

Transient frequency control with regional cooperation for power networks

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Abstract—This paper proposes a centralized and a distributed sub-optimal control strategy to maintain in safe regions the real-time transient frequencies of a given collection of buses, and simultaneously preserve asymptotic stability of the entire network. In a receding horizon fashion, the centralized control input is obtained by iteratively solving an open-loop optimization aiming to minimize the aggregate control effort over controllers regulated on individual buses with transient frequency and stability constraints. Due to the non-convexity of the optimization, we propose a convexification technique by identifying a reference control input trajectory. We then extend the centralized control to a distributed scheme, where each sub-controller can only access the state information within a local region. Simulations on a IEEE-39 network illustrate our results.

I. INTRODUCTION

Power network transient stability refers to the ability of an electric power network to remain synchronized after disturbances, during which system states should stay within safe bounds so that the entire system remains physically intact [1]. Due to the dynamics and interconnection of power networks, even if the power supply and consumption are re-balanced immediately after a failure, individual generators are still in the danger of overheating due to large transient frequency or voltage deviations, which in turn may trigger cascading failures. In practice, it is also common to treat the transient frequencies of some crucial generators as a key metric for evaluating system performance or as indexes for applying load-shedding strategies [2]. These considerations motivate us to design a frequency controller that mitigates the frequency overshoot observed in transients, and at the same time, preserves synchronization of the whole system.

Literature review: A body of work [3], [4] studies how network synchronization relates to factors such as network topology, parameter values, initial conditions, and power supply-demand balance. However, there is no guarantee that transient frequencies of individual buses do not exceed their physical limits, and thus, synchronization may not necessarily hold under transient frequency constraints. To improve transient behavior, various strategies have been proposed including power re-dispatch [5], power system stabilizer (PSS) [6], and virtual inertial placement [7]. Nonetheless, these strategies do not rigorously ensure that transient state stay within a safe region. Our previous work combines Lyapunov stability analysis and barrier function to propose a distributed controller [8] that simultaneously guarantees both synchronization and transient frequency safety. On the other hand, to account for the trade-offs between performance and control effort, [9], [10] investigate (distributed)

model predictive control (MPC) for networked system. The work [10] treats each subsystem as an independent system by considering the effect of interconnections as bounded uncertainties, resulting in a conservative approach to establishing stability. The work [9] shows that each subsystem having no knowledge of others' cost functions [11] leads to a non-cooperative game, and the control input trajectory may even diverge. In addition, to maintain the distributed nature of MPC, the predicted horizon is limited to a single step [9], [12] to restrict information sharing. As the horizon increases, the distributed control could require global information.

Statement of contribution: We develop a distributed MPC framework that meets the following requirements on system performance, control cost, and control structure: (i) all bus frequencies converge to the same (potentially unknown) frequency; (ii) for each bus of interest, if its initial frequency belongs to a desired safe frequency region, then its frequency trajectory stays in it for all subsequent time; (iii) for each region of the network, sub-controllers within it cooperatively achieve requirements (i) and (ii) by reducing their overall control efforts; and (iv) each sub-controller can only access system information within its underlying region. To achieve these goals, we start from considering the entire network as one region and design a centralized controller satisfying (i)-(iii). First, we consider an open-loop finite-horizon optimal control problem, aiming to minimize the overall accumulated control cost (to reflect requirement (iii)) under two hard constraints corresponding to requirements (i)-(ii). Due to the non-convex and non-smooth nature of the optimization problem, we then propose a convexification technique to obtain a sub-optimal control input trajectory. To close the loop, for each state, its control input is defined as the first step of the sub-optimal input trajectory. We show that the closed-loop system satisfies (i)-(ii). Finally, to achieve a distributed control structure, we divide the network into several regions and separately consider every region as a network. The distributed controller for each region is nothing but the centralized controller implemented on it. By carefully taking into account the power flow interconnections among the regions, we show that the closed-loop system also meets requirements (i)-(ii) under the distributed controller. For reasons of space, all proofs are omitted and will appear elsewhere.

II. PRELIMINARIES

We introduce notation and notions from graph theory.

1) *Notation:* Let \mathbb{N} , \mathbb{R} , $\mathbb{R}_{>}$, and \mathbb{R}_{\geq} denote the set of natural, real, positive, and nonnegative real numbers, respectively. Variables are assumed to belong to the Euclidean space if not specified otherwise. Denote $\mathbf{1}_n$ and $\mathbf{0}_n$ in \mathbb{R}^n as the vector of all ones and zeros, respectively. For $a \in \mathbb{R}$, denote $\lceil a \rceil$ as the biggest integer no bigger than a . We let $\|\cdot\|$ denote the

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2-norm on \mathbb{R}^n . For a vector $b \in \mathbb{R}^n$, denote b_i as its i th entry. For $A \in \mathbb{R}^{m \times n}$, let $[A]_i$ and $[A]_{i,j}$ denote its i th row and (i, j) th element. For any $c, d \in \mathbb{N}$, let $[c, d]_{\mathbb{N}} = \{x \in \mathbb{N} \mid c \leq x \leq d\}$.

2) *Algebraic graph theory*: We follow basic notions in algebraic graph theory from [13], [14]. An undirected graph is a pair $\mathcal{G} = (\mathcal{S}, \mathcal{E})$, where $\mathcal{S} = \{1, \dots, n\}$ is the vertex set and $\mathcal{E} = \{e_1, \dots, e_m\} \subseteq \mathcal{S} \times \mathcal{S}$ is the edge set. A path is an ordered sequence of vertices such that any pair of consecutive vertices in the sequence is an edge of the graph. A graph is connected if there exists a path between any two vertices. Two nodes are neighbors if there exists an edge linking them. Denote $\mathcal{N}(i)$ as the set of neighbors of node i . For each edge $e_k \in \mathcal{E}$ with vertices i, j , the orientation procedure consists of choosing either i or j to be the positive end of e_k and the other vertex to be the negative end. The incidence matrix $D = (d_{ki}) \in \mathbb{R}^{m \times n}$ associated with \mathcal{G} is defined as $d_{ki} = 1$ if i is the positive end of e_k , $d_{ki} = -1$ if i is the negative end of e_k , and $d_{ki} = 0$ otherwise. An induced subgraph $\mathcal{G}_\beta = (\mathcal{S}_\beta, \mathcal{E}_\beta)$ of the undirected graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ satisfies $\mathcal{S}_\beta \subseteq \mathcal{S}$, $\mathcal{E}_\beta \subseteq \mathcal{E}$, and $(i, j) \in \mathcal{E}_\beta$ if $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{S}_\beta$. Additionally, for each \mathcal{G}_β , let $\mathcal{E}'_\beta \subseteq \mathcal{S}_\beta \times (\mathcal{S} \setminus \mathcal{S}_\beta)$ be the collection of edges connecting \mathcal{G}_β and the rest of the network.

III. PROBLEM STATEMENT

In this section we introduce the model for the power network dynamics and state the control goals.

A. Power network model

The power network is described by a connected undirected graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$, where $\mathcal{S} = \{1, 2, \dots, n\}$ is the collection of buses and $\mathcal{E} = \{e_1, e_2, \dots, e_m\} \subseteq \mathcal{S} \times \mathcal{S}$ is the collection of transmission lines. For each node $i \in \mathcal{S}$, let $\omega_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ denote the shifted voltage frequency relative to the nominal frequency, and active power injection at node i , respectively. Given an arbitrary orientation on \mathcal{G} , for any edge with positive end i and negative end j , denote f_{ij} as its signed power flow. We partition buses into \mathcal{S}^u and $\mathcal{S} \setminus \mathcal{S}^u$, depending on whether an external control input is available to regulate the transient frequency behavior for some buses. The linearized power network dynamics [15] described by states of power flows and frequencies is

$$\dot{f}_{ij}(t) = b_{ij}(\omega_i(t) - \omega_j(t)), \quad \forall (i, j) \in \mathcal{E} \quad (1a)$$

$$M_i \dot{\omega}_i(t) = -E_i \omega_i(t) + q_i(t) + p_i(t) + u_i(t), \quad \forall i \in \mathcal{S}^u, \quad (1b)$$

$$M_i \dot{\omega}_i(t) = -E_i \omega_i(t) + q_i(t) + p_i(t), \quad \forall i \in \mathcal{S} \setminus \mathcal{S}^u, \quad (1c)$$

$$q_i(t) \triangleq \sum_{j:j \rightarrow i} f_{ji}(t) - \sum_{k:i \rightarrow k} f_{ik}(t), \quad (1d)$$

where $b_{ij} \in \mathbb{R}_{>}$ is the susceptance of the line connecting bus i and j , and $M_i \in \mathbb{R}_{\geq}$ and $E_i \in \mathbb{R}_{\geq}$ are the inertia and damping coefficient of bus $i \in \mathcal{S}$. For simplicity, we assume that they are strictly positive for every $i \in \mathcal{S}$. The term q_i stands for the aggregated electrical power injected to node i from its neighboring nodes, where $\{j: j \rightarrow i\}$ is the shorthand notation for $\{j: j \in \mathcal{N}(i) \text{ and } j \text{ is the positive end of } (i, j)\}$.

For convenience, let $f \in \mathbb{R}^m$ and $\omega \in \mathbb{R}^n$ denote the collection of f_{ij} and ω_i , respectively. Define $p \in \mathbb{R}^n$ as the collection of all p_i 's. Let $Y_b \in \mathbb{R}^{m \times m}$ be the diagonal matrix whose k th diagonal item represents the susceptance of the

transmission line e_k connecting bus i and j , i.e., $[Y_b]_{k,k} = b_{ij}$, for $k = 1, 2, \dots, m$. Let $M \triangleq \text{diag}(M_1, M_2, \dots, M_n) \in \mathbb{R}^{n \times n}$, $E \triangleq \text{diag}(E_1, E_2, \dots, E_n) \in \mathbb{R}^{n \times n}$, and $D \in \mathbb{R}^{m \times n}$ be the incidence matrix corresponding to the given orientation procedure. We re-write system (1) in compact form as

$$\dot{f}(t) = Y_b D \omega(t), \quad (2a)$$

$$M \dot{\omega}(t) = -E \omega(t) - D^T f(t) + p(t) + u(t), \quad (2b)$$

$$u(t) \in \mathbb{U} \triangleq \left\{ u \in \mathbb{R}^n \mid \forall w \in [1, n]_{\mathbb{N}}, [u]_w = \begin{cases} u_w & \text{if } w \in \mathcal{S}^u \\ 0 & \text{otherwise} \end{cases} \right\}.$$

For convenience, we use $x \triangleq (f, \omega) \in \mathbb{R}^{m+n}$ to denote the collection of all states. We consider power injections that satisfy the following assumption.

Assumption 3.1: (Finite-time convergence of active power injection). For each $i \in \mathcal{S}$, p_i is piecewise continuous and becomes a constant (denoted by p_i^*) after a finite time, i.e., there exists $0 \leq \bar{t} < \infty$ such that $p_i(t) = p_i^*$ for every $i \in \mathcal{S}$ and every $t \geq \bar{t}$.

This type of power injections generalizes the common constant injection assumption considered in the literature, e.g. [16], [17]. Under Assumption 3.1, one can show that [8] for system (2) with $u \equiv \mathbf{0}_n$, $(f(t), \omega(t))$ globally converges to the unique equilibrium point $(f_\infty, \omega_\infty \mathbf{1}_n)$ determined by power injection and network parameters, where ω_∞ is called synchronized frequency.

B. Control goal

Our goal is to design distributed state-feedback control strategy for each bus $i \in \mathcal{S}^u$ that ensure that the frequency of buses of a given targeted subset of \mathcal{S}^u stays within a safe bound. Furthermore, we require that the designed controller preserve the stability of the whole system. The above requirements are explicitly formalized as follows:

1) *Frequency invariance requirement*: Given $\mathcal{S}^\omega \subseteq \mathcal{S}^u$, for each $i \in \mathcal{S}^\omega$, let $\underline{\omega}_i \in \mathbb{R}$ and $\bar{\omega}_i \in \mathbb{R}$ be such that $\underline{\omega}_i < \bar{\omega}_i$. We require that $\omega_i(t)$ stays inside the safe region $[\underline{\omega}_i, \bar{\omega}_i]$ for any time $t > 0$, provided that the initial frequency $\omega_i(0)$ lies inside $[\underline{\omega}_i, \bar{\omega}_i]$.

2) *Asymptotic stability requirement*: Since the open-loop system state globally converges to $(f_\infty, \omega_\infty \mathbf{1}_n)$, we require that our controller only affects the system's transients such that the closed-loop also converges to the same equilibrium.

3) *Economic coordination requirement*: The controller u should achieve the above two requirements by having its sub-controller u_i for $i \in \mathcal{S}^u$ cooperate with others to reduce the overall control effort.

4) *Distributed feedback realization*: Each sub-controller can only use state and power injection information within a limited local region. This requirement makes the control implementable for large-scale power networks, as each sub-controller does not depend on global information.

Typically, the set \mathcal{S}^ω consists of generator nodes with over/underfrequency requirements, or nodes whose transient frequency behaviors play a fundamental role in evaluating system performance [2]. We have shown in [18] that for every node $i \in \mathcal{S}^\omega$, to guarantee its frequency invariance, an external control signal has to be available at node i , i.e.,

\mathcal{S}^ω must be a subset of \mathcal{S}^u . Nodes in $\mathcal{S}^u \setminus \mathcal{S}^\omega$, having no frequency requirement of their own, assist nodes in \mathcal{S}^ω to achieve frequency invariance.

To meet the above four requirements, our strategy is to first formulate an open-loop finite-horizon optimal control problem aiming to minimize the overall control effort, and at the same time, having two hard constraints reflecting frequency invariance and stability requirements, respectively. We then employ this formulation to design a centralized feedback controller, and finally employ graph decomposition to synthesize a distributed controller.

IV. OPEN-LOOP OPTIMAL CONTROL

In this section, we formulate the optimization problem of interest. Our goal is to minimize some cost function measuring control input effort, subject to system dynamics, and frequency invariance and asymptotic stability constraints. As the last constraints turn out to be non-convex and non-smooth, we propose a convexification strategy by generating a set of linear constraints. We build on this section later to design the centralized and the distributed controllers.

A. Open-loop finite-horizon optimal control

We here first introduce a robust asymptotic stability condition with respect to the open-loop equilibrium point.

Lemma 4.1: (Robust asymptotic stability condition). For system (2), suppose that the solution exists and is unique. For every $i \in \mathcal{S}^u$, let $\bar{\omega}_i^{\text{thr}} \in \mathbb{R}_{>}$ and $\underline{\omega}_i^{\text{thr}} \in \mathbb{R}_{<}$ be threshold values satisfying $\underline{\omega}_i^{\text{thr}} < \omega_\infty < \bar{\omega}_i^{\text{thr}}$. If for every $t \in \mathbb{R}_{\geq}$,

$$\omega_i(t)u_i(x(t), p(t)) \leq 0, \text{ if } \omega_i(t) \notin (\underline{\omega}_i^{\text{thr}}, \bar{\omega}_i^{\text{thr}}), \quad (3a)$$

$$u_i(x(t), p(t)) = 0, \text{ if } \omega_i(t) \in (\underline{\omega}_i^{\text{thr}}, \bar{\omega}_i^{\text{thr}}), \quad (3b)$$

then under Assumption 3.1, $(f(t), \omega(t)) \rightarrow (f_\infty, \omega_\infty)$ as $t \rightarrow \infty$. Furthermore, if $p(t)$ is time-invariant, then the closed-loop system is globally asymptotically stable.

Notice that the dependence of the robust asymptotic stability condition (3) on the equilibrium point $(f_\infty, \omega_\infty \mathbf{1}_n)$ is weak, as it requires neither any information regarding f_∞ , nor a priori knowledge of the exact value of ω_∞ . This reflects a practical consideration under which the controller should still ensure convergence and stability: although ideally ω_∞ is 0 when the load and supply are perfectly balanced (i.e., $\sum_{i=1}^n p_i^* = 0$), due to imperfect estimation on the load side and transmission loss, ω_∞ tends to slightly deviate from 0.

With the stability condition being set, we now are ready to formally introduce the finite-horizon optimal control problem, in which we aim to minimize some cost function of control effort over a finite time while respecting system dynamics, robust stability condition (3), as well as the frequency invariance condition. As the power injection p may not be precisely known a priori, instead, for every $t \in \mathbb{R}_{\geq}$, let a piecewise continuous signal $p_i^{\text{fcst}} : [t, t + \tilde{t}] \rightarrow \mathbb{R}^n$ be its forecasted value for the first \tilde{t} seconds, starting from t . We adopt the following assumption on the forecast.

Assumption 4.2: (Forecast reveals current true value at frequency-controlled nodes). For every $i \in \mathcal{S}^\omega$ and every $t \in \mathbb{R}_{\geq}$, assume that $p_{i,i}^{\text{fcst}}(t) = p_i(t)$, where $p_{i,i}^{\text{fcst}}(t)$ is the i th component of $p_i^{\text{fcst}}(t)$.

A simple way to meet Assumption 4.2 is to, at every $i \in \mathcal{S}^\omega$, first measure the power injection $p_i(t)$ at the current time t , and then let $p_{i,i}^{\text{fcst}}(\tau) = p_i(t)$ for every $\tau \in [t, t + \tilde{t}]$.

The optimization problem corresponding to the open-loop finite-horizon optimal control is as follows,

$$\min_{f, \omega, u} \sum_{i \in \mathcal{S}^u} \int_{\tau_0}^{\tau_0 + \tilde{t}} c_i u_i^2(\tau) d\tau$$

$$\text{s.t. } \dot{f}(\tau) = Y_b D \omega(\tau), \quad (4a)$$

$$M \dot{\omega}(\tau) = -E \omega(\tau) - D^T f(\tau) + p_\tau^{\text{fcst}}(\tau) + u(\tau), \quad (4b)$$

$$f(\tau_0) = f_0, \omega(\tau_0) = \omega_0, \quad (4c)$$

$$u(\tau) \in \mathbb{U}, \quad \forall \tau \in [\tau_0, \tau_0 + \tilde{t}], \quad (4d)$$

$$\underline{\omega}_i \leq \omega_i(\tau) \leq \bar{\omega}_i, \quad \forall i \in \mathcal{S}^\omega, \forall \tau \in [\tau_0, \tau_0 + \tilde{t}], \quad (4e)$$

$$(\omega, u) \in \Phi_{\text{cont}}, \quad (4f)$$

where for every $i \in \mathcal{S}^u$, $c_i \in \mathbb{R}_{>}$ corresponds to the cost weight for u_i ; constraints (4a)-(4c) represent system dynamics and initial state; constraint (4d) reflects available controlled bus indexes; constraint (4e) refers to the frequency invariance requirement, and

$$\Phi_{\text{cont}} \triangleq \{(\omega, u) \mid (3) \text{ holds } \forall t \in [\tau_0, \tau_0 + \tilde{t}], \forall i \in \mathcal{S}^u\}$$

refers to the stability condition established in Lemma 4.1.

We refer to the optimization problem (4) as $Q_{\text{cont}}(\mathcal{G}, \mathcal{S}^u, \mathcal{S}^\omega, p_\tau^{\text{fcst}}, f_0, \omega_0, \tau_0)$ to emphasize its dependence on the graph topology, controlled node indexes, transient-frequency-constrained node indexes, power injection, initial state, and initial time. If the context is clear, we may just denote it as Q_{cont} for brevity. We use the same notational logic for optimization problems defined along the paper. In addition, as we consider the hard frequency constraint (4e), we assume $(f_0, \omega_0) \in \Gamma$, where

$$\Gamma \triangleq \{(f, \omega) \mid \omega_i \leq \omega_i \leq \bar{\omega}_i, \forall i \in \mathcal{S}^\omega\}, \quad (5)$$

so that the problem is well-defined.

In practice, a convenient way to approximate the functional solution for Q_{cont} is by discretization. Specially, here we discretize the system periodically with time length $T \in \mathbb{R}_{>}$, and denote $N \triangleq \lceil \tilde{t}/T \rceil$ as the total number of steps. For every $k \in [0, N]_{\mathbb{N}}$, denote $\hat{f}(k), \hat{\omega}(k), \hat{u}(k), \hat{p}^{\text{fcst}}(k)$ as the approximation of $f(\tau_0 + kT), \omega(\tau_0 + kT), u(\tau_0 + kT)$ and $p_\tau^{\text{fcst}}(\tau_0 + kT)$, respectively, and let

$$\hat{F} \triangleq [\hat{f}(0), \hat{f}(1), \dots, \hat{f}(N)], \quad (6a)$$

$$\hat{\Omega} \triangleq [\hat{\omega}(0), \hat{\omega}(1), \dots, \hat{\omega}(N)], \quad (6b)$$

$$\hat{P}^{\text{fcst}} \triangleq [\hat{p}^{\text{fcst}}(0), \hat{p}^{\text{fcst}}(1), \dots, \hat{p}^{\text{fcst}}(N-1)], \quad (6c)$$

$$\hat{U} \triangleq [\hat{u}(0), \hat{u}(1), \dots, \hat{u}(N-1)], \quad (6d)$$

be the collection of power flow, frequency, predicted power injection, and control input discrete trajectories, respectively. We formulate the discrete version of Q_{cont} as follows,

$$\min_{\hat{F}, \hat{\Omega}, \hat{U}} g(\hat{U}) \triangleq \sum_{i \in \mathcal{S}^u} \sum_{k=0}^{N-1} c_i \hat{u}_i^2(k)$$

$$\text{s.t. } \hat{f}(k+1) = \hat{f}(k) + T Y_b D \hat{\omega}(k),$$

$$M\hat{\omega}(k+1) = M\hat{\omega}(k) + T\{-E\hat{\omega}(k) - D^T\hat{f}(k) + \hat{p}^{fst}(k) + \hat{u}(k)\}, \forall k \in [0, N-1]_{\mathbb{N}}, \quad (7a)$$

$$\hat{f}(0) = f_0, \quad \hat{\omega}(0) = \omega_0, \quad (7b)$$

$$\hat{u}(k) \in \mathbb{U}, \quad \forall k \in [0, N-1]_{\mathbb{N}}, \quad (7c)$$

$$\underline{\omega}_i \leq \hat{\omega}_i(k+1) \leq \bar{\omega}_i, \quad \forall i \in \mathcal{I}^\omega, \quad \forall k \in [0, N-1]_{\mathbb{N}}, \quad (7d)$$

$$(\hat{\Omega}, \hat{U}) \in \Phi_{disc}, \quad (7e)$$

where

$$\Phi_{disc} \triangleq \left\{ (\hat{\Omega}, \hat{U}) \mid \forall i \in \mathcal{I}^u, \quad \forall k \in [0, N-1]_{\mathbb{N}}, \text{ it holds that } \right. \\ \left. \begin{aligned} \hat{\omega}_i(k)\hat{u}_i(k) &\leq 0, \text{ if } \hat{\omega}_i(k) \notin (\underline{\omega}_i^{thr}, \bar{\omega}_i^{thr}), \\ \hat{u}_i(k) &= 0, \text{ if } \hat{\omega}_i(k) \in (\underline{\omega}_i^{thr}, \bar{\omega}_i^{thr}) \end{aligned} \right\}. \quad (8)$$

We refer to (7) as $Q_{disc}(\mathcal{G}, \mathcal{I}^u, \mathcal{I}^\omega, \hat{p}^{fst}, f_0, \omega_0, \tau_0)$.

B. Constraint convexification

From constraint (7e), one can see that the major problem solving Q_{disc} is to deal with the nonlinear and non-smooth feasible set Φ_{disc} . To this end, we propose a convexification method that seeks to identify a subset of Φ_{disc} consisting of only linear constraints. This method relies on the notion of reference trajectory, which is simply a trajectory $(\hat{F}, \hat{\Omega}, \hat{U})$ that satisfies (7). The following result formally states the convexification method using a reference trajectory.

Lemma 4.3: (Convexification of nonlinear constraints). For any reference trajectory $(\hat{F}^{ref}, \hat{\Omega}^{ref}, \hat{U}^{ref})$, let

$$\Phi_{cvx} \triangleq \left\{ (\hat{\Omega}, \hat{U}) \mid \forall i \in \mathcal{I}^u, \quad \forall k \in [0, N-1]_{\mathbb{N}}, \text{ it holds that } \right. \\ \left. \begin{aligned} \hat{\omega}_i(k) &\geq \bar{\omega}_i^{thr}, \quad \hat{u}_i(k) \leq 0, \text{ if } \hat{\omega}_i^{ref}(k) \geq \bar{\omega}_i^{thr}; \\ \hat{\omega}_i(k) &\leq \underline{\omega}_i^{thr}, \quad \hat{u}_i(k) \geq 0, \text{ if } \hat{\omega}_i^{ref}(k) \leq \underline{\omega}_i^{thr}; \\ \hat{u}_i(k) &= 0, \text{ if } \underline{\omega}_i^{thr} < \hat{\omega}_i^{ref}(k) < \bar{\omega}_i^{thr} \end{aligned} \right\}. \quad (9)$$

Then, Φ_{cvx} is convex and non-empty, and $\Phi_{cvx} \subseteq \Phi_{disc}$.

In light of Lemma 4.3, instead of directly solving Q_{disc} and given a reference trajectory, we alternatively solve its convexified version by replacing Φ_{disc} by Φ_{cvx} as follows,

$$\begin{aligned} \min_{\hat{F}, \hat{\Omega}, \hat{U}} \quad & g(\hat{U}) \\ \text{s.t.} \quad & (7a) - (7d) \text{ hold}, \end{aligned} \quad (10a)$$

$$(\hat{\Omega}, \hat{U}) \in \Phi_{cvx}. \quad (10b)$$

We refer to (10) as $Q_{cvx}(\mathcal{G}, \mathcal{I}^u, \mathcal{I}^\omega, \hat{p}^{fst}, f_0, \omega_0, \tau_0)$.

C. Reference trajectory generation

We see that the key problem of the convexification is to find a suitable reference trajectory to approximate Φ_{disc} characterized by nonlinear constraints by Φ_{cvx} containing only linear constraints. Based on our previous work [8], next we construct a specific reference trajectory.

Proposition 4.4: (Generate reference trajectory). For every $i \in \mathcal{I}^u$ and every $k \in [0, N-1]_{\mathbb{N}}$, suppose $\underline{\omega}_i < \underline{\omega}_i^{thr} < \omega_\infty < \bar{\omega}_i^{thr} < \bar{\omega}_i$, and $\bar{\gamma}_i, \underline{\gamma}_i \in \mathbb{R}_{>}$. Define \hat{u}_i^{ref} in (11) and let \hat{u}^{ref} be the collection of \hat{u}_i^{ref} over i . Define $\hat{U}^{ref} \triangleq [\hat{u}^{ref}(0), \hat{u}^{ref}(1), \dots, \hat{u}^{ref}(N-1)]$, and $(\hat{F}^{ref}, \hat{\Omega}^{ref})$ be the state trajectory uniquely determined by (7a) and (7b) using \hat{U}^{ref} as

input. If $\underline{\omega}_i \leq \hat{\omega}_i(0) \leq \bar{\omega}_i$ holds for every $i \in \mathcal{I}^\omega$, then there exists $\bar{T} \in \mathbb{R}_{>}$ such that for any $0 < T \leq \bar{T}$, $(\hat{F}^{ref}, \hat{\Omega}^{ref}, \hat{U}^{ref})$ is a reference trajectory.

From here on, we employ the specific reference trajectory defined in Proposition 4.4 in the convexification method. Notice that a small sampling length T reduces the discretization gap between Q_{cont} and Q_{disc} , as well as guarantees the qualification of $(\hat{F}^{ref}, \hat{\Omega}^{ref}, \hat{U}^{ref})$ defined in Proposition 4.4 as a reference trajectory. On the other hand, the number of constraints appearing in Q_{cvx} grows linearly with respect to $1/T$. Hence, it is of interest to understand the trade-offs among the discretization accuracy, reference trajectory qualification, and computational complexity.

V. FROM CENTRALIZED TO DISTRIBUTED CLOSED-LOOP RECEDING HORIZON FEEDBACK

In this section we close the loop on the system by defining the input at a given state $(f(t), \omega(t))$ at time t with a forecasted power injection p_i^{fst} as the first step of the optimal control input trajectory of $Q_{cvx}(\mathcal{G}, \mathcal{I}^u, \mathcal{I}^\omega, \hat{p}^{fst}, f(t), \omega(t), t)$. We first consider a centralized control strategy, where we assume that a single operator gathers global state information, computes the control law, and broadcasts it to corresponding sub-controllers. Based on this, we then propose a distributed control strategy.

A. Centralized control with stability and frequency invariance constraints

Formally, at time t , our centralized controller on one hand measures the current state $(f(t), \omega(t))$, and on the other, forecasts a power injection profile $p_i^{fst}(\tau)$ with $\tau \in [t, t + \bar{t}]$ as well as its corresponding discretization \hat{p}^{fst} (cf. (6c)). Let $(\hat{F}_{cvx}^*, \hat{\Omega}_{cvx}^*, \hat{U}_{cvx}^*)$ be the optimal solution of $Q_{cvx}(\mathcal{G}, \mathcal{I}^u, \mathcal{I}^\omega, \hat{p}^{fst}, f(t), \omega(t), t)$. The centralized control law is then given by

$$u(x(t), p_i^{fst}) \triangleq \hat{u}_{cvx}^*(0), \quad (12)$$

where $\hat{u}_{cvx}^*(0)$ is the first column of \hat{U}_{cvx}^* .

The following result states that the controller is able to guarantee frequency invariance and, meanwhile, stabilize the system without changing its open-loop equilibrium point.

Theorem 5.1: (Centralized control with stability and frequency invariance constraints). Given power injection p and any initial state $(f(0), \omega(0)) \in \Gamma$, under Assumption 3.1 and with sufficiently small sampling period T , the closed-loop system (2) with controller (12) satisfies:

- (i) $(f(t), \omega(t)) \rightarrow (f_\infty, \omega_\infty \mathbf{1}_n)$ as $t \rightarrow \infty$. Furthermore, if $p(t)$ is time-invariant, then the closed-loop system is asymptotically stable.
- (ii) For any $i \in \mathcal{I}^u$ and any $t \in \mathbb{R}_{\geq}$, $u_i(x(t), p_i^{fst}) = 0$ if $\omega_i(t) \in (\underline{\omega}_i^{thr}, \bar{\omega}_i^{thr})$.
- (iii) $u(x(t), p_i^{fst})$ converges to $\mathbf{0}_{|\mathcal{I}^u|}$ within a finite time.
- (iv) Further under Assumption 4.2, for any $t \in \mathbb{R}_{\geq}$ and every $i \in \mathcal{I}^\omega$, it holds $\omega_i(t) \in [\underline{\omega}_i, \bar{\omega}_i]$.

Note that to compute the centralized control signal defined in (12), the operator should complete the following procedures at every time: a) collect state information and forecast power injection of the entire network, b) determine

$$\hat{u}_i^{\text{ref}}(k) \triangleq \begin{cases} \min\{0, \frac{\gamma_i(\bar{\omega}_i - \hat{\omega}_i^{\text{ref}}(k))}{\bar{\omega}_i^{\text{ref}}(k) - \bar{\omega}_i^{\text{thr}}} - v_i(k)\} & \text{if } \hat{\omega}_i^{\text{ref}}(k) \geq \bar{\omega}_i^{\text{thr}}, \\ 0 & \text{if } \underline{\omega}_i^{\text{thr}} < \hat{\omega}_i^{\text{ref}}(k) < \bar{\omega}_i^{\text{thr}}, \\ \max\{0, \frac{\gamma_i(\underline{\omega}_i - \hat{\omega}_i^{\text{ref}}(k))}{\underline{\omega}_i^{\text{thr}} - \hat{\omega}_i^{\text{ref}}(k)} - v_i(k)\} & \text{if } \hat{\omega}_i^{\text{ref}}(k) \leq \underline{\omega}_i^{\text{thr}}, \end{cases} \quad \forall i \in \mathcal{I}^\omega, \forall k \in [0, N-1]_{\mathbb{N}}, \quad (11)$$

$$\hat{u}_i^{\text{ref}}(k) \triangleq 0, \quad \forall i \in \mathcal{I}^u \setminus \mathcal{I}^\omega, \forall k \in [0, N-1]_{\mathbb{N}},$$

$$v_i(k) \triangleq \sum_{j:j \rightarrow i} \hat{f}_{ji}^{\text{ref}}(k) - \sum_{l:i \rightarrow l} \hat{f}_{il}^{\text{ref}}(k) + \hat{p}_i^{\text{fcst}}(k) - E_i \hat{\omega}_i^{\text{ref}}(k), \quad \forall i \in \mathcal{I}^\omega, \forall k \in [0, N-1]_{\mathbb{N}}.$$

the optimal trajectory \hat{U}_{cvx}^* by solving Q_{cvx} , and c) broadcast the control signals to the corresponding sub-controllers. Since the time to complete any of these three steps grows with respect to the size of the network, it is impractical to implement it for large-scale power networks.

B. Distributed control using regional information

Here we describe our approach to design a distributed control strategy that retains the advantages of sub-controller cooperation with stability and frequency invariance constraints. The idea is to divide the network into smaller regions, and have each sub-controller make decisions based on the state and power injection prediction information within its region.

Assumption 5.2: (Controlled nodes in induced subgraphs). Let $\mathcal{G}_\beta = (\mathcal{S}_\beta, \mathcal{E}_\beta)$, $\beta \in [1, d]_{\mathbb{N}}$ be induced subgraphs of \mathcal{G} . Suppose

$$\mathcal{I}^u \subseteq \bigcup_{\beta=1}^d \mathcal{S}_\beta, \quad (13a)$$

$$\mathcal{I}_\alpha \cap \mathcal{I}_\beta \cap \mathcal{I}^u = \emptyset, \quad \forall \alpha, \beta \in [1, d]_{\mathbb{N}} \text{ with } \alpha \neq \beta. \quad (13b)$$

The induced subgraphs represent the regions of the network (note that each controlled node is contained in one and only one region). Our distributed control strategy is to implement the centralized control for every induced subgraph \mathcal{G}_β , where for every $(i, j) \in \mathcal{E}_\beta'$, i.e., line connecting \mathcal{G}_β and the rest of the network, we treat its power flow $f_{ij}(\tau)$ as an external power injection whose forecasted value is a constant equaling its current value $f_{ij}(t)$ for every $\tau \in [t, t + \tilde{t}]$. Formally, denote for every $i \in \mathcal{S}_\beta$,

$$p_{t,\beta,i}^{\text{fcst},f}(\tau) \triangleq \sum_{\substack{j \rightarrow i \\ (i,j) \in \mathcal{E}_\beta'}} f_{ij}(t) - \sum_{\substack{i \rightarrow j \\ (i,j) \in \mathcal{E}_\beta'}} f_{ij}(t), \quad \forall \tau \in [t, t + \tilde{t}], \quad (14)$$

as the forecasted (starting from the current time t) power flow from transmission lines in \mathcal{E}_β' injecting into node i . Let $p_{t,\beta}^{\text{fcst},f} : [t, t + \tilde{t}] \rightarrow \mathbb{R}^{|\mathcal{S}_\beta|}$ be the collection of all such $p_{t,\beta,i}^{\text{fcst},f}$'s with $i \in \mathcal{S}_\beta$. Also, let $p_{t,\beta}^{\text{fcst}} : [t, t + \tilde{t}] \rightarrow \mathbb{R}^{|\mathcal{S}_\beta|}$ be the collection of all $p_{t,i}^{\text{fcst}}$'s with $i \in \mathcal{S}_\beta$, and denote $p_{t,\beta}^{\text{fcst},o} \triangleq p_{t,\beta}^{\text{fcst},f} + p_{t,\beta}^{\text{fcst}}$ as the overall forecasted power injection for \mathcal{G}_β . Denote $\hat{p}_\beta^{\text{fcst},o}$ as its discretization. Define $\mathcal{I}_\beta^u \triangleq \mathcal{I}^u \cap \mathcal{S}_\beta$ (resp. $\mathcal{I}_\beta^\omega \triangleq \mathcal{I}^\omega \cap \mathcal{S}_\beta$) as the collection of nodes within \mathcal{G}_β with available sub-controllers (resp. with frequency constraints). Let $(f_\beta, \omega_\beta) \in \mathbb{R}^{|\mathcal{S}_\beta| + |\mathcal{E}_\beta' |}$ be the collection of state within \mathcal{G}_β .

We are now ready to define the distributed control law. Similar to (12), let $(\hat{F}_{\text{cvx},\beta}^*, \hat{\Omega}_{\text{cvx},\beta}^*, \hat{U}_{\text{cvx},\beta}^*)$ be the optimal solu-

tion of $Q_{\text{cvx}}(\mathcal{G}_\beta, \mathcal{I}_\beta^u, \mathcal{G}_\beta, \hat{p}_\beta^{\text{fcst},o}, f_\beta(t), \omega_\beta(t), t)$. The control law is then given by

$$u_i(x(t), p_t^{\text{fcst}}) \triangleq \hat{u}_{i,\text{cvx},\beta}^*(0), \quad \forall i \in \mathcal{I}^u. \quad (15)$$

where $u_{i,\text{cvx},\beta}^*(0)$ is the i th entry of $u_{\text{cvx},\beta}^*(0)$, which is the first column of $\hat{U}_{\text{cvx},\beta}^*$. Note that, for any given $i \in \mathcal{I}^u$, $u_i(x(t), p_t^{\text{fcst}})$ only requires state and forecasted power injection information within the corresponding induced subgraph \mathcal{G}_β and \mathcal{E}_β' : there is no need for neither communication between any two induced subgraphs nor a priori knowledge on topology or parameters of any other induced subgraphs.

We next state the properties of the control strategy.

Proposition 5.3: (Distributed control with stability and frequency invariance constraints). Given power injection p and initial state $(f(0), \omega(0)) \in \Gamma$, under Assumptions 3.1 and 5.2, with sufficiently small sampling period T , the following holds for system (2) with controller (15):

- (i) $(f(t), \omega(t)) \rightarrow (f_\infty, \omega_\infty \mathbf{1}_n)$ as $t \rightarrow \infty$. Furthermore, if $p(t)$ is time-invariant, then the closed-loop system is asymptotically stable.
- (ii) For any $i \in \mathcal{I}^u$ and any $t \in \mathbb{R}_{\geq}$, $u_i(x(t), p_t^{\text{fcst}}) = 0$ if $\omega_i(t) \in (\underline{\omega}_i^{\text{thr}}, \bar{\omega}_i^{\text{thr}})$.
- (iii) $u_i(x(t), p_t^{\text{fcst}})$'s converges to $\mathbf{0}_{|\mathcal{I}^u|}$ within a finite time.
- (iv) Further, under Assumption 4.2, for any $t \in \mathbb{R}_{\geq}$ and every $i \in \mathcal{I}^\omega$, it holds $\omega_i(t) \in [\underline{\omega}_i, \bar{\omega}_i]$.

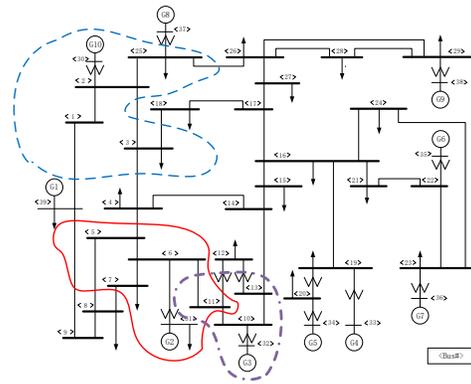


Fig. 1. IEEE 39-bus power network.

VI. SIMULATIONS

We illustrate the performance of the distributed controller in the IEEE 39-bus power network displayed in Fig. 1. The network consists of 46 transmission lines and 10 generators, serving a load of approximately 6GW. We take the values of susceptance b_{ij} and rotational inertia M_i for generator nodes from the Power System Toolbox [19]. We also use

this toolbox to assign the initial power injection $p_i(0)$ for every bus. We assign all non-generator buses a uniform small inertia $M_i = 0.1$. Let the damping parameter be $D_i = 1$ for all buses. The initial state $(f(0), \omega(0))$ is chosen to be the equilibrium with respect to the initial power injections. Let $\mathcal{S}^\omega = \{30, 31, 32\}$ be the three generators with transient frequency requirements. We assign each of them a region containing its 2-hop neighbors. Let $\mathcal{S}^u = \{3, 7, 25, 30, 31, 32\}$ be the collection of nodal indexes with sub-controllers. Notice that Assumption 5.2 holds in this

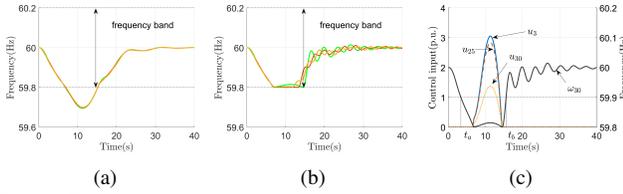


Fig. 2. Frequency and control input trajectories with and without distributed frequency controller (15).

scenario. To set up the optimization problem Q_{cvx} so as to define our controller (15), for every $i \in \mathcal{S}^u$, we set $\tilde{\gamma}_i = \gamma_i = 1$ required in (11), $c_i = 2$ if $i \in \mathcal{S}^\omega$ and $c_i = 1$ if $i \in \mathcal{S}^u \setminus \mathcal{S}^\omega$, $T = 0.001s$, $N = 200$ so that the predicted time horizon $\tilde{t} = 0.2s$. Let $\bar{\omega}_i = -\underline{\omega}_i = 0.2\text{Hz}$ and $\bar{\omega}_i^{\text{thr}} = -\underline{\omega}_i^{\text{thr}} = 0.1\text{Hz}$. The nominal frequency is 60Hz, and hence the safe frequency region is $[59.8\text{Hz}, 60.2\text{Hz}]$. We assume $p_i^{fst}(\tau) = p(\tau)$ for every $\tau \in [t, t + \tilde{t}]$ for simplicity.

We show that the proposed controller is able to maintain the targeted generator frequencies within the safe region, provided that these frequencies are initially in the safe region. We perturb all non-generator nodes by a sinusoidal power injection whose magnitude is proportional to the corresponding node's initial power injection. Specifically, for every $i \in \{1, 2, \dots, 29\}$, let $p_i(t) = (1 + \delta(t))p_i(0)$, where

$$\delta(t) = \begin{cases} 0 & \text{if } t \leq 0.5 \text{ or } t \geq 15.5, \\ 0.3 \sin(\pi/15(t - 0.5)) & \text{otherwise.} \end{cases} \quad (16)$$

For $i \in \{30, 31, \dots, 39\}$, let $p_i(t) \equiv p_i(0)$. Fig. 2(a) shows the open-loop frequency responses of the 3 generators without the controller, where all three trajectories exceed the lower bound around 8s. As a comparison, Fig. 2(b) shows the closed-loop response with the distributed controller, where all frequencies stay within the safe bounds and converge to 60Hz. Fig. 2(c) shows responses in the left-top region in Fig. 1 (similar results hold for the other two regions). Notice that all three control signals vanish within 20s. Since we assign a higher cost weight on u_{30} , and the same weight on u_{25} and u_3 , the latter two possess a similar trajectory, with magnitude higher than the first one. On the other hand, notice that u_{30} is always 0 while ω_{30} is above the lower frequency threshold denoted by the dashed line. All these observations are consistent with Proposition 5.3.

VII. CONCLUSIONS

We have proposed a centralized and a distributed frequency control on power networks to maintain bus transient

frequencies of interest within given safe frequency intervals. We have shown that the closed-loop system preserves the equilibrium point and convergence properties from the open-loop system, and the control input vanishes in finite time. Furthermore, in the distributed control framework, each sub-controller only requires regional information for feedback, and sub-controllers within a same region cooperatively achieve stability and frequency invariance by reducing the overall cost. Future work will investigate the extension to nonlinear power flow models and the incorporation of optimization-based and real-time control to reduce the computational time for controller implementation.

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