the previous section and responsible for whether solutions should slide above ($q = 1$) or below ($q = -1$) the shell when using the avoidance feedback. The value $q = 0$ is associated to the activation of the stabilizing feedback u_{s}. The control selection is summarized by the feedback law

$$u = \gamma(x,q) := (1-|q|)u_{\mathrm{s}}(x) + |q|u_{\mathrm{a}}(x,q). \tag{15}$$

The overall idea of the controller is to use the feedback law u_{s} until solutions enter the shell $\mathcal{S}(\delta)$. To ensure a robust switching between the local and global controllers, we exploit the h-hysteresis and orchestrate the switching of the logic variable q as follows:

$$\begin{aligned}
&\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_q(\xi) \end{bmatrix}, \quad \xi \in \mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0 && (16)\\
&\mathcal{D}_q := \left(\mathcal{S}_h(\delta) \cap \not\mathcal{S}_q\right) \times \{0\}, \quad q \in \{1,-1\} \\
&\mathcal{D}_0 := \overline{\mathbb{R}^n \setminus \mathcal{S}(\delta)} \times \{1,-1\}
\end{aligned}$$

$$G_q(\xi) := \begin{cases} 1, & \text{if } \xi \in \mathcal{D}_1 \setminus \mathcal{D}_{-1} \\ -1, & \text{if } \xi \in \mathcal{D}_{-1} \setminus \mathcal{D}_1 \\ \{1,-1\} & \text{if } \xi \in \mathcal{D}_1 \cap \mathcal{D}_{-1} \\ 0 & \text{if } \xi \in \mathcal{D}_0, \end{cases} \tag{17}$$

where, according to the representation in Figure 3, the two sets $\mathcal{D}_1$ and $\mathcal{D}_{-1}$ correspond to the upper and lower halves of the shell $\mathcal{S}_h(\delta)$. Note that these sets have a nonzero intersection, associated to the equator plane of the shell. To ensure suitable regularity properties of the jump map G in (17), we perform a set-valued selection, which allows for either $q^+ = 1$ or $q^+ = -1$. Note that this does not generate multiple simultaneous jumps because we impose $q = 0$ in the jump sets $\mathcal{D}_1 \cup \mathcal{D}_{-1}$, so that, once a decision has been made about whether sliding above or below the shell, this decision can not be changed.

The hybrid closed loop behavior is completed by the following flow dynamics, emerging from (1) and (15),

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = F(\xi) = \begin{bmatrix} Ax + B\gamma(x,q) \\ 0 \end{bmatrix}, \quad \xi \in \mathcal{C}, \tag{18}$$

where the flow set $\mathcal{C}$, is defined as the closed complement of all the jump sets defined above. In particular, using $\Xi := \mathbb{R}^n \times \{-1,0,1\}$, we select

$$\mathcal{C} := \overline{\Xi \setminus (\mathcal{D}_1 \cup \mathcal{D}_{-1} \cup \mathcal{D}_0)}, \tag{19}$$

which, using the fact that $\not\mathcal{S}_1 \cup \not\mathcal{S}_{-1} = \mathcal{S}(\delta) \supset \mathcal{S}_h(\delta)$, can also be expressed as

$$\begin{aligned}
\mathcal{C} =& \mathcal{C}_1 \cup \mathcal{C}_0 && (20)\\
:=& \{\xi : |q| = 1 \wedge x \in \mathcal{S}(\delta)\} \cup \{\xi : q = 0 \wedge x \in \overline{\mathbb{R}^n \setminus \mathcal{S}_h(\delta)}\}.
\end{aligned}$$

The selection above for the proposed jump sets has the important advantage that immediately after a jump the solution is in the interior of the flow set at a distance of at least $(1-h)\mu\delta/2$ from the jump set $\mathcal{D}$. Before our main result is given in the next section, we note that the following structural regularity conditions of the dynamical system are satisfied, whose proof is straightforward and therefore omitted.

Lemma 3: The closed-loop dynamics (15)–(19) satisfies the hybrid basic conditions in [4, Assumption 6.5] and all maximal solutions are complete. ⌟

B. Main result: GAS and local preservation

We now prove that the hybrid solution proposed in the previous section provides a solution to Problem 1 discussed in Section II. In particular, we provide quantitative information about a maximal size δ^* of the shell $\mathcal{S}(\delta)$, such that the hybrid control solution in (15)–(19) solves Problem 1 for any $\delta < \delta^*$. A trivial corollary of our result is that regardless of all the parameters, there always exists a small enough δ for which our solution is guaranteed to solve Problem 1.

To the end of providing the value δ^*, we need the following quantity

$$\zeta := -\frac{2|A_{\mathrm{s}}|}{\lambda_{\max}(A_{\mathrm{s}}^T + A_{\mathrm{s}})} > 0, \tag{21}$$

which is positive due to Assumption 1(d), ensuring that $A_{\mathrm{s}}^T + A_{\mathrm{s}}$ is negative definite. Then, we define δ^* as

$$\delta^* := \tfrac{1}{2}\Big(|x_{\mathrm{a}}| + \eta + \zeta - \sqrt{(|x_{\mathrm{a}}| + \eta + \zeta)^2 - 4|x_{\mathrm{a}}|\eta}\Big) > 0, \tag{22}$$

which is notably independent of μ and is well characterized in the next lemma.

Lemma 4: Under Assumption 1, given η in (4), the scalar δ^* in (22) is a positive real number, and for any value of δ satisfying $\delta < \delta^*$, we have $\delta < \eta$. ⌟

Proof: Since $\eta, \zeta > 0$ and $\eta < |x_{\mathrm{a}}|$, by expanding the squared terms, it is straightforward to verify the inequalities.

$$0 < (|x_{\mathrm{a}}| - \eta + \zeta)^2 < (|x_{\mathrm{a}}| + \eta + \zeta)^2 - 4|x_{\mathrm{a}}|\eta. \tag{23}$$

Taking the square root and adding 2η on both sides provides

$$|x_{\mathrm{a}}| - \eta + \zeta + 2\eta < \sqrt{(|x_{\mathrm{a}}| + \eta + \zeta)^2 - 4|x_{\mathrm{a}}|\eta} + 2\eta.$$

Finally, moving the square root to the left leads to the estimate

$$2\delta^* = |x_{\mathrm{a}}| + \eta + \zeta - \sqrt{(|x_{\mathrm{a}}| + \eta + \zeta)^2 - 4|x_{\mathrm{a}}|\eta} < 2\eta,$$

which shows the assertion $\delta^* < \eta$.

The proof is complete since $\delta^* \in \mathbb{R}_{>0}$ follows from (23), showing that the square root in (22) is positive. ■

The selection of δ^* in (22) is used in the following proposition.

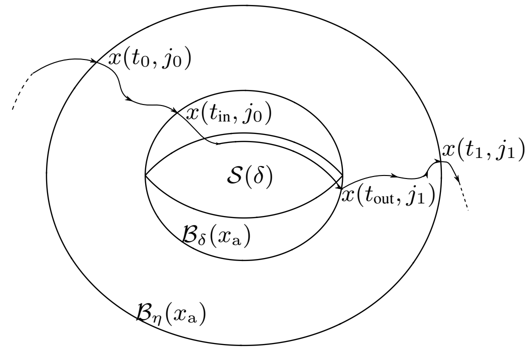

Fig. 4. The intuition behind the two statements of Proposition 3 and the hybrid times $(t_0, j_0) \le (t_{\rm in}, j_0) \le (t_{\rm out}, j_1) \le (t_1, j_1)$, characterized in its statement and its proof.

Proposition 3: Let Assumption 1 hold for the hybrid system (15)–(19). Let $\delta \in (0, \delta^*)$ be chosen such that $\delta < \frac{\eta}{1+\zeta}$. Then the following properties hold for solutions $\xi(\cdot,\cdot)$ starting at ξ_0.

(i) (*Wipeout property*) Let $\xi_0 \in \mathcal{B}_\delta(x_{\rm a}) \times \{-1,0,1\}$. Then there exists a time $(t^*, j^*) \in \mathrm{dom}(\xi)$ such that either $\xi(t^*, j^*) \in \partial\mathcal{B}_\eta(x_{\rm a}) \times \{-1,0,1\}$ or $\xi(t,j) \notin \mathcal{B}_\delta(x_{\rm a}) \times \{-1,0,1\}$ for all $(t,j) \ge (t^*, j^*)$.

(ii) (*Decrease property*) Let $\xi_0 \in \mathbb{R}^n \setminus \mathcal{S}(\delta) \times \{-1,0,1\}$. Additionally, consider any four times in the domain of $\xi(\cdot,\cdot)$, such that

$$(t_0, j_0) \le (t_{\rm in}, j_0) \le (t_{\rm out}, j_1) \le (t_1, j_1), \tag{24}$$

and

$$\begin{aligned} \xi(t_0, j_0), \xi(t_1, j_1) &\in \partial\mathcal{B}_\eta(x_{\rm a}) \times \{0\}, \\ \xi(t_{\rm in}, j_0), \xi(t_{\rm out}, j_1) &\in \partial\mathcal{B}_\delta(x_{\rm a}) \times \{0\}. \end{aligned} \tag{25}$$

Then either

$$|x(t_1, j_1)| < \min_{z \in \mathcal{S}(\delta)} |z| \quad \text{or} \quad |x(t_1, j_1)| \le |x(t_0, j_0)| - \varepsilon \tag{26}$$

for $\varepsilon > 0$, is satisfied. ⌟

The intuition behind the two items of Proposition 3 is illustrated in Figure 4. Item (i) ensures that any solution evolving with the avoidance controller $u_{\rm a}$ will switch to the stabilizing controller $u_{\rm s}$ and will not switch back to $u_{\rm a}$ unless its x component first reaches the set $\partial\mathcal{B}_\eta(x_{\rm a})$. Item (ii) ensures that any solution crossing $\partial\mathcal{B}_\eta(x_{\rm a}) \times \{0\}$ at some time (t_0, j_0) and then switching to the avoidance controller $u_{\rm a}$, if crossing again $\partial\mathcal{B}_\eta(x_{\rm a}) \times \{0\}$ at some later time (t_1, j_1), must satisfy (26), compensating for the increase in $|x|^2$ due to the avoidance controller. The two cases in (26) are helpful to prove asymptotic stability. If a solution enters and leaves the ball $\mathcal{B}_\eta(x_{\rm a})$ a decrease of at least ε in the Lyapunov function $V(x) = |x|$ is guaranteed. Otherwise, $x(t_1, j_1) < \min_{z \in \mathcal{S}(\delta)} |z|$ implies $q(t_1, j_1) = 0$ and $u = u_{\rm s}$ which leads to the fact that $\mathcal{B}_{|x(t_1,j_1)|}(0)$ is forward invariant and thus $x(t,j) \notin \mathcal{S}(\delta)$ for all $(t,j) \ge (t_1, j_1)$.

We make the following claim regarding the condition $\delta < \frac{\eta}{1+\zeta}$ which will be used in the proof of Proposition 3 to define ε.

Claim 1: (*Selection of* δ). Let η and ζ be defined through, (4) and (21), respectively. Then

$$\varepsilon(\delta) := \tfrac{2}{\zeta}(\eta - \delta) - 2\delta > 0 \;\; \forall \;\; \delta \in (0, \tfrac{\eta}{1+\zeta}). \tag{27}$$

Proof: Since ε is linear in δ, the assertion follows immediately from $\varepsilon(0) = \frac{2}{\zeta}\eta > 0$ and $\varepsilon(\frac{\eta}{1+\zeta}) = 0$. ■

Proof of Proposition 3.

Proof of item (i): Let us consider a solution $\xi(\cdot,\cdot)$, whose x-component $x(\cdot,\cdot)$, denoted by solution$_x$ in the following, is starting in $\mathcal{B}_\delta(x_{\rm a})$. Two cases may happen: either the solution$_x$ reaches $\partial\mathcal{B}_\eta(x_{\rm a})$ in finite time, or it never reaches it. In the first case the item is proven. In the second case, solution$_x$ must remain in the interior of $\mathcal{B}_\eta(x_{\rm a})$ for all times, and the derivations in Remark 1 apply for all times in $\mathrm{dom}(\xi)$. Let $x(t_1, j_1) \in \mathcal{B}_\delta(x_{\rm a})$ for some time $(t_1, j_1) \in \mathrm{dom}(\xi)$. Thus using the completeness of solutions from Lemma 3 and according to the estimate (7) applied with $\bar\eta = \delta$ (which satisfies $\bar\eta < \eta$ from Lemma 4), there exists $\underline{h}, \delta > 0$ such that for a $t^* > 2\delta\underline{h}_\delta + t_1$, $j^* \ge j_1$ it holds that $(t^*, j^*) \in \mathrm{dom}(\xi)$ and $x(t^*, j^*) \notin \mathcal{B}_\delta(x_{\rm a})$. Moreover, for all $(t,j) \ge (t^*, j^*)$, $(t,j) \in \mathrm{dom}(\xi)$ it holds that $\langle w_{x_{\rm a}}, x(t,j) - x(t^*, j^*)\rangle \ge 0$ due to the property (8). Thus $x(t,j) \notin \mathcal{B}_\delta(x_{\rm a})$ for all $(t,j) \ge (t^*, j^*)$, $(t,j) \in \mathrm{dom}(\xi)$.

Proof of item (ii): Consider any such solution $\xi(\cdot,\cdot)$ and first notice that due to the expression in (20) of the flow set, the solution$_x$ can only flow in $\mathcal{B}_\eta(x_{\rm a}) \setminus \mathcal{S}(\delta)$ if $q(t,j) = 0$. Let us now split the proof in two cases:

Case (a): For some $(t^*, j^*) \in \mathrm{dom}(\xi)$ satisfying $(t_0, j_0) \le (t^*, j^*) \le (t_1, j_1)$ we have $|x(t^*, j^*)| < \min_{z \in \mathcal{S}(\delta)} |z|$. Since from Assumption 1(d) the norm is contractive along flows with $q = 0$, the solution$_x$ satisfies $|x(t,j)| \le |x(t^*, j^*)| < \min_{z \in \mathcal{S}(\delta)} |z|$ for all $(t,j) \ge (t^*, j^*)$, which also includes (t_1, j_1), and the proof is complete.

Case (b): For all $(t,j) \in \mathrm{dom}(\xi)$ satisfying $(t_0, j_0) \le (t,j) \le (t_1, j_1)$ we have

$$|x(t,j)| \ge \min_{z \in \mathcal{S}(\delta)} |z| \ge |x_{\rm a}| - \delta, \tag{28}$$

where we used Lemma 2 in the last inequality. In this second case we will prove that $|x(t_1, j_1)| \le |x(t_0, j_0)| - \varepsilon$, where $\varepsilon := \varepsilon(\delta) > 0$ as defined in (27). In particular, due to the stated assumptions, the solution$_x$ must go through three phases characterized by the four hybrid times in (24), and corresponding to: 1) flow from $x(t_0, j_0) \in \partial\mathcal{B}_\eta(x_{\rm a})$ to $x(t_{\rm in}, j_0) \in \partial\mathcal{B}_\delta(x_{\rm a})$, 2) hit the boundary $\partial\mathcal{B}_\delta(x_{\rm a})$ at time $(t_{\rm in}, j_0)$ and reach $x(t_{\rm out}, j_1) \in \partial\mathcal{B}_\delta(x_{\rm a})$ again after some finite time, 3) flow from $x(t_{\rm out}, j_1) \in \partial\mathcal{B}_\delta(x_{\rm a})$ to $x(t_1, j_1) \in \partial\mathcal{B}_\eta(x_{\rm a})$.

We show below that $|x|$ increases at most by 2δ (corresponding to the rightmost term in (27)) during phase 2, and decreases by at least half of the remaining amount of $\varepsilon(\delta)$ in (27) during phases 1 and 3.

Phase 2. It holds that

$$|x_{\rm a}| - \delta = \min_{z \in \mathcal{B}_\delta(x_{\rm a})} |z| \quad \text{and} \quad |x_{\rm a}| + \delta \;=\; \max_{z \in \mathcal{B}_\delta(x_{\rm a})} |z|$$

and therefore we obtain the estimate

$$|x(t_{\rm out}, j_1)| - |x(t_{\rm in}, j_0)| \le |x_{\rm a}| + \delta - (|x_{\rm a}| - \delta) = 2\delta.$$

Phases 1 and 3. We will only address phase 1 because parallel arguments apply to Phase 3. Since $x(t,j)$ flows within $\mathcal{B}_\eta(x_\mathrm{a}) \setminus \mathcal{B}_\delta(x_\mathrm{a})$ for all (t,j) satisfying $(t_0,j_0) \le (t,j) < (t_\mathrm{in}, j_0)$, then $q(t,j)=0$ for all such (t,j) and the following inequality holds:

$$|\dot{x}(t,j)| \le |A_\mathrm{s}||x(t,j)| \le |A_\mathrm{s}|(|x_\mathrm{a}|+\eta). \tag{29}$$

Using $|x(t_\mathrm{in},j_0) - x(t_0,j_0)| \ge \eta - \delta$ (which holds because of the distance between $\partial\mathcal{B}_\eta(x_\mathrm{a})$ and $\partial\mathcal{B}_\delta(x_\mathrm{a})$), we obtain

$$t_\mathrm{in} - t_0 \ge \frac{\eta-\delta}{|A_\mathrm{s}|(|x_\mathrm{a}|+\eta)}. \tag{30}$$

Consider now the following upper bound of the decrease rate of the norm:

$$\overbrace{|x(t,j)|}^{\cdot} := \tfrac{d}{dt}\sqrt{|x(t,j)|^2} \le \frac{\lambda_{\max}(A_\mathrm{s}+A_\mathrm{s}^T)}{2|x(t,j)|}|x(t,j)|^2$$
$$\le \tfrac{1}{2}\lambda_{\max}(A_\mathrm{s}+A_\mathrm{s}^T)(|x_\mathrm{a}|+\eta),$$

which is well defined because $|x(t,j)| \ge |x_\mathrm{a}| - \delta$ according to (28). Integrating on both sides provides the estimate

$$|x(t_\mathrm{in},j_0)| - |x(t_0,j_0)| \le \tfrac{1}{2}(t_\mathrm{in}-t_0)\lambda_{\max}(A_\mathrm{s}+A_s^T)(|x_\mathrm{a}|+\eta)$$

and since A_s is Hurwitz, the right-hand side is negative. Thus we can use (30) to estimate the decrease

$$\begin{aligned} |x(t_\mathrm{in},j_0)| - |x(t_0,j_0)| &\le (t_\mathrm{in}-t_0)\lambda_{\max}(A_\mathrm{s}+A_s^T)(|x_\mathrm{a}|+\eta) \\ &\overset{(30)}{\le} \frac{\lambda_{\max}(A_\mathrm{s}+A_\mathrm{s}^T)}{2|A_s|}(\eta-\delta) \\ &= -\zeta^{-1}(\eta-\delta) \end{aligned} \tag{31}$$

which is a lower bound on the decrease in phase 1 and 3, and ζ was defined in (21).

Combining the increase and decrease bounds established in (29) and (31), we get

$$|x(t_1,j_1)| - |x(t_0,j_0)| = |x(t_1,j_1)| - |x(t_\mathrm{out},j_1)|$$
$$+ |x(t_\mathrm{out},j_1)| - |x(t_\mathrm{in},j_0)| + |x(t_\mathrm{in},j_0)| - |x(t_0,j_0)| \le -\varepsilon$$

from (27) and where $\varepsilon := \varepsilon(\delta)$. ■

Theorem 1: Let Assumption 1 be satisfied. Given any scalar $\delta \in (0, \min\{\delta^*, \frac{\eta}{1+\zeta}\})$, according to (22), any $\mu \in (0, 2/\sqrt{3})$, and $h \in (0,1)$, the hybrid controller (15)–(19) guarantees that

(i) the origin $\xi = (x,q) = (0,0)$ is uniformly globally asymptotically stable from Ξ;
(ii) for any initial condition $\xi(0,0) \in (\mathbb{R}^n \setminus \mathcal{S}(\delta)) \times \{-1,0,1\}$, all the arising solutions satisfy $|x(t,j)|_{x_\mathrm{a}} \ge h\frac{\mu\delta}{2}$ for all $(t,j) \in \mathrm{dom}(\xi)$.
(iii) for any initial condition $\xi(0,0) \in (\mathbb{R}^n \setminus \{x_\mathrm{a}\}) \times \{0\}$, all the arising solutions satisfy $x(t,j) \ne x_\mathrm{a}$ for all $(t,j) \in \mathrm{dom}(\xi)$. ⌟

Remark 3: In light of Theorem 1, we may conclude that the hybrid controller (15)–(19) solves Problem 1. In particular, from Lemma 2, and the fact that any positive value of $\delta < \min\{\delta^*, \frac{\eta}{1+\zeta}\}$ can be selected, it is possible to make set $\mathcal{S}(\delta)$ arbitrarily small, which in turn implies:

- semiglobal preservation, because from (20) no solution can flow with $q \ne 0$ outside $\mathcal{S}(\delta)$, and
- semiglobal x_a avoidance, because of Theorem 1, item (ii) combined with $\mathcal{S}(\delta) \subset \mathcal{B}_\delta(x_\mathrm{a})$, we conclude that solutions never enter the "safety neighborhood" of x_a corresponding to $\mathcal{B}_{h\frac{\mu\delta}{2}}(x_\mathrm{a})$.

Finally, GAS is guaranteed directly by Theorem 1(i). ⌟

Proof: **Proof of Item (i):** To prove uniform global asymptotic stability (UGAS) of the origin, we exploit the fact, established in Lemma 3(i), that the closed loop satisfies the hybrid basic conditions of [4] and all maximal solutions are complete. Then, using [4, Thm. 7.12], it is sufficient to prove (local) Lyapunov stability and global (not necessarily uniform) convergence to the origin, to obtain uniform global asymptotic stability. The two properties are proven below.

Local Stability: We first observe that η defined in (4) satisfies $\eta \le |x_\mathrm{a}|$, because $y=0$ trivially belongs to $\mathcal{E}$ in (2). Moreover, from Lemma 4 we have $\delta < \eta \le |x_\mathrm{a}|$, whose strict inequality, together with the inclusion $\mathcal{S}(\delta) \subset \mathcal{B}_\delta(x_\mathrm{a})$, established in Lemma 2, implies that $0 \notin \mathcal{S}(\delta)$. As a consequence, there exists an $r>0$, such that $\mathcal{B}_r(0) \cap \mathcal{S}(\delta) = \emptyset$. Moreover, for any initial state $\xi \in \mathcal{B}_r(0) \times \{-1,0,1\}$ the x- and q-components satisfy the following properties. If $q(0,0) \ne 0$, the dynamics will jump immediately to $q(0,1)=0$ (due to the definition of $\mathcal{D}_0$) and the x-component satisfies $x^+ = x$ across any jump. Thus, either after the first jump, or immediately, the solution $\xi(t,j)$ belongs to the interior of $\mathcal{C}_0$ and, from Assumption 1(d), it flows forever in the forward invariant set $\mathcal{B}_r(0) \times \{0\}$. Asymptotic stability then follows from the asymptotic stability of $\dot{x} = A_\mathrm{s}x$ and $q=0$.

Global Convergence: Consider any solution $\xi = (x,q)$, and based on the two possibilities in Proposition 3(i) we break the analysis in two cases.

Case (a): The solution never reaches $\partial\mathcal{B}_\eta(x_\mathrm{a})$. In this case, from Proposition 3(i) the solution remains in the stabilizing mode (i.e., $u = u_\mathrm{s}$ and $q=0$) on its tail. Then, the origin is asymptotically stable due to the dynamics $\dot{x} = A_\mathrm{s}x$ and Assumption 1(a).

Case (b): The solution reaches $\partial\mathcal{B}_\eta(x_\mathrm{a})$ at some time (t_0,j_0). In this second case, either there exists a finite time after which the solution does not evolve using the avoidance controller (i.e., using $u=u_\mathrm{a}$ and with $|q|=1$) anymore (and the analysis of case (a) applies), or there exists a sequence of times (t_k,j_k), $k \in \mathbb{N}$ satisfying $|x(t_{k+1},j_{k+1})| \le |x(t_k,j_k)| - \varepsilon$, according to (26), which leads to a contradiction.

Proof of Item (ii): Let h, δ and μ be fixed and consider $\xi = (x,q)$ with $x(0,0) \notin \mathcal{S}(\delta)$. By the definition of $\mathcal{S}_h(\delta)$ and from Lemma 2, it holds that $|x(0,0)|_{x_\mathrm{a}} \ge h\frac{\mu\delta}{2}$. If $q(0,0) \in \{-1,1\}$, then due to the definition of the jump set, the solution must jump to $q^+(0,0)=0$.

Let us consider without loss of generality, $q(0,0)=0$. Since $\mathrm{int}(\mathcal{S}(\delta) \setminus \mathcal{S}_h(\delta)) \ne \emptyset$, the solution must flow with $|x(t,0)|_{x_\mathrm{a}} \ge h\frac{\mu\delta}{2}$ for all $t \ge 0$ (which would prove the item), or otherwise until $x(t_1,0) \in \partial\mathcal{S}_h(\delta)$ for some $t_1 > 0$ when the solution jumps from $x(t_1,0) \in \mathcal{D}$. Depending on whether $x(t_1,0) \in \mathcal{D}_{-1}$ or $x(t_1,0) \in \mathcal{D}_1$, u switches to the avoidance controller u_a and $x(t,1)$ satisfies $|x(t,1)|_{x_\mathrm{a}} \ge h\frac{\mu\delta}{2}$ for all $t \ge t_1$ such that $(t,1) \in \mathrm{dom}(\xi)$ due to the properties of the local controller established in Proposition 2. If the solution jumps again to $q^+ = 0$, then, by definition of the jump set $\mathcal{D}_0$ in (16), it must be outside $\mathcal{S}_h(\delta)$ and the reasoning above can be repeated for the subsequent evolution, so that it is impossible for the solution to enter the interior of $\mathcal{S}_h(\delta)$, thus proving the item (ii).

Proof of Item (iii): We consider two cases.

Case (a): Let $x \in \mathbb{R}^n \setminus \mathcal{S}_h(\delta)$ and $q(0,0) = 0$. Then the same arguments as in item (ii) imply that $|x(t,j)|_{x_\mathrm{a}} \geq h\frac{\mu\delta}{2}$, i.e., $x(t,j) \neq x_\mathrm{a}$ for all $(t,j) \in \operatorname{dom} x$.

Case (b): Let $x \in \mathcal{S}_h(\delta) \setminus \{x_\mathrm{a}\}$ and $q(0,0) = 0$. Due to the definition of the jump sets $\mathcal{D}_q$, $q \in \{-1,1\}$ the solution immediately jumps to

$$\xi(0,1) = \begin{bmatrix} x(0,0) \\ q(0,1) \end{bmatrix}$$

for $q(0,1) \in \{-1,1\}$. We assume without loss of generality that $x(0,0) \in \mathcal{S}_1$ and $q(0,1) = 1$. Moreover, it holds that

$$|x(0,1) - (x_\mathrm{a} - \delta_\mu B)| > \delta_\mu = |x_\mathrm{a} - (x_\mathrm{a} - \delta_\mu B)|.$$

Due to the properties of the avoidance controller (11) established in Proposition 2, it holds that

$$\delta_\mu < |x(0,1) - (x_\mathrm{a} - \delta_\mu B)| = |x(t,1) - (x_\mathrm{a} - \delta_\mu B)|$$

for all $t \geq 0$ such that $(t,1) \in \operatorname{dom}(\xi)$, which particularly implies that $x(t,1) \neq x_\mathrm{a}$ for all $t \geq 0$ and $(t,1) \in \operatorname{dom} x$. By definition of the jump set $\mathcal{D}_0$ in (16), the solution will jump back to $q^+ = 0$ only when its x component is outside $\mathcal{S}_h(\delta)$, and then the analysis carried out in case (a) applies. ■

VI. Numerical examples

To illustrate our results we simulate the controller for the simple two-dimensional system defined by

$$A = \begin{bmatrix} -1.0 & 1.5 \\ -1.5 & -1.0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{32}$$

and one obstacle $x_\mathrm{a} = [0\ 1]^T$, which does not belong to the subspace $\mathcal{E}$ in (2). The eigenvalues of the matrix A are given by $\sigma(A) = \{-1+1.5i, -1-1.5i\}$ and the matrix $A + A^T$ satisfies $\sigma(A + A^T) = \{-2,-2\}$, which implies that for $u_\mathrm{s} = 0$ and $A_\mathrm{s} = A$ the origin of the closed-loop system is asymptotically stable and $V(x) = |x|^2$ is a Lyapunov function. The optimization problem (4) provides a value of $\eta = 0.8321$, we set $\mu = 1.15 < 2/\sqrt{3}$ leading to $\zeta = 1.8028$ and $\delta^* = 0.2455$. For the hysteresis we use a value of $h = 0.9$ and $\delta = \delta^*$ (even though the condition $\delta < \delta^*$ is not satisfied). The simulation results for 50 initial values with $|x_0| = 2$ (and $q_0 = 0$) are shown in Figure 5, where the subspace $\mathcal{E}$ is shown as a red line. As one might expect from the theoretical results, all initial values asymptotically approach the origin while avoiding the neighborhood around the unsafe point.

VII. Conclusions

In this paper we proposed a hybrid controller ensuring GAS of the origin and avoidance of a neighborhood around a given point $x_\mathrm{a} \neq 0$ representing an obstacle. In this respect, an explicit formula for the control law as well as for the size of the neighborhood are given. Even though the result are conservative with respect to the size of the neighborhood and only a single obstacle is considered, the results are presented in such a way that an exension to multiple obstacles and more general system dynamics is straightforward.

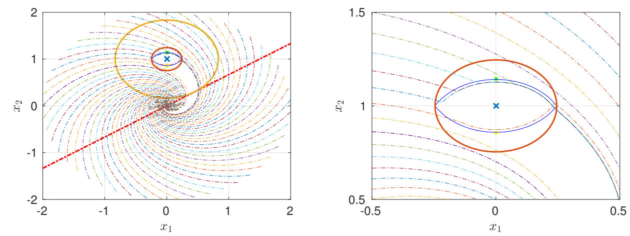

Fig. 5. Avoidance of the unsafe point $x_\mathrm{a} = [0\ \ 1]^T$ and closed-loop solutions corresponding to initial values with $|x_0| = 2$ (and $q_0 = 0$). Additionally, the shell $\mathcal{S}(\delta)$, the η- and δ-ball, and the subspace $\mathcal{E}$ are shown.

References

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876, 2017.

[2] P. Braun and C. M. Kellett. On (the existence of) control Lyapunov barrier functions. Preprint, https://eref.uni-bayreuth.de/40899/.

[3] R. W. Brockett. Asymptotic stability and feedback stabilization. *Differential Geometric Control Theory*, 27(1):181–191, 1983.

[4] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[5] A. M. Lyapunov. The general problem of the stability of motion. *International Journal of Control*, 55(3):531–534, 1992. (Original in Russian, Math. Soc. of Kharkov, 1892).

[6] C. G. Mayhew and A. R. Teel. Hybrid control of planar rotations. In *Proc. of IEEE American Control Conference (ACC)*, pages 154–159. IEEE, 2010.

[7] K. B. Ngo, R. Mahony, and Z.-P. Jiang. Integrator backstepping using barrier functions for systems with multiple state constraints. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 8306–8312, Seville, Spain, December 2005.

[8] C. Prieur. Uniting local and global controllers with robustness to vanishing noise. *Mathematics of Controls, Signals and Systems*, 14:143–172, 2001.

[9] C. Prieur and A. R. Teel. Uniting local and global output feedback controllers. *IEEE Transactions on Automatic Control*, 56(7):1636–1649, 2011.

[10] M. Z. Romdlony and B. Jayawardhana. Stabilization with guaranteed safety using control Lyapunov-barrier function. *Automatica*, 66(Supplement C):39 – 47, 2016.

[11] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. In *2006 American Control Conference*, pages 6 pp.–, 2006.

[12] R. G. Sanfelice and C. Prieur. Robust supervisory control for uniting two output-feedback hybrid controllers with different objectives. *Automatica*, 49(7):1958–1969, 2013.

[13] E. D. Sontag. A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13(2):117–123, 1989.

[14] K. P. Tee, S. S. Ge, and E. H. Tay. Barrier Lyapunov functions for the control of output-constrainted nonlinear systems. *Automatica*, 45(4):918–927, 2009.

[15] A. R. Teel and N. Kapoor. Uniting local and global controllers. In *Proceedings of the European Control Conference*, number FR-A-G3, Bruxelles, Belgium, 1997.

[16] S. E. Tuna, R. G. Sanfelice, M. J. Messina, and A. R. Teel. *Hybrid MPC: Open-Minded but Not Easily Swayed*, pages 17–34. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[17] P. Wieland and F. Allgöwer. Constructive safety using control barrier functions. *IFAC Proceedings Volumes*, 40(12):462–467, 2007.