# Distributed Average Consensus under Quantized Communication via Event-Triggered Mass Summation 

Apostolos I. Rikos and Christoforos N. Hadjicostis


#### Abstract

We study distributed average consensus problems in multi-agent systems with directed communication links that are subject to quantized information flow. The goal of distributed average consensus is for the nodes, each associated with some initial value, to obtain the average (or some value close to the average) of these initial values. In this paper, we present and analyze a distributed averaging algorithm which operates exclusively with quantized values (specifically, the information stored, processed and exchanged between neighboring agents is subject to deterministic uniform quantization) and relies on event-driven updates (e.g., to reduce energy consumption, communication bandwidth, network congestion, and/or processor usage). We characterize the properties of the proposed distributed averaging protocol on quantized values and show that its execution, on any time-invariant and strongly connected digraph, will allow all agents to reach, in finite time, a common consensus value represented as the ratio of two integer that is equal to the exact average. We conclude with examples that illustrate the operation, performance, and potential advantages of the proposed algorithm.


Index Terms- Quantized average consensus, event-triggered, distributed algorithms, quantization, digraphs, multi-agent systems.

## I. INTRODUCTION

In recent years, there has been a growing interest for control and coordination of networks consisting of multiple agents, like groups of sensors [1] or mobile autonomous agents [2]. A problem of particular interest in distributed control is the consensus problem where the objective is to develop distributed algorithms that can be used by a group of agents in order to reach agreement to a common decision. The agents start with different initial values/information and are allowed to communicate locally via inter-agent information exchange under some constraints on connectivity. Consensus processes play an important role in many problems, such as leader election [3], motion coordination of multivehicle systems [2], [4], and clock synchronization [5].

One special case of the consensus problem is distributed averaging, where each agent (initially endowed with a numerical value) can send/receive information to/from other agents in its neighborhood and update its value iteratively, so that eventually, it is able to compute the average of all initial values. Average consensus is an important problem [4], [6][12] and has been studied extensively in settings where each agent processes and transmits real-valued states with infinite precision.

[^0]More recently, researchers have also studied the case when network links can only allow messages of limited length to be transmitted between agents (presumably due to constraints on their capacity), effectively extending techniques for average consensus towards the direction of quantized consensus. Various probabilistic strategies have been proposed, allowing the agents in a network to reach quantized consensus with probability one [13]-[18]. Furthermore, in many types of communication networks it is desirable to update values infrequently to avoid consuming valuable network resources. Thus, there is an increasing need for novel event-triggered algorithms for cooperative control, which aim at more efficient usage of network resources [19]-[21].

In this paper, we present a novel distributed average consensus algorithm that combines the both of the features mentioned above. More specifically, the processing, storing, and exchange of information between neighboring agents is "event-driven" and subject to uniform quantization. Following [15], [18] we assume that the states are integer-valued (which comprises a class of quantization effects). We note that most work dealing with quantization has concentrated on the scenario where the agents have real-valued states but can transmit only quantized values through limited rate channels (see, e.g., [17], [22]). By contrast, our assumption is also suited to the case where the states are stored in digital memories of finite capacity (as in [15], [18], [23]) and the control actuation of each node is event-based, which enables more efficient use of available resources. The main result of this paper shows that the proposed algorithm will allow all agents to reach quantized consensus in finite time by reaching a value represented as the ratio of two integer values that is equal to the average.

## II. PRELIMINARIES

The sets of real, rational, integer and natural numbers are denoted by $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$, respectively. The symbol $\mathbb{Z}_{+}$ denotes the set of nonnegative integers.

Consider a network of $n(n \geq 2)$ agents communicating only with their immediate neighbors. The communication topology can be captured by a directed graph (digraph), called communication digraph. A digraph is defined as $\mathcal{G}_{d}=$ $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}-\left\{\left(v_{j}, v_{j}\right) \mid v_{j} \in \mathcal{V}\right\}$ is the set of edges (selfedges excluded). A directed edge from node $v_{i}$ to node $v_{j}$ is denoted by $m_{j i} \triangleq\left(v_{j}, v_{i}\right) \in \mathcal{E}$, and captures the fact that node $v_{j}$ can receive information from node $v_{i}$ (but not the other way around). We assume that the given digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ is static (i.e., does not change over time) and
strongly connected (i.e., for each pair of nodes $v_{j}, v_{i} \in \mathcal{V}$, $v_{j} \neq v_{i}$, there exists a directed path from $v_{i}$ to $v_{j}$ ). The subset of nodes that can directly transmit information to node $v_{j}$ is called the set of in-neighbors of $v_{j}$ and is represented by $\mathcal{N}_{j}^{-}=\left\{v_{i} \in \mathcal{V} \mid\left(v_{j}, v_{i}\right) \in \mathcal{E}\right\}$, while the subset of nodes that can directly receive information from node $v_{j}$ is called the set of out-neighbors of $v_{j}$ and is represented by $\mathcal{N}_{j}^{+}=\left\{v_{l} \in \mathcal{V} \mid\left(v_{l}, v_{j}\right) \in \mathcal{E}\right\}$. The cardinality of $\mathcal{N}_{j}^{-}$ is called the in-degree of $v_{j}$ and is denoted by $\mathcal{D}_{j}^{-}$(i.e., $\left.\mathcal{D}_{j}^{-}=\left|\mathcal{N}_{j}^{-}\right|\right)$, while the cardinality of $\mathcal{N}_{j}^{+}$is called the outdegree of $v_{j}$ and is denoted by $\mathcal{D}_{j}^{+}$(i.e., $\mathcal{D}_{j}^{+}=\left|\mathcal{N}_{j}^{+}\right|$).

We assume that each node is aware of its out-neighbors and can directly (or indirectly ${ }^{17}$ ) transmit messages to each out-neighbor; however, it cannot necessarily receive messages from them. In the randomized version of the protocol, each node $v_{j}$ assigns a nonzero probability $b_{l j}$ to each of its outgoing edges $m_{l j}$ (including a virtual self-edge), where $v_{l} \in \mathcal{N}_{j}^{+} \cup\left\{v_{j}\right\}$. This probability assignment can be captured by a column stochastic matrix $\mathcal{B}=\left[b_{l j}\right]$. A very simple choice would be to set

$$
b_{l j}= \begin{cases}\frac{1}{1+\mathcal{D}_{j}^{+}}, & \text {if } v_{l} \in \mathcal{N}_{j}^{+} \cup\left\{v_{j}\right\} \\ 0, & \text { otherwise }\end{cases}
$$

Each nonzero entry $b_{l j}$ of matrix $\mathcal{B}$ represents the probability of node $v_{j}$ transmitting towards the out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$ through the edge $m_{l j}$, or performing no transmission ${ }^{2}$

In the deterministic version of the protocol, each node $v_{j}$ also assigns a unique order in the set $\left\{0,1, \ldots, \mathcal{D}_{j}^{+}-1\right\}$ to each of its outgoing edges $m_{l j}$, where $v_{l} \in \mathcal{N}_{j}^{+}$. The order of link $\left(v_{l}, v_{j}\right)$ for node $v_{j}$ is denoted by $P_{l j}$ (such that $\left.\left\{P_{l j} \mid v_{l} \in \mathcal{N}_{j}^{+}\right\}=\left\{0,1, \ldots, \mathcal{D}_{j}^{+}-1\right\}\right)$. This unique predetermined order is used during the execution of the proposed distributed algorithm as a way of allowing node $v_{i}$ to transmit messages to its out-neighbors in a round-robir ${ }^{3}$ fashion.

## III. PROBLEM FORMULATION

Consider a strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$, where each node $v_{j} \in \mathcal{V}$ has an initial (i.e., for $k=0$ ) quantized value $y_{j}[0]$ (for simplicity, we take $y_{j}[0] \in \mathbb{Z}$ ). In this paper, we develop a distributed algorithm that allows nodes (while processing and transmitting quantized information via available communication links between nodes) to eventually obtain, after a finite number of steps, a quantized fraction $q^{s}$ which is equal to the average $q$ of the initial values of the

[^1]nodes, where
\[

$$
\begin{equation*}
q=\frac{\sum_{l=1}^{n} y_{l}[0]}{n} \tag{1}
\end{equation*}
$$

\]

Remark 1: Following [15], [18] we assume that the state of each node is integer valued. This abstraction subsumes a class of quantization effects (e.g., uniform quantization).

The quantized average $q^{s}$ is defined as the ceiling $q^{s}=\lceil q\rceil$ or the floor $q^{s}=\lfloor q\rfloor$ of the true average $q$ of the initial values. Let $S \triangleq 1^{\mathrm{T}} y[0]$, where $\mathbf{1}=[1 \ldots 1]^{\mathrm{T}}$ is the vector of ones, and let $y[0]=\left[y_{1}[0] \ldots y_{n}[0]\right]^{\mathrm{T}}$ be the vector of the quantized initial values. We can write $S$ uniquely as $S=$ $n L+R$ where $L$ and $R$ are both integers and $0 \leq R<n$. Thus, we have that either $L$ or $L+1$ may be viewed as an integer approximation of the average of the initial values $S / n$ (which may not be integer in general).

The algorithm we will develop will be iterative. With respect to quantization of information flow, we have that at time step $k \in \mathbb{Z}_{+}$(where $\mathbb{Z}_{+}$is the set of nonnegative integers), each node $v_{j} \in \mathcal{V}$ maintains the state variables $y_{j}^{s}, z_{j}^{s}, q_{j}^{s}$, where $y_{j}^{s} \in \mathbb{Z}, z_{j}^{s} \in \mathbb{N}$ and $q_{j}^{s}$ (where $q_{j}^{s}=$ $\left.\frac{y_{j}^{s}}{z_{j}^{s}}\right)$, and the mass variables $y_{j}, z_{j}$ where $y_{j} \in \mathbb{Z}$ and $z_{j} \in \mathbb{N}_{0}$. The aggregate states are denoted by $y^{s}[k]=$ $\left[y_{1}^{s}[k] \ldots y_{n}^{s}[k]\right]^{\mathrm{T}} \in \mathbb{Z}^{n}, z^{s}[k]=\left[z_{1}^{s}[k] \ldots z_{n}^{s}[k]\right]^{\mathrm{T}} \in$ $\mathbb{N}^{n}, q^{s}[k]=\left[\begin{array}{lll}q_{1}^{s}[k] & \ldots & \left.q_{n}^{s}[k]\right]^{\mathrm{T}} \in \mathbb{Q}^{n} \text { and } y[k]= \\ =\end{array}\right.$ $\left[y_{1}[k] \ldots y_{n}[k]\right]^{\mathrm{T}} \in \mathbb{Z}^{n}, z[k]=\left[z_{1}[k] \ldots z_{n}[k]\right]^{\mathrm{T}} \in \mathbb{N}^{n}$ respectively.

Following the execution of the proposed distributed algorithm, we argue that $\exists k_{0}$ so that for every $k \geq k_{0}$ we have

$$
\begin{equation*}
y_{j}^{s}[k]=\frac{\sum_{l=1}^{n} y_{l}[0]}{\alpha} \text { and } z_{j}^{s}[k]=\frac{n}{\alpha} \tag{2}
\end{equation*}
$$

where $\alpha \in \mathbb{N}$. This means that

$$
\begin{equation*}
q_{j}^{s}[k]=\frac{\left(\sum_{l=1}^{n} y_{l}[0]\right) / \alpha}{n / \alpha}=q \tag{3}
\end{equation*}
$$

for every $v_{j} \in \mathcal{V}$ (i.e., for $k \geq k_{0}$ every node $v_{j}$ has calculated $q$ as the ratio of two integer values).

## IV. RANDOMIZED QUANTIZED AVERAGING ALGORITHM

In this section we propose a distributed information exchange process in which the nodes transmit and receive quantized messages so that they reach quantized average consensus on their initial values after a finite number of steps. The operation of the proposed distributed algorithm is summarized below.
Initialization: Each node $v_{j}$ selects a set of probabilities $\left\{b_{l j} \mid v_{l} \in \mathcal{N}_{j}^{+} \cup\left\{v_{j}\right\}\right\}$ such that $0<b_{l j}<1$ and $\sum_{v_{l} \in \mathcal{N}_{j}^{+} \cup\left\{v_{j}\right\}} b_{l j}=1$ (see Section II). Each value $b_{l j}$, represents the probability for node $v_{j}$ to transmit towards outneighbor $v_{l} \in \mathcal{N}_{j}^{+}$(or perform no transmission), at any given time step (independently between time steps). Each node has some initial value $y_{j}[0]$, and also sets its state variables, for time step $k=0$, as $z_{j}[0]=1, z_{j}^{s}[0]=1$ and $y_{j}^{s}[0]=y_{j}[0]$, which means that $q_{j}^{s}[0]=y_{j}[0] / 1$.
The iteration involves the following steps:

Step 1. Transmitting: According to the nonzero probabilities $b_{l j}$, assigned by node $v_{j}$ during the initialization step, it either transmits $z_{j}[k]$ and $y_{j}[k]$ towards out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$or performs no transmission. If it performs a transmission towards an out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$, it sets $y_{j}[k]=0$ and $z_{j}[k]=0$.
Step 2. Receiving: Each node $v_{j}$ receives messages $y_{i}[k]$ and $z_{i}[k]$ from its in-neighbors $v_{i} \in \mathcal{N}_{j}^{-}$, and it sums them along with its stored messages $y_{j}[k]$ and $z_{j}[k]$ as

$$
y_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] y_{i}[k],
$$

and

$$
z_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] z_{i}[k],
$$

where $w_{j i}[k]=0$ if no message is received from in-neighbor $v_{i} \in \mathcal{N}_{j}^{-}$; otherwise $w_{j i}[k]=1$.
Step 3. Processing: If $z_{j}[k+1] \geq z_{j}^{s}[k]$, node $v_{j}$ sets $z_{j}^{s}[k+$ $1]=z_{j}[k+1], y_{j}^{s}[k+1]=y_{j}[k+1]$ and

$$
q_{j}^{s}[k+1]=\frac{y_{j}^{s}[k+1]}{z_{j}^{s}[k+1]}
$$

Then, $k$ is set to $k+1$ and the iteration repeats (it goes back to Step 1).

The probabilistic quantized mass transfer process is detailed as Algorithm 1 below (for the case when $b_{l j}=$ $1 /\left(1+\mathcal{D}_{j}^{+}\right)$for $v_{l} \in \overline{\mathcal{N}}_{j}^{+} \cup\left\{v_{j}\right\}$ and $b_{l j}=0$ otherwise).

Example 1: Consider the strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ shown in Fig. 1, with $\mathcal{V}=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $\mathcal{E}=\left\{m_{21}, m_{31}, m_{42}, m_{13}, m_{23}, m_{34}\right\}$, where each node has initial quantized values $y_{1}[0]=5, y_{2}[0]=3, y_{3}[0]=7$, and $y_{4}[0]=2$ respectively. The average $q$ of the initial values of the nodes, is equal to $q=\frac{17}{4}$.


Fig. 1. Example of digraph for probabilistic quantized averaging.
Each node $v_{j} \in \mathcal{V}$ follows the Initialization steps (1-2) in Algorithm 1, assigning to each of its outgoing edges $v_{l} \in$ $\mathcal{N}_{j}^{+} \cup\left\{v_{j}\right\}$ a nonzero probability value $b_{l j}$ equal to $b_{l j}=$ $\frac{1}{1+\mathcal{D}_{j}^{+}}$. The assigned values can be seen in the following matrix

$$
\mathcal{B}=\left[\begin{array}{llll}
\frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{array}\right]
$$

while the initial mass and state variables are shown in Table
For the execution of the proposed algorithm, suppose that at time step $k=0$, nodes $v_{1}, v_{3}$ and $v_{4}$ transmit to nodes $v_{2}, v_{1}$ and $v_{3}$, respectively, whereas node $v_{2}$, performs no

## Algorithm 1 Probabilistic Quantized Average Consensus Input

1) A strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ with $n=|\mathcal{V}|$ nodes and $m=|\mathcal{E}|$ edges.
2) For every $v_{j}$ we have $y_{j}[0] \in \mathbb{Z}$.

## Initialization

Every node $v_{j} \in \mathcal{V}$ :

1) Assigns a nonzero probability $b_{l j}$ to each of its outgoing edges $m_{l j}$, where $v_{l} \in \mathcal{N}_{j}^{+}$, as follows

$$
b_{l j}= \begin{cases}\frac{1}{1+\mathcal{D}_{j}^{+}}, & \text {if } l=j \text { or } v_{l} \in \mathcal{N}_{j}^{+} \\ 0, & \text { if } l \neq j \text { and } v_{l} \notin \mathcal{N}_{j}^{+}\end{cases}
$$

2) Sets $z_{j}[0]=1, z_{j}^{s}[0]=1$ and $y_{j}^{s}[0]=y_{j}[0]$ (which means that $\left.q_{j}^{s}[0]=y_{j}[0] / 1\right)$.

## Iteration

For $k=0,1,2, \ldots$, each node $v_{j} \in \mathcal{V}$ does the following:

1) It either transmits $y_{j}[k]$ and $z_{j}[k]$ towards a randomly chosen out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$(according to the nonzero probability $b_{l j}$ ) or performs no transmission (according to the nonzero probability $b_{j j}$ ). If it transmitted towards an outneighbor, it sets $y_{j}[k]=0$ and $z_{j}[k]=0$.
2) It receives $y_{i}[k]$ and $z_{i}[k]$ from its in-neighbors $v_{i} \in \mathcal{N}_{j}^{-}$ and sets

$$
y_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] y_{i}[k],
$$

and

$$
z_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] z_{i}[k]
$$

where $w_{j i}[k]=1$ if node $v_{j}$ receives values from node $v_{i}$ (otherwise $w_{j i}[k]=0$ ).
3 ) If the following condition holds,

$$
\begin{equation*}
z_{j}[k+1] \geq z_{j}^{s}[k] \tag{4}
\end{equation*}
$$

it sets $z_{j}^{s}[k+1]=z_{j}[k+1], y_{j}^{s}[k+1]=y_{j}[k+1]$, which means that $q_{j}^{s}[k+1]=\frac{y_{j}^{s}[k+1]}{z_{j}^{s}[k+1]}$.
4) It repeats (increases $k$ to $k+1$ and goes back to Step 1).
transmission. The mass and state variables for $k=1$ are shown in Table II.

TABLE I
Initial Mass and State Variables for Fig.

| Nodes <br> $v_{j}$ | Mass and State Variables for $k=0$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $y_{j}[0]$ | $z_{j}[0]$ | $y_{j}^{s}[0]$ | $z_{j}^{s}[0]$ | $q_{j}^{s}[0]$ |  |
| $v_{1}$ | 5 |  |  |  |  |
| $v_{2}$ | 3 | 1 | 5 | 1 | $5 / 1$ |
| $v_{3}$ | 7 | 1 | 3 | 1 | $3 / 1$ |
| $v_{4}$ | 2 | 1 | 2 | 1 | $7 / 1$ |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

TABLE II
Mass and State Variables for Fig. 1 For $k=1$

| Nodes | Mass and State Variables for $k=1$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | $y_{j}[1]$ | $z_{j}[1]$ | $y_{j}^{s}[1]$ | $z_{j}^{s}[1]$ | $q_{j}^{s}[1]$ |
| $v_{1}$ | 7 |  |  |  |  |
| $v_{2}$ | 8 | 1 | 7 | 1 | $7 / 1$ |
| $v_{3}$ | 2 | 2 | 8 | 2 | $8 / 2$ |
| $v_{4}$ | 0 | 0 | 2 | 1 | $2 / 1$ |
|  |  |  |  |  |  |

It is important to notice here that nodes $v_{1}$ and $v_{3}$ have mass variables $y_{1}[1]=y_{3}[0]=7, z_{1}[1]=z_{3}[0]=1$ and $y_{3}[1]=y_{4}[0]=2, z_{3}[1]=z_{4}[0]=1$ (and update their state variables), while node $v_{2}$ has mass variables $y_{2}[1]=$ $y_{1}[0]+y_{2}[0]=8, z_{2}[1]=z_{1}[0]+z_{2}[0]=2$ (also updating its state variables). In the latter case we can say that the mass variables of nodes $v_{1}$ and $v_{2}$ will "merge".

Suppose now that at time step $k=1$, nodes $v_{1}$ and $v_{2}$ transmit to nodes $v_{3}$ and $v_{4}$. Node $v_{3}$, does not perform a transmission while node $v_{4}$ has no mass to transmit. The mass and state variables for $k=2$ are shown in Table III.

TABLE III
Mass and State Variables for Fig. 1 For $k=2$

| Nodes |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | Mass and State Variables for $k=2$ |  |  |  |  |
| $y_{j}[2]$ | $z_{j}[2]$ | $y_{j}^{s}[2]$ | $z_{j}^{s}[2]$ | $q_{j}^{s}[2]$ |  |
|  | $v_{1}$ | 0 |  |  |  |
|  |  |  |  |  |  |
| $v_{2}$ | 0 | 0 | 7 | 1 | $7 / 1$ |
| $v_{3}$ | 9 | 2 | 9 | 2 | $8 / 2$ |
| $v_{4}$ | 8 | 2 | 8 | 2 | $9 / 2$ |
|  |  |  |  | 2 | $8 / 2$ |

Then, suppose that at time step $k=2$, node $v_{4}$ transmits to node $v_{3}$, while node $v_{3}$, does not perform a transmission (nodes $v_{1}$ and $v_{2}$ have no mass to transmit). The mass and state variables for $k=3$ are shown in Table IV]

We can see that, at time step $k=3$ all the initial mass variables are "merged" in node $v_{3}$ (i.e., $y_{3}[3]=y_{1}[0]+$ $y_{2}[0]+y_{3}[0]+y_{4}[0]$ and $\left.z_{3}[3]=z_{1}[0]+z_{2}[0]+z_{3}[0]+z_{4}[0]\right)$. Now suppose that during time steps $k=3,4,5$ the following transmissions take place: " $v_{3}$ transmits to $v_{1}$ ", " $v_{1}$ transmits to $v_{2}$ ", " $v_{2}$ transmits to $v_{4}$ ". The mass and state variables for $k=5$ are shown in Table $\nabla$

TABLE IV
Mass and State Variables for Fig. 1 for $k=3$

| Nodes |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | Mass and State Variables for $k=3$ |  |  |  |  |
|  | $y_{j}[3]$ | $z_{j}[3]$ | $y_{j}^{s}[3]$ | $z_{j}^{s}[3]$ | $q_{j}^{s}[3]$ |
| $v_{1}$ | 0 | 0 |  |  |  |
| $v_{2}$ | 0 | 0 | 8 | 1 | $7 / 1$ |
| $v_{3}$ | 17 | 4 | 17 | 4 | $8 / 2$ |
| $v_{4}$ | 0 | 0 | 8 | 2 | $8 / 2$ |

TABLE V
Mass and State Variables for Fig. 1 for $k=5$

| Nodes |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | Mass and State Variables for $k=3$ |  |  |  |  |
|  | $y_{j}[5]$ | $z_{j}[5]$ | $y_{j}^{s}[5]$ | $z_{j}^{s}[5]$ | $q_{j}^{s}[5]$ |
| $v_{1}$ | 0 | 0 | 17 | 4 | $17 / 4$ |
| $v_{2}$ | 0 | 0 | 17 | 4 | $17 / 4$ |
| $v_{3}$ | 0 | 0 | 17 | 4 | $17 / 4$ |
| $v_{4}$ | 17 | 4 | 17 | 4 | $17 / 4$ |

From Table V , we can see that for $k \geq 5$ it holds that

$$
q_{j}^{s}[k]=q=\frac{17}{4}
$$

for every $v_{j} \in \mathcal{V}$, which means that every node $v_{j}$ will eventually obtain a quantized fraction $q_{j}^{s}$, which is equal to the average $q$ of the initial values of the nodes.

Remark 2: From the previous example, it is important to notice that, once the initial mass variables "merge" at time step $k=3$, they remain "merged" during the operation of Algorithm 1 for every time step $k \geq 3$.

We are now ready to prove that during the operation of Algorithm 1 each agent obtains two integer values $y^{s}$ and $z^{s}$, the ratio of which is equal to the average $q$ of the initial values of the nodes.

Proposition 1: Consider a strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ with $n=|\mathcal{V}|$ nodes and $m=|\mathcal{E}|$ edges, and $z_{j}[0]=1$ and $y_{j}[0] \in \mathbb{Z}$ for every node $v_{j} \in \mathcal{V}$ at time step $k=0$. Suppose that each node $v_{j} \in \mathcal{V}$ follows the Initialization and Iteration steps as described in Algorithm 1 . Let $\mathcal{V}^{+}[k] \subseteq \mathcal{V}$ be the set of nodes $v_{j}$ with positive mass variable $z_{j}[k]$ at iteration $k$ (i.e., $\mathcal{V}^{+}[k]=\left\{v_{j} \in \mathcal{V} \mid z_{j}[k]>\right.$ $0\}$ ). During the execution of Algorithm 11, for every $k \geq 0$, we have that

$$
1 \leq\left|\mathcal{V}^{+}[k+1]\right| \leq\left|\mathcal{V}^{+}[k]\right| \leq n
$$

Proof: During the Iteration Steps 1 and 2 of Algorithm 1. at time step $k$, we have that each node $v_{j} \in \mathcal{V}$ transmits $z_{j}[k]$ and $y_{j}[k]$ towards a randomly chosen outneighbor $v_{l} \in \mathcal{N}_{j}^{+}$, or performs no transmission. Then, it receives $y_{i}[k]$ and $z_{i}[k]$ from its in-neighbors $v_{i} \in \mathcal{N}_{j}^{-}$and sets $y_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] y_{i}[k]$, and $z_{j}[k+1]=$ $\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] z_{i}[k]$. The Iteration Steps 1 and 2 of Algorithm 1, during time step $k$, can be expressed according to the following equations

$$
\begin{equation*}
y[k+1]=\mathcal{W}[k] y[k] \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
z[k+1]=\mathcal{W}[k] z[k] \tag{6}
\end{equation*}
$$

where $y[k]=\left[y_{1}[k] \ldots y_{n}[k]\right]^{\mathrm{T}}, z[k]=\left[z_{1}[k] \ldots z_{n}[k]\right]^{\mathrm{T}}$ and $\mathcal{W}[k]=\left[w_{l j}[k]\right]$ is an $n \times n$ binary (i.e., for every $k, w_{l j}[k]$ is either equal to 1 or 0 , for every $\left.\left(v_{l}, v_{j}\right) \in \mathcal{E}\right)$, column stochastic matrix.

Focusing on (6), during time step $k_{0}$, let us assume without loss of generality that $z\left[k_{0}\right]=\left[z_{1}\left[k_{0}\right] \ldots z_{p_{0}}\left[k_{0}\right] 0 \ldots 0\right]^{\mathrm{T}}$, which means that we have $z_{i}\left[k_{0}\right]>0, \forall v_{i} \in\left\{v_{1}, \cdots, v_{p_{0}}\right\}$ and $z_{l}\left[k_{0}\right]=0, \forall v_{l} \in \mathcal{V}-\left\{v_{1}, \cdots, v_{p_{0}}\right\}$. We can assume without loss of generality that the nodes with zero mass do not transmit (transmit to themselves). Let us consider the scenario where $\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}\left[k_{0}\right]=1, \forall v_{j} \in \mathcal{V}$ (i.e., for every row of $\mathcal{W}\left[k_{0}\right]$ exactly one element is equal to 1 and all the other are equal to zero). This means that every node $v_{j}$ will receive exactly one mass variable $z_{i}\left[k_{0}\right]$ (the bottom $n-p_{0}$ nodes receive their own mass). Since, at time step $k_{0}$, we have $p_{0}$ nodes with nonzero mass variables, we have that at time step $k_{0}+1$, exactly $p_{0}$ nodes have a nonzero mass variable. As a result, for this scenario, we have $\left|\mathcal{V}^{+}\left[k_{0}+1\right]\right|=$ $\left|\mathcal{V}^{+}\left[k_{0}\right]\right|$.

Without loss of generality, let us consider the scenario where $w_{j i_{1}}\left[k_{0}\right]=1, w_{j i_{2}}\left[k_{0}\right]=1$ (where $v_{i_{1}}, v_{i_{2}} \in \mathcal{N}_{j}^{-} \cup$
$\left\{v_{j}\right\}$ ) and $w_{j i}\left[k_{0}\right]=0, \forall v_{i} \in\left\{\mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}\right\}-\left\{v_{i_{1}}, v_{i_{2}}\right\}$ (i.e., the $j^{\text {th }}$ row of matrix $\mathcal{W}\left[k_{0}\right]$ has exactly 2 elements equal to 1 and all the other equal to zero). Also, let us assume that $\sum_{v_{i} \in \mathcal{N}_{l}^{-} \cup\left\{v_{l}\right\}} w_{l i}\left[k_{0}\right] \leq 1, \forall v_{l} \in \mathcal{V}-\left\{v_{j}\right\}$ (i.e., for every row of $\mathcal{W}\left[k_{0}\right]$ (except row $j$ ) at most one element is equal to 1 and all the other are equal to zero). The above assumptions, regarding matrix $\mathcal{W}$, mean that, during time step $k_{0}$, only node $v_{j}$ will receive two mass variables (from nodes $v_{i_{1}}$ and $v_{i_{2}}$ ) and all the other nodes will receive at most one mass variable. We have that $z_{j}\left[k_{0}+1\right]=z_{i_{1}}\left[k_{0}\right]+z_{i_{2}}\left[k_{0}\right]$ and $z_{l}\left[k_{0}+1\right]=z_{i}\left[k_{0}\right]$, for $v_{l} \in \mathcal{V}-\left\{v_{j}\right\}$ and some $v_{i} \in \mathcal{V}-$ $\left\{v_{i_{1}}, v_{i_{2}}\right\}$ (i.e., node $v_{j}$ received two nonzero mass variables while all the other nodes received at most one nonzero mass variable, also including its own mass variable). Since, at time step $k_{0}$, we had $p_{0}$ nodes with nonzero mass variables and at time step $k_{0}+1$ node $v_{j}$ received (and summed) two nonzero mass variables, while all the other nodes received at most one nonzero mass variable, this means that, at time step $k_{0}+1$, we have $p_{0}-1$ nodes with nonzero mass variables. This means that $\left|\mathcal{V}^{+}\left[k_{0}+1\right]\right|<\left|\mathcal{V}^{+}\left[k_{0}\right]\right|$.

By extending the above analysis for scenarios where each row of $\mathcal{W}[k]$, at different time steps $k$, has multiple elements equal to 1 (but $\mathcal{W}[k]$ remains column stochastic) we can see that the number of nodes $v_{j}$ with nonzero mass variable $z_{j}[k]>0$ is non-increasing and thus we have $\left|\mathcal{V}^{+}[k+1]\right| \leq$ $\left|\mathcal{V}^{+}[k]\right|, \forall k \in \mathbb{N}$.

Proposition 2: Consider a strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ with $n=|\mathcal{V}|$ nodes and $m=|\mathcal{E}|$ edges and $z_{j}[0]=1$ and $y_{j}[0] \in \mathbb{Z}$ for every node $v_{j} \in \mathcal{V}$ at time step $k=0$. Suppose that each node $v_{j} \in \mathcal{V}$ follows the Initialization and Iteration steps as described in Algorithm 1 . With probability one, we can find $k_{0} \in \mathbb{N}$, so that for every $k \geq k_{0}$ we have

$$
y_{j}^{s}[k]=\sum_{l=1}^{n} y_{l}[0] \quad \text { and } \quad z_{j}^{s}[k]=n
$$

which means that

$$
q_{j}^{s}[k]=\frac{\sum_{l=1}^{n} y_{l}[0]}{n}
$$

for every $v_{j} \in \mathcal{V}$ (i.e., for $k \geq k_{0}$ every node $v_{j}$ has calculated $q$ as the ratio of two integer values).

Proof: From Proposition 1 we have that $\left|\mathcal{V}^{+}[k+1]\right| \leq$ $\left|\mathcal{V}^{+}[k]\right|$ (i.e., the number of nonzero mass variables is nonincreasing). We will show that the number of nonzero mass variables is decreasing after a finite number of steps, until, at some $k_{0} \in \mathbb{N}$, we have $y_{j}\left[k_{0}\right]=\sum_{l=1}^{n} y_{l}[0]$ and $z_{j}\left[k_{0}\right]=n$, for some node $v_{j} \in \mathcal{V}$, and $y_{i}\left[k_{0}\right]=0$ and $z_{i}\left[k_{0}\right]=0$, for each $v_{i} \in \mathcal{V}-\left\{v_{j}\right\}$ ). In this scenario, (2) and (3) hold for each node $v_{j}$ for the case where $\alpha=1$.

The Iteration Steps 1 and 2 of Algorithm 1, during time step $k$, can be expressed according to (5) and (6), where
 $\mathcal{W}[k]=\left[w_{l j}[k]\right]$ is an $n \times n$ binary, column stochastic matrix. Focusing on (6), suppose that, during time step $k_{0}$, we have $z_{i}\left[k_{0}\right]>0, z_{j}\left[k_{0}\right]>0$ and $w_{l i}\left[k_{0}\right]=1$, $w_{l j}\left[k_{0}\right]=1$. This scenario will occur with probability equal
to $\left(1+\mathcal{D}_{i}^{+}\right)^{-1}\left(1+\mathcal{D}_{j}^{+}\right)^{-1}$ (i.e., as long as nodes $v_{i}$ and $v_{j}$ both transmit towards node $v_{l}$ ). Furthermore, we have that the mass variables of $v_{i}$ and $v_{j}$ will not "merge" in $v_{l}$ with probability $1-\left(1+\mathcal{D}_{i}^{+}\right)^{-1}\left(1+\mathcal{D}_{j}^{+}\right)^{-1}$. By extending the above analysis we have that, every $n$ time steps, the probability that two nonzero mass variables "merge" is positive and lower bounded by $\left(\prod_{j=1}^{n}\left(1+\mathcal{D}_{j}^{+}\right)^{-1}\right)^{2}$ (i.e., $\left.\mathrm{P}_{\text {merge }} \geq\left(\prod_{j=1}^{n}\left(1+\mathcal{D}_{j}^{+}\right)^{-1}\right)^{2}\right)$.

Thus, from the execution of Algorithm 1, we have that the probability that all nonzero mass variables "merge" will be arbitrarily close to 1 for a sufficiently large $k$. This means that $\exists k_{0} \in \mathbb{N}$ for which $y_{j}\left[k_{0}\right]=\sum_{l=1}^{n} y_{l}[0]$, and $z_{j}\left[k_{0}\right]=n$, for some node $v_{j} \in \mathcal{V}$, and $y_{i}\left[k_{0}\right]=0$, and $z_{i}\left[k_{0}\right]=0$, for each $v_{i} \in \mathcal{V}-\left\{v_{j}\right\}$. Once this "merging" of all nonzero mass variables occurs, we have that the nonzero mass variables of node $v_{j}$ will update the state variables of every node $v_{i} \in \mathcal{V}$ (because it eventually will be forward to all other nodes) which means that $\exists k_{1} \in \mathbb{N}$ (where $k_{1}>k_{0}$ ) for which $y_{i}^{s}\left[k_{1}\right]=\sum_{l=1}^{n} y_{l}[0]$ and $z_{i}^{s}\left[k_{1}\right]=n$, for every node $v_{i} \in \mathcal{V}$. This means that after a finite number of steps, (2) and (3) will hold for every node $v_{j} \in \mathcal{V}$ for the case where $\alpha=1$.

Remark 3: It is interesting to note that during the operation of Algorithm [1, after a finite number of steps $k_{0}$, the state variables of each node $v_{j} \in \mathcal{V}$, become equal to $y_{j}^{s}[k]=\sum_{l=1}^{n} y_{l}[0], z_{j}^{s}[k]=n$, so that

$$
q_{j}^{s}[k]=\frac{\sum_{l=1}^{n} y_{l}[0]}{n}
$$

for $k \geq k_{0}$. This means that (2) and (3) will hold for each node $v_{j}$ for the case where $\alpha=1$. However, this does not necessarily hold for the distributed algorithm presented in the following section.

Remark 4: It is also worth pointing out that during the operation of Algorithm (1) once (2) and (3) hold for each node $v_{j}$ for the case where $\alpha=1$, then each node also obtains knowledge regarding the total number of nodes in the digraph, since $z_{j}^{s}[k]=n, \forall v_{j} \in \mathcal{V}$, which may be useful for determining the number of agents in the network.

## V. EVENT-TRIGGERED QUANTIZED AVERAGING ALGORITHM

In this section we propose a distributed algorithm in which the nodes receive quantized messages and perform transmissions according to a set of deterministic conditions, so that they reach quantized average consensus on their initial values. The operation of the proposed distributed algorithm is summarized below.
Initialization: Each node $v_{j}$ assigns to each of its outgoing edges $v_{l} \in \mathcal{N}_{j}^{+}$a unique order $P_{l j}$ in the set $\left\{0,1, \ldots, \mathcal{D}_{j}^{+}-\right.$ $1\}$, which will be used to transmit messages to its outneighbors in a round-robin fashion. Node $v_{j}$ has initial value $y_{j}[0]$ and sets its state variables, for time step $k=0$, as $z_{j}[0]=1, z_{j}^{s}[0]=1$ and $y_{j}^{s}[0]=y_{j}[0]$, which means that $q_{j}^{s}[0]=y_{j}[0] / 1$. Then, it chooses an out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$ (according to the predetermined order $P_{l j}$ ) and transmits
$z_{j}[0]$ and $y_{j}[0]$ to that particular neighbor. Then, it sets $y_{j}[0]=0$ and $z_{j}[0]=0$ (since performed a transmission).
The iteration involves the following steps:
Step 1. Receiving: Each node $v_{j}$ receives messages $y_{i}[k]$ and $z_{i}[k]$ from its in-neighbors $v_{i} \in \mathcal{N}_{j}^{-}$and sums them along with its stored messages $y_{j}[k]$ and $z_{j}[k]$ to obtain

$$
y_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] y_{i}[k],
$$

and

$$
z_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] z_{i}[k],
$$

where $w_{j i}[k]=0$ if no message is received from in-neighbor $v_{i} \in \mathcal{N}_{j}^{-}$; otherwise $w_{j i}[k]=1$.
Step 2. Event-Triggered Conditions: Node $v_{j}$ checks the following conditions:

1) It checks whether $z_{j}[k+1]$ is greater than $z_{j}^{s}[k]$,
2) If $z_{j}[k+1]$ is equal to $z_{j}^{s}[k]$, it checks whether $y_{j}[k+1]$ is greater than (or equal to) $y_{j}^{s}[k]$.
If one of the above two conditions holds, it sets $y_{j}^{s}[k+1]=$ $y_{j}[k+1], z_{j}^{s}[k+1]=z_{j}[k+1]$ and $q_{j}^{s}[k+1]=\frac{y_{j}^{s}[k+1]}{z_{j}^{s}[k+1]}$.
Step 3. Transmitting: If the event-trigger conditions above do not hold, no transmission is performed. Otherwise, if the event-trigger conditions above hold, node $v_{j}$ chooses an outneighbor $v_{l} \in \mathcal{N}_{j}^{+}$according to the order $P_{l j}$ (in a roundrobin fashion) and transmits $z_{j}[k+1]$ and $y_{j}[k+1]$. Then, since it transmitted its stored mass, it sets $y_{j}[k+1]=0$ and $z_{j}[k+1]=0$. Then, $k$ is set to $k+1$ and the iteration repeats (it goes back to Step 1).

This event-based quantized mass transfer process is summarized as Algorithm 2, where each node $v_{j}$ at time step $k$ maintains mass variables $y_{j}[k]$ and $z_{j}[k]$ and state variables $y_{j}^{s}[k]$ and $z_{j}^{s}[k]$ (and $q_{j}^{s}[k]=y_{j}^{s}[k] / z_{j}^{s}[k]$ ). Note that the event trigger conditions effectively imply that no transmission is performed if $z_{j}[k]=0$.

We now analyze the functionality of the distributed algorithm and we prove that it allows all agents to reach quantized average consensus after a finite number of steps. Depending on the graph structure and the initial mass variables of each node, we have the following two possible scenarios:
A. Full Mass Summation (i.e., there exists $k_{0} \in \mathbb{N}$ where we have $y_{j}\left[k_{0}\right]=\sum_{l=1}^{n} y_{l}[0]$ and $z_{j}\left[k_{0}\right]=n$, for some node $v_{j} \in \mathcal{V}$, and $y_{i}\left[k_{0}\right]=0$ and $z_{i}\left[k_{0}\right]=0$, for each $\left.v_{i} \in \mathcal{V}-\left\{v_{j}\right\}\right)$. In this scenario (2) and (3) hold for each node $v_{j}$ for the case where $\alpha=1$.
B. Partial Mass Summation (i.e., there exists $k_{0} \in \mathbb{N}$ so that for every $k \geq k_{0}$ there exists a set $\mathcal{V}^{p}[k] \subseteq \mathcal{V}$ in which we have $y_{j}[k]=y_{i}[k]$ and $z_{j}[k]=z_{i}[k]$, $\forall v_{j}, v_{i} \in \mathcal{V}^{p}[k]$ and $y_{l}[k]=0$ and $z_{l}[k]=0$, for each $v_{l} \in \mathcal{V}-\mathcal{V}^{p}[k]$ ). In this scenario (2) and (3) hold for each node $v_{j}$ for the case where $\alpha=\left|\mathcal{V}^{p}[k]\right|$.
An example regarding the scenario of "Partial Mass Summation" is given below.

## Algorithm 2 Deterministic Quantized Average Consensus Input

1) A strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ with $n=|\mathcal{V}|$ nodes and $m=|\mathcal{E}|$ edges.
2) For every $v_{j}$ we have $y_{j}[0] \in \mathbb{Z}$.

## Initialization

Every node $v_{j} \in \mathcal{V}$ :

1) Assigns to each of its outgoing edges $v_{l} \in \mathcal{N}_{j}^{+}$a unique order $P_{l j}$ in the set $\left\{0,1, \ldots, \mathcal{D}_{j}^{+}-1\right\}$.
2) Sets $z_{j}[0]=1, z_{j}^{s}[0]=1$ and $y_{j}^{s}[0]=y_{j}[0]$ (which means that $\left.q_{j}^{s}[0]=y_{j}[0] / 1\right)$.
3) Chooses an out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$according to the predetermined order $P_{l j}$ (i.e., it chooses $v_{l} \in \mathcal{N}_{j}^{+}$such that $\left.P_{l j}=0\right)$ and transmits $z_{j}[0]$ and $y_{j}[0]$ to this out-neighbor. Then, it sets $y_{j}[0]=0$ and $z_{j}[0]=0$.

## Iteration

For $k=0,1,2, \ldots$, each node $v_{j} \in \mathcal{V}$ does the following: 1) It receives $y_{i}[k]$ and $z_{i}[k]$ from its in-neighbors $v_{i} \in \mathcal{N}_{j}^{-}$ and sets

$$
y_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] y_{i}[k],
$$

and

$$
z_{j}[k+1]=\sum_{v_{i} \in \mathcal{N}_{j}^{-} \cup\left\{v_{j}\right\}} w_{j i}[k] z_{i}[k],
$$

where $w_{j i}[k]=0$ if no message is received (otherwise $w_{j i}[k]=1$ ).
2) Event triggered conditions: If one of the following two conditions hold, node $v_{j}$ performs Steps 3 and 4 below, otherwise it skips Steps 3 and 4.
Condition 1: $z_{j}[k+1]>z_{j}^{s}[k]$.
Condition 2: $z_{j}[k+1]=z_{j}^{s}[k]$ and $y_{j}[k+1] \geq y_{j}^{s}[k]$.
3) It sets $z_{j}^{s}[k+1]=z_{j}[k+1]$ and $y_{j}^{s}[k+1]=y_{j}[k+1]$ which implies that

$$
q_{j}^{s}[k+1]=\frac{y_{j}^{s}[k+1]}{z_{j}^{s}[k+1]}
$$

4) It chooses an out-neighbor $v_{l} \in \mathcal{N}_{j}^{+}$according to the order $P_{l j}$ (in a round-robin fashion) and transmits $z_{j}[k+1]$ and $y_{j}[k+1]$. Then it sets $y_{j}[k+1]=0$ and $z_{j}[k+1]=0$. 5) It repeats (increases $k$ to $k+1$ and goes back to Step 1).

Example 2: Consider a strongly connected digraph $\mathcal{G}_{d}=$ $(\mathcal{V}, \mathcal{E})$, shown in Fig. 2, with $\mathcal{V}=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $\mathcal{E}=\left\{m_{21}, m_{32}, m_{43}, m_{14}\right\}$ where each node has an initial quantized value $y_{1}[0]=9, y_{2}[0]=3, y_{3}[0]=9$ and $y_{4}[0]=3$ respectively. We have that the average of the initial values of the nodes, is equal to $q=\frac{24}{4}$.

At time step $k=0$ the initial mass and state variables for nodes $v_{1}, v_{2}, v_{3}, v_{4}$ are shown in Table VI.

TABLE VI
Initial Mass and State Variables for Fig. 2


Fig. 2. Example of digraph for partial mass summation.

| Nodes | Mass and State Variables for $k=0$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | $y_{j}[0]$ | $z_{j}[0]$ | $y_{j}^{s}[0]$ | $z_{j}^{s}[0]$ | $q_{j}^{s}[0]$ |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
| $v_{1}$ | 9 | 1 | 9 | 1 | $9 / 1$ |
| $v_{2}$ | 3 | 1 | 3 | 1 | $3 / 1$ |
| $v_{3}$ | 9 | 1 | 9 | 1 | $9 / 1$ |
| $v_{4}$ | 3 | 1 | 3 | 1 | $3 / 1$ |

Then, during time step $k=0$, every node $v_{j}$ will transmit its mass variables $y_{j}[0]$ and $z_{j}[0]$ (since the event-triggered conditions hold for every node). The mass and state variables of every node at $k=1$ are shown in Table VII.

It is important to notice here that, for time step $k=1$, nodes $v_{1}$ and $v_{3}$ have mass variables equal to $y_{1}[1]=3$, $z_{1}[1]=1$ and $y_{3}[1]=3, z_{3}[1]=1$ but the corresponding state variables are equal to $y_{1}^{S}[1]=9, z_{1}^{S}[1]=1$ and $y_{3}^{S}[1]=$ $9, z_{3}^{s}[1]=1$. This means that at time step $k=1$, the eventtriggered conditions do not hold for nodes $v_{1}$ and $v_{3}$; thus, these nodes will not transmit their mass variables (i.e., they will not execute Steps 3 and 4 of Algorithm 22. The mass and state variables of every node at $k=2$ are shown in Table VIII

TABLE VII
Mass and State Variables for Fig. 2for $k=1$

| Nodes | Mass and State Variables for $k=1$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | $y_{j}[1]$ | $z_{j}[1]$ | $y_{j}^{s}[1]$ | $z_{j}^{s}[1]$ | $q_{j}^{s}[1]$ |
|  |  |  |  |  |  |
| $v_{1}$ | 3 | 1 | 9 | 1 | $9 / 1$ |
| $v_{2}$ | 9 | 1 | 9 | 1 | $9 / 1$ |
| $v_{3}$ | 3 | 1 | 9 | 1 | $9 / 1$ |
| $v_{4}$ | 9 | 1 | 9 | 1 | $9 / 1$ |

TABLE VIII
Mass and State Variables for Fig. 2 For $k=2$

| Nodes | Mass and State Variables for $k=2$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | $y_{j}[2]$ | $z_{j}[2]$ | $y_{j}^{s}[2]$ | $z_{j}^{s}[2]$ | $q_{j}^{s}[2]$ |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
| $v_{1}$ | 12 | 2 | 12 | 2 | $12 / 2$ |
| $v_{2}$ | 0 | 0 | 9 | 1 | $9 / 1$ |
| $v_{3}$ | 12 | 2 | 12 | 2 | $12 / 2$ |
| $v_{4}$ | 0 | 0 | 9 | 1 | $9 / 1$ |

During time step $k=2$ we can see that the event-triggered conditions hold for nodes $v_{1}$ and $v_{3}$ which means that they will transmit their mass variables towards nodes $v_{2}$ and $v_{4}$ respectively. The mass and state variables of every node for $k=3$ are shown in Table IX

TABLE IX
Mass and State Variables for Fig. 2for $k=3$

| Nodes | Mass and State Variables for $k=3$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{j}$ | $y_{j}[3]$ | $z_{j}[3]$ | $y_{j}^{s}[3]$ | $z_{j}^{s}[3]$ | $q_{j}^{s}[3]$ |
|  |  |  |  |  |  |
| $v_{1}$ | 0 | 0 | 12 | 2 | $12 / 2$ |
| $v_{2}$ | 12 | 2 | 12 | 2 | $12 / 2$ |
| $v_{3}$ | 0 | 0 | 12 | 2 | $12 / 2$ |
| $v_{4}$ | 12 | 2 | 12 | 2 | $12 / 2$ |

Following the algorithm operation we have that, for $k=3$, the event-trigger conditions hold for nodes $v_{2}$ and $v_{4}$ which means that they will transmit their masses to nodes $v_{1}$ and $v_{3}$ respectively. As a result we have, for $k=4$, that the mass variables for nodes $v_{1}$ and $v_{3}$ are $y_{1}[4]=y_{4}[3]=$ $12, z_{1}[4]=z_{4}[3]=2$ and $y_{3}[4]=y_{2}[3]=12, z_{3}[4]=$ $z_{2}[3]=2$ respectively. Then, during time step $k=4$, we have that the event-triggered conditions hold for nodes $v_{1}$ and $v_{3}$ which means that they will transmit their mass variables to nodes $v_{1}$ and $v_{3}$. We can easily notice that, during the execution of Algorithm 2 for $k \geq 3$, we have $\mathcal{V}^{p}[k]=\mathcal{V}^{p}[k+$ 2] (where $\mathcal{V}^{p}[3]=\left\{v_{2}, v_{4}\right\}$ and $\mathcal{V}^{p}[4]=\left\{v_{1}, v_{3}\right\}$ ), which means that the exchange of mass variables between the nodes will follow a periodic behavior and the mass variables will never "merge" in one node (i.e., $\exists k_{0}$ for which $y_{j}\left[k_{0}\right]=$ $\sum_{l=1}^{n} y_{l}[0]$ and $z_{j}\left[k_{0}\right]=n$, for some node $v_{j} \in \mathcal{V}$, and $y_{i}\left[k_{0}\right]=0$ and $z_{i}\left[k_{0}\right]=0$, for each $v_{i} \in \mathcal{V}-\left\{v_{j}\right\}$ ).

As a result, from Table IX, we can see that for $k \geq 3$ it holds that

$$
q_{j}^{s}[k]=q=\frac{24 / \alpha}{4 / \alpha}
$$

for every $v_{j} \in \mathcal{V}$, for $\alpha=\left|\mathcal{V}^{p}[k]\right|=2$. This means that, after a finite number of steps, every node $v_{j}$ will obtain a quantized fraction $q_{j}^{s}$ which is equal to the average $q$ of the initial values of the nodes.

Remark 5: Note that the periodic behavior in the above graph is not only a function of the graph structure but also of the initial conditions. Also note that, in general, the priorities will also play a role because they determine the order in which nodes transmit to their out-neighbors (in the example, priorities do not come into play because each node has exactly one out-neighbor).
Proposition 3: Consider a strongly connected digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ with $n=|\mathcal{V}|$ nodes and $m=|\mathcal{E}|$ edges. The execution of Algorithm 2 will allow each node $v_{j} \in \mathcal{V}$ to reach quantized average consensus after a finite number of steps, bounded by $n^{5}$.

## VI. SIMULATION RESULTS

In this section, we present simulation results and comparisons. Specifically, we present simulation results of the proposed distributed algorithms for the digraph $\mathcal{G}_{d}=(\mathcal{V}, \mathcal{E})$ (borrowed from [24]), shown in Fig. 3 with $\mathcal{V}=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\right\}$ and $\mathcal{E}=\left\{m_{21}, m_{51}, m_{12}, m_{52}, m_{13}, m_{53}, m_{24}, m_{54}, m_{65}, m_{75}\right.$, $\left.m_{36}, m_{47}, m_{67}\right\}$, where each node has initial quantized values $y_{1}[0]=5, y_{2}[0]=4, y_{3}[0]=8, y_{4}[0]=3, y_{5}[0]=5$, $y_{6}[0]=2$, and $y_{7}[0]=7$, respectively. The average $q$ of the initial values of the nodes, is equal to $q=\frac{34}{7}$.

In Figure 4 we plot the state variable $q_{j}^{s}[k]$ of every node $v_{j} \in \mathcal{V}$ as a function of the number of iterations $k$ for


Fig. 3. Example of digraph for comparison of Algorithms 1 and 2
the digraph shown in Fig. 3 The plot demonstrates that the proposed distributed algorithms are able to achieve a common quantized consensus value to the average of the initial states after a finite number of iterations.


Fig. 4. Comparison between Algorithm 1 and Algorithm 2 for the digraph shown in Fig. 3 Top figure: Node state variables plotted against the number of iterations for Algorithm 1 Bottom figure: Node state variables plotted against the number of iterations for Algorithm 2

## VII. CONCLUSIONS

We have considered the quantized average consensus problem and presented one randomized and one deterministic distributed averaging algorithm in which the processing, storing and exchange of information between neighboring agents is subject to uniform quantization. We analyzed the operation of the proposed algorithms and established that they will reach quantized consensus after a finite number of iterations.

In the future we plan to investigate the dependence of the graph structure with full and partial mass summation of the initial values. Furthermore, we plan to extend the operation of the proposed algorithm to more realistic cases, such as transmission delays over the communication links and the presence of unreliable links over the communication network.

## REFERENCES

[1] L. Xiao, S. Boyd, and S. Lall, "A scheme for robust distributed sensor fusion based on average consensus," Proceedings of the International Symposium on Information Processing in Sensor Networks, pp. 63-70, April 2005.
[2] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1520-1533, September 2004.
[3] N. Lynch, Distributed Algorithms. San Mateo: CA: Morgan Kaufmann Publishers, 1996.
[4] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," Proceedings of the IEEE Conference on Decision and Control, pp. 2996-3000, 2005.
[5] L. Schenato and G. Gamba, "A distributed consensus protocol for clock synchronization in wireless sensor network," Proceedings of the IEEE Conference on Decision and Control, pp. 2289-2294, 2007.
[6] C. N. Hadjicostis, A. D. Domínguez-García, and T. Charalambous, "Distributed averaging and balancing in network systems, with applications to coordination and control," Foundations and Trends $®$ in Systems and Control, vol. 5, no. 3-4, 2018.
[7] S. Sundaram and C. N. Hadjicostis, "Distributed function calculation and consensus using linear iterative strategies," IEEE Journal on Selected Areas in Communications, vol. 26, no. 4, pp. 650-660, May 2008.
[8] T. Charalambous, Y. Yuan, T. Yang, W. Pan, C. N. Hadjicostis, and M. Johansson, "Decentralised minimum-time average consensus in digraphs," Proceedings of the IEEE Conference on Decision and Control (CDC), pp. 2617-2622, 2013.
[9] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," Systems and Control Letters, vol. 53, no. 1, pp. 65-78, September 2004.
[10] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione, "Gossip algorithms for distributed signal processing," Proceedings of the IEEE, vol. 98, no. 11, pp. 1847-1864, November 2010.
[11] J. Liu, S. Mou, A. S. Morse, B. D. O. Anderson, and C. Yu, "Deterministic gossiping," Proceedings of the IEEE, vol. 99, no. 9, pp. 1505-1524, September 2011.
[12] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, Cambridge, 1984.
[13] T. C. Aysal, M. Coates, and M. Rabbat, "Distributed average consensus using probabilistic quantization," IEEE/SP Workshop on Statistical Signal Processing, pp. 640-644, 2007.
[14] J. Lavaei and R. M. Murray, "Quantized consensus by means of gossip algorithm," IEEE Transactions on Automatic Control, vol. 57, no. 1, pp. 19-32, January 2012.
[15] A. Kashyap, T. Basar, and R. Srikant, "Quantized consensus," Automatica, vol. 43, no. 7, pp. 1192-1203, 2007.
[16] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri, "Average consensus on networks with quantized communication," International Journal on Robust and Nonlinear Control, vol. 19, no. 16, pp. 1787-1816, November 2009.
[17] M. E. Chamie, J. Liu, and T. Basar, "Design and analysis of distributed averaging with quantized communication," IEEE Transactions on Automatic Control, vol. 61, no. 12, pp. 3870-3884, December 2016.
[18] K. Cai and H. Ishii, "Quantized consensus and averaging on gossip digraphs," IEEE Transactions on Automatic Control, vol. 56, no. 9, pp. 2087-2100, September 2011.
[19] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," Automatica, vol. 49, no. 1, pp. 245-252, January 2013.
[20] C. Nowzari and J. Cortés, "Distributed event-triggered coordination for average consensus on weight-balanced digraphs," Automatica, August 2014.
[21] Z. Liu, Z. Chen, and Z. Yuan, "Event-triggered average-consensus of multi-agent systems with weighted and direct topology," Journal of Systems Science and Complexity, vol. 25, no. 5, pp. 845-855, October 2012.
[22] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, "Communication constraints in the average consensus problem," Automatica, vol. 44, no. 3, pp. 671-684, 2008.
[23] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. Tsitsiklis, "On distributed averaging algorithms and quantization effects," IEEE Transactions on Automatic Control, vol. 54, no. 11, pp. 2506-2517, November 2009.
[24] A. I. Rikos, T. Charalambous, and C. N. Hadjicostis, "Distributed weight balancing over digraphs," IEEE Transactions on Control of Network Systems, vol. 1, no. 2, pp. 190-201, June 2014.


[^0]:    The authors are with the Department of Electrical and Computer Engineering at the University of Cyprus, Nicosia, Cyprus. Emails: \{arikos01, chadjic\}@ucy.ac.cy.

[^1]:    ${ }^{1}$ Indirect transmission could involve broadcasting a message to all outneighbors while including in the message header the ID of the out-neighbor it is intended for.
    ${ }^{2}$ From the definition of $\mathcal{B}=\left[b_{l j}\right]$ we have that $b_{j j}=\frac{1}{1+\mathcal{D}_{j}^{+}}, \forall v_{j} \in \mathcal{V}$. This represents the probability that node $v_{j}$ will not perform a transmission to any of its out-neighbors $v_{l} \in \mathcal{N}_{j}^{+}$(i.e., it will transmit to itself).
    ${ }^{3}$ When executing the deterministic protocol, each node $v_{j}$ transmits to its out-neighbors by following a predetermined order. The next time it needs to transmit to an out-neighbor, it will continue from the outgoing edge it stopped the previous time and cycle through the edges in a round-robin fashion according to the predetermined ordering.

