I&I Adaptive Control for Systems with Varying Parameters

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***Abstract*— This paper combines the so-called *congelation of variables* method with *adaptive immersion and invariance* (*I&I*) to adaptively control systems with varying parameters. A *dynamic scaling* estimator without overparameterization is proposed. This does not require solving partial differential equations and removes other restrictive assumptions in the classical *I&I* estimator design. A controller that guarantees input-to-state stability of the closed-loop system is then used. The joint estimator-controller design guarantees global stability of the adaptive closed-loop system, convergence of the plant state and global boundedness of the estimator state. A design example for the position/force control of a series elastic actuator is discussed. This exploits the idea that bounded nonlinearities in the system dynamics can be viewed as time-varying parameters. Simulation results show a well-damped transient response and illustrate the theoretical results.**

I. Introduction

Adaptive control has under gone extensive research in the past 30 years (see e.g. [1], [2], [3], [4]), though only a few works have been performed for systems with time-varying parameters. The early works on adaptive control for time-varying systems (see e.g. [5]) exploit *persistency of excitation* to guarantee stability. Later works (see e.g. [6], [7]) remove the restriction of *persistency of excitation* and only require bounded and slow (in an average sense) parameter variations.

More recent works mainly belong to two trends. One of the trends is based on the *robust adaptive law* (RAL) [3], where a switching parameter update law called σ*-modification* is applied. In this scheme asymptotic tracking is achieved when the parameters are constant, otherwise the tracking error is related to the rates of the parameter variations [8]. In [9] and [10] the parameter variations are modelled via the superposition of known structured parameter variations and unknown unstructured variations. The residual tracking error is only related to the rates of the unstructured parameter variations.

The other trend is based on *filtered transformations*, see [11], [12] and [13]. These methods can guarantee asymptotic tracking provided that the parameters are bounded in a compact set, their derivatives are $\mathscr{L}_1$ and the disturbance on the state evolution is additive and $\mathscr{L}_2$. In addition *a priori* knowledge on parameter variations is not needed, and the residual tracking error is independent of the rates of parameter variations.

In [14] a method called the *congelation of variables* has been proposed. This method substitutes the time-varying parameters in the Lyapunov function used for controller design with constant unknown parameters. The controller design is then divided into a traditional adaptive control design with constant unknown parameters and a damping design to counteract the perturbation caused by the substitution of parameters. This method achieves adaptive regulation with state feedback and output feedback as well as removes common restrictions on the derivatives of the unknown parameters.

This paper intends to combine the *congelation of variables* method with the *adaptive immersion and invariance (I&I)* scheme introduced in [15] and developed in [16], [4] and [17]. In the *adaptive I&I* scheme the parameter estimate is composed of a dynamic updated part and a static part (the β function). The major feature of this scheme is that it avoids the cancellation of parameter estimation error terms and renders $\mathscr{L}_2$ the inner product of the regressor and the parameter estimation error, while in most other schemes the parameter estimation error terms are cancelled and one can only conclude boundedness of the estimation error. This feature, though does not necessarily guarantee the convergence of the parameter estimation error, typically achieves good transient performance due to the extra damping effect given by the β function.

In this paper the only restriction on parameter variations is the following natural assumption.

Assumption 1: The vector of q unknown time-varying parameters θ satisfies, $\forall t \geq 0$, the box constraint

$$\underline{\theta} \leq \theta(t) \leq \bar{\theta}, \quad \underline{\theta}, \bar{\theta} \in \mathbb{R}^q, \tag{1}$$

where the sign "$\leq$" is to be understood element-wise. Only the "radius" of the compact set $\delta = \frac{1}{2}|\bar{\theta} - \underline{\theta}|$ is assumed to be known while $\underline{\theta}$ and $\bar{\theta}$ may be unknown. $\diamond$

II. An Introductory Example

In order to study the combination of the *adaptive I&I* scheme with the *congelation of variables* method in the presence of time-varying parameters, we first consider a scalar system, the time-invariant case of which has been discussed in Section 3.2 of [4], namely the system

$$\dot{x} = u + x^2\theta, \tag{2}$$

where the state $x(t) \in \mathbb{R}$, the input $u(t) \in \mathbb{R}$, and the unknown time-varying parameter $\theta(t) \in \mathbb{R}$. Consider a nominal con-

This work has been partially supported by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No 739551 (KIOS CoE).

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troller with a constant parameter ℓ, that is

$$u=-k_1x-k_2x^3-x^2\ell, \tag{3}$$

where $\Delta(t)=\ell-\theta(t)$ is the perturbation term caused by the *congelation of variables*. Due to Assumption 1, it is always possible to select $\ell=\frac{1}{2}(\bar{\theta}+\underline{\theta})$ such that $|\Delta(t)|\leq\delta$, $\forall t\geq 0$. Consider now the Lyapunov function candidate $V_x(x)=\frac{1}{2}x^2$. Taking the time derivative of V_x along the trajectories of the system yields

$$\begin{aligned}\dot{V}_x=&-k_1x^2-k_2x^4-x^3\Delta\\ \leq&-(k_1-\frac{1}{2})x^2-(k_2-\frac{1}{2}\delta^2)x^4\leq 0,\end{aligned} \tag{4}$$

for some selection of $k_1>0$ and $k_2>0$. This confirms that the system is stabilizable. Note the fact that ℓ is unknown and not implementable: one needs to use adaptive controller to "recover" the nominal controller. To this end, define the off-the-manifold error $z=\hat{\theta}-\ell+\beta(x)$. If one regards $\hat{\theta}+\beta(x)$ as the parameter estimate, then the estimation error is $z+\Delta=\hat{\theta}-\theta+\beta(x)$. Based on the nominal stabilizing controller, we construct the adaptive control law

$$u=-k_1x-k_2x^3-x^2\big(\hat{\theta}+\beta(x)\big) \tag{5}$$

and the update law

$$\dot{\hat{\theta}}=-\frac{\partial\beta}{\partial x}\left(x^2\big(\hat{\theta}+\beta(x)\big)+u\right), \tag{6}$$

with $k_1>\frac{3}{2}$ and $k_2>\frac{3}{2}\delta^2$, which yields the x-dynamics

$$\dot{x}=-k_1x-k_2x^3-x^2(z+\Delta) \tag{7}$$

and the z-dynamics

$$\dot{z}=-\frac{\partial\beta}{\partial x}x^2(z+\Delta). \tag{8}$$

Consider V_x again and its time derivatives along the trajectories of the system. This yields

$$\begin{aligned}\dot{V}_x=&-k_1x^2-k_2x^4-x^3(z+\Delta)\\ \leq&-(k_1-\frac{3}{2})x^2-(k_2-\frac{1}{2}\delta^2)x^4+\frac{1}{4}x^4z^2.\end{aligned} \tag{9}$$

To dominate the positive term x^4z^2 let

$$\frac{\partial\beta}{\partial x}=\gamma x^2, \tag{10}$$

with $\gamma>0$. Consider the Lyapunov function candidate $V_{xz}(x,z)=V_x(x)+\dfrac{1}{2\gamma}z^2$. Its time derivative along the trajectories of the closed-loop system is such that

$$\begin{aligned}\dot{V}_{xz}=&\dot{V}_x-zx^4(z+\Delta)\\ \leq&-(k_1-\frac{3}{2})x^2-(k_2-\frac{3}{2}\delta^2)x^4-\frac{1}{2}x^4z^2\leq 0.\end{aligned} \tag{11}$$

It can be concluded, invoking LaSalle-Yoshizawa theorem, that $\lim_{t\to\infty}x(t)=0$ and $z(t)$ is globally uniformly bounded. Additionally, as a feature of the *adaptive I&I* scheme, $x^2(t)z(t)\in\mathscr{L}_2$.

Compared to the introductory example in [4], the control law (5) contains an extra nonlinear damping term $-k_2x^3$ to counteract the perturbation caused by the parameter variations. This idea is similar to the methods used in [14].

Remark 1: It can be seen from this simple example that the method of *congelation of variables* is essentially designing a nominal robust controller with constant unknown parameters for a system with vanishing perturbation terms and then designing an adaptive controller to "recover" the nominal controller. $\diamond$

Note that for the scalar case one only needs to solve for $\beta(x)$ by integrating both sides of (10): $\beta(x)=\gamma\int_0^x\chi^2\mathrm{d}\chi=\frac{1}{3}\gamma x^3$. In general equation (10) reduces to a partial differential equation (PDE) and constructing β by integration requires restrictive assumptions, as discussed in [4] and [17].

III. Dynamic Scaling Estimator

In this section we consider a linearly parameterized nonlinear system[1] described by the equation

$$\dot{x}=f_u(x,u)+\Phi^\top(x)\theta, \tag{12}$$

where the state $x(t)\in\mathbb{R}^n$, the input $u(t)\in\mathbb{R}^m$, and the vector of unknown time-varying parameters $\theta(t)\in\mathbb{R}^q$. To avoid impractically restrictive assumptions, the *dynamic scaling* technique (which has been originally developed for high-gain observers [18]) is applied and the off-the-manifold error $z(t)\in\mathbb{R}^q$ is defined as

$$z=\frac{\hat{\theta}-\ell+\beta(x,\hat{x})}{r}, \tag{13}$$

where $r(t)\in\mathbb{R}$ is the scaling factor, $\hat{\theta}(t)$ is the dynamic part of the parameter estimate $\hat{\theta}+\beta(x,\hat{x})$, $\ell\in\mathbb{R}^q$ is the constant vector of congealed parameters, $\beta(x,\hat{x})$ is the static part of the parameter estimate and

$$\beta(x,\hat{x})=\Gamma\Phi(\hat{x})x, \tag{14}$$

with $\Gamma=\Gamma^\top\succ 0$. The auxiliary state $\hat{x}$ is updated using the filter

$$\dot{\hat{x}}=f_u(x,u)+\Phi^\top(x)\big(\hat{\theta}+\beta(x,\hat{x})\big)-L(x,r,\tilde{x})\tilde{x}, \tag{15}$$

where $\tilde{x}=\hat{x}-x$, and $L(x,r,\tilde{x})$ is the injection gain to be designed later. As a result the $\tilde{x}$-dynamics is

$$\dot{\tilde{x}}=-L(x,r,\tilde{x})\tilde{x}+r\Phi^\top(x)(z+\frac{\Delta}{r}). \tag{16}$$

Applying the parameter update law:

$$\dot{\hat{\theta}}=-\frac{\partial\beta}{\partial x}\left(f_u(x,u)+\Phi^\top(x)\big(\hat{\theta}+\beta(x,\hat{x})\big)\right)-\frac{\partial\beta}{\partial\hat{x}}\dot{\hat{x}} \tag{17}$$

yields the z-dynamics

$$\dot{z}=-\Gamma\Phi(\hat{x})\Phi^\top(x)(z+\frac{\Delta}{r})-\frac{\dot{r}}{r}z. \tag{18}$$

Note that $\Phi(\hat{x})=\Phi(x)+D(x,\tilde{x})(I_n\otimes\tilde{x})$ with some $D(x,\tilde{x})\in\mathbb{R}^{q\times n^2}$ due to the smoothness of $\Phi(\cdot)$, where $\otimes$ denotes the

[1]If not otherwise stated, all functions and mappings are assumed to be $\mathscr{C}^\infty$ (and therefore also locally Lipschitz), though such regularity assumptions can be further relaxed.

Kronecker product. Finally the dynamics of z can be written as

$$\dot{z} = -\Gamma\big(\Phi(x) + D(x,\tilde{x})(I_n \otimes \tilde{x})\big)\Phi^\top(x)(z + \frac{\Delta}{r}) - \frac{\dot{r}}{r}z. \quad (19)$$

Lemma 1: Consider the plant (12) and the *dynamic scaling* estimator with

$$\dot{r} = c|D(x,\tilde{x})(I_n \otimes \tilde{x})|_{\mathrm{F}}^2 r, \quad (20)$$

where $r(0) = 1$, $cI_q \succeq 2\Gamma$, $|\cdot|_{\mathrm{F}}$ the Frobenius norm, and

$$L(x,r,\tilde{x}) = \lambda r^2 I_n + \bar{L}(x,r,\tilde{x}), \quad (21)$$

with $\lambda > 0$, $\bar{L}(x,r,\tilde{x}) = \varepsilon c r^2 \mathrm{diag}\big(|D(x,\tilde{x})(I_n \otimes e_j)|_{\mathrm{F}}^2\big)$, $\varepsilon > 0$, and e_j the $j-th$ unit vector in $\mathbb{R}^n$, for $j = 1,\dots,n$. The Lyapunov function candidate $V_{z\tilde{x}r}(z,\tilde{x},r) = 2z^\top\Gamma^{-1}z + \frac{1}{2}\lambda|\tilde{x}|^2 + \frac{1}{2}\lambda\varepsilon r^2$ has a time derivative along the trajectories of the system that satisfies

$$\dot{V}_{z\tilde{x}r} \leq -|\Phi^\top(x)z|^2 - \frac{1}{2}\lambda^2|\tilde{x}|^2 + 6\delta^2|\Phi(x)|_{\mathrm{F}}^2. \quad (22)$$

Moreover, if $\delta = 0$, i.e. all unknown parameters are constant, $z(t) \in \mathscr{L}_\infty$, $r(t) \in \mathscr{L}_\infty$, $\tilde{x}(t) \in \mathscr{L}_\infty \cap \mathscr{L}_2$, and $\Phi^\top\big(x(t)\big)z(t) \in \mathscr{L}_2$. ⋄

Remark 2: The selection of $D(x,\tilde{x})$ is not unique. One obvious selection is

$$D(x,\tilde{x}) = \frac{\Phi(x+\tilde{x}) - \Phi(x)}{|\tilde{x}|^2}(I_n \otimes \tilde{x})^\top, \quad (23)$$

which is always well-defined due to the smoothness of $\Phi(\cdot)$. In some cases, "smart" selections can lead to much simpler estimators. For example, consider the regressor $\Phi(x) = [0,\ x_1 + x_2]$. The design following (23) yields $D(x,\tilde{x}) = [0,\ 0,\ \frac{\tilde{x}_1^2+\tilde{x}_1\tilde{x}_2}{\tilde{x}_1^2+\tilde{x}_2^2},\ \frac{\tilde{x}_2^2+\tilde{x}_1\tilde{x}_2}{\tilde{x}_1^2+\tilde{x}_2^2}]$. However, it is easy to see that a much simpler selection is $D = [0,\ 0,\ 1,\ 1]$. ⋄

Remark 3: Compared to its counterpart in [17], Lemma 1 does not use overparameterization, which makes the result applicable to non-overparameterized controllers. In addition, Lemma 1 uses the Frobenius norm instead of the induced 2-norm, which can be turned into pre-computed expressions without online norm computation. Due to the same reason, the Kronecker products in the D-terms are not implemented in practice and do not complicate the estimator design. ⋄

Although the estimator design cannot guarantee the boundedness of the estimator states alone, this problem can be solved by a joint estimator-controller design, as discussed in the next section.

IV. ISS Controller

Consider a linearly parameterized, input affine, nonlinear system described by the equation

$$\dot{x} = f(x) + g(x)u + \Phi^\top(x)\theta, \quad (24)$$

which is a special form of (12) with $f_u(x,u) = f(x) + g(x)u$. Consider a nominal control law $v(x,\ell)$, which is a function of the state $x(t)$ and a constant vector of parameters ℓ (assumed to be known). The resulting closed-loop system is

$$\dot{x} = f(x) + g(x)v(x,\ell) + \Phi^\top(x)(\ell - \Delta) = f_\ell(x). \quad (25)$$

To be able to conclude stability properties one typically needs to make a structural assumption based on the plant and the nominal controller.

Assumption 2: The system (25) has a globally asymptotically stable equilibrium at $x = x_*$. ⋄

Remark 4: Assumption 2 means that the system can be robustly stabilized in the presence of perturbation $\Delta(t)$ even when the controller does not incorporate $\theta(t)$ directly, but only uses a constant ℓ (typically selected as $\frac{1}{2}(\underline{\theta} + \bar{\theta})$) related to $\theta(t)$. This is the fundamental difference between nominal controllers of classical certainty-equivalence adaptive schemes and the nominal controller in the *congelation of variables* scheme. It can also be seen that when θ is constant, $\ell = \theta$ and $\Delta = 0$, and the nominal controller reduces to the classical case. ⋄

Since ℓ is unknown in adaptive control scenarios, we replace ℓ with the parameter estimate $\hat{\theta} + \beta$, which yields an adaptive control law of the form[2] $v(x, \hat{\theta} + \beta)$ and the closed-loop dynamics

$$\begin{aligned}\dot{x} &= f(x) + g(x)v(x,\hat{\theta}+\beta) + \Phi^\top(x)(\hat{\theta} + \beta - rz - \Delta)\\ &= f_{\hat{\theta}\beta}(x),\end{aligned} \quad (26)$$

Proposition 1: Consider the system (24) and the *dynamic scaling* estimator given by (14), (15), (17), (20), (21). Assume that Assumption 2 holds and there exists a positive definite (centered at $x = x_*$) and radially unbounded function V_x and a control law $v(x, \hat{\theta} + \beta)$ such that the time derivative of V_x along the trajectories of the system satisfies

$$\dot{V}_x = \frac{\partial V_x}{\partial x} f_{\hat{\theta}\beta}(x) \leq -U(x) - 6\delta^2|\Phi(x)|_{\mathrm{F}}^2 + \frac{1}{2}|\Phi^\top(x)z|^2, \quad (27)$$

where $U(x)$ is a positive-definite function centered at $x = x_*$. Then $\lim_{t\to\infty} x(t) = x_*$ and all other states are globally uniformly bounded. ⋄

Remark 5: From (27) we can see that the aim of designing a controller that guarantees plant-controller input-to-state stability is to use strengthened damping terms to construct the stabilizing term $\delta^2|\Phi(x)|_{\mathrm{F}}^2$ in $\dot{V}_x$ to dominate the positive term $\delta^2|\Phi(x)|_{\mathrm{F}}^2$ in $\dot{V}_{z\tilde{x}r}$ resulting from the estimator design, and treat $\Phi^\top(x)z$ as an exogenous input. The stability of the whole plant-controller-estimator system is then guaranteed by the property of the estimator that $\Phi^\top(x)z$ is $\mathscr{L}_2$.

In practice, this ISS controller is applicable to at least three types of systems. The first type is given by systems satisfying the *matching condition*, which is the case considered in the introductory example in Section II. The second type is given by systems satisfying the *extended matching condition*, which will be discussed in the design example in Section V. The third type is given by systems in *parametric strict-feedback* form. This type of systems requires the overparameterized and lower triangular estimator-controller design of the *I&I* scheme. In this case, $\Phi(x)$ and z in Proposition 1 become

[2] Note that $v(x, \hat{\theta} + \beta)$ may not be the nominal control law evaluated with the parameter estimate. For example. in adaptive backstepping design one has to add terms to compensate for the terms $\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\theta} + \beta)$ caused by the substitution of $\hat{\theta} + \beta$ for ℓ.

$\phi_i(x_1,\ldots,x_i)$ and z_i, $i=1,\ldots,n$ due to overparameterization. The design procedures are elaborated in [4], [16], [17] and the modifications for time-varying parameters can be performed in the same spirit as the method shown in this paper. $\diamond$

Remark 6: The result in [15] on linearly parameterized plant implicitly requires the number of unknown parameters not to be larger than the dimension of the state, i.e. $q \le n$. This restriction is removed in this paper by designing an ISS controller with respect to the input $\Phi^\top(x)z$. $\diamond$

V. A Design Example on Series Elastic Actuators

In this section we provide an illustration of the proposed ideas designing a controller for the so-called series elastic actuators (SEAs) [19]. SEAs are widely used in robotics: they turn a force control problem into a position control problem using the elastic characteristic of the link due to the well-known Hooke's law.

Control problems arise in SEA due to the extra dynamics caused by the elastic linkage compared to traditional servo problems. A variety of control methods have been applied on SEAs, including PID control [19], PD control with a disturbance observer [20], adaptive control [21], and sliding mode control [22]. In most works, the elastic linkage is modelled as a linear spring with known stiffness and the force exerted on the load is determined by the relative position between the load and the actuator. However, in general, the elastic linkage has nonlinear elastic characteristic. Such nonlinearity is either designed on purpose [23], [24], or unavoidable due to the property of elastic material [25].

Assume that the nonlinearity can be described by the model

$$F_s = K_s(d)d, \tag{28}$$

where F_s is the elastic force of the spring, K_s is the apparent stiffness parameter, and d is the deflection of the spring. Since $d(t)$ is time-varying, $K_s(d)$ is also a function of time that can be written as $K_s(t)$ with a slight abuse of notation. This allows to view K_s as a time-varying parameter and apply the adaptive control scheme introduced above to the SEA position/force control problem.

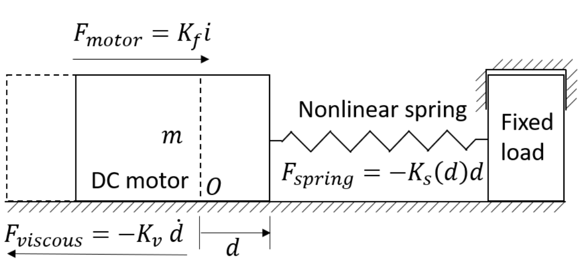

Fig. 1. The SEA with a fixed load.

We now consider the SEA connected with a fixed load as shown in Fig. 1. This is the scenario in which the end-effector is in contact with the object to be manipulated and gradually exerts force on the object (e.g. the egg-grasping task). The goal of the control is to let the DC motor drive the moving end of the spring to a desired deflection d_* such that the force exerted on the load is the desired value $K_s(d_*)$. The transient stage should behave in an over-damped way so that the force on the end-effector does not cause damages. The physical model of the SEA with fixed load driven by a translational DC motor (a compound of DC motor, gearbox, and linkages that turn the rotary motion into translational motion) is given by the differential equations

$$\begin{aligned} m\ddot{d} &= -K_s(d)d - K_v\dot{d} + K_f i, \\ L\dot{i} &= -Ri - K_b\dot{d} + V_{in}, \end{aligned} \tag{29}$$

where m is the apparent mass of the moving parts (the total inertia of the rotor of the motor, the gearbox and other linkages), K_v is the viscous friction constant, K_f is the force constant, K_b is the back-electromotive-force constant, L is the inductance of the armature, R is the resistance of the armature, i is the current through the armature, and V_{in} is the voltage on the armature. Here we assume that the constants of the DC motor are known and the only unknown "parameter" is $K_s\big(d(t)\big)$.

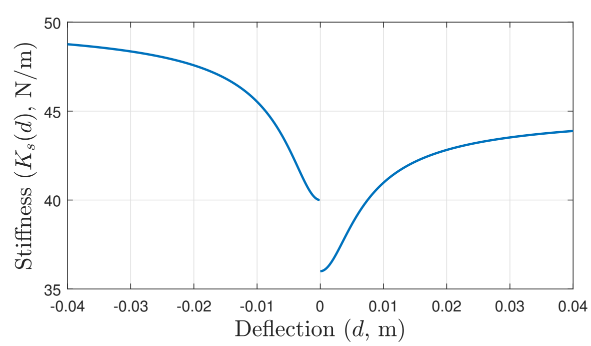

Fig. 2. The stiffness $K_s(d)$ of the nonlinear spring.

Consider an asymmetric nonlinear spring[3], the stiffness of which (plotted in Fig. 2) is given by

$$K_s(d) = \begin{cases} 2K_{s1}\big(1 - \frac{l_{01}}{\sqrt{d^2+l_1^2}}\big), & d \ge 0 \\ 2K_{s2}\big(1 - \frac{l_{02}}{\sqrt{d^2+l_2^2}}\big), & d < 0, \end{cases} \tag{30}$$

where K_{s1} and K_{s2} are the stiffness constants of the linear springs used to realize the nonlinear spring device, l_{01}, l_{02}, l_1, l_2 are parameters related to geometric configurations such that $l_{01} \le l_1$, $l_{02} \le l_2$. The system (29) has an equilibrium at $d = d_*$, $\dot{d} = 0$, $i = i_*$, with input $V_{in} = V_{in*}$. In a regulator problem we want to shift the origin of the state variables to the desired set point. To this end, define the shifted elastic characteristic $K_{s*}(d)$ such that $K_{s*}(d - d_*) = K_s(d)$. This allows writing (29) into the 3-dimensional state space model

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \phi_2\theta - a x_2 + x_3, \ \ \phi(x_1) = -x_1, \\ \dot{x}_3 &= u, \end{aligned} \tag{31}$$

[3]The realization of the nonlinearity using linear springs is shown in Fig. 8 (b) of [23] and the realization of the asymmetry is shown in Fig. 4 of [24].

where $x_1(t) = d - d_*$, $x_2(t) = \dot{d}$, $x_3(t) = \frac{K_f}{m}(i - i_*)$, $u(t) = \frac{K_f}{L}\big((V_{in} - Ri - K_b\dot{d}) - (V_{in*} - Ri_*)\big)$, $a = \frac{K_v}{m}$ and

$$\theta(x_1) = \frac{1}{m}\left(K_{s*}(x_1) + \frac{K_{s*}(x_1) - K_{s*}(0)}{x_1}d_*\right). \qquad (32)$$

Due to the boundedness of $K_{s*}(x_1)$ and the Lipschitz continuity of $K_{s*}(x_1)$ at $x_1 = 0$, $\theta\big(x_1(t)\big)$ can be treated as a bounded time-varying parameter $\theta(t)$, with a slight abuse of notation.

Unlike the system (2) in the introductory example, (31) does not satisfy the *matching condition*, thus requiring the use of the adaptive backstepping techniques [2].

Step 1. Let $\xi_1 = x_1$, $\xi_2 = x_2 - \alpha_1$ be the first two error variables in the backstepping design, and let the first virtual control law be

$$\alpha_1 = -\sigma_1, \qquad (33)$$

which yields

$$\dot{\xi}_1 = \xi_2 + \alpha_1 = -\sigma_1 + \xi_2. \qquad (34)$$

Step 2. Let $\xi_3 = x_3 - \alpha_2$ and

$$\alpha_2 = -\sigma_2 - \xi_1 - \phi_2(\hat{\theta} + \beta) + ax_2 + \frac{\partial \alpha_1}{\partial x_1}x_2. \qquad (35)$$

Then the dynamics of ξ_2 becomes

$$\dot{\xi}_2 = \xi_3 + \alpha_2 = -\sigma_2 - \xi_1 + \xi_3 + \phi_2(rz + \Delta). \qquad (36)$$

Step 3. Let the actual control law

$$\begin{aligned} u =& -\sigma_3 - \xi_2 + \frac{\partial \alpha_2}{\partial x_1}x_2 + \frac{\partial \alpha_2}{\partial x_2}\big((\hat{\theta} + \beta) - ax_2 + x_3\big) \\ &+ \frac{\partial \alpha_2}{\partial r}\dot{r} + \frac{\partial \alpha_2}{\partial \hat{x}}\dot{\hat{x}} + \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}}. \end{aligned} \qquad (37)$$

This yields the dynamics of the third error variable

$$\dot{\xi}_3 = -\sigma_3 - \xi_2 + \frac{\partial \alpha_2}{\partial x_2}\phi_2(rz + \Delta). \qquad (38)$$

Proposition 2: Consider system (31) with the *dynamic scaling* estimator given by (14), (15), (17), (20), (21), and the controller (37). Select the damping terms as

$$\sigma_1 = (k_1 + \frac{13}{2}\delta^2)\xi_1, \qquad (39)$$

$$\sigma_2 = (k_2 + r^2 + 1)\xi_2, \qquad (40)$$

$$\sigma_3 = \big(k_3 + (\frac{\partial \alpha_2}{\partial x_2})^2(r^2 + 1)\big)\xi_3, \qquad (41)$$

with $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$. Then $\lim_{t\to\infty} x(t) = 0$, in particular, $d \to d_*$ as $t \to \infty$, and all other states are globally uniformly bounded. ◇

Proof: Consider the Lyapunov function candidate $V_\xi = \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\xi_3^2$. The time derivative of V_ξ along the trajectories of the system is such that

$$\begin{aligned} \dot{V}_\xi =& -\sigma_1\xi_1 + \xi_1\xi_2 - \sigma_2\xi_2 - \xi_1\xi_2 + \xi_2\xi_3 + \xi_2\phi_2(rz + \Delta) \\ & -\sigma_3\xi_3 - \xi_2\xi_3 + \xi_3\frac{\partial \alpha_2}{\partial x_2}\phi_2(rz + \Delta) \\ \leq& -\sigma_1\xi_1 - \sigma_2\xi_2 - \sigma_3\xi_3 + \xi_2^2(r^2 + 1) \\ & + \xi_3^2(\frac{\partial \alpha_2}{\partial x_2})^2(r^2 + 1) + \frac{1}{2}(\phi_2 z)^2 + \frac{1}{2}\delta^2\phi_2^2. \end{aligned} \qquad (42)$$

Using the damping terms (39)-(41) yields

$$\dot{V}_\xi \leq -k_1\xi_1^2 - k_2\xi_2^2 - k_3\xi_3^2 - 6|\phi_2|^2 + \frac{1}{2}|\phi_2 z|^2. \qquad (43)$$

It can be concluded from Proposition 1 that $\lim_{t\to\infty} \xi(t) = 0$ and all other states are globally uniformly bounded. Using a standard argument for stability in the backstepping scheme, $\lim_{t\to\infty} \xi(t) = 0$ implies $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} \alpha_1(t) = 0$, which gives $\lim_{t\to\infty} x_2(t) = 0$, since $\lim_{t\to\infty} \xi_2(t) = 0$. In the same way we can prove that $\lim_{t\to\infty} x_3(t) = 0$ and this completes the proof. ■

Remark 7: In practice it is not necessary to use δ directly in the controller since in most cases the parameter variations are not in the worst case. Typically a discounted variation radius $\delta_d < \delta$ can be implemented to avoid large control amplitude and other robustness issues caused by a high-gain controller. ◇

Consider now the SEA with the parameters: $K_{s1} = 45$N/m, $K_{s2} = 50$N/m, $l_{01} = l_{02} = 1 \times 10^{-3}$m, $l_1 = l_2 = 5 \times 10^{-3}$m, $m = 1$kg, $R = 3\Omega$, $L = 1.5 \times 10^{-4}$H, $K_f = 1.5$N/A, $K_b = 2.5$V·s/m, $K_v = 0.01$N·s/m, and the estimator-controller setting: $\Gamma = \gamma I_1 = 1 \times 10^3$, $\lambda = \varepsilon = 5$, $c = 100$, $k_1 = 2$, $k_2 = 2$, $k_3 = 2$, $\delta_d^2 = 2$. Let $d_* = 2 \times 10^{-2}$m, and the initial condition $x = [-0.04,\ 0,\ -0.85]^\top$ (the third element enforces zero initial armature current).

As shown in Fig. 3 the actuation is intentionally tuned to be overdamped so that the force is smoothly exerted on the load thus preventing damages. Fig. 5 shows the variation of $\theta(t)$ during the transient stage. Note that $\theta(t)$ is bounded as is stated in the discussion on (32).

VI. Conclusions and Future Work

In this paper the *adaptive I&I* scheme has been modified with the *congelation of variables* method to cope with time-varying parameters. A non-overparameterized *dynamic scaling* estimator is proposed to avoid solving PDEs in the estimator design of the *adaptive I&I* scheme. An ISS controller is then designed. This works jointly with the estimator to guarantee global stability of the system, convergence of the plant state and global boundedness of the estimator state. A design example for an SEA in which the bounded nonlinearity in the system dynamics is regarded as a time-varying parameter is discussed. The simulation results show a well-damped transient response that fulfills the position/force control task.

The proposed adaptive control scheme has still limitations in general set-point regulation and reference tracking tasks. These will be investigated in the future.

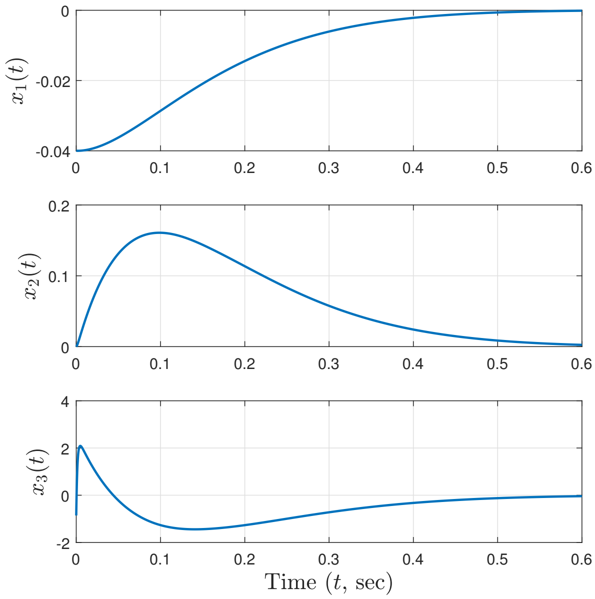

Fig. 3. Time trajectories of the states of the closed-loop system.

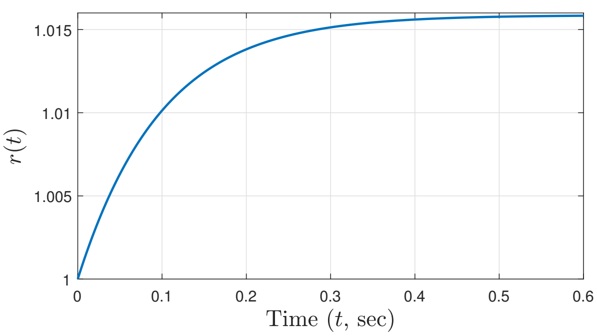

Fig. 4. Time history of $r(t)$.

ACKNOWLEDGMENT

The authors would like to thank Mr. Dongjun Wu for his inspiration on the explicit form of $D(x,\tilde{x})$ in (19) and also the reviewers for their helpful comments.

REFERENCES

[1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Prentice Hall, 1989.

[2] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st ed. New York: John Wiley & Sons, Inc, 1995.

[3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. PTR Prentice-Hall Upper Saddle River, NJ, 1996.

[4] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and Adaptive Control with Applications*. Springer Science & Business Media, 2007.

[5] G. C. Goodwin and E. K. Teoh, "Adaptive control of a class of linear time varying systems," *IFAC Proc. Volumes*, vol. 16, no. 9, pp. 1–6, 1983.

[6] G. Kreisselmeier, "Adaptive control of a class of slowly time-varying plants," *Syst. Control Lett.*, vol. 8, no. 2, pp. 97–103, 1986.

[7] R. H. Middleton and G. C. Goodwin, "Adaptive control of time-varying linear systems," *IEEE Trans. Automat. Contr.*, vol. 33, no. 2, pp. 150–155, 1988.

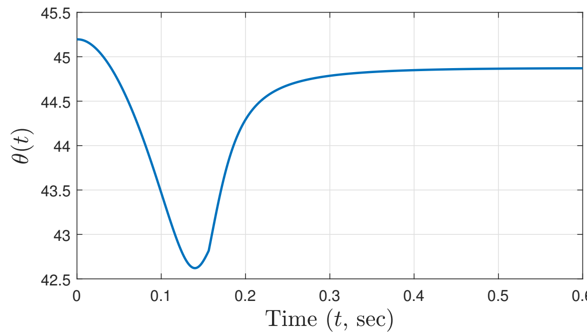

Fig. 5. Time history of $\theta(t)$.

[8] K. Tsakalis and P. A. Ioannou, "Adaptive control of linear time-varying plants," *Automatica*, vol. 23, no. 4, pp. 459–468, 1987.

[9] Y. Zhang and P. A. Ioannou, "Adaptive control of linear time varying systems," in *Proc. 35th IEEE Conf. on Decision and Control*, vol. 1. IEEE, 1996, pp. 837–842.

[10] Y. Zhang, B. Fidan, and P. A. Ioannou, "Backstepping control of linear time-varying systems with known and unknown parameters," *IEEE Trans. Automat. Contr.*, vol. 48, no. 11, pp. 1908–1925, 2003.

[11] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear systems. I. Linear parameterization," *IEEE Trans. Automat. Contr.*, vol. 38, no. 1, pp. 17–32, 1993.

[12] ——, "Robust adaptive regulation of linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1301–1311, 2000.

[13] ——, "Adaptive control of linear time-varying systems," *Automatica*, vol. 39, no. 4, pp. 651–659, 2003.

[14] K. Chen and A. Astolfi, "Adaptive control of linear systems with time-varying parameters," in *Proc. 2018 Amer. Control Conf.* IEEE, 2018, pp. 80–85.

[15] A. Astolfi and R. Ortega, "Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 4, pp. 590–606, 2003.

[16] D. Karagiannis and A. Astolfi, "Nonlinear adaptive control of systems in feedback form: An alternative to adaptive backstepping," *Syst. Control Lett.*, vol. 57, no. 9, pp. 733–739, 2008.

[17] D. Karagiannis, M. Sassano, and A. Astolfi, "Dynamic scaling and observer design with application to adaptive control," *Automatica*, vol. 45, no. 12, pp. 2883–2889, 2009.

[18] L. Praly, "Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 1103–1108, 2003.

[19] G. A. Pratt and M. M. Williamson, "Series elastic actuators," in *Proc. 1995 IEEE/RSJ Int. Conf. on Intell. Robots and Syst.*, vol. 1. IEEE, 1995, pp. 399–406.

[20] K. Kong, J. Bae, and M. Tomizuka, "Control of rotary series elastic actuator for ideal force-mode actuation in human–robot interaction applications," *IEEE/ASME Trans. Mechatron.*, vol. 14, no. 1, pp. 105–118, 2009.

[21] J. A. Blaya and H. M. Herr, "Adaptive control of a variable-impedance ankle-foot orthosis to assist drop-foot gait," *IEEE Trans. Neural Syst. Rehab. Eng.*, vol. 12, no. 1, pp. 24–31, 2004.

[22] J. Bae, K. Kong, and M. Tomizuka, "Gait phase-based smoothed sliding mode control for a rotary series elastic actuator installed on the knee joint," in *Proc. 2010 Amer. Control Conf.* IEEE, 2010, pp. 6030–6035.

[23] J. Yamaguchi and A. Takanishi, "Development of a biped walking robot having antagonistic driven joints using nonlinear spring mechanism," in *Proc. 1997 IEEE Int. Conf. Robotics and Automat.*, vol. 1. IEEE, 1997, pp. 185–192.

[24] E. J. Rouse, L. M. Mooney, and H. M. Herr, "Clutchable series-elastic actuator: Implications for prosthetic knee design," *The Int. J. of Robotics Research*, vol. 33, no. 13, pp. 1611–1625, 2014.

[25] D. Rollinson, S. Ford, B. Brown, and H. Choset, "Design and modeling of a series elastic element for snake robots," in *Proc. ASME 2013 Dynamic Syst. Control Conf.* ASME, 2013, p. V001T08A002.