# Robust Kalman Filtering: Asymptotic Analysis of the Least Favorable Model 

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#### Abstract

We consider a robust filtering problem where the robust filter is designed according to the least favorable model belonging to a ball about the nominal model. In this approach, the ball radius specifies the modeling error tolerance and the least favorable model is computed by performing a Riccati-like backward recursion. We show that this recursion converges provided that the tolerance is sufficiently small.


## I. Introduction

Consider the problem of estimating a state process whose state-space model is known only imperfectly. In such a situation the standard Kalman filter may perform poorly. Robust filtering seeks to find a state estimate which takes the model uncertainty into account.

In this paper, we consider the robust filtering approach proposed in [12], see also [11], [8]. The actual state-space model is assumed to belong to a ball centered about the nominal state-space model. The ball is formed by placing a bound on the Kullback-Leibler divergence between the actual and the nominal state-space model, and the ball radius represents the modeling error tolerance. Then, the robust filter is designed according to the least favorable model in the ball. The resulting filter obeys a Kalman-like recursion which makes it very appealing for online applications [16]. Interestingly, if the ball is selected by using the $\tau$-divergence instead of the Kullback-Leibler divergence, the resulting filter still obeys a Kalman-like recursion [18], [20]. In [21], [19] it was shown that when the tolerance is sufficienly small, the robust filter converges. Finally, it worth noting that this robust filter represents a generalization of risk-sensitive filters [15], [1], [13], [9] where large errors are severely penalized by selecting a risk-sensitivity parameter.

It is also important to characterize the least favorable model corresponding to the robust filter because it can be used to evaluate the performance of an arbitrary filter under this least favorable situation. In [12] it was shown that the least favorable model can be computed over a finite interval by first evaluating the robust filter over the interval and then performing a backward recursion to generate the least favorable model dynamics. In this paper, we show that this backward recursion is a Riccati-like equation of the form

$$
X_{t}=A\left(X_{t+1}+R\right)^{-1} A^{T}+Q
$$

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which converges provided the tolerance is sufficiently small. As a consequence, the least favorable model is a state-space model with constant parameters in steady state. The convergence of discrete-time Riccati equations with $R$ positive definite or semi-definite has been studied in detail [3], [4], [2], [7]. But in the equation considered here, $R$ is negative definite, and in this case only a few results are available, see [17].

The outline of the paper is as follows. In Section II we introduce the robust filtering problem, in particular the backward least favorable model recursion. In section III we prove that the recursion converges when the tolerance is sufficiently small. In Section IV we show that the estimation error of an arbitrary stable estimator under the least favorable model is bounded. In Section $\square$ some simulation results are presented. Finally, conclusions are presented in Section VI

In this paper we will use the following notation. $(a, b]$ denotes an interval which is left-open and right-closed. Given a matrix $A \in \mathbb{R}^{n \times n}$, its spectrum is denoted by $\lambda(A)$ and its spectral radius is denoted by $\sigma(A)$. We say that $A$ is (Schur) stable if $\sigma(A)<1 . \mathcal{Q}_{n}$ denotes the vector space of symmetric matrices of dimension $n$. Given $X \in \mathcal{Q}_{n}, X>0$ $(X \geq 0)$ indicates that $X$ is positive definite (semi-definite). Given two functions $f$ and $g, f(x)=o(g(x))$ around $x=\alpha$ means that $\lim _{x \rightarrow \alpha} f(x) / g(x)=0$.

## II. Robust Kalman filtering

Consider a nominal state-space model of the form

$$
\begin{align*}
x_{t+1} & =A x_{t}+B v_{t} \\
y_{t} & =C x_{t}+D v_{t} \tag{1}
\end{align*}
$$

where $x_{t} \in \mathbb{R}^{n}$ is the state process, $y_{t} \in \mathbb{R}^{p}$ the observation process, $v_{t} \in \mathbb{R}^{m}$ is a white Gaussian noise (WGN) with unit variance, i.e. $\mathbb{E}\left[v_{t} v_{s}^{T}\right]=I \delta_{t-s}$ and $\delta_{t}$ denotes the Kronecker delta function. We assume that $v_{t}$ is independent of the initial state vector $x_{0} \sim \mathcal{N}\left(0, P_{0}\right)$, and that the pairs $(A, B)$ and $(A, C)$ are reachable and observable, respectively. Without loss of generality, we assume that $B D^{T}=0$. Indeed, if this is not the case we can always rewrite (1) with $\check{A}=A-A B D^{T}\left(D D^{T}\right)^{-1} C, \check{B}$ such that $\check{B} \check{B}^{T}=B\left(I-D^{T}\left(D D^{T}\right)^{-1} D\right) B^{T}, \check{C}=C$ and $\check{D}=$ $D$. The nominal model (1) is completely characterized by the transition probability density $\phi_{t}\left(x_{t+1}, y_{t} \mid x_{t}\right)$ and by the probability density $f\left(x_{0}\right)$ of $x_{0}$. Let $\tilde{\phi}_{t}\left(x_{t+1}, y_{t} \mid x_{t}\right)$ denote the transition probability density of the actual model. We assume that the actual and nominal densities of initial state $x_{0}$ coincide, whereas $\tilde{\phi}_{t}$ belongs to a ball centered about
$\phi_{t}$ with radius $c>0$, hereafter called tolerance, which is specified by

$$
\begin{equation*}
\mathcal{B}_{t}=\left\{\tilde{\phi}_{t} \text { s.t. } D_{K L}\left(\phi_{t}, \tilde{\phi}_{t}\right) \leq c\right\} \tag{2}
\end{equation*}
$$

Here $D_{K L}$ denotes the Kullback-Leibler divergence [10] between $\phi_{t}$ and $\tilde{\phi}_{t}$. Note that $D_{K L}\left(\phi_{t}, \tilde{\phi}_{t}\right)$ is finite only if matrix $\left[B^{T} D^{T}\right]^{T}$ has full row rank. Accordingly, without loss of generality we assume that $\left[B^{T} D^{T}\right]^{T}$ is square and invertible, so that $m=n+p$. Indeed it is always possible to compress the column space of this matrix and remove the noises which do not affect model (1). Let $Y_{t}=\left\{y_{s}, s \leq t\right\}$ and $g_{t}\left(Y_{t}\right)$ be an estimator of $x_{t+1}$ given $Y_{t}$. Adopting the minimax approach described in [12], a robust estimator of $x_{t+1}$ is obtained by solving:

$$
\begin{equation*}
\hat{x}_{t+1}=\underset{g_{t} \in \mathcal{G}_{t}}{\operatorname{argmin}} \max _{\tilde{\phi}_{t} \in \mathcal{B}_{t}} \tilde{\mathbb{E}}\left[\left\|x_{t+1}-g_{t}\left(Y_{t}\right)\right\|^{2} \mid Y_{t-1}\right] \tag{3}
\end{equation*}
$$

where $\tilde{\mathbb{E}}$ denotes the expectation operator taken with respect to the joint probability density of the actual model and $\mathcal{G}_{t}$ denotes the class of estimators with finite second-order moments with respect to $\tilde{\phi}_{t} \in \mathcal{B}_{t}$. In [12], it was shown that the robust estimator satisfies a Kalman-like recursion of the form

$$
\begin{align*}
G_{t} & =A V_{t} C^{T}\left(C V_{t} C^{T}+D D^{T}\right)^{-1} \\
\hat{x}_{t+1} & =A \hat{x}_{t}+G_{t}\left(y_{t}-C \hat{x}_{t}\right) \\
P_{t+1} & =A\left(V_{t}^{-1}+C^{T}\left(D D^{T}\right)^{-1} C\right)^{-1} A^{T}+B B^{T} \\
V_{t+1} & =\left(P_{t+1}^{-1}-\theta_{t} I\right)^{-1} \tag{4}
\end{align*}
$$

where $\theta_{t}>0$ is the unique solution to the equation $c=$ $\gamma\left(P_{t+1}, \theta_{t}\right)$. The function $\gamma$ is given by

$$
\begin{equation*}
\gamma(\theta, P)=\frac{1}{2}\left[\log \operatorname{det}(I-\theta P)+\operatorname{tr}\left[(I-\theta P)^{-1}\right]-n\right] \tag{5}
\end{equation*}
$$

The initial conditions of the recursion are $\hat{x}_{0}=0$ and $V_{0}=$ $P_{0}$. The least favorable prediction error $e_{t}=x_{t}-\hat{x}_{t}$ of the robust estimator has zero mean and covariance matrix $V_{t}$.

The following result is proved in [21, Proposition 3.5], see also [19].

Proposition 2.1: There exists $c_{M A X}>0$ such that if $c \in$ $\left(0, c_{M A X}\right]$, then for any $P_{0}>0$ the sequence $P_{t}, t \geq 0$, generated by (4) converges to a unique solution $P>0$, $\theta_{t} \rightarrow \theta$ with $\theta>0, V_{t} \rightarrow V$ with $V>0$ and the limit $G$ of the filtering gain $G_{t}$ as $t \rightarrow \infty$ is such that $A-G C$ is stable. Moreover, $P$ is the unique solution of the algebraic Riccati-like equation

$$
\begin{equation*}
P=A\left(P^{-1}-\theta I+C^{T}\left(D D^{T}\right)^{-1} C\right)^{-1} A^{T}+B B^{T} \tag{6}
\end{equation*}
$$

It is possible to show that the least favorable model obtained by solving (3) is given by [12]

$$
\begin{align*}
\xi_{t+1} & =\tilde{A}_{t} \xi_{t}+\tilde{B}_{t} \varepsilon_{t} \\
y_{t} & =\tilde{C}_{t} \xi_{t}+\tilde{D}_{t} \varepsilon_{t} \tag{7}
\end{align*}
$$

where

$$
\begin{align*}
\tilde{A}_{t} & =\left[\begin{array}{cc}
A & B H_{t} \\
0 & A-G_{t} C+\left(B-G_{t} D\right) H_{t}
\end{array}\right] \\
\tilde{B}_{t} & =\left[\begin{array}{c}
B \\
B-G_{t} D
\end{array}\right] L_{t} \\
\tilde{C}_{t} & =\left[\begin{array}{cc}
C & D H_{t}
\end{array}\right], \tilde{D}_{t}=D L_{t} \\
H_{t} & =\tilde{K}_{t}\left(B-G_{t} D\right)^{T}\left(\Omega_{t+1}^{-1}+\theta_{t} I\right)\left(A-G_{t} C\right) \\
\tilde{K}_{t} & =\left[I-\left(B-G_{t} D\right)^{T}\left(\Omega_{t+1}^{-1}+\theta_{t} I\right)\left(B-G_{t} D\right)\right]^{-1} \tag{8}
\end{align*}
$$

and $L_{t}$ is such that $\tilde{K}_{t}=L_{t} L_{t}^{T}$. In this model $\varepsilon_{t}$ is a WGN with unit variance, and $\Omega_{t+1}^{-1}$ is computed by the backward recursion

$$
\begin{align*}
\Omega_{t}^{-1}= & \left(A-G_{t} C\right)^{T}\left[\left(\Omega_{t+1}^{-1}+\theta_{t} I\right)^{-1}-\left(B-G_{t} D\right) \times\right. \\
& \left.\times\left(B-G_{t} D\right)^{T}\right]^{-1}\left(A-G_{t} C\right) \tag{9}
\end{align*}
$$

where if $T$ denotes the simulation horizon, the initial condition is $\Omega_{T}^{-1}=0$.

In summary, the least favorable model (7) is obtained in two steps:

1) The Riccati equation (4) for $P_{t}$ is propagated forward in time over $[0, T]$ and used to compute $G_{t}$ and $\theta_{t}$.
2) The model $\left(\tilde{A}_{t}, \tilde{B}_{t}, \tilde{C}_{t}, \tilde{D}_{t}\right)$ is obtained by propagating (9) backward in time to evaluate $\Omega_{t}^{-1}$ over interval $[0, T]$.
It is clear that the least favorable model depends on the length $T$ of the simulation interval. Let $\alpha, \beta$ such that $0<$ $\alpha<\beta<1$. Then, the interval $[\alpha T, \beta T]$ is contained in $[0, T]$. In the next section we show that when $c>0$ is sufficient small, then $\Omega_{t}^{-1}$ converges over the interval $[\alpha T, \beta T]$ as $T$ tends to infinity. As a consequence, the least favorable model (7) is constant over this interval.

Before establishing the convergence of the backward recursion (9), it is worth considering the limit case $c=0$ when the nominal and the actual models coincide. In this case, the robust filter (4) reduces to the usual Kalman filter and $\theta_{t}=0$ for all $t$. Hence the limit of $\theta_{t}$ is $\theta=0$. By using the matrix inversion lemma, the backward recursion (9) with $\theta_{t}=0$ can be rewritten as

$$
\begin{aligned}
\Omega_{t}^{-1}= & \left(A-G_{t} C\right)^{T}\left[\Omega_{t+1}^{-1}-\Omega_{t+1}^{-1}\left(B-G_{t} D\right) \times\right. \\
& \left.\times S_{t}\left(B-G_{t} D\right)^{T} \Omega_{t+1}^{-1}\right]\left(A-G_{t} C\right)
\end{aligned}
$$

where

$$
S_{t}=\left[\left(B-G_{t} D\right)^{T} \Omega_{t+1}^{-1}\left(B-G_{t} D\right)-I\right]^{-1}
$$

Therefore, if $\Omega_{t+1}^{-1}=0$ then $\Omega_{t}^{-1}=0$. Since $\Omega_{T}^{-1}=0$, we conclude that $\Omega_{t}^{-1}=0$ for all $t \in[0, T]$. Accordingly, $H_{t}=0$ and $L_{t}=I$. Substituting these expressions inside (8), it is then easy to verify that the least favorable model coincides with the nominal model.

## III. Convergence of the backward recursion

Suppose that the condition of Proposition 2.1 is satisfied. Then as $t \rightarrow \infty$ the backward recursion (9) becomes

$$
\begin{equation*}
\Omega_{t}^{-1}=\bar{A}^{T}\left[\left(\Omega_{t+1}^{-1}+\theta I\right)^{-1}-\bar{B} \bar{B}^{T}\right]^{-1} \bar{A} \tag{10}
\end{equation*}
$$

where the matrix $\bar{A}:=A-G C$ is stable, and $\bar{B}:=B-$ $G D$. To ease the exposition, we assume that $T$ is finite and we study the convergence of (10) as $t$ tends to $-\infty$. This is equivalent to studying the convergence in $[\alpha T, \beta T]$ as $T$ tends to $\infty$. Adding $\theta I$ on both sides and defining $X_{t}:=$ $\Omega_{t}^{-1}+\theta I$ yields the equivalent recursion

$$
\begin{equation*}
X_{t}=\bar{A}^{T}\left(X_{t+1}^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A}+\theta I \tag{11}
\end{equation*}
$$

with terminal value $X_{T}=\theta I$. It has the form of a Riccati equation, but an important difference, compared to the standard case, is that in the inverse we add to $X_{t+1}^{-1}$ the negative definite matrix $-\bar{B} \bar{B}^{T}$. This difference makes the convergence analysis nontrivial. At this point, it is useful to introduce the following map defined for $0<X<\left(\bar{B} \bar{B}^{T}\right)^{-1}$

$$
\begin{equation*}
\Theta(X):=\bar{A}^{T}\left(X^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A}+\theta I \tag{12}
\end{equation*}
$$

Note that $\bar{B} \bar{B}^{T}$ is an invertible matrix since

$$
\begin{align*}
\bar{B} \bar{B}^{T} & =(B-G D)(B-G D)^{T} \\
& =B B^{T}+G D D^{T} G^{T} \geq B B^{T} \tag{13}
\end{align*}
$$

where $B B^{T}$ is invertible because $B \in \mathbb{R}^{n \times n+p}$ has full rowrank. Accordingly, the recursion (11) can be rewritten as

$$
\begin{equation*}
X_{t}=\Theta\left(X_{t+1}\right) \tag{14}
\end{equation*}
$$

Proposition 3.1: For any $0<X<\left(\bar{B} \bar{B}^{T}\right)^{-1}$, we have $\Theta(X) \geq \theta I$.

Proof: We have

$$
\begin{equation*}
\Theta(X)-\theta I=\bar{A}^{T}\left(X^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A} \tag{15}
\end{equation*}
$$

where the right hand side is positive semi-definite.
Proposition 3.2: The map $\Theta$ preserves the partial order of positive semi-definite matrices, so if $X_{1}, X_{2}$ are such that $0<X_{1} \leq X_{2}<\left(\bar{B} \bar{B}^{T}\right)^{-1}$, we have

$$
\Theta\left(X_{1}\right) \leq \Theta\left(X_{2}\right)
$$

Proof: The first variation of $\Theta(X)$ along the direction $\delta X \in \mathcal{Q}_{n}$ can be expressed as

$$
\begin{align*}
\delta \Theta(X ; \delta X)= & \bar{A}^{T}\left(X^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} X^{-1} \delta X \times \\
& \times X^{-1}\left(X^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A} \tag{16}
\end{align*}
$$

Thus $\delta \Theta(X ; \delta X) \geq 0$ for any $\delta X \geq 0$, so the map is nondecreasing.

Before stating the next property of $\Theta$, we prove the following lemmas.

Lemma 3.1: It is always possible to select $c \in\left(0, c_{M A X}\right]$ such that $\theta$ is arbitrarily small.

Proof: In [21], [19] it was shown that

$$
\begin{align*}
& \gamma\left(P, \theta_{1}\right)>\gamma\left(P, \theta_{2}\right), \quad \forall \theta_{1}>\theta_{2} \text { s.t. } P \geq 0, P \neq 0  \tag{17}\\
& \gamma\left(P_{1}, \theta\right) \geq \gamma\left(P_{2}, \theta\right), \quad \forall P_{1} \geq P_{2}  \tag{18}\\
& \gamma(P, 0)=0, \quad \forall P \geq 0  \tag{19}\\
& \gamma\left(P,\left[0, \sigma(P)^{-1}\right)\right)=[0, \infty), \quad \forall P>0 \tag{20}
\end{align*}
$$

where (20) means that the image of $\left[0, \sigma(P)^{-1}\right)$ under $\gamma(P, \cdot)$ is $[0, \infty)$. Since $c \in\left(0, c_{M A X}\right]$, by Proposition 2.1 we have that $P_{t} \rightarrow P, c_{t} \rightarrow c, \theta_{t} \rightarrow \theta$ where $c$ and $\theta$ are
related by $c=\gamma(P, \theta)$. Here $P$ solves the algebraic form of Riccati equation (4), so $P \geq B B^{T}$. In view of (17)-(20) it follows that $\theta \leq \tilde{\theta}$ where $\tilde{\theta}$ is the unique solution of equation $c=\gamma\left(B B^{T}, \tilde{\theta}\right)$. Furthermore, the map

$$
\begin{align*}
\mu:\left[0, \sigma\left(B B^{T}\right)^{-1}\right) & \rightarrow[0, \infty) \\
\tilde{\theta} & \mapsto \gamma\left(B B^{T}, \tilde{\theta}\right) \tag{21}
\end{align*}
$$

is injective and continuous. Accordingly, the inverse map $\mu^{-1}:[0, \infty) \rightarrow\left[0, \sigma\left(B B^{T}\right)^{-1}\right)$ exists and is continuous, in particular $\mu^{-1}(0)=0$. This means that we can always select $c>0$ such that $\tilde{\theta}$ is arbitrarily small. Since $\theta \leq \tilde{\theta}$, the statement follows.
It is worth noting that $\bar{A}$ and $\bar{B}$ depend on $c$ through $\theta$. Throughout the paper we make the following assumption.

Assumption 1: The map

$$
\begin{gather*}
\gamma:[0, \check{\theta}] \rightarrow \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \\
\theta \mapsto(\bar{A}, \bar{B}) \tag{22}
\end{gather*}
$$

is continuous for $\check{\theta}$ sufficiently small.
Even though Assumption 1 may appear restrictive, it holds under mild conditions on system $(A, B, C, D)$. Indeed, for $c \in\left(0, c_{M A X}\right]$ the unique solution of (6) is $P=X Y^{-1}$ where $\left[X^{T} Y^{T}\right]^{T}$ spans the stable deflating subspace of regular matrix pencil $s L-M$ [14], where
$L=\left[\begin{array}{cc}A^{T} & 0 \\ -B B^{T} & I\end{array}\right], \quad M=\left[\begin{array}{cc}I & C^{T}\left(D D^{T}\right)^{-1} C-\theta I \\ 0 & A\end{array}\right]$.
Conditions for the continuity of such subspaces are given in [6]. Accordingly, the map $\theta \mapsto P$ is continuous over $[0 \check{\theta}]$ with $\check{\theta}$ small enough. Since the map $P \mapsto(\bar{A}, \bar{B})$ is continuous, we conclude that $\gamma$ is continuous for $\theta$ sufficiently small.

Lemma 3.2: For $c \in\left(0, c_{M A X}\right]$ sufficiently small, there exists $\rho \in\left(1, \sigma(\bar{A})^{-1}\right)$ such that

$$
\begin{equation*}
\left(1-\rho^{-2}\right) \Sigma_{q}^{-1}-\bar{B} \bar{B}^{T} \geq 0 \tag{23}
\end{equation*}
$$

where $\Sigma_{\rho}$ is the unique solution of the algebraic Lyapunov equation (ALE)

$$
\begin{equation*}
\Sigma_{\rho}=\rho^{2} \bar{A}^{T} \Sigma_{\rho} \bar{A}+\theta I \tag{24}
\end{equation*}
$$

Proof: First, note that $\rho \bar{A}$ is a stable matrix. Then, the solution of (24) is given by

$$
\begin{equation*}
\Sigma_{\rho}=\theta \sum_{k \geq 0} \rho^{2 k}\left(\bar{A}^{T}\right)^{k} \bar{A}^{k} \tag{25}
\end{equation*}
$$

which is positive definite. Note that

$$
\Sigma_{\rho} \leq \theta \sum_{k \geq 0} \rho^{2 k} \sigma(\bar{A})^{2 k} I=\frac{\theta}{1-\rho^{2} \sigma(\bar{A})^{2}} I
$$

and thus $\Sigma_{\rho}^{-1} \geq\left(1-\rho^{2} \sigma(\bar{A})^{2}\right) / \theta I$. In view of Assumption 1 , for $\theta$ sufficiently small we have

$$
\sigma(\bar{A})^{2}=\sigma\left(\bar{A}_{0}\right)^{2}+o(1)
$$

where $\bar{A}_{0}=A-G_{0} C, G_{0}=A P^{(0)} C^{T}\left(C P^{(0)} C^{T}+\right.$ $\left.D D^{T}\right)^{-1}$ and $P^{(0)}$ is the unique solution of (6) with $\theta=0$. As a consequence,

$$
\begin{equation*}
\left(1-\rho^{-2}\right) \Sigma_{\rho}^{-1} \geq\left(1-\rho^{-2}\right) \frac{1-\rho^{2}\left(\sigma\left(\bar{A}_{0}\right)+o(1)\right)^{2}}{\theta} I \tag{26}
\end{equation*}
$$

We can always choose $\rho$ in the range $\left(1, \sigma\left(\bar{A}_{0}\right)^{-1}\right)$ such that $\left(1-\rho^{-2}\right)\left(1-\rho^{2} \sigma^{2}\left(\bar{A}_{0}\right)\right)$ is positive. By Lemma3.1, we can also select $c \in\left[0, c_{M A X}\right]$ sufficiently small so that $\theta$ is small enough that the scaled identity matrix on the right hand side of (26) upper bounds $\bar{B} \bar{B}^{T}$.

Let $\bar{c} \in\left(0, c_{M A X}\right]$ be a value of $c$ such that Lemma 3.2 is satisfied, so that (23) holds for a certain $\rho$ and $\theta$. Then it is useful to observe that for any $c \in(0, \bar{c})$, the equation (23) still holds with the same value for $\rho$ but with a smaller value for $\theta$.

Corollary 3.1: For any $c \in(0, \bar{c}]$, we have $\Sigma_{\rho}<$ $\left(\bar{B} \bar{B}^{T}\right)^{-1}$.

Proof: Since (23) holds for a suitable $\rho>0$, we have

$$
\Sigma_{\rho}^{-1} \geq \rho^{-2} \Sigma_{\rho}^{-1}+\bar{B} \bar{B}^{T}>\bar{B} \bar{B}^{T}
$$

which implies $\Sigma_{\rho}<\left(\bar{B} \bar{B}^{T}\right)^{-1}$.
We are now ready to state the third property of the map $\Theta$.

Proposition 3.3: Consider the compact set

$$
\mathcal{C}=\left\{X \in \mathcal{Q}_{n} \text { s.t. } \theta I \leq X \leq \Sigma_{\rho}\right\}
$$

where $\Sigma_{\rho}$ is computed as in Lemma 3.2. If $c \in(0, \bar{c}]$ then $\Theta(X) \in \mathcal{C}$ for any $X \in \mathcal{C}$.

Proof: First, observe that $\mathcal{C}$ is a nonempty set. Indeed, by (25) we have $\Sigma_{\rho} \geq \theta I$, so that $\theta I \in \mathcal{C}$. Since $c \in$ ( $0, \bar{c}$ ], by Lemma 3.2 the inequality (23) holds for some $\rho \in\left(1, \sigma(\bar{A})^{-1}\right)$, and thus

$$
\begin{align*}
& \Sigma_{\rho}^{-1}-\bar{B} \bar{B}^{T} \geq \rho^{-2} \Sigma_{\rho}^{-1} \\
& \left(\Sigma_{\rho}^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \leq \rho^{2} \Sigma_{\rho} \\
& \bar{A}^{T}\left(\Sigma_{\rho}^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A}+\theta I \leq \rho^{2} \bar{A}^{T} \Sigma_{\rho} \bar{A}+\theta I \\
& \Theta\left(\Sigma_{\rho}\right) \leq \Sigma_{\rho} \tag{27}
\end{align*}
$$

Assume that $X \in \mathcal{C}$. Since $X \leq \Sigma_{\rho}$, the nondecreasing property of $\Theta$ and (27) imply

$$
\Theta(X) \leq \Theta\left(\Sigma_{\rho}\right) \leq \Sigma_{\rho}
$$

Since $X \geq \theta I$, we have

$$
\begin{equation*}
\Theta(X) \geq \Theta(\theta I) \geq \theta I \tag{28}
\end{equation*}
$$

where we exploited again the nondecreasing property of $\Theta$ and Proposition 3.1 We conclude that $\Theta(X) \in \mathcal{C}$.

Proposition 3.4: Consider the sequence $X_{t}$ satisfying the backward recursion

$$
\begin{equation*}
X_{t}=\Theta\left(X_{t+1}\right), \quad X_{T}=\theta I \tag{29}
\end{equation*}
$$

For $c \in(0, \bar{c}]$, the sequence belongs to $\mathcal{C}$ and is nondecreasing. Thus as $t \rightarrow-\infty, X_{t}$ converges to $X \in \mathcal{C}$ which is a solution of the algebraic Riccati equation

$$
\begin{equation*}
X=\bar{A}^{T}\left(X^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \bar{A}+\theta I \tag{30}
\end{equation*}
$$

Proof: We prove the first two statements by induction. We start by showing that $X_{t} \in \mathcal{C}$ for any $t$. We know that $X_{T} \in \mathcal{C}$ because $\mathcal{C}$ contains $\theta I$. Assume that $X_{t+1} \in \mathcal{C}$, then Proposition 3.3 implies that $X_{t}=\Theta\left(X_{t+1}\right) \in \mathcal{C}$. This proves the first claim.

Next we show that the sequence is nondecreasing. We observe that

$$
\begin{equation*}
X_{T-1}=\Theta\left(X_{T}\right)=\Theta(\theta I) \geq \theta I=X_{T} \tag{31}
\end{equation*}
$$

where we exploited the nondecreasing property of $\Theta$, see Propositions 3.2 and 3.1 Assume that $X_{t} \geq X_{t+1}$, then

$$
\begin{equation*}
X_{t-1}=\Theta\left(X_{t}\right) \geq \Theta\left(X_{t+1}\right)=X_{t} \tag{32}
\end{equation*}
$$

so by induction the sequence is nondecreasing.
The convergence follows from the fact that the sequence is nondecreasing and belongs to a compact set.

Since $X_{t}=\Omega_{t}^{-1}+\theta I$, we have the following result.
Corollary 3.2: For $c \in(0, \bar{c}]$, the sequence $\Omega_{t}^{-1}$ generated by (10) converges to $\Omega^{-1}$ as $t \rightarrow-\infty$ where $\Omega^{-1}$ is such that $0 \leq \Omega^{-1} \leq \Sigma_{\rho}-\theta I$ for some $\rho \in\left(1, \sigma(\bar{A})^{-1}\right)$ satisfying (23). Furthermore

$$
\begin{align*}
& H_{t} \rightarrow H, \quad \tilde{K}_{t} \rightarrow \tilde{K}, \quad L_{t} \rightarrow L \\
& \tilde{A}_{t} \rightarrow \tilde{A}, \quad \tilde{B}_{t} \rightarrow \tilde{B} \\
& \tilde{C}_{t} \rightarrow \tilde{C}, \quad \tilde{D}_{t} \rightarrow \tilde{D} \tag{33}
\end{align*}
$$

It is worth noting that the algebraic equation (30) may admit several positive definite solutions. Indeed, in the scalar case, equation (30) becomes

$$
\begin{equation*}
x=\frac{\bar{a}^{2}}{x^{-1}-\bar{b}^{2}}+\theta \tag{34}
\end{equation*}
$$

or equivalently

$$
\bar{b}^{2} x^{2}-\left(1-\bar{a}^{2}+\bar{b}^{2} \theta\right) x+\theta=0
$$

For small $\theta>0$, the discriminant of this equation is positive, so the equation has two positive real solutions since the coefficient $1-\bar{a}^{2}-\bar{b}^{2} \theta$ is positive. For $\bar{a}=0.1, \bar{b}=1$ and $\theta=0.1$ we obtain the two solutions $x_{1} \approx 0.99$ and $x_{2} \approx 0.10$. It is not difficult to see that (34) can be rewritten as a Lyapunov equation

$$
\begin{equation*}
x=(\bar{a}-j \bar{b})^{2} x+\bar{b}^{2}-j^{2} \tag{35}
\end{equation*}
$$

where $j=\bar{a} x \bar{b} /\left(\bar{b}^{2} x-1\right)$. Let $f:=\bar{a}-j \bar{b}$ be the "feedback" matrix and $f_{1}, f_{2}$ denote the values corresponding to $x_{1}$ and $x_{2}$, respectively. Then we have $f_{1} \approx 8.9$ and $f_{2} \approx 0.11$. In view of (35), this means that $x_{1}$ is a stabilizing solution of (11) whereas $x_{2}$ corresponds to an unstable one. Accordingly, the limit of the sequence (29) is $x_{2}$. In the general case (i.e., for $n>1$ ) the algebraic Riccati equation (30) can be rewritten as

$$
X=\left(\bar{A}-\bar{B} J^{T}\right)^{T} X\left(\bar{A}-\bar{B} J^{T}\right)+\bar{B} \bar{B}^{T}-J J^{T}
$$

where $J=\bar{A}^{T} X \bar{B}\left(\bar{B} X \bar{B}^{T}-I\right)^{-1}$. However, the reasoning used in the scalar case cannot be applied since the matrix $\bar{B} \bar{B}^{T}-J J^{T}$ is indefinite.

Proposition 3.5: For $c \in(0, \bar{c}]$ sufficiently small, the limit $X$ of (29) is a stabilizing solution of (30) in the sense that the matrix $\bar{A}^{T}-J \bar{B}^{T}$ is stable.

Proof: Let $X_{\theta}$ be the limit of the sequence in (29) where we made explicit its dependence on $\theta$. Notice that $\rho$ does not depend on $\theta$. Indeed, if a certain $\rho$ satisfies (23) for a given $\theta$, then the same $\rho$ satisfies (23) with $\theta^{\prime}$ such that $0<\theta^{\prime} \leq \theta$. Since $X_{\theta} \in \mathcal{C}$, we have that $\theta I \leq X_{\theta} \leq \theta \sum_{k>0} \rho^{2 k}\left(\bar{A}^{T}\right)^{k} \bar{A}^{k}$. Let $Q_{\theta}$ be such that $X_{\theta}=\theta Q_{\theta}$. Hence $Q_{\theta} \geq I$. Observe that

$$
\begin{align*}
M_{\theta} & :=\bar{A}^{T}-J \bar{B}^{T} \\
& =\bar{A}^{T}\left[X_{\theta}-X_{\theta} \bar{B}\left(\bar{B} X_{\theta} \bar{B}^{T}-I\right)^{-1} \bar{B}^{T} X_{\theta}\right] X_{\theta}^{-1} \\
& =\bar{A}^{T}\left(X_{\theta}^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} X_{\theta}^{-1} \\
& =\bar{A}^{T}\left(\theta^{-1} Q_{\theta}^{-1}-\bar{B} \bar{B}^{T}\right)^{-1} \theta^{-1} Q_{\theta}^{-1} \\
& =\bar{A}^{T}\left(Q_{\theta}^{-1}-\theta \bar{B} \bar{B}^{T}\right)^{-1} Q_{\theta}^{-1} . \tag{36}
\end{align*}
$$

For $\theta$ sufficiently small, by Assumption 1 we have $\bar{B} \bar{B}^{T}=$ $\bar{B}_{0} \bar{B}_{0}^{T}+o(1)$ where $\bar{B}_{0}=B-G_{0} D$ and $G_{0}$ has been defined in the proof of Lemma 3.2. Accordingly,

$$
\begin{equation*}
\left(Q_{\theta}^{-1}-\theta \bar{B} \bar{B}^{T}\right)^{-1}=Q_{\theta}+o(1) \tag{37}
\end{equation*}
$$

which after substitution inside (36) gives

$$
\begin{equation*}
M_{\theta}=\bar{A}^{T}+o(1) . \tag{38}
\end{equation*}
$$

The map $\theta \mapsto \lambda\left(M_{\theta}\right)$ is a continuous function for $\theta>0$ since the mapping from the entries of a matrix to its spectrum is continuous. Hence for $\theta$ sufficiently small, the matrix $M_{\theta}$ is stable. By Lemma 3.1 we conclude that if we select $c \in$ ( $\left.0, c_{M A X}\right]$ sufficiently small, the matrix $M_{\theta}$ will be stable.

## IV. PERFORMANCE ANALYSIS

We want to evaluate the performance of an arbitrary estimator

$$
\begin{equation*}
\hat{x}_{t+1}^{\prime}=A \hat{x}_{t}^{\prime}+G_{t}^{\prime}\left(y_{t}-C \hat{x}_{t}^{\prime}\right) \tag{39}
\end{equation*}
$$

under the least favorable model (7) in steady state, i.e. with $\tilde{A}_{t}, \tilde{B}_{t}, \tilde{C}_{t}$ and $\tilde{D}_{t}$ constant. Note that the steady state condition is guaranteed under the assumption that $c \in(0, \bar{c}]$. Recall that $e_{t}$ denotes the least favorable prediction error of the robust filter (4). Let $e_{t}^{\prime}=x_{t}-\hat{x}_{t}^{\prime}$ be the prediction error of filter (39). Let $\mathbf{e}_{t}=\left[e_{t}^{\prime T} e_{t}^{T}\right]^{T}$. In [12] it was shown that the dynamics of $\mathbf{e}_{t}$ are given by

$$
\begin{equation*}
\mathbf{e}_{t+1}=F_{t} \mathbf{e}_{t}+M_{t} \varepsilon_{t} \tag{40}
\end{equation*}
$$

where

$$
F_{t}:=\tilde{A}-\left[\begin{array}{c}
G_{t}^{\prime} \\
0
\end{array}\right] \tilde{C}, \quad M_{t}:=\quad \tilde{B}-\left[\begin{array}{c}
G_{t}^{\prime} \\
0
\end{array}\right] \tilde{D}
$$

and $\varepsilon_{t}$ is a WGN with unit variance. Then the covariance matrix $\Pi_{t}$ of $\mathbf{e}_{t}$ obeys the Lyapunov equation

$$
\begin{equation*}
\Pi_{t+1}=F_{t} \Pi_{t} F_{t}^{T}+M_{t} M_{t}^{T} \tag{41}
\end{equation*}
$$

with initial condition $\Pi_{0}=I_{2} \otimes V_{0}$.

From (40) it is clear that the mean of the prediction error $e_{t}^{\prime}$ is zero. Next, we show that the covariance matrix of $e_{t}^{\prime}$ converges to a constant matrix and is bounded provided that $c$ is sufficiently small. To do so, we use the following result [5, Theorem 1].

Lemma 4.1: Consider the time-varying Lyapunov equation

$$
Y_{t+1}=\mathcal{F}_{t} Y_{t} \mathcal{F}_{t}^{T}+\mathcal{R}_{t}
$$

where $\mathcal{F}_{t}$ and $\mathcal{R}_{t}$ converges to $\mathcal{F}$ and $\mathcal{R}$, respectively, as $t \rightarrow \infty$ with $\mathcal{F}$ stable. Then $Y_{t}$ converges to the unique solution $Y$ of the Lyapunov equation:

$$
Y=\mathcal{F} Y \mathcal{F}^{T}+\mathcal{R}
$$

Proposition 4.1: Assume that the gain $G_{t}^{\prime}$ in (39) converges to a matrix $G^{\prime}$ such that $A-G^{\prime} C$ is stable. Then, for $c \in(0, \bar{c}]$ sufficiently small the recursion (41) converges to the solution $\Pi$ of the Lyapunov equation

$$
\Pi=F \Pi F^{T}+M M^{T}
$$

where

$$
F:=\tilde{A}-\left[\begin{array}{c}
G^{\prime} \\
0
\end{array}\right] \tilde{C}, \quad M:=\left(\tilde{B}-\left[\begin{array}{c}
G^{\prime} \\
0
\end{array}\right] \tilde{D}\right)
$$

Proof: First, we prove that the matrix

$$
F=\left[\begin{array}{cc}
A-G^{\prime} C & \left(B-G^{\prime} D\right) H  \tag{42}\\
0 & A-G C+(B-G D) H
\end{array}\right]
$$

is stable. Since $F$ is an upper block-triangular matrix, it is sufficient to show that its two diagonal blocks are stable. The matrix $A-G^{\prime} C$ is stable by assumption. Next, by recalling that $\bar{A}=A-G C, \bar{B}=B-G D, H=\tilde{K} \bar{B}^{T} X \bar{A}$ and $\tilde{K}=\left(I-\bar{B}^{T} X \bar{B}\right)^{-1}$, the $(2,2)$ block of $F$ can be expressed as

$$
\begin{align*}
\bar{A} & +\bar{B}\left(I-\bar{B} X \bar{B}^{T}\right)^{-1} \bar{B}^{T} X \bar{A} \\
& =\bar{A}-\bar{B}\left(\bar{B} X \bar{B}^{T}-I\right)^{-1} \bar{B}^{T} X \bar{A} \\
& =\bar{A}-\bar{B} J^{T} \tag{43}
\end{align*}
$$

which has the same eigenvalues of $\bar{A}^{T}-J \bar{B}^{T}$. By Proposition 3.5 this matrix is stable provided that $c$ is sufficiently small. The conditions of Proposition 4.1 are satisfied since $F_{t}$ converges to $F$ with $F$ stable and $M_{t}$ converges to $M$, and thus $M_{t} M_{t}^{T}$ converges to $M M^{T}$ as $t \rightarrow \infty$. Hence $\Pi_{t}$ converges to $\Pi$.

Corollary 4.1: Under the assumption that $c \in(0, \bar{c}]$ is sufficiently small, the prediction error $e_{t}^{\prime}$ of the filter (39) under the least favorable model (in steady state) has zero mean and bounded variance.

## V. Simulation Example

Consider the state-space model

$$
\begin{align*}
& A=\left[\begin{array}{cc}
0.1 & 1 \\
0 & 1.2
\end{array}\right], B=0.01 I_{2} \\
& C=\left[\begin{array}{cc}
1 & -1
\end{array}\right], D=0.04 \tag{44}
\end{align*}
$$



Fig. 1. Minimum eigenvalue of matrix $\left(1-\rho^{-2}\right) \Sigma_{\rho}^{-1}-\bar{B} \bar{B}^{T}$ as a function of $\rho$ for $c=c_{M A X}$.

Note that the pairs $(A, B)$ and $(A, C)$ are reachable and observable, respectively. Using the procedure of [21, Proposition 3.5], it results that the robust filter (4) converges for $c \in\left(0, c_{M A X}\right]$, with $c_{M A X}=0.1879$.

The minimum eigenvalue of $\left(1-\rho^{-2}\right) \Sigma_{\rho}^{-1}-\bar{B} \bar{B}^{T}$ is depicted in Fig. 1 as a function of $\rho$ for $c=c_{M A X}$. For $c=c_{M A X}$, we see that when $\rho=1.382$, the minimum eigenvalue is $4.02 \cdot 10^{-5}$, so the matrix is positive definite and $\bar{c}=c_{M A X}$. Consider the sequence generated by (9) for $c=c_{M A X}$. We have

$$
\Sigma_{\rho} \approx 10^{2} \cdot\left[\begin{array}{cc}
5.89 & -5.03 \\
-5.03 & 4.31
\end{array}\right]
$$

and iteration (9) converges to

$$
\Omega^{-1} \approx 10^{2} \cdot\left[\begin{array}{cc}
4.56 & -3.90 \\
-3.90 & 3.34
\end{array}\right]
$$

Furthermore, the matrix $\bar{A}^{T}-J \bar{B}^{T}$ has for eigenvalues $0.8373,0.0892$, so it is stable. Finally, Figures 2 and 3 depict the variances of the first and second component of prediction error of the Kalman filter and robust filter Kalman for the steady-state least favorable model. As expected, both variances converge to a constant value and for both components, the performance of the robust filter is approximately 1.5 dB lower than that of the Kalman filter.

## VI. Conclusion

We have considered a robust filtering problem, where the minimum variance estimator is designed according to the least favorable model belonging to a ball about the nominal model and with a certain radius corresponding to the modeling tolerance. We showed that as long as the model tolerance does not exceed a maximum value $\bar{c}$, the least favorable model converges to a constant model. Furthermore, as long as the tolerance is sufficiently small, the covariance matrix of the prediction error for any stable filter remains bounded when applied to the steady-state least favorable model.


Fig. 2. Variance (in decibel) of the first component of the prediction error of the Kalman and robust filters for the least favorable model.


Fig. 3. Variance (in decibel) of the second component of the prediction error of the Kalman and robust filters for the least favorable model.

## REFERENCES

[1] R. N. Banavar and J. L. Speyer. Properties of risk-sensitive filters/estimators. IEE Proc.-Control Theory Appl., 145, January 1998.
[2] R. R. Bitmead, M. R. Gevers, I. R. Petersen, and R. J. Kaye. Monotonicity and stabilizability properties of solutions of the Riccati difference equation: Propositions, lemmas, theorems, fallacious conjectures and counterexamples. Systems and Control Letters, 5:309315, 1985.
[3] P. E. Caines and D. Q. Mayne. On the discrete time matrix Riccati equation of optimal control. Int. J. Control, 12(5):785-794, 1970.
[4] P. E. Caines and D. Q. Mayne. On the discrete time matrix Riccati equation of optimal control- a correction. Int. J. Control, 14(1):205207, 1971.
[5] F. S. Cativelli and A. H. Sayed. Diffusion strategies for distributed Kalman filtering and smoothing. IEEE Trans. Automat. Control, 55(9):2069-2084, 2010.
[6] J. W. Demmel and B. Kagstrom. Computing stable eigendecompositions of matrix pencils. Linear Algebra and its Appl., 88/89:139-186, 1987.
[7] A. Ferrante and L. Ntogramatzidis. The generalized discrete algebraic Riccati equation in linear-quadratic optimal control. Automatica, 49:471-478, 2013.
[8] L. P. Hansen and T. J. Sargent. Robustness. Princeton University Press, Princeton, NJ, 2008.
[9] B. Hassibi, A. H. Sayed, and T. Kailath. Indefinite-Quadratic Estimation and Control- A Unified Approach to $H^{2}$ and $H^{\infty}$ Theories. Soc. Indust. Appl. Math., Philadelphia, 1999.
[10] S. Kullback. Information Theory and Statistics. J. Wiley \& Sons, New York, 1959. Reprinted by Dover Publ., Mineola, NY, 1968.
[11] B. C. Levy and R. Nikoukhah. Robust least-squares estimation with a relative entropy constraint. IEEE Trans. Informat. Theory, 50(1):89104, January 2004.
[12] B. C. Levy and R. Nikoukhah. Robust state-space filtering under incremental model perturbations subject to a relative entropy tolerance. IEEE Trans. Automat. Control, 58:682-695, March 2013.
[13] B. C. Levy and M. Zorzi. A contraction analysis of the convergence of risk-sensitive filters. SIAM J. Control and Optimiz., 54(4):2154-2173, 2016.
[14] T. Pappas, A. J. Laub, and N. R. Sandell. On the numerical solution of the discrete-time algebraic Riccati equation. IEEE Trans. Automat. Control, 25:631-641, 1980.
[15] P. Whittle. Risk-sensitive Optimal Control. J. Wiley, Chichester, England, 1980.
[16] A. Zenere and M. Zorzi. Model predictive control meets robust kalman filtering. IFAC PapersOnLine, 50(1):3774-3779, 2017.
[17] Y. Zhou. Convergence of the discrete-time matrix Riccati equation with indefinite weighting matrix. In Proc. 33rd IEEE Conference on Decision and Control (CDC), pages 1517-1518, Lake Buena Vista, FL, December 1994.
[18] M. Zorzi. Robust Kalman filtering under model perturbations. IEEE Trans. Automat. Control, 62(6):2902-2907, June 2016.
[19] M. Zorzi. Convergence analysis of a family of robust Kalman filters based on the contraction principle. SIAM J. Control and Optimiz., 55(5):3116-3131, 2017.
[20] M. Zorzi. On the robustness of the Bayes and Wiener estimators under model uncertainty. preprint, arXiv:1508.01904v2, May 2017.
[21] M. Zorzi and B. C. Levy. On the convergence of a risk sensitive like filter. In Proc. 54th IEEE Conference on Decision and Control (CDC), pages 4990-4995, Osaka, Japan, December 2015.

