# Random Fixed Points, Limits and Systemic risk 

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#### Abstract

We consider vector fixed point (FP) equations in large dimensional spaces involving random variables, and study their realization-wise solutions. We have an underlying directed random graph, that defines the connections between various components of the FP equations. Existence of an edge between nodes $i, j$ implies the $i$-th FP equation depends on the $j$-th component. We consider a special case where any component of the FP equation depends upon an appropriate aggregate of that of the random 'neighbour' components. We obtain finite dimensional limit FP equations (in a much smaller dimensional space), whose solutions approximate the solution of the random FP equations for almost all realizations, in the asymptotic limit (number of components increase). Our techniques are different from the traditional mean-field methods, which deal with stochastic FP equations in the space of distributions to describe the stationary distributions of the systems. In contrast our focus is on realization-wise FP solutions. We apply the results to study systemic risk in a large financial heterogeneous network with many small institutions and one big institution, and demonstrate some interesting phenomenon.


## 1 Introduction

Random fixed points (FPs) are generalization of classical deterministic FPs, and arise when one considers systems with uncertainty. Broadly one can consider two types of such fixed points. There is considerable literature that considers stochastic FP equations on the space of probability distributions (e.g., [3, 4]). These equations typically arise as a limit of some iterative schemes, or as asymptotic (stationary) distribution of stochastic systems. Alternatively, one might be interested in sample path wise FPs (e.g., [5, 6]). For each realization of the random quantities describing the system, we have one deterministic FP equation. These kind of equations can arise when the performance/status of an agent depends upon that of a number of other agents. For example, a financial network with any given liability graph is affected by individual/common random economic shocks received by the agents. The amount cleared
(full/fraction of liability) by an agent depends upon: a) the shocks it receives; and b) the liabilities cleared by the other agents. Our primary focus in this paper is on the second type of equations. Existing literature primarily considers the existence of measurable FP, given the existence of realization-wise FPs (e.g., [5, 6]). In [6] (and reference therein) authors consider the idea of random proximity points.

To the best of our knowledge, there are no (common) techniques that provide 'good' solutions to (some special types of) these equations. We consider a special type of FP equations, which are quite common, and provide a procedure to compute the approximate solutions. We have FP equations in which the performance/status of an agent is influenced only by the aggregate performance/status of its neighbours. A random graph describes the neighbours, while a set of FP equations (one per realization of the random quantities) describe the performance vectors. The key idea is to study these FPs, asymptotically as the number of agents increase. Towards this, we first study the aggregate influence factors, with an aim to reduce the dimensionality of the problem. But due to random connections, the aggregate influence factors can also depend upon the nodes. However the aggregates might converge towards the same limit (e.g., as in law of large numbers). We precisely consider such scenarios and show that the random FPs converge to that of a limit system, under certain conditions. The performance of the agents in the limit system, depends upon finitely many 'aggregate' limits. We could also obtain closed form expressions for approximate solutions of some examples.

The mean field theory (MFT) is close to this approach: MFT approximates many body problem with a one body problem and our result is also similar in nature. However there are significant differences. The mean field theory also deals with a system of large number of agents, wherein the state/behaviour of an individual agent is influenced by its own (previous) state and the mean (aggregate) field seen by it (e.g., [7] and reference therein). The mean field is largely described in terms of occupation (empirical) measures representing the fraction of agents in different states. The theory shows the convergence of the state trajectories as well as the stationary (time limit) distributions of the original system towards that of a limit deterministic system. The stationary distribution can be described by FP equations in the space of distributions (e.g., [7]). While we directly have a set of FP equations, which are defined realization-wise and depend upon the realization-wise 'mean' performance. Further, as already mentioned the mean influence factor is not common to all the agents. In [8] and references therein, authors consider the mean field analysis with 'random' aggregate influence factors like in our case. They consider the first-order approximation, wherein the joint expected values are approximated by the product of the marginal expected values etc. Thus the moments of the joint distributions representing the FP solutions are asymptotically proven to be product of marginal moments. Some authors also consider second-order approximations or moment closure techniques, where the joint states of triplets are assumed to have a specific distribution (see [8] for relevant discussions). Our FP solutions are also proved to be asymptotically
independent, however the asymptotic solutions are independent (infinite dimensional) random vectors.

We consider FP equations with possibly multiple solutions. We show that any sequence of the chosen FPs, converges to the unique FP of the limit system almost surely under sufficiently general conditions.

We apply our results to study the systemic risk in a large financial network with many financial institutions. The institutions borrow/lend money from/to other institutions, and will have to clear their obligations at a later time point. These systems are subjected to economic shocks, because of which some entities default (do not clear their obligations). Because of interdependencies, this can lead to further defaults and the cascade of these reactions can lead to (partial/full) collapse of the system. After the financial crisis of 2007-2008, there is a surge of activity towards studying systemic risk (e.g., [10],[9],[11]). The focus in these papers has been on several aspects including, measures to capture systemic risk, influence of network structure on systemic risk, phase transitions etc. These papers primarily discuss homogeneous systems, although heterogeneity is a crucial feature of real world networks. As already mentioned the clearing vectors are represented by FP equations and may have multiple FP solutions. Thus our asymptotic solution can be useful in this context. We consider one stylized example of heterogeneous financial network, that of one big bank and numerous small banks. Our key contribution is that we develop a methodology to arrive at simplified asymptotic representation to large bank networks. This allows easy resolution of many practical what-if scenarios. For instance, in a simple framework we observe that having a big bank in an economy well connected to the small banks can stabilize the small banks even when the big bank itself faces shocks. However, the reverse may not be true. The proposed methodology can be similarly used to provide insights into many other practical scenarios. We analyze these in future. To summarise, our analysis helps identify important patterns in a complex structure, since the structure simplifies when large number of constituents are involved.

## 2 System Model

Consider a random graph with $n+1$ vertices $\{1,2, \cdots, n, b\}$ whose directed edges, given by random weights $\left\{W_{i, j}\right\}$, represent the influence factors. The node $b$ is a 'big node', and is highly influential. There is an edge between any two of the 'small' nodes (nodes in $\{1,2, \cdots, n\}$ ) with probability $p_{s s}$ independently of the others and let $\left\{I_{i, j}\right\}_{i \leq n, j \leq n}$ be the corresponding indicators. Then the weights from a small node $j$ are the fractions ${ }^{1}$ defined as below:

$$
\begin{equation*}
W_{j, b}=\eta_{j}^{s b} \text { and } W_{j, i}=\frac{I_{j, i}\left(1-\eta_{j}^{s b}\right)}{\sum_{i^{\prime} \leq n} I_{j, i^{\prime}}}, \tag{1}
\end{equation*}
$$

[^0]where $\left\{\eta_{j}^{s b}\right\}_{j}$ are IID (independent, identically distributed) random variables with values between 0,1 . These fractions, for example, can represent random fractions of some resources shared between various nodes. From small node $j$, there is a dedicated fraction $\eta_{j}^{s b}$ towards the b-node while the remaining $\left(1-\eta_{j}^{s b}\right)$ fraction is equally shared by the other connected small nodes. The weights from $b$-node are the fractions,
$$
W_{b, j}=\frac{\eta_{j}^{b s}}{\sum_{i} \eta_{i}^{b s}},
$$
where $\left\{\eta_{j}^{b s}\right\}_{j}$ are IID random variables again. We are interested in some performance of the nodes, which depends upon the weighted average of the performance of other nodes with weights as given by $\left\{W_{i, j}\right\}$. We consider the following fixed point (FP) equation (in $R^{n+1}$ ) constructed using functions $\left(f^{s}, f^{b}\right)$, which in turn depend upon weighted averages $\left\{\bar{X}_{i}^{s}\right\}_{i}$ and $\bar{X}^{b}$, and whose FP ( $i$-th component) represents important performance measure of the nodes (node- $i$ ) as below:
\[

$$
\begin{align*}
X_{i}^{s} & =f^{s}\left(G_{i}, \bar{X}_{i}^{s}, \eta_{i}^{b s} X^{b}\right) \text { for each } i \leq n,  \tag{2}\\
X^{b} & =f^{b}\left(\bar{X}^{b}\right) \text { with aggregates }  \tag{3}\\
\bar{X}_{i}^{s} & :=\sum_{j \leq n} X_{j}^{s} W_{j, i} \text { and } \bar{X}^{b}:=\frac{1}{n} \sum_{j \leq n} X_{j}^{s} W_{j, b} .
\end{align*}
$$
\]

In the above, $\left\{G_{i}\right\}$ is an IID sequence and the performance of the big node $X^{b}$ is defined per small node (performance divided by $n$ ). For any $n$ define mapping $\mathbf{f}:=\left(f^{b}, f^{s}, \cdots f^{s}\right)$, with $\mathbf{x}:=\mathbf{x}^{n}:=\left(x_{1}^{n}, x_{2}^{n}, \cdots, x_{n}^{n}\right)$, component wise:

$$
\begin{aligned}
& f_{1}\left(\mathbf{x}, x_{b}\right):=f^{b}\left(\bar{x}_{b}\right), \bar{x}_{b}:=\frac{1}{n} \sum_{j \leq n} x_{j} W_{j, b} \text { and } \\
& f_{i}\left(\mathbf{x}, x_{b}\right):=f^{s}\left(G_{i}, \bar{x}_{i}, \eta_{i}^{b s} x_{b}\right), \bar{x}_{i}:=\sum_{j \leq n} x_{j} W_{j, i} \forall i>1,
\end{aligned}
$$

which represents the FP of the random operator (2)-(3). We assume the following:
A. 1 The functions $f^{s}, f^{b}$ are non-negative, continuous and are bounded by an $y<\infty$,

$$
0 \leq f^{s}\left(g, x, x_{b}\right), f^{b}\left(x_{b}\right) \leq y \text { for all } g, x, x_{b}
$$

Under the above assumption, by well known Brouwers fixed point theorem, FP solution exists for almost all realizations of $\left\{G_{i}\right\},\left\{W_{j, i}\right\}$ and for any $x_{b}$. Thus we have a random (measurable) FP $\left(\mathbf{X}^{*}, X_{b}^{*}\right)$ for each $n$ for the random operator (2)-(3]) (see [5]). To be precise we have:

Lemma 1 For any $n$ define mapping $\mathbf{f}:=\left(f^{b}, f^{s}, \cdots f^{s}\right)$, with $\mathbf{x}:=\mathbf{x}^{n}:=\left(x_{1}^{n}, x_{2}^{n}, \cdots, x_{n}^{n}\right)$, component wise:

$$
\begin{aligned}
f_{1}\left(\mathbf{x}, x_{b}\right) & :=f^{b}\left(\bar{x}_{b}\right), \bar{x}_{b}:=\frac{1}{n} \sum_{j \leq n} x_{j} W_{j, b} \text { and } \\
f_{i}\left(\mathbf{x}, x_{b}\right) & :=f^{s}\left(G_{i}, \bar{x}_{i}, \eta_{i}^{b s} x_{b}\right), \bar{x}_{i}:=\sum_{j \leq n} x_{j} W_{j, i} \forall i>1 .
\end{aligned}
$$

Each component is a mapping from $[0, y]^{n+1} \rightarrow[0, y]$ for almost all $\left\{G_{i}\right\},\left\{W_{j, i}\right\}$ and for any $x_{b}$. Further by continuity of $\mathbf{f}$, we have a deterministic fixed point for almost all $\left\{G_{i}\right\}$ and $\left\{I_{j, i}\right\}$ under A.1. Then we have (almost sure) random fixed point $\left(\mathbf{X}^{*}, X_{b}^{*}\right)$ for each $n$ (see [5]).

Assumptions on the graph structure: We require that the number of nodes influencing any given node, grows asymptotically linearly for almost all sample paths: A. 2 Consider $p_{s s}>0$, and only graphs for which,

$$
\lim _{n \rightarrow \infty} \sum_{j \leq n}\left|\frac{1}{\sum_{i} I_{j, i}}-\frac{1}{n p_{s s}}\right|=0 \text { almost surely (a.s.), for any } i .
$$

### 2.1 Aggregate fixed points

One can rewrite the fixed point equations for the weighted averages $\left\{\bar{X}_{i}^{s}\right\}_{i}, \bar{X}^{b}$ and we begin with their analysis. Define the following random variables, that depend upon real constants $\left(x, x_{b}\right)$ :

$$
\begin{equation*}
\xi_{i}\left(x, x_{b}\right):=f^{s}\left(G_{i}, x, \eta_{i}^{b s} x_{b}\right), \tag{4}
\end{equation*}
$$

and assume:
A. $3\left|\xi_{i}\left(x, x_{b}\right)-\xi_{i}\left(u, u_{b}\right)\right| \leq \sigma\left(|x-u|+\left|x_{b}-u_{b}\right|\right)$ with $\sigma \leq 1$.

Consider the following operators on infinite sequence spac\& ${ }^{2}$ / $s^{\infty}$, one for each $n$ :

$$
\begin{align*}
\overline{\mathbf{f}}^{n}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right) & =\left(\bar{f}_{b}^{n}, \bar{f}_{1}^{n}, \bar{f}_{2}^{n} \cdots\right) \text { where }  \tag{5}\\
\bar{f}_{i}^{n}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right) & = \begin{cases}\sum_{j \leq n} \xi_{j}\left(\bar{x}_{j}, x_{b}\right) W_{j, i} & \text { if } i \leq n \\
0 & \text { else, }\end{cases} \\
\bar{f}_{b}^{n}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right) & :=\frac{1}{n} \sum_{j \leq n} \xi_{j}\left(\bar{x}_{j}, x_{b}\right) W_{j, b} \text { with } x_{b}:=f^{b}\left(\bar{x}_{b}\right) .
\end{align*}
$$

It is clear that the fixed points of the above operators equal the aggregate vectors, $\left(\left\{\bar{X}_{i}^{s}\right\}_{i \leq n}, \bar{X}_{b}\right)$. Define the 'limit' operator $\overline{\mathbf{f}}^{\infty}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right)=\left(\bar{f}_{b}^{\infty}, \bar{f}_{1}^{\infty}, \bar{f}_{2}^{\infty} \cdots\right)$ :

$$
\begin{equation*}
\bar{f}_{i}^{\infty}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right):=\lim \sup _{n} \bar{f}_{i}^{n}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right) \text { for all } i \in\{b, 1,2, \cdots\} . \tag{6}
\end{equation*}
$$

The idea is to show that the fixed point of this operator equals that of a 'limit' system and that the fixed points of the original system converge towards these fixed points. Recall that the weights sum up to one, i.e., $\sum_{i} W_{j, i}=1$ for any $i$. Thus we require the fixed point of the operator:

$$
\overline{\mathbf{f}}^{n} \text { where } \mathbf{f}^{n}:[0, y] \times s^{\infty} \rightarrow[0, y] \times s^{\infty},
$$

[^1]where the $s^{\infty}$ is defined in footnote 2. Idea is to derive a kind of mean field analysis where the aggregates will be approximated by their expected values.

When we consider constant sequence, i.e., if $\overline{\mathbf{x}}=(\bar{x}, \bar{x}, \cdots)$ the limit superiors in the definition of the limit system $\overline{\mathbf{f}}^{\infty}$ are actually limits by $\mathbf{A} .2$ and Law of large numbers (LLN) and equal (with $x_{b}$ as in (5))

$$
\begin{align*}
& \bar{f}_{i}^{\infty}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right)=E_{G_{i}, \eta_{i}^{b_{s}}}\left[\xi_{i}\left(\bar{x}, x_{b}\right)\right]\left(1-E\left[\eta_{1}^{s b}\right]\right) \text { and } \\
& \bar{f}_{b}^{\infty}\left(\overline{\mathbf{x}}, \bar{x}_{b}\right)=E_{G_{i}, \eta_{i}^{b_{s}}}\left[\xi_{i}\left(\bar{x}, x_{b}\right)\right] E\left[\eta_{1}^{s b}\right] . \tag{7}
\end{align*}
$$

In the above, $E_{X, Y}$ represents the expectation with respect to $X, Y$. The random variables are IID and hence the first equation is the same function for all $i$. By Theorem 11, given below, one such constant sequence would be the almost sure limit of the solutions of the aggregate fixed point equations (5). Thus one will have to solve a two-dimensional fixed point equation corresponding to the above function (7). And then random fixed points (2)-(3) are asymptotically independent depending upon the other nodes only via the aggregate fixed point, as given by the theorem below.

Theorem 1 Assume either $0<E\left[\eta_{1}^{s b}\right]<1$ or $\sigma<1$ in A.3. The aggregates of the random system, which are FPs of (4)-(5), denoted by $\left(\overline{\mathbf{X}}^{*}, \bar{X}_{b}^{*}\right)(n):=\left(\left\{\bar{X}_{i}^{s}\right\}_{i}, \bar{X}_{b}^{*}\right)(n)$ converge as $n \rightarrow \infty$ along a sub-sequence. That is there exist a $k_{n} \rightarrow \infty$ such that:

$$
\begin{equation*}
\bar{X}_{i}^{s}\left(k_{n}\right) \rightarrow \bar{x}^{\infty *} \text { for all } i \text { and } \bar{X}_{b}^{*}\left(k_{n}\right) \rightarrow \bar{x}_{b}^{\infty *} \text { almost surely (a.s.), } \tag{8}
\end{equation*}
$$

where $\left(\bar{x}_{b}^{\infty *}, \overline{\mathbf{x}}^{\infty *}\right)$ with $\overline{\mathbf{x}}^{\infty *}:=\left(\bar{x}^{\infty *}, \bar{x}^{\infty *}, \cdots\right)$ is the FP of the limit system given by (7). Further (any sequence of) FPs of the original system (2)- (3) converge almost surely (along the sub-sequence of (8), i.e., as $k_{n} \rightarrow \infty$ ):

$$
\begin{align*}
& X^{b}\left(k_{n}\right) \rightarrow X_{b}^{\infty *}:=f^{b}\left(\bar{x}_{b}^{\infty *}\right) \text { as } n \rightarrow \infty \text { and }  \tag{9}\\
& X_{i}^{s}\left(k_{n}\right) \rightarrow f^{s}\left(G_{i}, \bar{x}^{\infty *}, \eta_{i}^{b s} X_{b}^{\infty *}\right) .
\end{align*}
$$

Proof: This is a special case of [12, Theorem 1] and the proof is available in [12].
Thus the fixed points of the finite $n$ system converge to that of the limit system. The fixed points are asymptotically independent and depend upon the other nodes only via an almost sure constant $\bar{x}_{i}^{\infty *}$ which is common for all $i$. Another important point to observe here is that, the aggregate fixed points need not be unique, however any sequence of fixed points (one for each $n$ ) converges towards that of the limit system (when it has unique fixed point).

Remarks: Another interesting observation is that the result does not depend upon the precise probability $p_{s s}$ of the connection between small nodes. It only depends upon the fact that every node can potentially influence every other node directly or indirectly (i.e., $p_{s s}>0$ ). It is straight forward to generalize to the case where $\left\{I_{i, j}\right\}$ are any IID random variables and the weights are formed in a similar way. Further one may have finite number of groups of small
nodes, nodes within a group are identical stochastically (identical $\left\{G_{i}\right\},\left\{\eta_{i}^{s b}\right\}$ and $\left\{\eta_{i}^{b s}\right\}$ ), and any typical small node can be of group $i$ with probability $q_{i}$ independent of others. With this one can study a wider variety of heterogeneous situations. For example one can consider a financial network with many big and small banks. One can also generalize to the case when we have more than one (but finite) distinct limits for the aggregate fixed points. The results are true even when $\left\{X_{i}^{s}\right\}$ are finite dimensional, with dimension greater than one. One can then consider banks with different levels of connectivity.

## 3 Financial Network

Consider a huge financial network with $n$ small banks and one big bank. The assets (shares, bonds etc.) of the big bank are large compared to any small bank and the small banks are similar in nature. At time $T=0$ the banks invest in a project by taking loans from one another or from outside the financial network. At time $T=1$, the banks anticipate some returns from their investments, which is used to clear their obligations (e.g., [11, 9]). But the investments are risky, there are chances of economic shocks, the returns might be lower than anticipated, because of which some banks may not be able to (fully/partially) pay the liability. We say these banks defaulted. The defaulted banks increase the shocks to other connected banks, because of which we may have more defaults. And this can continue and the system can 'collapse'. Systemic risk precisely studies these aspects. The defaulted banks break the bonds, those that they invested at time $T=0$, and try to clear their obligations using the partial returns obtained after breaking. At the end of second time period $T=2$ the survived banks obtain a return $A^{s}$ (respectively $n A^{b}$ for the big bank) while the defaulted ones obtain obtain $\rho^{s} A^{s} / \rho^{b} n A^{b}$ (at time period $T=1$ ) where $\rho^{s} / \rho^{b}<1$ (usually much less than 1). Our main aim is to study the influence of economic shocks on the stability of the network, wherein stability is understood in terms of the fraction of defaults, for a given realization of the shocks or in terms of the expected surplus after the shocks etc.

Liabilities: We model the financial network using a directed weighted graph, with the banks as the nodes and the weighted edges represent the liability fractions and directions. The weight $W_{i, j}:=l_{i, j} / Y_{i}$ represents the fraction of the liability, where $l_{i, j}$ is the amount the bank $i$ is liable to bank $j$ and $Y_{i}:=\sum_{j} l_{i, j}+l_{i, b}$ is the total liability of the small bank $i$. The small banks are liable to big bank with proportionality factors $\left\{\eta_{i}^{s b}\right\}_{i}$ and, a small bank is liable to another small bank with probability $p_{s s}>0$. Let $I_{i, j}$ be 1 if small bank $i$ is liable to small bank $j$ and then the fractions of liability would be:

$$
W_{i, j}:=\frac{I_{i, j}\left(1-\eta_{i}^{s b}\right)}{\sum_{j^{\prime}} I_{i, j^{\prime}}} \text { and } W_{i, b}=\eta_{i}^{s b} .
$$

The fraction of liability of big bank towards small bank $i$ equals $W_{b, i}=\eta_{i}^{b s} / \bar{\eta}$, and $W_{b, o}=\eta^{o} / \bar{\eta}$ is the fraction that it is liable to sources outside the network, with $\bar{\eta}:=\sum_{j \leq n} \eta_{j}^{b s}+n \eta^{o}$. Let


Figure 1: First sub-figure: only BB defaults; Second sub-figure: only SBs default; Third sub-figure: both default.


Figure 2: Small bank-major shocks $\left(Z_{i} \sim \operatorname{Bin}(0.4,8)\right.$ ), big bank-minor shocks $\left(Z_{c}=2, Z_{b}=\right.$ 2) connection with big bank (best at 0.9778 ) improves the surplus.
$n Y^{b}$ represent the total liability of big bank.
Shocks: Let $K_{i}^{s}$ be the amount of money small bank $i$ is expecting as return (plus its liquid cash) at time $T=1$, let $Z_{i}^{s}$ represent the individual/independent shock experienced by small bank $i$, and let $Z_{c}$ represent the shock that is commonly received by all the small banks. The big bank receives a shock of magnitude $n \delta Z_{c}$ along with its independent shock $n Z_{b}$. After the shocks the small bank $i$ receives $\left(K_{i}^{s}-Z_{c}-Z_{i}^{s}\right)^{+}$at $T=1$ while the big bank receives $n\left(K^{b}-\delta Z_{c}-Z^{b}\right)^{+}$, where $n K^{b}$ is the shock free return anticipated at time $T=1$.

Clearing Vectors: Let $X_{i}^{s}$ represent the maximum possible part of the total liability, eventually cleared by small bank $i$, and let $X_{b}$ (per small bank) represent the same for big bank. The vector $\left(X^{b},\left\{X_{i}^{s}\right\}_{s}\right)$ is referred to as clearing vector (e.g., [11, 9]) and we make the following commonly made assumptions (as in [11, 9] etc.) for computing the same.

When a bank (say bank $i$ ) defaults, it may not be able to clear its liabilities completely. However it repays the maximum possible, and the amount cleared to another bank (say bank
$j$ ) is proportional to the fraction $W_{i, j}$. Thus small bank $i$ receives $\bar{X}_{i}^{s}:=\sum_{j} W_{j, i} X_{j}^{s}$ at time period $T=1$ when the other small banks try to clear their liabilities. In a similar way, it receives $X^{b} W_{b, i}$ from big bank. It also receives $\left(K_{i}^{s}-Z_{c}-Z_{i}^{s}\right)^{+}$at time period $T=1$ (after shock) from outside investments. The liabilities are paid, only after paying the operational costs/taxes $v^{s}$ ( $n v^{b}$ for big bank). Thus the bank $i$ at maximum can clear $\left(\left(K_{i}^{s}-Z_{c}-Z_{i}^{s}\right)^{+}+\bar{X}_{i}^{s}-v^{s}\right)^{+}$and if this amount is less than $Y_{i}$ it breaks its bonds which are supposed to mature at time period $T=2$. The amount cleared by big bank is also computed in a similar manner. Thus, in all, the total amount cleared by small bank $i$ and big bank respectively equals,

$$
\begin{aligned}
X_{i}^{s} & =\min \left\{\Phi_{i}^{s}\left(\bar{X}_{i}^{s}, X^{b}\right), Y_{i}\right\}, \text { and } \\
X^{b} & =\frac{1}{\bar{\eta}} \min \left\{n \Phi^{b}\left(\bar{X}^{b}\right), n Y^{b}\right\} \text { with aggregates } \\
\bar{X}^{b} & =\frac{\sum_{j \leq n} X_{j}^{s} W_{j, b}}{n}, \bar{X}_{i}^{s}:=\sum_{j} W_{j, i} X_{j}^{s}, \text { where } \\
\Phi_{i}^{s}\left(\bar{X}_{i}^{s}, X^{b}\right) & :=\left(\left(K_{i}^{s}-Z_{c}-Z_{i}^{s}\right)^{+}+\bar{X}_{i}^{s}+\rho^{s} A^{s}+\eta_{i}^{b s} X^{b}-v^{s}\right)^{+}, \\
\Phi^{b}\left(\bar{X}_{b}\right) & :=\left(\left(K^{b}-Z_{c} \delta-Z^{b}\right)^{+}+\rho^{b} A^{b}-v^{b}+\bar{X}^{b}\right)^{+} .
\end{aligned}
$$

By Law of large numbers (LLN), $\bar{\eta} / n \rightarrow E\left[\eta^{b s}\right]+\eta^{o}$ a.s. The rest of the system is exactly like the general system (2)-(3), for any given realization of $\left(Z_{c}, Z_{b}, K^{b}, Y^{b}\right)$. Assumptions A. 1 and A. 3 are clearly satisfied and Theorem 1 is applicable if we assume A.2. By Theorem 1 and equation (7), for any given realization $\left(Z_{c}, Z_{b}, K^{b}, Y^{b}\right)=\left(z_{c}, z_{b}, k^{b}, y^{b}\right)$, the aggregate vectors are approximately (accurate for large $n$ ) the solutions of the following fixed point equations:

$$
\begin{align*}
& \bar{x}^{\infty *}=E_{Z_{i}, Y_{i}, K_{i}, \eta_{i}^{s b}}\left[\min \left\{\Phi_{i}^{s}\left(\bar{x}^{\infty *}, x_{b}^{\infty *}\right), Y_{i}\right\}\right]\left(1-E\left[\eta_{1}^{s b}\right]\right), \\
& \bar{x}_{b}^{\infty *}=\bar{x}^{\infty *} \frac{E\left[\eta_{1}^{s b}\right]}{1-E\left[\eta_{1}^{s b}\right]}, \quad x_{b}^{\infty *}=\frac{\min \left\{\Phi^{b}\left(\bar{x}_{b}^{\infty * *}\right), y^{b}\right\}}{E\left[\eta_{1}^{b s}\right]+\eta^{o}} . \tag{10}
\end{align*}
$$

Once these fixed point equations are solved, the clearing vectors are approximately given by (9) of Theorem 1. These are asymptotically independent, as now the aggregates $\bar{X}_{i}^{s}$ are almost sure constants and are common for all $i$. We now compute relevant asymptotic performance measures.

### 3.1 Performance measure to study Systemic risk

Once these fixed point are available for each pair of shock, initial value, total liability realizations ( $k^{b}, y^{b}, z_{c}, z_{b}$ ), one can obtain various performance measures as below:

1) Expected surplus till time $T=1$ per small bank: This is the total income of the network (big bank as well as small banks) after paying away the liabilities and taxes ( $v$ ) divided by number of small banks. We are currently using the following expression which has to be
proved as in [11, 9]

$$
\begin{align*}
E[S(1)]:= & E\left[\left(\Psi^{s}\right)^{+}+\left(\Psi^{s}+\rho^{s} A^{s}\right)^{+} ; \Psi^{s}<0\right] \\
& +\left(\Psi^{b}\right)^{+}+\left(\Psi^{b}+\rho^{b} A^{b}\right)^{+} 1_{\left\{\Psi^{b}<0\right\}}, \text { with } \\
\Psi_{i}^{s}:= & \left(K_{i}^{s}-Z_{c}-Z_{i}^{s}\right)^{+}+\bar{x}^{\infty *}+x_{b}^{\infty *} \eta_{i}^{b s}-v^{s}-Y_{i}^{s} \\
\Psi^{b}:= & \left(K^{b}-Z_{c} \delta-Z_{b}\right)^{+}+\bar{x}^{\infty *} \frac{E\left[\eta_{1}^{s b}\right]}{1-E\left[\eta_{1}^{s b}\right]} \\
& \quad-v^{b}-Y^{b} . \tag{11}
\end{align*}
$$

The expectations are with respect to $\left(Z_{i}, \eta_{i}^{b s}, Y_{i}, K_{i}^{s}\right)$ and are conditioned over $\left(K^{b}, Z_{c}, Z_{b}, Y^{b}\right)$.
2) Expected number of defaults: The fraction of small banks that defaulted (by bounded convergence theorem) and the indicator that the big bank defaults asymptotically equal:

$$
\begin{equation*}
P_{D}^{s}:=P\left(\Psi_{i}^{s}\left(\bar{x}^{\infty *}, x_{b}^{\infty *}\right)<Y_{i}\right), P_{D}^{b}:=1_{\left\{\Psi^{b}\left(\bar{x}_{b}^{\infty *}\right)<y^{b}\right\}} \text { a.s. } \tag{12}
\end{equation*}
$$

3) Expected surplus till time $T=2$ per small bank: The banks invest and the liability structure is defined at time period $T=0$. The first installment is returned to the banks at $T=1$ (as in [11, 9]) and we are studying the situation when there are shocks to these returns. The surplus per small bank till time period $T=1$ is given by $E[S(1)]$. The banks that defaulted at time $T=1$, break their bonds/investments. If this did not happen a small bank receives amount $A^{s}$ on maturity at $T=2$, while the big bank receives $n A^{b}$ amount (if it did not default). Thus the surplus at $T=2$ equals:

$$
\begin{equation*}
E[S(2)]=E[S(1)]+\left(1-P_{D}^{s}\right) A^{s}+\left(1-P_{D}^{b}\right) A^{b} . \tag{13}
\end{equation*}
$$

Regular networks: In these networks, the total claim of any bank equals its total liability, i.e., $\sum_{j} l_{j i}+l_{b i}=\sum_{j} l_{i j}+l_{i b}$ for all $i$. We consider these networks for further study, as they ensure the initial wealth ${ }^{3}$ of the network remains the same once the characteristics of $K^{b}, K^{s}, Z_{i}, Z_{c}, Z_{b}$ remain the same (stochastically). This allows fair comparison of stability of the network for different values of the parameters, importantly the big bank-small bank connection parameter $E\left[\eta^{s b}\right]$. In our network the liabilities are random and equal:

$$
l_{j i}=Y_{j} \frac{I_{j i}\left(1-\eta_{j}^{s b}\right)}{\sum_{i^{\prime}} I_{j, i^{\prime}}}, l_{j b}=\eta_{j}^{s b} Y_{j} \text { and } l_{b j}=\frac{1}{\bar{\eta}} \eta_{j}^{b s} n Y^{b} .
$$

Thus clearly $\sum_{i} l_{j i}+l_{j b}=Y_{j}$, and as $n \rightarrow \infty$ by LLN and A.2.

[^2]

Figure 3: When big bank has major shocks ( $Z_{c}=2, Z_{b}=8$ ), risk of small bank is minor $\left(Z_{i} \sim \operatorname{Bin}(0.2,2)\right.$ ). Connection with big bank does not help, surplus at $p_{b s}=0$ equals that at $p_{b s}=1$.


Figure 4: With big common shock, setting as in Figure 3 but for $z_{c}=8$. Now the connection with big bank improves the performance as in Figure 2. The improvement is more pronounced, but big-bank performance remains the same.

$$
\begin{aligned}
\sum_{i} l_{i j} & =\sum_{i} Y_{i} \frac{I_{i, j}\left(1-\eta_{i}^{s b}\right)}{\sum_{j} I_{i, j}} \rightarrow E\left[Y_{i}\right] E\left[1-\eta_{i}^{s b}\right], \text { thus } \\
\sum_{i} l_{i j}+l_{b j} & \rightarrow E\left[Y_{i}\right]\left(1-E\left[\eta_{i}^{s b}\right]\right)+\frac{1}{E\left[\eta^{b s}\right]+\eta^{o}} \eta_{j}^{b s} Y^{b}
\end{aligned}
$$

Since the liabilities are random, we require equality in stochastic sense or atleast in expected sense ( $\stackrel{m}{=}$ ), i.e., we need:

$$
E\left[Y_{i}\right]\left(1-E\left[\eta_{i}^{s b}\right]\right)+\frac{1}{E\left[\eta^{b s}\right]+\eta^{o}} \eta_{i}^{b s} Y^{b} \stackrel{m}{=} Y_{i}
$$

We also need that the liabilities and claims of the big bank match, i.e., $\sum_{j} l_{b j}=\sum_{j} l_{j b}$. That is we need (at limit),

$$
\lim _{n \rightarrow \infty} \frac{\sum_{j} l_{b j}}{n}=\lim _{n \rightarrow \infty} \frac{\sum_{j} \eta_{j}^{b s}}{n E\left[\eta^{b s}\right]+\eta^{o}} Y^{b} \stackrel{m}{=} \lim _{n \rightarrow \infty} \frac{\sum_{j} \eta_{j}^{s b} Y_{j}}{n}
$$

### 3.2 Example Case studies

We consider two scenarios of regular networks and compute the performance measures (11)(13).

### 3.3 No external links, $\eta^{o}=0$

To keep things simple yet sufficiently interesting, we consider a deterministic $Y_{i}^{s} \equiv y, Y^{b} \equiv y^{b}$ and $K_{i}^{s} \equiv k^{s}$. We then consider a given scenario $\left(z_{c}, z_{b}, k^{b}\right)$ as discussed before. This is the case with identical small banks in terms of initial wealth, investments at time $T=0$ and when they receive common shock $z_{c}$ as well individual independent shocks $\left\{Z_{i}^{s}\right\}$. The total shock of the big bank equals $\delta z_{c}+z_{b}$. To have regular networks we set:

$$
\eta_{i}^{b s} \stackrel{d}{=} \eta_{i}^{s b} \text { and } y^{b}=y E\left[\eta^{b s}\right] \text { for all } i,
$$

and so $E\left[\eta_{i}^{s b}\right]=E\left[\eta_{i}^{b s}\right]=p_{b s}$. We immediately have the following for the limit system when $\eta_{i}^{b s}$ are indicators. Let $\underline{k}^{s}, \bar{k}^{s}$ respectively represent the worst and best returns $\left(\left(k_{i}^{s}-z_{c}-Z_{i}^{s}\right)^{+}\right)$ of a small bank, given $z_{c}$. Then

Lemma 2 (i) If $y p_{b s} \leq\left(\underline{k}^{s}-v^{s}\right)$ then none of the small banks default, i.e, $P_{D}^{s}=0$.
(ii) If $y p_{b s}>\left(\bar{k}^{s}+x_{b}^{\infty *}-v^{s}\right)$, then all small banks default, i.e., $P_{D}^{s}=1$. Thus if $y p_{b s}>$ $\left(\bar{k}^{s}+y-v^{s}\right)$, then $P_{D}^{s}=1$.
(iii) If $0<y p_{b s}<\left(\bar{k}^{s}+x_{b}^{\infty *}-v^{s}\right)$, then atleast some small banks do not default, as $P_{D}^{s} \leq$ $1-(1-w) p b s<1$.

Proof: The small banks never default if for all scenarios (realizations of $Z_{i}, \eta_{i}^{b s}$ ):

$$
\left(k^{s}-z_{c}-Z_{i}\right)^{+}+y\left(1-p_{b s}\right)-v^{s}+\eta_{i}^{b s} x_{b} \geq y .
$$

The worst scenario is with worst shock $\underline{k}^{s}$ and with $\eta_{i}^{b s}=0$ and hence $P_{D}^{s}=0$ when $y p_{b s} \leq$ $\underline{k}^{s}-v^{s}$ proving (i). In a similar way consider the best scenario to obtain part (ii).

Thus we identified the conditions for zero and all defaults. As long as $p_{b s}<\left(\underline{k}^{s}-v^{s}\right) / y$, none of the small banks default. But (for example) when $p_{b s}$ increases beyond $\left(\underline{k}^{s}-v^{s}\right) / y$, there can be a 'phase transition' in the fraction of defaults, $P_{D}^{s}$. At this point it probably would jump from 0 to some non-zero value. One need more analysis to understand this possible 'phase transition'. We derive more such details for the special case with binary shocks.

Consider that $Z_{i} \sim \operatorname{Bin}(w, \epsilon)$, i.e., binary $(0, \epsilon)$ shocks with $P\left(Z_{i}=\epsilon\right)=w$. Then $\underline{k}^{s}=$ $\left(k^{s}-z_{c}-\epsilon\right)^{+}$and $\bar{k}^{s}=\left(k^{s}-z_{c}\right)^{+}$.

## With two time periods

We begin with analysis with two time period, $T=0,1$. Thus $\rho^{s} \rho^{b}$ and $A^{s} A^{b}$ are not applicable. We compute only the expected fraction of defaults. We have the closed form expressions for the clearing vectors as well as the asymptotic fraction of defaults, for the sub-case when the big bank does not default. These expressions approximately equal the corresponding quantities for system with large number of small banks.

## When the big bank does not default

Lemma 3 Let $y>\epsilon$. With binary shocks, the a.s. limit fraction of defaults equal (with $\bar{K}_{Z}^{s}:=$ $\left.E\left[\left(k^{s}-z_{c}-Z_{i}\right)^{+}\right]\right)$:

$$
\begin{aligned}
& P_{D}^{s}\left(p_{b s}\right)=\left[\begin{array}{l}
P_{D 1} \\
P_{D 2} \\
P_{D 3} \\
P_{D 4} \\
P_{D 5}
\end{array}\right]= \begin{cases}0 & \text { if } b_{0}<y p_{b s} \leq b_{1} \\
w\left(1-p_{b s}\right) & \text { if } b_{1}<y p_{b s} \leq b_{2} \\
1-p_{b s} & \text { if } b_{2}<y p_{b s} \leq b_{3} \\
1-p_{b s}(1-w) & \text { if } b_{3}<y p_{b s} \leq b_{4} \\
1 & \text { else },\end{cases} \\
& b_{i}:=c_{i}\left(1-p_{b s}\right) P_{D i}+d_{i}\left(1-\left(1-p_{b s}\right) P_{D i}\right), i>0, b_{0}=0, \\
& {\left[\begin{array}{l}
c_{1} \\
c_{2} \\
c_{3} \\
c_{4}
\end{array}\right]=\left[\begin{array}{l}
0 \\
\frac{k^{s}-v^{s}}{\bar{K}_{Z}^{s}-v^{s}} \\
\frac{d_{1}}{\bar{K}_{Z}^{s}\left(1-p_{b s}\right)+\left(k^{s}+y\right) w p_{b s}} \\
1-p_{b s}(1-w) \\
d_{2} \\
d_{3} \\
d_{4}
\end{array}\right]=\left[\begin{array}{l}
\frac{k^{s}-v^{s}}{\bar{k}^{s}-v^{s}} \\
\frac{k^{s}}{}-v^{s}+y \\
\bar{k}^{s}-v^{s}+y
\end{array}\right] .}
\end{aligned}
$$

Let $b_{5}:=y, c_{5}:=\bar{K}_{Z}^{s}-v^{s}+y p_{b s}$ and $P_{D 5}\left(p_{b s}\right)=1$. The common clearing aggregate (when $\left.b_{i-1}<y p_{b s}<b_{i}\right)$ :

$$
{\overline{x_{s}}}^{\infty}=y\left(1-p_{b s}\right)-\frac{\left(y p_{b s}-c_{i}\right)\left(1-p_{b s}\right) P_{D i}\left(p_{b s}\right)}{1-\left(1-p_{b s}\right) P_{D i}\left(p_{b s}\right)} \forall i \leq 5 .
$$

The above is true when the big bank does not default, i.e., if

$$
\left(k^{b}-\delta z_{c}-z_{b}\right)^{+}-v^{b}+{\overline{x_{s}}}^{\infty} \frac{p_{b s}}{1-p_{b s}}>y p_{b s}
$$

Proof : available in Appendix.
The above result indicates the 'phase transitions' with respect to the connection parameter $p_{b s}$. This lemma is true for the sub-case when big bank does not default (one example scenario, when $\left.\left(k^{b}-\delta z_{c}-z_{b}\right)^{+}-v^{b}>y\right)$. However some of the 'phase transition' results mentioned below are also true for the other case, by Lemma 2 .

As already discussed when the connectivity parameter $p_{b s}$ is below $b_{1} / y$, the network of all small banks remains stable (the fraction of defaults is zero). But as soon as the connection parameter crosses $b_{1} / y=\left(\underline{k}^{s}-v^{s}\right) / y$ the small banks start defaulting, and we see a sharp jump of size (see $P_{D 1}$ and $P_{D 2}$ in Lemma 3):

$$
w\left(1-p_{b s}\right)=\frac{w\left(y-\underline{k}^{s}+v^{s}\right)}{y} \underline{\text { at exactly }} p_{b s}=\frac{\underline{k}^{s}-v^{s}}{y} .
$$

These kind of phase-transitions can also be seen in Figures 1.5.5. When $p_{b s}$ is increased further, when it crosses the threshold $b_{2} / y, P_{D}^{s}$ has another jump/phase-transition. At $p_{b s}=b_{2} / y$, we notice a sharp jump of size (see $P_{D 2}$ and $P_{D 3}$ in Lemma 3):

$$
\left(1-\frac{b_{2}}{y}\right)(1-w) .
$$

For this case, one needs to solve the equation (Lemma 3)

$$
\begin{aligned}
p_{b s} & =\frac{b_{2}}{y} \\
& =\frac{\left(\underline{k}^{s}-v^{s}\right)\left(1-p_{b s}\right)^{2} w+\left(\bar{k}^{s}-v^{s}\right)\left(1-\left(1-p_{b s}\right)^{2} w\right)}{y}
\end{aligned}
$$

to get the exact point of phase transition. In a similar way from Lemma 3, we see four possible phase transitions with respect to parameter $p_{b s}$. Since all the co-efficients also depend upon the shock realizations ( $z_{c}, z_{b}$ ) one can also obtain phase transitions with respect to shock sizes. Same is the case with other parameters.

More general observation from Lemma 3 is that, we have a possibility of small fraction of defaults when $p_{b s}$ is near $0\left(P_{D}^{s}\right.$ can be 0 ) or when $p_{b s}$ is near 1 with $b_{4}>1\left(P_{D}^{s} \propto\left(1-p_{b s}\right)\right.$ or $\propto\left(1-p_{b s}(1-w)\right)$. The coefficient $b_{4}$ with $p_{b s}$ close to one, approximately equals $d_{4}=$ $\bar{k}^{s}-v^{s}+y$, is mostly bigger than one. Thus either small connectivity or large connectivity is better, but intermediate connectivity may not be good. This can also be observed in the figures.

## With three time periods

From equation 10p, If $\underline{k}^{s}+\rho^{s} A^{s}>v^{s}+y p_{b s}$ then $\bar{x}^{\infty *}=y\left(1-p_{b s}\right)$ and further $k^{b}>\delta z_{c}+z_{b}+v^{b}$ implies $x_{b}^{\infty *}=y$. Thus the banks (small) may default, but they are able to clear their obligations completely by breaking the bonds. One can easily verify the following in this case.

Lemma 4 Assume $k^{b}>\delta z_{c}+z_{b}+v^{b}, \epsilon<y$ and $\underline{k}^{s}+\rho^{s} A^{s}>v^{s}+y p_{b s}$. Then

$$
P_{D}^{s}= \begin{cases}w\left(1-p_{b s}\right) & \text { if } \quad \frac{k^{s}-v^{s}}{y}<p_{b s} \leq \frac{\bar{k}^{s}-v^{s}}{y} \\ 1-p_{b s} & \text { if } \frac{\bar{k}^{s}-v^{s}}{y}<p_{b s} \leq \frac{\underline{k}^{s}+y-v^{s}}{y} \\ 1-p_{b s}+w p_{b s} & \text { if } \frac{k^{s}+y-v^{s}}{y}<p_{b s} \leq \frac{\bar{k}^{s}+y-v^{s}}{y} \\ 1 & \text { if } \frac{\bar{k}^{s}+y-v^{s}}{y}<p_{b s}\end{cases}
$$

Proof: When $y>\epsilon$ (i.e., $x_{b}>\epsilon$ ), $\bar{k}^{s}<\underline{k}^{s}+x_{b}$ and one can derive the result considering various scenarios as in the previous lemma.
We continue with sub-case considered of Lemma 4 for which $P_{D}^{b}=0$. Define,

$$
P_{D}^{* s}:=\inf _{p_{b s}} P_{D}^{s}
$$

the minimum 'expected defaults' possible. It is clear that for the sub-case considered in Lemma 4 it equals the following:

$$
\begin{aligned}
& P_{D}^{* s}= \\
& \begin{cases}\min \left\{w\left(1-\frac{\bar{k}^{s}-v^{s}}{y}\right), \frac{\left(v^{s}-\underline{k}^{s}\right)^{+}}{y}\right\} & \text { if } v^{s}<\bar{k}^{s} \\
\min \left\{w+(1-w) \frac{v^{s}-\bar{k}^{s}}{y}, \frac{\left(v^{s}-\underline{k}^{s}\right)}{y}, 1\right\} & \text { if } v^{s}>\bar{k}^{s}\end{cases}
\end{aligned}
$$

Further the assumptions of Lemma 4 for any $p_{b s}=E\left[\eta^{b s}\right]$, we have $\bar{x}^{\infty *}=y\left(1-p_{b s}\right), x_{b}^{\infty *}=y$, and hence

$$
\Psi^{s}=\left(k^{s}-z_{c}-Z_{i}^{s}\right)^{+}+y\left(\eta_{i}^{b s}-E\left[\eta^{b s}\right]\right)-v^{s}
$$

and $\Psi^{b}=\psi^{b}=k^{b}-z_{c} \delta-z_{b}-v^{b}>0$. Further $\Psi^{s}+\rho^{s} A^{s} \geq 0$ almost surely and thus from equations (11)-(13) the expected surplus equals

$$
\begin{aligned}
& E[S(2)]=E\left[\Psi^{s}+\rho^{s} A^{s} ; \Psi^{s}<0+A^{s} ; \Psi^{s} \geq 0\right]+\psi^{b} \\
& \quad=E\left[\left(k^{s}-z_{c}-Z_{i}^{s}\right)^{+}\right]-v^{s}+A^{s}-\left(1-\rho^{s}\right) A^{s} P_{D}^{s}+\psi^{b} .
\end{aligned}
$$

Therefore the expected surplus is maximized at the same $p_{b s}^{*}$ which minimizes $P_{D}^{s}$ and then the optimal surplus is obtained by substituting $P_{D}^{s} *$ in equation (13).

Influence of connectivity, shocks: We study the influence of connectivity parameter $p_{b s}=$ $E\left[\eta^{\overline{b s}}\right]=E\left[\eta^{s b}\right]$ for various shock scenarios. We begin with the case when $p_{b s}=0^{+}$, i.e., as $p_{b s}$ approaches 0 (or is $\approx 0$ ). There is negligible connection between the small banks and big bank, and the small banks are primarily liable to other small banks. The limit FP equations are: $x_{b}=0^{+}$and

$$
\bar{x}^{\infty *}=E\left[\min \left\{\left(k^{s}-z_{c}-Z_{i}^{s}\right)^{+}+\bar{x}^{\infty *}+\rho^{s} A^{s}+\eta_{i}^{b s} 0^{+}-v^{s}, y\right\}\right] .
$$

When $\underline{k}^{s}>v^{s}$, it is clear that $\bar{x}^{\infty *}=y$, i.e., the small banks do not default at $p_{b s}=0^{+}$. If the big bank encounters big shock, i.e., if $\left(k^{z}-\delta z_{c}-z_{b}\right)^{+}<v^{b}$, then it defaults near $p_{b s}=0^{+}$. In


Figure 5: For different shock realizations: $v^{s}=v^{b}=12, Z_{i} \sim \operatorname{Bin}(.4,20),\left(y, k^{b}, k^{s}\right)=(80,55,25)$, $\left(\delta, p_{b s}\right)=(.4, .9)$
fact from equation 10 , the big bank defaults for any $p_{b s}>0$ under these conditions ${ }^{4}$. Therefore we conclude, 'when the big bank receives large shocks, it always defaults, the connection with the small banks does not help'. Further some of the small banks can also default leading to an increased fraction of defaults as $p_{b s}$ increases. We considered one such example in the first sub-figure of Figure 1, which reaffirms our observation: the fraction of defaults $P_{D}^{s}=0$ for small $p_{b s}($ till 0.1$)$ and near $p_{b s}=1$. The defaults are more for intermediate $p_{b s}$.

We consider the reverse situation now. Say the big bank does not default at $p_{b s}=0^{+}$. This is because it received small shocks such that $\left(k^{z}-\delta z_{c}-z_{b}\right)^{+}>v^{b}$. Say some small banks receive big shocks such that $\underline{k}^{s}<v^{s}$. Then from Lemma 3, $P_{D}^{s} \geq w$ at $p_{b s}=0^{+}$. The connection with big bank can improve the fraction of defaults, for example, if we manage to chose a large enough $p_{b s}$ which is between $b_{2} / y$ and $b_{3} / y$ of Lemma 3 (and if further the big bank does not default). In this case the asymptotic fraction of defaults could be smaller than $w$. We consider one such example in the second sub-figure of Figure 1. We notice that $P_{D}^{s}$ reduces as $p_{b s}$ increases beyond 0.8 , in fact $P_{D}^{s}=0$ (i.e, no small bank defaults) at $p_{b s}=1$. Thus we conclude 'the big bank (with small shocks) can help the small banks'.

We consider a third example in Figure 1, where big bank and some small banks receive big shocks to default at $p_{b s}=0^{+}$. The big bank continues to default at all $p_{b s}$, however $P_{D}^{s}$ decrease with increase in $p_{b s}$. Thus for the chosen example of economy with one big bank and many small banks, 'small banks can be stabilized by connecting to big bank, even if the later defaults, however they can't help the big bank'.

Influence of shocks: The banks can face two types of shocks. The idiosyncratic shocks ( $\left\{Z_{i}^{s}\right\}, Z^{b}$ ) are bank specific shocks, while the common shock $\left(Z_{c}\right)$ affects all the banks. We aim to study the role of magnitude of these shocks on the cascading of defaults in Figure 5 for a fixed $p_{b s}=0.9$. We observe two phase transitions in the fraction of defaults, $P_{D}^{s}$ remains at $0.1\left(1-p_{b s}\right)$ for some region of $\left(Z_{b}, Z_{c}\right)$, jumps to $0.46\left(1-p_{b s}(1-w)\right)$ and then to 1 , and, one phase transition with respect to big bank. This behaviour can also be explained using the barrier constants $\left\{b_{i}\right\}$ of Lemma 3, which depend upon these shocks. One can observe significant sharp jumps at the phase transition points, and these points are very important for any financial network. These jumps and transitions points can be studied either using Lemma 3 or by studying the simplified FP equations (10) numerically.

[^3]
## Conclusions

We considered a random graph, with edges representing the influence factors between a big (highly influential) node and numerous small nodes. The performance/status of individual nodes is resultant of these influences, which are represented by fixed point (FP) equations. We showed that the solution of the random FP equations converge almost surely to that of a limit system and these solutions are asymptotically independent. One may have multiple solutions for finite graphs, however any sequence of them converge to the unique FP of the limit system (if it has unique FP). Thus we have a procedure to solve the large dimensional FP equations, using mean-field kind of techniques. The proposed solution requires solving of 'aggregate' FPs in a much smaller dimensional space and is accurate asymptotically.

The clearing vectors (the fraction of liabilities eventually cleared) in a financial network are generally represented by random FP equations and we studied the same using our results. We study an example heterogeneous financial network with one big bank and many small banks. We have reduced the overall economy problem in this set-up to a two node problem - one big bank and one aggregate small bank, thus facilitating big picture analysis. We observe some interesting phase transitions, one can easily study the nature of these phase transitions using the approximate solutions of the involved FPs. When small banks invest more in big banks, lesser fraction of them default and this is true even when all of them face large idiosyncratic shocks. These observations could be specific to the example considered by us, however we now have a procedure to study complex networks and a more elaborate study would help us derive more concrete observations. One can easily generalize the results by relaxing many of the assumptions, one can apply this approach to more applications and these two would be the topics of future interest.

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## 4 Markov decision process (MDP) - state aggregation

Say we have finite number of actions in $\mathcal{A}$. There exists finite number of groups $\left\{G_{l}\right\}_{l \leq L}$ and if the states of an infinite MDP aggregate in the following manner

$$
\begin{aligned}
& \left|\sum_{j \in G_{l}} p^{(n)}(j \mid i, a)-p(l \mid k, a)\right| \rightarrow 0, \text { i.e., } \\
& \left|p^{(n)}\left(G_{l} \mid i, a\right)-p(l \mid k, a)\right| \text { for all } l \in G_{k}
\end{aligned}
$$

and if the immediate rewards also converge

$$
r^{(n)}(i, a)=r(k, a) \text { for all } i \in G_{k}
$$

Then using our theorem we can show that the value functions

$$
v^{(n)}(i)=\min _{a \in \mathcal{A}}\left\{r(i, a)+\lambda \sum_{j} p^{(n)}(j \mid i, a) v^{(n)}(j)\right\}
$$

converges

$$
v^{(n)}(i) \rightarrow v(k) \text { for all } i \in G_{k},
$$

and also the optimal strategy converges

$$
a^{(n) *}(i) \rightarrow a^{*}(k) .
$$

This will be true if the limit system has unique optimizer.
Idea is to use convergence of aggregates

$$
\bar{v}_{i, a}^{(n)}:=\sum_{j} p^{(n)}(j \mid i, a) v^{(n)}(j)=\sum_{j} p^{(n)}(j \mid i, a) \xi_{j}\left(\overline{\mathbf{v}}_{j}^{(n)}\right)
$$

where the vector of aggregate for any $i$ is defined as:

$$
\overline{\mathbf{v}}_{i}^{(n)}:=\left\{\bar{v}_{i, a}^{(n)}\right\}_{a},
$$

and then for all $j \in G_{l}$

$$
\left.\xi_{j}\left(\overline{\mathbf{v}}_{j}^{(n)}\right):=\min _{a}\left\{r(l, a)+\bar{v}_{j, a}^{(n)}\right)\right\} .
$$

## Appendix: Proof of Lemma 3

Proof of Lemma 3: The big bank does not default hence $x_{b}^{\infty *}=y$. Therefore we can rewrite the FP equation representing aggregate clearing vector $\bar{x}_{s}{ }^{\infty}$ as below:

$$
\begin{aligned}
\frac{\bar{x}_{s}}{1-p_{b s}}= & \min \left\{\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}, y\right\} w\left(1-p_{b s}\right)+\min \left\{\bar{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}, y\right\}(1-w)\left(1-p_{b s}\right) \\
& +\min \left\{\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}+y, y\right\} w p_{b s}+\min \left\{\bar{k}^{s}-v^{s}+\bar{x} s^{\infty}+y, y\right\}(1-w) p(1.4)
\end{aligned}
$$

It is clear that $\underline{k}^{s}<\bar{k}^{s}$ and $\underline{k}^{s}<\underline{k}^{s}+y$ etc. If $y>\epsilon$ then we also have $\bar{k}^{s}<\underline{k}^{s}+y$. Then the above FP equation has a natural order in the following sense: the terms are arranged in increasing order when the corresponding probabilities are not considered. For example the third term, $\min \left\{\underline{k}^{s}-v^{s}+{\overline{x_{s}}}^{\infty}+y, y\right\} \leq \min \left\{\bar{k}^{s}-v^{s}+{\overline{x_{s}}}^{\infty}+y, y\right\}$, the fourth term. The best scenario is with fourth term (small banks receive zero shock and connect with big bank) while the worst is with the first term (small bank face negative shock and are not connected to big bank).

Case 1: There is no default even in the worst scenario i.e. if

$$
\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}>y .
$$

Then none of the small banks default leading to $P_{D}^{s}=P_{D 1}=0$, and hence the clearing vector satisfies $\frac{\bar{x}_{s} \infty}{1-p_{b s}}=y$, or equivalently $\bar{x}_{s}^{\infty}=y\left(1-p_{b s}\right)$. That is Case 1 holds as long as:

$$
\begin{equation*}
\underline{k}^{s}-v^{s}+y\left(1-p_{b s}\right)>y \text { or equivalently as long as } y p_{b s}<\underline{k}^{s}-v^{s} . \tag{15}
\end{equation*}
$$

However as $p_{b s}$ increases, the above may not be true and this gives us the bound $b_{1}=\underline{k}^{s}-v^{s}$. Case 2 When there is default only in the first term of (14), i.e., when $P_{D}^{s}\left(p_{b s}\right)=P_{D 2}\left(p_{b s}\right)=$ $w\left(1-p_{b s}\right)$. The aggregate clearing vector in this case satisfies:

$$
\begin{array}{r}
\frac{\overline{x_{s}}}{1-p_{b s}}=\left(\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}\right) P_{D 2}+y\left(1-P_{D 2}\right) \\
\Rightarrow \bar{x}_{s}
\end{array}
$$

The Case 2 holds as long as

$$
\begin{equation*}
\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}<y \text { and } \bar{k}^{s}-v^{s}+{\overline{x_{s}}}^{\infty}>y . \tag{16}
\end{equation*}
$$

Once again as $p_{b s}$ increases, the above may not be true (the second inequality can fail) and this gives us the bound $b_{2}$. The bound $b_{2}$ can be obtained:

$$
\begin{gathered}
\bar{k}^{s}-v^{s}+\bar{x} s^{\infty}=y \Rightarrow y p_{b s}=\left(\underline{k}^{s}-v^{s}\right) P_{D 2}\left(1-p_{b s}\right)+\left(\bar{k}^{s}-v^{s}\right)\left(1-\left(1-p_{b s}\right) P_{D 2}\right) \\
\Rightarrow y p_{b s}=c_{2} P_{D 2}\left(1-p_{b s}\right)+d_{2}\left(1-\left(1-p_{b s}\right) P_{D 2}\right) \text { where } d_{2}=\bar{k}^{s}-v^{s} .
\end{gathered}
$$

Thus bound, $b_{2}=c_{2} P_{D 2}\left(1-p_{b s}\right)+d_{2}\left(1-\left(1-p_{b s}\right) P_{D 2}\right)$.
Case 3 When there is default only in the first two terms of (14), i.e., when $P_{D}^{s}\left(p_{b s}\right)=$ $P_{D 3}\left(p_{b s}\right)=\left(1-p_{b s}\right)$. The aggregate clearing vector in this case satisfies:

$$
\frac{\bar{x}_{s}^{\infty}}{1-p_{b s}}=\left(\underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}\right) w\left(1-p_{b s}\right)+\left(\bar{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}\right)(1-w)\left(1-p_{b s}\right)+y\left(1-P_{D 3}\right) .
$$

This implies

$$
\begin{equation*}
\bar{x}_{s}^{\infty}=y\left(1-p_{b s}\right)-\frac{\left(y p_{b s}-c_{3}\right)\left(1-p_{b s}\right) P_{D 3}\left(p_{b s}\right)}{1-\left(1-p_{b s}\right) P_{D 3}\left(p_{b s}\right)} \text { where } c_{3}=\bar{K}_{Z}^{s}-v^{s} . \tag{17}
\end{equation*}
$$

The Case 3 holds as long as

$$
\begin{equation*}
\bar{k}^{s}-v^{s}+\bar{x}_{s}{ }^{\infty}<y \text { and } \underline{k}^{s}-v^{s}+\bar{x}_{s}{ }^{\infty}+y>y . \tag{18}
\end{equation*}
$$

Using (17) one can easily show that that $\overline{x_{s}}{ }^{\infty}$ is decreasing with increase in $p_{b s}$. Thus again as $p_{b s}$ increases, the above inequalities (second one) may not be true and this gives us the bound $b_{3}$. The bound $b_{3}$ can be obtained:

$$
\begin{aligned}
& \underline{k}^{s}-v^{s}+\bar{x}_{s}^{\infty}+y=y \Rightarrow y p_{b s}=c_{3} P_{D 3}\left(1-p_{b s}\right)+d_{3}\left(1-\left(1-p_{b s}\right) P_{D 3}\right) \\
& \Rightarrow y p_{b s}=c_{3} P_{D 3}\left(1-p_{b s}\right)+d_{3}\left(1-\left(1-p_{b s}\right) P_{D 3}\right) \text { where } d_{3}=\underline{k}^{s}-v^{s}+y .
\end{aligned}
$$

Thus bound, $b_{3}=c_{3} P_{D 3}\left(1-p_{b s}\right)+d_{3}\left(1-\left(1-p_{b s}\right) P_{D 3}\right)$.
Continuing this way one can obtain all the sub-cases of the lemma.


[^0]:    ${ }^{1}$ Note that $\sum_{i} W_{j, i}+W_{j, b}=1$ for all $j$.

[^1]:    ${ }^{2}$ Here $s^{\infty}$ is the space (subset) of bounded sequences equipped with $l^{\infty}$ norm $|\overline{\mathbf{x}}|_{\infty}:=\sup _{i}\left|x_{i}\right|$,

    $$
    s^{\infty}:=\left\{\overline{\mathbf{x}}=\left(x_{1}, x_{2}, \cdots\right): x_{i} \in[0, y] \text { for all } i\right\} .
    $$

[^2]:    ${ }^{3}$ This is proportional to the amount anticipated without shocks at time $T=1$ plus the amount anticipated by the returns from other banks minus the amount it has to pay to other banks, all at time $T=1$. For small banks and big bank (per small bank) it is proportional respectively to:

    $$
    K_{i}^{s}+\sum_{j} l_{j, i}+l_{b, i}-\sum_{j} l_{i, j}-l_{i, b} \text { and } K^{b}+\sum_{j} l_{j, b}-\sum_{j} l_{b, j} .
    $$

[^3]:    ${ }^{4}$ Clearly $\bar{x}^{\infty *} \leq y\left(1-p_{b s}\right), y^{b}=y\left(1-p_{b s}\right)$ and so $x_{b}<y$.

