Necessary conditions involving Lie brackets for impulsive optimal control problems*

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Abstract— We obtain higher order necessary conditions for a minimum of a Mayer optimal control problem connected with a nonlinear, control-affine system, where the controls range on an m-dimensional Euclidean space. Since the allowed velocities are unbounded and the absence of coercivity assumptions makes big speeds quite likely, minimizing sequences happen to converge toward impulsive, namely discontinuous, trajectories. As is known, a distributional approach does not make sense in such a nonlinear setting, where instead a suitable embedding in the graph space is needed. We will illustrate how the chance of using impulse perturbations makes it possible to derive a Higher Order Maximum Principle which includes both the usual needle variations (in space-time) and conditions involving iterated Lie brackets. An example, where a third order necessary condition rules out the optimality of a given extremal, concludes the paper.

I. INTRODUCTION

In this paper we aim to investigate necessary optimality conditions for an optimal process of the following minimum problem:

$$(P) \begin{cases} \text{Minimize } \Psi(T, x(T)) \\ \text{over the set of processes } (T, u, x) \text{ satisfying} \\ \frac{dx}{dt} = f(x) + \sum_{i=1}^{m} g_i(x)u^i, \\ x(0) = \check{x}, \quad (T, x(T)) \in \mathfrak{T}. \end{cases}$$

The *target* \mathfrak{T} is given by

$$\mathfrak{T} := \{ (t, x) : \varphi_i(t, x) \le 0, \ \psi_j(t, x) = 0, \\ i = 1, \dots, r_1, \ j = 1, \dots, r_2 \}.$$

where $\varphi_i, \psi_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are functions of class C^1 . We assume that the state variable ranges over \mathbb{R}^n and that the control *u* takes values in \mathbb{R}^m : in particular, *u* is allowed to be *unbounded*. The cost function $\Psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is assumed of class C^1 , while $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$ are the vector fields of class C^∞ . However, we refer to Remark 3.2 for comments on the possibility of drastically reducing these

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³F. Rampazzo, Dipartimento di Matematica, Università di Padova, Padova 35121, Italy rampazzo@math.unipd.it regularity hypotheses and replacing the state space \mathbb{R}^n with a manifold, as done in [1].

Let us point out that problem (P) has an *impulsive* character, i.e. minimizing sequences generally fail to converge to an absolutely continuous path and, in fact, they may happen to approach a discontinuous path. It is well-known that, because of the nonlinearity of the dynamics, a measuretheoretical approach, with u to be interpreted as a Radon measure, does not verify basic well-posedness conditions [12]. Different but substantially equivalent approaches take care of this crucial point (see, among others, [7], [17], [16]). We choose here to adopt the so-called graph-completion point of view and embed the original problem into the spacetime problem (P^e) below: the extended state variable is now $(y^0, y) := (t, x)$, and the extended trajectories are (t, x)paths which are (reparameterized) C^0 -limits of graphs of the original trajectories.

More precisely, we consider the optimization problem

$$(P^e) \begin{cases} \begin{array}{l} \text{Minimize } \Psi(y^0(S), y(S)) \\ \text{over the set of processes } (S, w^0, w, y^0, y) \text{ verifying} \\ \\ \frac{dy^0}{ds} = w^0, \\ \\ \frac{dy}{ds} = f(y)w^0 + \sum_{i=1}^m g_i(y)w^i, \\ \\ (y^0, y)(0) = (0, \check{x}), \qquad (y^0(S), y(S)) \in \mathfrak{T}, \end{array} \end{cases}$$

where the controls (w^0, w) are functions from a pseudo-time interval [0, S] into the set

$$\mathcal{W} := \{ (w^0, w) \in \mathbb{R} \times \mathbb{R}^m : w^0 \ge 0, \ w^0 + |w| = 1 \}.$$
(1)

Notice that, unlike the controls u, the control pairs (w^0, w) are now bounded. A process (T, u, x) of the original system is identified with a process (S, w^0, w, y^0, y) of the space-time system through the reparameterization

$$\begin{aligned} \sigma(t) &:= \int_0^t (1 + |u(\tau)|) \, d\tau, \quad y^0 := \sigma^{-1} : [0, S] \to [0, T], \\ w^0(s) &:= (1 + |u(y^0(s))|)^{-1}, \quad w(s) := w^0(s) \, u(y^0(s)), \\ y(s) &:= x(y^0(s)), \end{aligned}$$

while the actual *impulsive* processes –namely the ones that are not reparameterizations of original processes– are the five-tuples (S, w^0, w, y^0, y) with $w^0 = 0$ on some non trivial subinterval $[s_1, s_2] \subseteq [0, S]$. Unlike the original problem (P), the extended problem often admits an optimal process (provided the target can be reached by one trajectory), so that

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it is natural to look for necessary conditions for the extended problem (P^e) .

The first part of the paper (Section II) is devoted to the consistency of minimum problems (P) and (P^e) , in their local version. Actually, it turns out that a process $(\bar{T}, \bar{u}, \bar{x})$ of the original system is locally optimal for the original problem (P) if and only if its space-time representation $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$ is locally optimal for the restriction of the extended problem (P^e) to processes (S, w^0, w, y^0, y) verifying $w^0 > 0$ almost everywhere.

In Theorem 3.1 we state an abridged version of a Higher Order Maximum Principle for (P^e) , where necessary optimality conditions involving iterated Lie brackets of the nondrift vector fields $\{g_1, \ldots, g_m\}$ are presented. In particular, this result generalizes to impulsive trajectories a result that, for minimum time problems, has been established for absolutely continuous processes with unbounded controls (see [9]). The detailed proof of the result's main point is rather long and technical, and is provided --under much weaker regularity hypotheses- in [1]. Instead, here we just give some hints of the idea lying behind the stated higher order conditions. Further relations, which involve the drift f, are derived in Corollary 3.3. The paper concludes with an Example, in Section IV, where the optimality of a spacetime process verifying the standard maximum principle is ruled out by our higher order conditions.

Several papers, an incomplete list of which includes [22], [20], [16], deal with First Order Maximum Principles for impulsive systems. As for *higher-order* necessary conditions —see e.g. [6], [11], [13], [14], [23], [21] for the bounded control case— we are aware only of results for the *commutative case*, i.e. when $[g_i, g_j] \equiv 0$ for all $i, j = 1, \ldots, m$, and up to the second order (see e.g. [4], [5], [10]). Instead, our Higher Order Maximum Principle is established for the generic, non-commutative, case, and involves iterated brackets of any order.

A. Notation and definitions

Let N be a natural number. For every $i \in \{1, \ldots, N\}$, we write \mathbf{e}_i for the *i*-th element of the canonical basis of \mathbb{R}^N , $\mathbb{B}_N(\check{x})$ for the closed ball $\{x \in \mathbb{R}^N : |x - \check{x}| \leq 1\}$, and \mathbb{B}_N when $\check{x} = 0$. $\partial \mathbb{B}_N := \{x \in \mathbb{R}^N : |x| = 1\}$. A subset $K \subseteq \mathbb{R}^N$ is called *cone* if $\alpha x \in K$ whenever $\alpha > 0$, $x \in K$. Given a real interval I and $X \subseteq \mathbb{R}^N$, we write AC(I, X) for the space of absolutely continuous functions, $C^0(I, X)$ for the space of continuous functions, $L^1(I, X)$ for the space of L^1 -functions, and $L^{\infty}(I, X)$ for space of measurable, bounded functions, respectively, defined on I with values in X. As customary, we shall use $\|\cdot\|_{\infty}, \|\cdot\|_1$ to denote the sup-norm and the L^1 -norm, respectively, where domain and codomain are omitted when obvious.

The Lie bracket of two vector fields F_1, F_2 is the vector field $[F_1, F_2]$ defined by

$$[F_1, F_2](x) := DF_2(x) \cdot F_1(x) - DF_1(x) \cdot F_2(x),$$

where D denotes differentiation. By repeating the bracketing procedure we obtain the so-called iterated brackets.

For a given real interval I, let us consider the L^1 -norm operator $\nu : L^1(I, \mathbb{R}^m) \to AC(I, [0, +\infty))$ defined by

$$\nu[u](t) := \int_0^t |u(\tau)| \, d\tau, \qquad \text{for } t \in I.$$
(3)

II. THE OPTIMIZATION PROBLEM AND ITS EXTENSION

In this section we introduce the optimization problem over L^1 controls and its embedding in an impulsive problem in detail.

A. The original optimal control problem

We define the set of strict sense controls as

$$\mathcal{U} := \bigcup_{T>0} \{T\} \times L^1([0,T], \mathbb{R}^m)$$

Definition 2.1: For any $(T, u) \in \mathcal{U}$ we say that (T, u, x) is a *strict-sense process* if x is the unique Carathéodory solution to

$$\begin{cases} \frac{dx}{dt}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u^i(t) \\ x(0) = \check{x} \end{cases}$$

$$\tag{4}$$

corresponding to the control u and defined on [0, T].¹ Furthermore, we say that a process (T, u, x) is *feasible* if it agrees with the final constraint, i.e. $(T, x(T)) \in \mathfrak{T}$.

Let us fix an integer q and let us define a distance by setting for all $\tau_1, \tau_2 \in (0, +\infty)$ and for any pair $(z_1, z_2) \in C^0([0, \tau_1], \mathbb{R}^q) \times C^0(\tau_2([0, \tau_2], \mathbb{R}^q))$,

$$d((\tau_1, z_1), (\tau_2, z_2)) := |\tau_1 - \tau_2| + \|\tilde{z}_1 - \tilde{z}_2\|_{\infty}, \quad (5)$$

where, for every map $z \in C^0([0, \tau], \mathbb{R}^q)$ we have used \tilde{z} to denote its continuous constant extension to $[0, +\infty)$.

Definition 2.2: We say that a feasible strict sense process $(\overline{T}, \overline{u}, \overline{x})$ is a strict sense L^{∞} -local minimizer of (P) if there exists $\delta > 0$ such that

$$\Psi(\bar{T}, \bar{x}(\bar{T})) \le \Psi(T, x(T)) \tag{6}$$

for every feasible strict sense process (T, u, x) verifying

$$d\Big((T, x, \nu[u]), (\bar{T}, \bar{x}, \nu[\bar{u}])\Big) < \delta.$$

If relation (6) is satisfied for all admissible strict sense processes, we say that $(\bar{T}, \bar{u}, \bar{x})$ is a global strict sense minimizer.

B. The space-time optimal control problem

Define the set of space-time controls

$$W := \bigcup_{S>0} \{S\} \times L^{\infty}([0,S], \mathcal{W}), \tag{7}$$

where \mathcal{W} is as in (1).

¹Under our assumptions on the control system, for any strict-sense control (T, u), there exists a unique solution of (4) which is defined in general on a maximal interval of definition $[0, \tau) \subseteq [0, T]$.

Definition 2.3: For any space-time control $(S, w^0, w) \in$ W, we say that (S, w^0, w, y^0, y) is a space-time process if (y^0, y) is the unique Carathéodory solution to

$$\begin{cases} \frac{dy^{0}}{ds} = w^{0}, \\ \frac{dy}{ds} = f(y)w^{0} + \sum_{i=1}^{m} g_{i}(y)w^{i}, \quad \text{a.e. } s \in [0, S], \\ (y^{0}, y)(0) = (0, \check{x}), \end{cases}$$
(8)

corresponding to the control (w^0, w) and defined on [0, S]. As before, we say that a space-time process (S, w^0, w, y^0, y) is feasible if $(y^0(S), y(S)) \in \mathfrak{T}$.

Definition 2.4: A feasible space-time process $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$ is said to be an L^{∞} -local minimizer for the space-time problem (P^e) if there exists $\delta > 0$ such that

$$\Psi\left((\bar{y}^0, \bar{y})(\bar{S})\right) \le \Psi\left((y^0, y)(S)\right) \tag{9}$$

for all feasible space-time processes (S, w^0, w, y^0, y) satisfying

$$d\Big((y^{0}(S), y, \nu[w]), (\bar{y}^{0}(\bar{S}), \bar{y}, \nu[\bar{w}])\Big) < \delta.$$
 (10)

If (9) is satisfied for all feasible space-time processes, we say that $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$ is a global space-time minimizer.

C. The space-time embedding

Next lemma, being an easy consequence of the chain rule, shows that the space-time problem (P^e) restricted to the controls (w_0, w) with $w_0 > 0$ a.e. is an equivalent formulation of the original problem (P). Since every L^1 equivalence class contains Borel measurable representatives, here and in the sequel we tacitly assume that all L^1 maps are Borel measurable, when necessary.

Lemma 2.5: (i) If (T, u, x) is a strict sense process, then

$$\mathcal{I}(T, u, x) := (S, w_0, w, y^0, y),$$

where (S, w_0, w, y^0, y) is defined as in (2), is a space-time process with $w_0 > 0$ a.e. on [0, S].

(ii) Vice-versa, if (S, w_0, w, y^0, y) is a space-time process with $w_0 > 0$ a.e. on [0, S], then

$$\mathcal{I}^{-1}(S, w_0, w, y^0, y) := (T, u, x),$$

where

$$\sigma(t) := (y^0)^{-1}(t), \quad T := y^0(S),$$

$$x(t) := y(\sigma(t)), \quad \text{for } t \in [0, T],$$

$$u(t) := \frac{w(\sigma(t))}{w_0(\sigma(t))} \quad \text{a.e. } t \in [0, T].$$

is a strict sense process.

(....

Furthermore, in both cases one has

$$\Psi(T, x(T)) = \Psi((y^0, y)(S))$$

Notice that the impulsive extension consists in allowing subintervals $I \subseteq [0, S]$ where $w_0 \equiv 0$. Then the state y

evolves on I in zero t-time, driven by the non-drift dynamics $\sum_{i=1}^{m} g_i(y(s))w^i(s).$

Remark 2.6: Let us point out that a reparameterization like the one utilized above is made possible by the fact that our localization of the problem implies that we are looking for minima among controls u with uniformly bounded L^1 norms. Of course, by relaxing this constraint, larger classes of controls can be considered. This, however, leads to the consideration of much more structured processes, in the direction e.g. of [3], [2], [19], [8] or [15].

Actually, the notion of space-time L^{∞} -local minimizer is consistent with the definition of strict-sense L^{∞} -local minimizer, as stated in the following result:

Proposition 2.7: A feasible strict sense process $(\overline{T}, \overline{u}, \overline{x})$ is a strict sense L^{∞} -local minimizer for problem (P) if and only if $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y}) := \mathcal{I}(\bar{T}, \bar{u}, \bar{x})$ is an L^{∞} -local minimizer for problem (P^e) among the feasible space-time processes (S, w^0, w, y^0, y) with $w^0 > 0$ a.e.. Moreover,

$$\Psi((\bar{y}^0, \bar{y})(\bar{S})) = \Psi(\bar{T}, \bar{x}(\bar{T})).$$

Proof: We have to prove the following assertion:

(a) there is $\delta > 0$ such that $\Psi(\overline{T}, \overline{x}(\overline{T})) < \Psi(T, x(T))$ for all feasible strict-sense processes (T, u, x) verifying

$$d\left((T, x, \nu[u]), (\bar{T}, \bar{x}, \nu[\bar{u}])\right) < \delta \tag{11}$$

if and only if

(b) there is $\delta' > 0$ such that $\Psi((\bar{y}^0, \bar{y})(\bar{S})) \leq \Psi((y^0, y)(S))$ for all feasible space-time processes (S, w^0, w, y^0, y) verifying

$$d\Big((y^{0}(S), y, \nu[w]), (\bar{y}^{0}(\bar{S}), \bar{y}, \nu[\bar{w}])\Big) < \delta'.$$
 (12)

Let us show that (a) \implies (b). For any space-time process (S, w^0, w, y^0, y) with $w^0 > 0$ a.e., let us consider the strictsense process $(T, u, x) := \mathcal{I}^{-1}(S, w^0, w, y^0, y)$, as defined in Lemma 2.5. In the rest of the proof we still denote $x, \bar{x}, \nu[u]$, $\nu[\bar{u}], y, \bar{y}, \nu[w], \text{ and } \nu[\bar{w}]$ the constant continuous extensions of these functions to $[0, +\infty)$. Instead, the maps y^0 , \bar{y}^0 are extended to $[0, +\infty)$ by setting –we do not rename–

$$y^0 := (T+s-S)\chi_{(S,+\infty)}, \quad \bar{y}^0 := (\bar{T}+s-\bar{S})\chi_{(\bar{S},+\infty)}.$$

Let $\sigma, \bar{\sigma}: [0, +\infty) \to [0, +\infty)$ be the continuous, increasing, one-to-one maps given by $\sigma := (y^0)^{-1}$, $\bar{\sigma} := (\bar{y}^0)^{-1}$. For every t > 0, we set

$$s := \sigma(t), \quad \bar{s} := \bar{\sigma}(t),$$

so that $|s - \bar{s}| = |\sigma(t) - \bar{\sigma}(t)|$ (and $y^0(s) = t = \bar{y}^0(\bar{s})$). Recalling the definition of \mathcal{I} , we have

$$\begin{aligned} |T - T| + |(\bar{x}, \nu[\bar{u}])(t) - (x, \nu[u])(t)| \\ &= |\bar{y}^0(\bar{S}) - y^0(S)| + |(\bar{y}, \nu[\bar{w}])(\bar{s}) - (y, \nu[w])(s)|. \end{aligned}$$

Moreover,

$$\nu[w](s) - \nu[\bar{w}](\bar{s}) = s - \bar{s} + \bar{y}^0(\bar{s}) - y^0(s) = s - \bar{s}$$

and the Lipschitz continuity of \bar{y} also implies that

$$|\bar{y}(\bar{s}) - y(s)| \le L|\bar{s} - s| + |\bar{y}(s) - y(s)|$$
(13)

for some L > 0. Let us set $\bar{t} := \bar{y}^0(s)$, so that $\bar{\sigma}(\bar{t}) = s = \sigma(t)$. Then

$$|s - \bar{s}| = |\sigma(t) - \bar{\sigma}(t)| = |\bar{\sigma}(\bar{t}) - \bar{\sigma}(t)| \le \omega_{\bar{\sigma}}(|\bar{t} - t|) = \omega_{\bar{\sigma}}(|\bar{y}^{0}(s) - y^{0}(s)|) = \omega_{\bar{\sigma}}(|\nu[\bar{w}](s) - \nu[w](s)|),$$

where $\omega_{\bar{\sigma}}$ is the modulus of continuity of $\bar{\sigma}$, which exists since $\bar{\sigma} = (\bar{\varphi}^0)^{-1}$ is absolutely continuous and thus uniformly continuous. In conclusion, we obtain

$$\begin{aligned} |T - T| + |(\bar{x}, \nu[\bar{u}])(t) - (x, \nu[u])(t)| \\ &\leq |\bar{y}^0(\bar{S}) - y^0(S)| + \|\bar{y} - y\|_{\infty} \\ &+ (1 + L)\omega_{\bar{\sigma}}(\|\nu[\bar{w}] - \nu[w]\|_{\infty}), \end{aligned}$$

which implies assertion (b), as soon as we choose $\delta' > 0$ verifying $2\delta' + (1+L)\omega_{\bar{\sigma}}(\delta') < \delta$. Indeed, we obtain

$$\begin{split} \Psi((\bar{y}^0,\bar{y})(\bar{S})) &= \Psi(\bar{T},\bar{x}(\bar{T})) \\ &\leq \Psi(T,x(T)) = \Psi((y^0,y)(S)), \end{split}$$

for all feasible space-time processes (S, w^0, w, y^0, y) with $w^0 > 0$ a.e. and satisfying (12) for such δ' .

Let us now prove that (b) \implies (a). For any feasible strict sense process (T, u, x) let us set $(S, w^0, w, y^0, y) := \mathcal{I}(T, u, x)$. Once again, we consider the functions extended to $[0, +\infty)$ as described above.

For any $s \ge 0$, let us set $t := y^0(s)$ and $\bar{s} := (\bar{y}^0)^{-1}(t)$. We define $\sigma := (y^0)^{-1}$ and $\bar{\sigma} := (\bar{y}^0)^{-1}$. Then, using L > 0 to denote the Lipschitz constant of \bar{y} , we get

$$\begin{aligned} |\bar{y}^{0}(\bar{S}) - y^{0}(S)| + |(\bar{y},\nu[\bar{w}])(s) - (y,\nu[w])(s)| \\ &\leq |\bar{T} - T| + |(\bar{y},\nu[\bar{w}])(s) - (\bar{y},\nu[\bar{w}])(\bar{s})| \\ &+ |(\bar{y},\nu[\bar{w}])(\bar{s}) - (y,\nu[w])(s)| \end{aligned}$$

 $\leq |\bar{T} - T| + (1 + L)(|\nu[\bar{u}](t) - \nu[u](t)|) + |\bar{x}(t) - x(t)|,$

where the last inequality holds, because

$$|\bar{s} - s| = |\bar{\sigma}(t) - \sigma(t)| = |\nu[\bar{u}](t) - \nu[u](t)|.$$

At this point, we derive assertion (a) as soon as we choose δ such that $2\delta + (1+L)\delta < \delta'$, since

$$\begin{split} \Psi(\bar{T}, \bar{x}(\bar{T})) &= \Psi((\bar{y}^0, \bar{y})(\bar{S})) \\ &\leq \Psi((y^0, y)(S)) = \Psi(T, x(T)), \end{split}$$

for all strict sense feasible processes verifying (11) for such δ .

III. A HIGHER ORDER MAXIMUM PRINCIPLE

Let us consider the unmaximized Hamiltonian

$$H(x, p_0, p, \lambda, w^0, w) := p_0 w^0 + p \cdot \left(f(x) w^0 + \sum_{i=1}^m g_i(x) w^i \right)$$

and the Hamiltonian $\mathbf{H}:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ defined by

$$\mathbf{H}(x, p_0, p, \lambda) := \max_{(w^0, w) \in \mathcal{W}} H(x, p_0, p, \lambda, w^0, w),$$

where \mathcal{W} is as in (1). For any continuous vector field F: $\mathbb{R}^n \to \mathbb{R}^n$, let us introduce the classical *F*-Hamiltonian

$$\mathbf{H}_F(x,p) := p \cdot F(x) \quad \text{for } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Furthermore, let us define the the *polar cone* of \mathfrak{T} at a point $(T, x) \in \mathfrak{T}$ as the set

$$N_{(T,x)}\mathfrak{T} := \operatorname{span}_+ \Big\{ D\varphi_\ell(T,x) : \ \ell \in I(T,x) \Big\} \\ + \operatorname{span} \Big\{ D\psi_j(T,x) : \ j = 1 \dots, r_2 \Big\},$$

where $I(T, x) \subseteq \{1, ..., r_1\}$ is the subsets of indexes ℓ such that $\varphi_{\ell}(T, x) = 0$.

Theorem 3.1: [HIGHER ORDER MAXIMUM PRINCIPLE] Let $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$ be an L^{∞} -local minimizer for the space-time problem (P^e) . Then there exists a multiplier $(p_0, p, \lambda) \in \mathbb{R} \times AC([0, \bar{S}], \mathbb{R}^n) \times [0, +\infty)$ such that the following conditions hold true:

(i) (NON-TRIVIALITY)

$$(p_0, p, \lambda) \neq (0, 0, 0);$$
 (14)

(ii) (NON-TRANVERSALITY)

$$(p_0, p(\bar{S})) \in -\lambda D\Psi\Big((\bar{y}^0, \bar{y})(\bar{S})\Big) - N_{(\bar{y}^0, \bar{y})(\bar{S})}\mathfrak{T};$$
(15)

(iii) (ADJOINT EQUATION) the path p solves on $[0, \overline{S}]$ the adjoint equation

$$\frac{dp}{ds} = -p \cdot \left(Df(\bar{y}))\bar{w}^0 + \sum_{i=1}^m Dg_i(\bar{y})\bar{w}^i \right); \quad (16)$$

(iv) (FIRST ORDER MAXIMIZATION) for a. e. $s \in [0, \bar{S}]$, one has

$$H\left(\bar{y}(s), p_0, p(s), \lambda, \bar{w}^0(s), \bar{w}(s)\right) = \\ \mathbf{H}\left(\bar{y}(s), p_0, p(s), \lambda\right);$$
(17)

(V) (VANISHING OF HAMILTONIANS)

$$\mathbf{H}\Big(\bar{y}(s), p_0, p(s), \lambda\Big) = 0 \qquad \text{for all } s \in [0, \bar{S}]; \quad (18)$$

$$\mathbf{H}_{g_i}(\bar{y}(s), p(s)) = 0 \text{ for all } s \in [0, \bar{S}], \ i = 1, \dots, m;$$
(19)

(vi) (VANISHING OF HIGHER ORDER HAMILTONIANS)

$$\mathbf{H}_B(\bar{y}(s), p(s)) = 0 \quad \text{for all } s \in [0, \bar{S}], \quad (20)$$

for every iterated bracket B of the vector fields g_1, \ldots, g_m .

Furthermore, if the trajectory \bar{y} is not instantaneous, namely, if $\bar{y}^0(\bar{S}) > 0$, then (14) can be strengthened to

$$(p,\lambda) \neq (0,0) . \tag{21}$$

The existence of a multiplier verifying the first order conditions (i)–(v) and of the strengthened non-triviality condition (21) has been already proved in [18] as direct consequence of the standard Maximum Principle. Instead, the fact that the same multiplier verifies the higher order relations in (vi) needs a proof that exceeds the space limits of the present paper. A proof of a stronger version of this theorem, including very low regularity assumptions on both the vector fields and the target, can be found in [1].

Remark 3.2: The higher order condition (20) in (vi) has been obtained in [9] for the special case of non-impulsive (but unbounded) optimal time trajectories. We are able to prove it for possibly impulsive trajectories due to the fact that one can construct *instantaneous* approximations of Lie brackets. Incidentally let us remark that, unlike what is done in [9], we do not assume the constancy of the rank of the Lie Algebra generated by g_1, \ldots, g_m . Actually, in the result proved in [1] we do not even assume that the vector fields g_1, \ldots, g_m are C^{∞} : the only regularity required is the continuity of the involved Lie bracket *B*.

We get immediately further higher order conditions involving the drift f as well.

Corollary 3.3: Let $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$ be an L^{∞} -local minimizer for the space-time problem (P^e) . Then there exists a multiplier $(p_0, p, \lambda) \in \mathbb{R} \times AC([0, \bar{S}], \mathbb{R}^n) \times [0, +\infty)$ such that, besides verifying conditions conditions (i)-(vi) of Theorem 3.1 makes the relation

$$\mathbf{H}_{[f,B]}(\bar{y}(s), p(s))\bar{w}^0(s) = 0 \quad \text{a.e.} \quad s \in [0, \bar{S}],$$
(22)

hold true for every iterated bracket B of the vector fields g_1, \ldots, g_m .

Proof: Condition (22) can be obtained by simply differentiating (20) and recalling that the derivative of p verifies (16). Indeed, for a. e. $s \in [0, \overline{S}]$ one has

$$\begin{aligned} \frac{d}{ds} \left(p(s) \cdot B(\bar{y}(s)) \right) \\ &= \frac{dp}{ds}(s) \cdot B(\bar{y}(s)) + p(s) \cdot D B(\bar{y}(s)) \frac{d\bar{y}}{ds}(s) \\ &= p(s) \cdot \left([f, B](\bar{y}(s)) \bar{w}^0(s) + \sum_{i=1}^m [g_i, B](\bar{y}(s)) \right) \\ &= p(s) \cdot [f, B](\bar{y}(s)) \bar{w}^0(s) = 0. \end{aligned}$$

IV. AN EXAMPLE

This example shows how higher order necessary conditions may be useful to rule out the optimality of a spacetime process for which there exists a multiplier verifying all the first order conditions (i)–(v) of Theorem 3.1, namely, the usual maximum principle.

Consider the problem

in w

$$\begin{cases} \text{Minimize } \Psi(x(1)), \\ \text{over } (u, x) : [0, 1] \to \mathbb{R}^2 \times \mathbb{R}^5 \text{ s. t.} \\ \frac{dx}{dt} = f(x) + \sum_{i=1}^2 g_i(x)u^i \text{ a.e. } t \in [0, 1], \\ x(0) = (1, 0, 0, 0, 0), \quad x(1) \in \mathfrak{T}, \end{cases}$$
which $\Psi(x) := x^3 + x^4,$

$$(23)$$

$$\mathfrak{T} := \{ (x^1, \dots, x^5) \in \mathbb{R}^5 : \ x^1 = x^2 = 0, \ x^3 \ge -1 \},\$$

$$f(x) := \frac{1}{2} \Big((x^2)^2 + (x^3)^2 + (1 - x^1 - x^5)^2 \Big) \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5},$$
$$g_1(x) := \frac{\partial}{\partial x^1} - \frac{1}{2} (x^2)^2 \frac{\partial}{\partial x^3}, \qquad g_2(x) := \frac{\partial}{\partial x^2}.$$

The corresponding space-time problem reads

$$\begin{array}{l} \text{Minimize} \quad \Psi(y(S)),\\ \text{over } S > 0, \ (w^0, w, y^0, y) : [0, S] \to \mathbb{R}^9 \text{ s. t.}\\ \\ \frac{dy^0}{ds} = w^0,\\ \\ \frac{dy}{ds} = f(y)w^0 + \sum_{i=1}^2 g_i(y)w^i \text{ a. e. } s \in [0, S],\\ y^0(0) = 0, \ y(0) = (1, 0, 0, 0, 0), \ (y^0, y)(S) \in \{1\} \times \mathfrak{T}. \end{array}$$

$$(24)$$

Let us consider the feasible space-time process $(\hat{S}, \hat{w}^0, \hat{w}, \hat{y}^0, \hat{y})$, where $\hat{S} = \sqrt{2}$,

$$(\hat{w}^0, \hat{w}^1, \hat{w}^2) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right),$$

and

and

$$(\hat{y}^0, \hat{y}^1, \dots, \hat{y}^5) = \left(\frac{\sqrt{2}s}{2}, 1 - \frac{\sqrt{2}s}{2}, 0, 0, 0, \frac{\sqrt{2}s}{2}\right)$$

on $[0, \sqrt{2}]$. In the sequel, we prove that $(\hat{S}, \hat{w}^0, \hat{w}, \hat{y}^0, \hat{y})$ is an *extremal* for the space-time problem (24) –that is, it verifies conditions (i)–(v) of Theorem 3.1 for some multiplier– but there is no non-zero multiplier for which all the necessary conditions in Theorem 3.1 are met.

The adjoint equation and the non-transversality condition read

$$\begin{cases} \frac{dp_0}{ds} = 0, \\ \frac{dp_1}{ds} = p_4(1 - \hat{y}^1 - \hat{y}^5)\hat{w}^0 \\ \frac{dp_2}{ds} = p_3\hat{y}^2\hat{w}^1 - p_4\hat{y}^2\hat{w}^0 \\ \frac{dp_3}{ds} = -p_4\hat{y}^3\hat{w}^0 \\ \frac{dp_4}{ds} = 0 \\ \frac{dp_5}{ds} = p_4(1 - \hat{y}^1 - \hat{y}^5)\hat{w}^0 \\ (p_0, p(\sqrt{2})) = -\lambda(0, 0, 0, 1, 1, 0) - \mathbb{R}^3 \times \{(0, 0, 0)\}, \end{cases}$$

with $\lambda \ge 0$. Therefore, p_0 , p_1 , and p_3 are arbitrary real constants, while $p_3 = p_4 = -\lambda$ and $p_5 = 0$. Moreover, by (the first order conditions) (17) and (18) we get

 $p_0\hat{w}^0 + p_1\hat{w}^1 + p_2\hat{w}^2 = p_0\frac{\sqrt{2}}{2} - p_1\frac{\sqrt{2}}{2} = 0$

$$p_0 w^0 + p_1 w^1 + p_2 w^2 \le 0$$

for all $(w^0, w^1, w^2) \in [0, +\infty) \times \mathbb{R}^2$ with $w^0 + |w| = 1$. Hence there exists a non-trivial multiplier (p_0, p, λ) , necessarily of the form

$$(p_0, p, \lambda) = (0, 0, 0, -\lambda, -\lambda, 0, \lambda), \qquad \lambda > 0$$

However, since

$$\left[[g_1,g_2],g_2\right] = -\frac{\partial}{\partial x^3},$$

the higher-order condition (20) implies $p_3 \equiv 0$, thus $\lambda = 0$, and then $(p_0, p, \lambda) = (0, 0, 0)$. Therefore, we can conclude that $(\hat{S}, \hat{w}^0, \hat{w}, \hat{y}^0, \hat{y})$ is not a minimizer, since there is no non-zero multiplier verifying the higher order necessary conditions in Theorem 3.1.

Actually, it is not difficult to see that the space-time process $(\bar{S}, \bar{w}^0, \bar{w}, \bar{y}^0, \bar{y})$, where $\bar{S} = \sqrt{2} + 4\sqrt[3]{2}$,

$$I_0 := [0, \sqrt{2}], \ I_i := \sqrt{2} + ((j-1)\sqrt[3]{2}, j\sqrt[3]{2}], \ j = 1, \dots, 4,$$

$$(\bar{w}^0, \bar{w}^1, \bar{w}^2) = \begin{cases} \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right) & I_0 \\ (0, -1, 0) & I_1 \\ (0, 0, -1) & I_2 \\ (0, 1, 0) & I_3 \\ (0, 0, 1) & I_4 \end{cases}$$

and

$$(\bar{y}^0, \bar{y}^1, \dots, \bar{y}^5) =$$

$$\left(\begin{array}{c} \left(\frac{\sqrt{2}}{2}s, 1 - \frac{\sqrt{2}}{2}s, 0, 0, 0, \frac{\sqrt{2}}{2}s\right) \\ \left(1 - \sqrt{2}s, 0, 0, 0, \frac{\sqrt{2}}{2}s\right) \end{array} \right)$$
 I_0

$$\begin{pmatrix} 1, \sqrt{2} - s, 0, 0, 0, 1 \\ 1, -\sqrt[3]{2}, \sqrt{2} + \sqrt[3]{2} - s, 0, 0, 1 \end{pmatrix}$$
 I_1

$$\begin{pmatrix} (1, \sqrt{2}, \sqrt{2} + \sqrt{2} - 3, (3, 7)) \\ (1, s - \sqrt{2} - 3\sqrt[3]{2}, -\sqrt[3]{2}, \frac{(\sqrt[3]{2})^2}{2} (\sqrt{2} + 2\sqrt[3]{2} - s), 0, 1 \end{pmatrix} I_3$$

$$(1, 0, s - \sqrt{2} - 4\sqrt[3]{2}, -1, 0, 1)$$
 $I_4,$

is a global minimizer. In particular, this process steers the state (1,0,0,0,0) to (0,0,0,0,1) by a uniform rectilinear motion until time t = 1. After that, the state jumps instantly to (0,0,-1,0,1).

The non-transversality condition (15) now reads

$$\begin{pmatrix} p_0, p(\sqrt{2} + 4\sqrt[3]{2}) \\ = -\lambda(0, 0, 0, 1, 1, 0) - \mathbb{R}^3 \times (-\infty, 0] \times \{(0, 0)\}, \end{cases}$$

with $\lambda \geq 0$. This yields

$$(p_0, p(\sqrt{2} + 4\sqrt[3]{2})) = (c_0, c_1, c_2, -\lambda + c_3, -\lambda, 0)$$

with $c_0, c_1, c_2 \in \mathbb{R}$ and $c_3 \ge 0$. Choosing $c_0 = c_1 = c_2 = 0$, $c_3 = 1$ and $\lambda = 1$, we get the non-trivial, constant multiplier

$$(p_0, p_1, p_2, p_3, p_4, p_5, \lambda) = (0, 0, 0, 0, -1, 0, 1),$$

which satisfies all the necessary conditions in Theorem 3.1 (and agrees with the strengthened non-triviality condition (21), i.e. $(p, \lambda) \neq (0, 0)$, that is in force, for $\bar{y}^0(\sqrt{2}+4\sqrt[3]{2}) = 1 > 0$). In particular, the higher order condition (20) is trivially verified, since the vector fields g_1, g_2 , and all the elements of the Lie algebra generated by $\{g_1, g_2\}$ have the fourth component equal to zero.

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