Optimal control of MIMO input-quadratic nonlinear systems

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Abstract— **We study the infinite-horizon optimal control problem for nonlinear, multi-input, input-quadratic systems. It is shown that optimality of the input-quadratic closed-loop system is intimately related to the property that an auxiliary input-affine system possesses a $\mathcal{L}_2$-gain smaller than one. Such equivalence is established, or approximated, by relying on (a combination of) three alternative sets of technical conditions based *(i)* on the inclusion of the gradient of the underlying storage function in a certain co-distribution, *(ii)* on verifying specific algebraic inequalities, *(iii)* or achieved *dynamically* by considering the immersion of the original nonlinear plant into a system defined on an augmented state-space.**

I. INTRODUCTION

While the class of input-affine nonlinear systems has been thoroughly studied in the past decades (see, *e.g.*, [3], [4], [13]), the literature concerning more general systems is instead mainly focused on plants that exhibit simultaneously a generic nonlinear dependence both on the state and on the control input, hence typically losing the particularly interesting structural and constructive insight acquired for input-affine systems. It is then not surprising that results dealing with input-quadratic nonlinear systems are rather limited and mostly hinging upon the notion of *Control Lyapunov Function* [5], [14], [6], despite the fact that the study of such a class of systems is significantly motivated by practical applications, including for instance magnetic systems [6] and micro-electromechanical systems (MEMS), based on electromagnetic or electrostatic actuation forces [9]. Moreover, quadratic inputs may appear in intermediate steps of the popular *back-stepping* stabilizing procedure, see *e.g.* [1] for the ball and beam example or [2] for the transient stabilization problem in multimachine power systems.

The main contribution of this paper consists in formulating and addressing the infinite-horizon optimal control problem for nonlinear, *multi-input*, input-quadratic systems. In particular, the property of optimality of the closed-loop system is shown to be intimately related to the property that an auxiliary input-affine system possesses a $\mathcal{L}_2$-gain smaller or equal to one and provided that the gradient of the corresponding *storage function* belongs to a certain co-distribution. Alternatively, the explicit solution of the underlying Hamilton-Jacobi partial differential equation and the requirement of the following inclusion may be relaxed to a pair of algebraic inequalities by relying on the use of a dynamic extension, thus obtaining an approximate solution to the optimal control problem. Note that the case of single-input models, which is dealt with in [10], does not require any additional technical condition to establish the above equivalence. Therefore, with respect to [10], herein the analysis of the inclusion of the gradient of the storage function to the annihilator of a specific distribution is expanded and the constructive scheme based on algebraic solutions and dynamic extension is introduced.

The rest of the paper is organized as follows. A few preliminary results are briefly reviewed in Section II, while the considered problem is formulated in Section III. The main results concerning optimality of the underlying input-quadratic system and its approximation via immersion and dynamic extension are the topic of Sections IV and V, respectively. Finally, a simple numerical example illustrates the theory in Section VI.

II. NOTATION AND PRELIMINARIES

The aim of this section consists in briefly reviewing a few basic definitions and results that are instrumental for the following derivations. Towards this end, consider a nonlinear, input-affine system described by equations of the form

$$\dot{x} = f(x) + g(x)v + p(x)w, \qquad (1)$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ denotes the state of the system, $v(t) \in \mathcal{V} \subseteq \mathbb{R}^m$ is a control input and $w(t) \in \mathcal{W} \subseteq \mathbb{R}^d$ is an *exogenous* signal. Assume that f, g_i, $i = 1, ..., m$ and p_j, $j = 1, ..., d$ are smooth vector fields mapping $x \mapsto T_x\mathcal{X}$, where $T_x\mathcal{X}$ denotes the tangent space to $\mathcal{X}$ at x. Moreover, let $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^q$ denote the output of system (1).

Assumption 1: The origin of $\mathbb{R}^n$, contained in $\mathcal{X}$, is an equilibrium point of the system (1) with $v(t) = 0$ and $w(t) = 0$, for all $t \geqslant 0$, namely $f(0) = 0$. ∘

Given a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the following definitions are employed in the rest of the paper. The notation $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ is used to denote a function that is *positive definite* around the origin, namely a function such that $V(0) = 0$, while $V(x)$ is locally strictly positive. Moreover, the notation $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ denotes a locally *positive semi-definite function*, namely such that $V(x) \geqslant 0$.

Definition 1: A nonlinear system $\dot{x} = f(x) + g(x)v$ with output y is said to be *zero-state detectable* from the output y if for any trajectory such that $v(t) \equiv 0$, the condition $y(t) \equiv 0$ implies $\lim_{t \to \infty} x(t) = 0$. ∘

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Assuming initially that $v(t) = 0$ for all $t \geqslant 0$, consider, on the space $\mathbb{R}^d \times \mathbb{R}^q$ of the external variables of (1), a function $s : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$, referred to as *supply rate*.

Definition 2: [13] The system (1), with $v(t) \equiv 0$, is said to be *dissipative* with respect to the *supply rate* s if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, called *storage function*, such that for all $x_0 = x(0) \in \mathcal{X}$, $T \geqslant 0$ and inputs w

$$V(x(T)) \leqslant V(x_0) + \frac{1}{2} \int_0^T s(w(\tau), y(\tau)) d\tau. \quad (2)$$

Moreover, if (2) holds with the equality sign, then system (1) is *lossless* with respect to s. ○

Definition 3: [13] Let $\gamma > 0$. The system (1), with $v(t) \equiv 0$, has $\mathcal{L}_2$-gain less than or equal to γ if it is dissipative with respect to the supply rate $s(w, y) = \gamma^2 \|w\|^2 - \|y\|^2$. ○

The following classical result relates the possibility of imposing, via feedback, a desired $\mathcal{L}_2$-gain to the existence of a solution to a certain first-order quadratic partial differential equation, the so-called *Hamilton-Jacobi* (HJ) equation.

Proposition 1: [12] Consider the nonlinear system (1) and let $\gamma > 0$. Suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ of the *Hamilton-Jacobi* equation

$$
\begin{aligned}
0 = &V_x f(x) + \frac{1}{2} c(x)^\top c(x) \\
&+ \frac{1}{2} V_x \left(\frac{1}{\gamma^2} p(x) p(x)^\top - g(x) g(x)^\top \right) V_x^\top,
\end{aligned}
\quad (3)
$$

with $V(0) = 0$. Then system (1) in closed loop with $v = -g(x)^\top V_x^\top(x)$ has $\mathcal{L}_2$-gain less than or equal to γ from the input from w to the output $[y, v]^\top$, with V as a *storage function*. ◇

III. PROBLEM DEFINITION

Consider a nonlinear, multi-input system, quadratic in the control input, described by equations of the form

$$
\dot{x} = f(x) + g(x)u + \frac{1}{2} \begin{bmatrix} u^\top h_1(x) u \\ \vdots \\ u^\top h_n(x) u \end{bmatrix}, \quad (4)
$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system and $u(t) \in \mathbb{R}^m$ is the control input. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^{m \times m}$, $i = 1, ..., n$ are assumed to be sufficiently smooth and such that $h_i(x) = h_i(x)^\top$ for all x. Throughout the paper we assume that $f(0) = 0$, namely the nonlinear system (4) possesses an equilibrium at the origin with $u = 0$. The optimal control problem of interest can be formulated as follows.

Problem 1: Consider the nonlinear, input-quadratic, system (4) together with the cost functional

$$
\begin{aligned}
J_{x_0}(u) &\triangleq \int_0^\infty \left(q(x(\tau)) + \|u(\tau)\|^2 \right) d\tau \\
&= \int_0^\infty \left(k(x(\tau))^\top k(x(\tau)) + \|u(\tau)\|^2 \right) d\tau
\end{aligned}
\quad (5)
$$

where $q : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is a smooth positive semi-definite function, $q(0) = 0$, and $k : \mathbb{R}^n \to \mathbb{R}^n$ is such that $q(x) = k(x)^\top k(x)$, $\forall x \in \mathbb{R}^n$. The *infinite-horizon optimal control* problem consists in determining a state-feedback control law $u^\star = \alpha(x)$, $\alpha : \mathbb{R}^n \to \mathbb{R}^m$, $\alpha(0) = 0$, smooth mapping, such that $J_{x_0}(u^\star) \leqslant J_{x_0}(u)$ for any u and all x_0. ○

In the following, we refer to the function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$, defined as $V(x_0) = J_{x_0}(u^\star)$, $\forall x_0 \in \mathbb{R}^n$, as the *value function* of the optimal control problem. Consider now the following standing assumption, which is supposed to hold throughout the entire paper.

Assumption 2: The system (4) with output $y = k(x)$ is *zero-state detectable*. ○

Since the presence of the input nonlinearity in (4) may render the solution to Problem 1 a daunting task, a *relaxed* formulation is provided. In particular, the simplification is twofold: on one hand we allow for the presence of an additional running cost while, on the other hand, the design of a dynamic control law, instead of a static state feedback, is permitted, as summarized in the following statement.

Problem 2: Consider the nonlinear, input-quadratic, system (4) together with the cost functional (5) and Assumption 2. The *infinite-horizon dynamic optimal control problem with stability* consists in determining an integer $\nu \geqslant 0$, a positive semi-definite function $\varrho : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}_{\geqslant 0}$, a dynamic control law described by the equations

$$
\begin{aligned}
\dot{\xi} &= \alpha(x, \xi), & \text{(6a)} \\
u &= \beta(x, \xi), & \text{(6b)}
\end{aligned}
$$

with $\xi(t) \in \mathbb{R}^\nu$, $\alpha : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^\nu$, $\beta : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^m$ smooth mappings, $\alpha(0, 0) = 0$, $\beta(0, 0) = 0$, and an open set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^\nu$ containing the origin such that:

(i) the zero equilibrium of the interconnected system (4), (6) is asymptotically stable with region of attraction containing $\mathcal{U}$;

(ii) for any $\tilde{u}(x, \xi)$ and any (x_0, ξ_0) such that the trajectory of system (4), (6a) interconnected by $\tilde{u}$ remains in $\mathcal{U}$ the inequality $\tilde{J}_{x_0, \xi_0}(u) \leqslant \tilde{J}_{x_0, \xi_0}(\tilde{u})$ holds, with the augmented cost $\tilde{J}$ defined as

$$
\tilde{J}_{x_0, \xi_0} \triangleq \int_0^\infty \left(q(x(\tau)) + \varrho(x(\tau), \xi(\tau)) + \|u(\tau)\|^2 \right) d\tau.
\quad (7)
$$

○

For convenience the following notation is introduced. Let the function $h_k^{ij} : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, $j = 1, ..., m$, $k = 1, ..., n$, denote the (i, j) entry of the matrix-valued function h_k, namely

$$
h_k(x) = \begin{bmatrix} h_k^{11}(x) & \dots & h_k^{1m}(x) \\ \vdots & \ddots & \vdots \\ h_k^{m1}(x) & \dots & h_k^{mm}(x) \end{bmatrix}, \quad (8)
$$

and define the mappings $\mu_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mu_{i,j}(x) = \begin{bmatrix} h_1^{i,j}(x) \\ \vdots \\ h_n^{i,j}(x) \end{bmatrix}. \tag{9}$$

Note that, by symmetry of the matrix-valued functions h_i in (4), it follows that $\mu_{i,j} = \mu_{j,i}$ for all $i = 1, ..., m$ and $j = 1, ..., m$. The section is concluded by stating a property of the vector field f, the matrix-valued functions g, and h_i and the positive semi-definite function q that is instrumental for the derivations in the following sections.

Fact 1: There exist non-negative integers n_p, n_d and n_l and smooth mappings $p : \mathbb{R}^n \to \mathbb{R}^{n \times n_p}$, $d : \mathbb{R}^n \to \mathbb{R}^{n \times n_d}$ and $l : \mathbb{R}^n \to \mathbb{R}^{n \times n_l}$ such that

$$f(x) \sum_{i=1}^m \mu_{i,i}(x)^\top + \sum_{i=1}^m \mu_{i,i}(x) f(x)^\top$$
$$+ \frac{1}{2} q(x) \sum_{i=1}^m \sum_{j=i+1}^m (\mu_{i,i}(x)\mu_{j,j}(x)^\top + \mu_{j,j}(x)\mu_{i,i}(x)^\top$$
$$- 2\mu_{i,j}(x)\mu_{i,j}(x)^\top) = p(x)p(x)^\top - d(x)d(x)^\top, \tag{10}$$

and

$$l(x)l(x)^\top = g(x)g(x)^\top + d(x)d(x)^\top, \tag{11}$$

for all $x \in \mathbb{R}^n$. $\circ$

Note that Fact 1 essentially summarizes, in (10), the property of a symmetric matrix-valued function to be decomposed in its positive and negative semi-definite parts, respectively, and, in (11), the property of a positive semi-definite matrix-valued function to be decomposable as the external product of a certain vector field $l(x)$ of suitable dimensions related to the rank of the matrices on the right-hand side.

IV. OPTIMALITY OF MIMO INPUT-QUADRATIC SYSTEMS

In this section we explore the relation between the properties of *optimality* for the underlying input-quadratic system (4) and of *dissipativity* of an auxiliary *input-affine* nonlinear system. To provide concise statements of the following propositions, given the matrix-valued functions h_i, $i = 1, ..., n$ and a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, define the matrix-valued function $\mathcal{M} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$ as

$$\mathcal{M}(x, V_x) \triangleq I + V_{x_1} h_1(x) + ... + V_{x_n} h_n(x). \tag{12}$$

Then, by distinguishing between the set of *diagonal* vector-valued functions $\mu^d \triangleq \{\mu_{1,1}, ..., \mu_{m,m}\}$ and the *off-diagonal* ones $\mu^o \triangleq \{\mu_{1,2}, ..., \mu_{1,m}, \mu_{2,3}, ...\mu_{2,m}, ...\mu_{m-1,m}\}$, without

repetitions, it can be shown that

$$\det(\mathcal{M}(x, V_x)) = 1 + \sum_{i=1}^m L_{\mu_{i,i}} V$$
$$+ \sum_{i=1}^m \sum_{j=i+1}^m \left(L_{\mu_{i,i}} V L_{\mu_{j,j}} V - (L_{\mu_{i,j}} V)^2 \right)$$
$$+ \mathcal{D}_1(V_x, \mu^d) + \mathcal{D}_2(V_x, \mu^d, \mu^o), \tag{13}$$

with $\mathcal{D}_1 \triangleq \sum_{i_1=0}^1 \cdots \sum_{i_m=0}^1 \mathcal{I}_\mathcal{E}\left(\sum_{k=1}^m i_k\right)\left((L_{\mu_{1,1}} V)^{i_1} ...(L_{\mu_{m,m}} V)^{i_m}\right)$, where $\mathcal{I}_\mathcal{E} : \mathbb{N} \to \{0, 1\}$ denotes the *indicator function* of the set $\mathcal{E} \triangleq \{r \in \mathbb{N} : r \geqslant 3\}$, and with $\mathcal{D}_2$ such that $\mathcal{D}_2(V_x, \mu^d(x), 0) = 0$, for any $x \in \mathbb{R}^n$. Note that the terms $\mathcal{D}_1$ and $\mathcal{D}_2$ are identically equal to zero if $m \leqslant 2$. Moreover

$$\mathcal{M}^{\mathrm{ad}}(x, V_x) = I + \Psi(V_x, \mu^d, \mu^o), \tag{14}$$

where $\mathcal{M}^{\mathrm{ad}}$ denotes the adjoint matrix of $\mathcal{M}$ and the matrix-valued function Ψ is such that

$$\Psi(V_x, \mu^d(x), 0) = \mathrm{blkdiag}\left\{\mathcal{P}_i(V_x, \mu^d)\right\}_{i=1,...,m}, \tag{15}$$

with $\mathcal{P}_j \triangleq \sum_{i_1=0}^1 \cdots \sum_{i_{j-1}=0}^1 \sum_{i_{j+1}=0}^1 \cdots \sum_{i_m=0}^1 ((L_{\mu_{1,1}} V)^{i_1} ...(L_{\mu_{1,1}} V)^{i_{j-1}} (L_{\mu_{1,1}} V)^{i_{j+1}} ...(L_{\mu_{m,m}} V)^{i_m})$.

The following result provides the characterization of the optimal solution to Problem 1 in a somewhat trivial case. Nonetheless, its proof permits the illustration of the main ideas on which also the proofs of the following results are based, hence it is explicitly reported.

Proposition 2: Consider the nonlinear, input-quadratic, system (4) together with the cost functional (5) and Assumption 2. Consider the *input-affine* system

$$\dot{x} = f(x) + g(x)v, \tag{16}$$

where $v(t) \in \mathbb{R}^m$ is a control input, and suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ to the HJ equation

$$V_x f(x) + \frac{1}{2} q(x) - \frac{1}{2} V_x g(x) g(x)^\top V_x^\top = 0. \tag{17}$$

Suppose, in addition, that the *value function* $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ is such that[1] $L_{\mu_{i,j}} V = 0$, for all $i = 1, ..., m$, $j = 1, ..., m$. Then there exists $\mathcal{U} \subseteq \mathbb{R}^n$, containing the origin, such that the state-feedback

$$u^\star(x) = -g(x)^\top V_x^\top \tag{18}$$

solves Problem 1 for any $x \in \mathcal{U}$. Moreover, V is the corresponding *value function*. $\diamond$

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[1] Given a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ and a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $L_f \lambda$ denotes the Lie derivative of λ along f, namely $L_f \lambda(x) = \frac{\partial \lambda}{\partial x} f(x)$.

Proof: Consider the input-quadratic system (4) together with the cost functional (5). The corresponding Hamilton-Jacobi-Bellman (HJB) partial differential equation is

$$0 = \min_u \{\mathcal{HJB}(x,u)\} \triangleq \min_u \left\{ \frac{1}{2}q(x) + \frac{1}{2}u^\top u \right.$$

$$\left. + V_x f(x) + V_x g(x)u + \frac{1}{2}\sum_{i=1}^n V_{x_i} u^\top h_i(x)u \right\}$$

$$= \min_u \left\{ V_x f(x) + V_x g(x)u + \frac{1}{2}q(x) + \frac{1}{2}u^\top \mathcal{M}(x, V_x)u \right\},$$
$$(19)$$

in the unknown $V : \mathbb{R}^n \to \mathbb{R}$, $V(0) = 0$. Then, by recalling (13) and by continuity of the involved functions, there exists a non-empty neighborhood of the origin $\mathcal{W}$ such that $\det(\mathcal{M}(x, V_x)) > 0$ for any $x \in \mathcal{W}$. Therefore, the minimum with respect to the control input of $\mathcal{HJB}(x,u)$ is continuously achieved at $\hat{u}(x) = -\mathcal{M}(x, V_x)^{-1}g(x)^\top V_x^\top$. Replacing then $\hat{u}(x)$ into (19), the latter reduces to

$$0 = V_x f(x) + \frac{1}{2}q(x) - \frac{1}{2}V_x g(x)\mathcal{M}^{-1}(x, V_x)g(x)^\top V_x^\top$$

$$= \left(V_x f(x) + \frac{1}{2}q(x)\right)\det(\mathcal{M}(x, V_x))$$

$$- \frac{1}{2}V_x g(x)\mathcal{M}^{\mathrm{ad}}(x, V_x)g(x)^\top V_x^\top .$$
$$(20)$$

Therefore, by relying on the equations (13) and (14) and by recalling the decomposition in (10), the partial differential equation (20) can be equivalently arranged as

$$0 = V_x \bar{f}(x) + \frac{1}{2}V_x \left(p(x)p(x)^\top - l(x)l(x)^\top\right)V_x^\top$$
$$+ \frac{1}{2}k(x)^\top k(x) + \mathcal{R}(V_x, \mu^d, \mu^o) \tag{21}$$

with $\bar{f}(x) = f(x) + (1/2)q(x)\sum_{i=1}^m \mu_{i,i}(x)$ and

$$\mathcal{R}(V_x, \mu^d, \mu^o) \triangleq \left(V_x f(x) + \frac{1}{2}q(x)\right)(\mathcal{D}_1 + \mathcal{D}_2)$$

$$+ V_x f(x)\sum_{i=1}^m \sum_{j=i+1}^m \left(L_{\mu_{i,i}}V L_{\mu_{j,j}}V - (L_{\mu_{i,j}}V)^2\right) \tag{22}$$

$$- \frac{1}{2}V_x g(x)\Psi g(x)^\top V_x^\top .$$

It is evident that the equation (21) reduces to (17) provided the solution to the latter satisfies $L_{\mu_{i,j}}V = 0$, for all $i = 1, ..., m$ and $j = 1, ..., m$, thus concluding the proof. $\square$

Example 1. To illustrate the constructions of Proposition 2 consider a nonlinear, input-quadratic system described by

$$\dot{x}_1 = x_2 + u_2 + \frac{1}{2}x_2\sigma_1(x)u_1^2 + x_2\sigma_2(x)u_1 u_2$$
$$+ \frac{1}{2}x_2\sigma_3(x)u_2^2,$$

$$\dot{x}_2 = u_1 - \frac{1}{2}x_1\sigma_1(x)u_1^2 - x_1\sigma_2(x)u_1 u_2 - \frac{1}{2}x_1\sigma_3(x)u_2^2, \tag{23}$$

which possesses the form (4), with $f(x) = Ax$, $g(x) = B$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{24}$$

and with

$$h_1 = x_2 \begin{bmatrix} \sigma_1(x) & \sigma_2(x) \\ \sigma_2(x) & \sigma_3(x) \end{bmatrix}, \quad h_2 = -x_1 \begin{bmatrix} \sigma_1(x) & \sigma_2(x) \\ \sigma_2(x) & \sigma_3(x) \end{bmatrix}, \tag{25}$$

for some continuous functions $\sigma_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, 3$. Consider the cost functional (5) with $q(x) = x_1^2 + 2x_1 x_2 + x_2^2$, which is positive semi-definite. Then, recalling the auxiliary input-affine (linear) system defined in (16), namely

$$\dot{x} = Ax + Bv, \tag{26}$$

it can be easily shown that the equation (17) reduces to the classic Algebraic Riccati Equation (ARE) $0 = PA + A^\top P + Q - PBB^\top P$, with $Q \in \mathbb{R}^{2\times 2}$ such that $q(x) = x^\top Qx$, which admits the positive definite solution $P = I$. Therefore, letting $V(x) = (1/2)x^\top Px$, since $V_x = [x_1, x_2]$ and noting that by definition

$$\mu_{1,1} = \sigma_1 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \mu_{1,2} = \sigma_2 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \mu_{2,2} = \sigma_3 \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \tag{27}$$

it follows that $L_{\mu_{i,j}}V = 0$, for any $x \in \mathbb{R}^2$ and for any $i = 1, 2$, $j = 1, 2$. Thus, the optimal solution of the auxiliary problem for the linear system, *i.e.* $v^\star(x) = [-x_2, -x_1]^\top$, constitutes also the optimal solution for the family of input-quadratic nonlinear systems defined in (23). $\circ$

The statement of the previous proposition entails that the optimal solution to Problem 1 is derived by a value function for the system (30) provided the latter function satisfies additional conditions in terms of its Lie derivatives along the vector fields $\mu_{i,j}$. The latter technical condition can be circumvented if a certain function of the state and of the storage function is (locally) negative semi-definite.

Proposition 3: Consider the nonlinear, input-quadratic, system (4) together with the cost functional (5) and Assumption 2. Consider the *input-affine* system

$$\dot{x} = \bar{f}(x) + l(x)v + p(x)w, \tag{28}$$

with $\bar{f}(x) = f(x) + (1/2)q(x)\sum_{i=1}^m \mu_{i,i}(x)$, where $v(t) \in \mathbb{R}^{n_l}$ is a control input and $w(t) \in \mathbb{R}^{n_p}$ is a disturbance input. Suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ to the HJ equation

$$V_x \bar{f}(x) + \frac{1}{2}q(x) + \frac{1}{2}V_x(p(x)p(x)^\top - l(x)l(x)^\top)V_x^\top = 0, \tag{29}$$

with the property that there exists an open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, such that $\mathcal{R}(V_x, \mu^d, \mu^o) \leqslant 0$ for any $x \in \mathcal{U}$, with $\mathcal{R}$ defined in (22). Then there exists $\hat{\mathcal{U}} \subseteq \mathcal{U}$ such that the state-feedback $u^\star(x) = -\mathcal{M}(x, V_x)^{-1}g(x)^\top V_x^\top$ solves Problem 2 for any $x \in \hat{\mathcal{U}} \subseteq \mathcal{U}$, with $\nu = 0$ and $\varrho(x) = -\mathcal{R}(V_x, \mu^d, \mu^o) \geqslant 0$. $\diamond$

Proof: The claim is proved by following the same arguments as those employed in the proof of Proposition 2. In

particular, it can be easily shown that a solution to the partial differential equation (29) with the additional property that $\mathcal{R}(V_x, \mu^d, \mu^o) \leqslant 0$ for all x in a neighborhood of the origin is such that also (21), hence (19), holds with the inequality sign. As a consequence, Problem 2 is solved by a static feedback and with the additional running cost provided by the positive semi-definite term $-\mathcal{R}$, with the set $\mathcal{U}$ obtained by considering the intersection of $\mathcal{W}$, defined in the proof of Proposition 2, $\mathcal{U}$ and a level-set of the function V. $\square$

Remark 1: By inspecting the structure of the auxiliary input-affine system (28) and of the partial differential equation (29) in the statement of Proposition 3, it is evident that the optimality properties of the original nonlinear, input-quadratic system (4) with respect to (7) are in fact related to the property that the input-affine system (28) in closed loop with $v = -l(x)^\top V_x^\top$ possesses an $\mathcal{L}_2$-gain from the (virtual) disturbance input w to the output $[y, v]$, with $y = k(x)$, less than or equal to one, provided the underlying *storage function* satisfies an additional technical condition. ▲

The following result, instead, combines (relaxed) versions of the conditions introduced in the two previous statements. To this end, consider a modified version of the decompositions in Fact 1, namely define $\hat{p}$ and $\hat{l}$ such that $\hat{p}(x) = p(x)|_{\mu^o=0}$ and $\hat{l}(x) = l(x)|_{\mu^o=0}$. Note that the *off-diagonal* functions μ^o are equal to zero if the matrix-valued functions h_i are diagonal, namely if (4) does not contain mixed terms $u_i u_j$ for all i and j.

Proposition 4: Consider the nonlinear, input-quadratic, system (4) together with the cost functional (5) and Assumption 2. Consider the *input-affine* system

$$\dot{x} = \bar{f}(x) + \hat{l}(x)v + \hat{p}(x)w \,, \tag{30}$$

with $\bar{f}(x) = f(x) + (1/2)q(x)\sum_{i=1}^{m}\mu_{i,i}(x)$, where $v(t) \in \mathbb{R}^{n_l}$ is a control input and $w(t) \in \mathbb{R}^{n_p}$ is a disturbance input. Suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ to the HJ inequality

$$V_x\bar{f}(x) + \frac{1}{2}q(x) - \frac{1}{2}V_x(\hat{p}(x)\hat{p}(x)^\top - \hat{l}(x)\hat{l}(x)^\top)V_x^\top \leqslant 0 \,, \tag{31}$$

namely system (30) in closed loop with $v = -\hat{l}(x)^\top V_x^\top$ has $\mathcal{L}_2$- gain less than or equal to one from w to $[y, v]^\top$. Suppose, in addition, that the *storage function* $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ is such that

(i) $L_{\mu_{i,j}}V = 0$, for any mapping $\mu_{i,j}$ belonging to $\mu^o(x)$;
(ii) there exists an open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, such that $\mathcal{R}_r(V_x, \mu^d) \leqslant 0$, with

$$\mathcal{R}_r(V_x, \mu^d) \triangleq \left(V_x f(x) + \frac{1}{2}q(x) \right) \mathcal{D}_1$$
$$+ V_x f(x) \sum_{i=1}^{m} \sum_{j=i+1}^{m} L_{\mu_{i,i}}V L_{\mu_{j,j}}V \tag{32}$$
$$- \frac{1}{2}\sum_{i=1}^{m} V_x g_i(x)\mathcal{P}_i(V_x, \mu^d)g_i(x)^\top V_x^\top \,,$$

with $g_i : \mathbb{R}^n \to \mathbb{R}^n$ denoting the i-th column of g, for all $x \in \mathcal{U}$. Then there exists $\hat{\mathcal{U}} \subseteq \mathcal{U}$ such that the state-feedback

$$u^\star(x) = -\left(1 + \sum_{i=1}^{m} V_x\mu_{i,i} + \sum_{i=1}^{m}\sum_{j=i+1}^{m}(V_x\mu_{i,i}V_x\mu_{j,j}) \right.$$
$$\left. + \mathcal{D}_1(V_x, \mu^d) \right)^{-1} g(x)^\top V_x^\top \tag{33}$$

solves Problem 2 for any $x \in \hat{\mathcal{U}}$. $\diamond$

Remark 2: By introducing the *distribution*

$$\Delta_{\mu^o}(x) \triangleq \mathrm{span}\{\mu_{1,2}, ..., \mu_{1,m}, \mu_{2,3}, ...\mu_{2,m}, ...\mu_{m-1,m}\}, \tag{34}$$

spanned by the mappings in μ^o, the requirement entailed by item *(i)* of Proposition 4 can be equivalently formulated as the inclusion $\mathrm{d}V \in \Delta_{\mu^o}^\perp(x)$, for all $x \in \mathcal{U}$, where $\Delta_{\mu^o}^\perp$ denotes the *left annihilator* of Δ_{μ^o}. Moreover, the existence of at least one *exact co-vector* in $\Delta_{\mu^o}^\perp(x)$ is guaranteed provided the distribution Δ_{μ^o} is *nonsingular* around the origin and its *involutive closure* has dimension smaller than n. Note that in general the distribution Δ_{μ^o} contains $m(m-1)/2$ vector fields in $\mathbb{R}^n$, hence it is reasonable to expect that the distribution is nonsingular whenever $n > m(m-1)/2$. ▲

V. DYNAMIC SOLUTION VIA SYSTEM IMMERSION

While the discussion in Remark 2 deals with the requirement in item *(i)*, the following statement tackles the inequality condition in item *(ii)*. In particular, the rationale behind the following result is that - instead of satisfying the inequality $\mathcal{R}_r(V_x, \mu^d) \leqslant 0$ by implementing the static state feedback (33) - a similar inequality is dynamically enforced via the selection of the time-evolution of a dynamic extension ξ, as suggested in (6). More precisely, by relying on constructions similar to those introduced in [11] and then further extended in [7] and [8], the following statement yields a characterization of the solution to Problem 2 based only on *algebraic* conditions rather than partial differential equations. To this end, consider first the partial differential equation the solution of which - provided it satisfies item *(i)* of Proposition 4 - should be approximated via an algebraic solution and the immersion of the system into an auxiliary one defined on an extended state-space, *i.e.*

$$0 = V_x\bar{f}(x) + \frac{1}{2}V_x\left(p(x)p(x)^\top - l(x)l(x)^\top\right)V_x^\top$$
$$+ \frac{1}{2}q(x) + \mathcal{R}_r(V_x, \mu^d) \,. \tag{35}$$

Following [7], the matrix-valued function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $P(x) = P(x)^\top \geqslant 0$ for any $x \in \mathbb{R}^n$, is an *approximate* algebraic solution of (35) if there exists a matrix-valued function $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with $\Sigma(0) > 0$ and $\Sigma(x) \geqslant 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$, such that

$$0 \geqslant P(x)\bar{F}(x) + \bar{F}(x)^\top P(x) + Q(x)$$
$$+ P(x)(p(x)p(x)^\top - l(x)l(x)^\top)P(x) + \Sigma(x) \,, \tag{36}$$

where $\bar{F} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is any matrix-valued function such that $\bar{f}(x) = \bar{F}(x)x$, for all x. Moreover, to provide a concise statement of the following result, define the function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$V(x, \xi) = \frac{1}{2} x^\top P(\xi) x + \|x - \xi\|_R^2, \qquad (37)$$

with $R = R^\top > 0$ such that V is locally positive definite, with the property that its partial derivatives satisfy

$$V_x = x^\top P(\xi) + (x - \xi)^\top R \triangleq x^\top P(x) + \delta(x, \xi)^\top, \quad (38a)$$

$$V_\xi = \frac{1}{2} x^\top \nabla_\xi (P(\xi) x) - (x - \xi)^\top R. \qquad (38b)$$

Finally, let the function $\pi : \mathbb{R}^n \to \mathbb{R}^n$ be such that $V_\xi|_{\xi = \pi(x)} = 0$ for any $x \in \mathbb{R}^n$.

Proposition 5: Consider the nonlinear, input-quadratic, system (4) together with the cost functional (5) and Assumption 2. Consider the *input-affine* system (30), where $v(t) \in \mathbb{R}^{n_l}$ is a control input and $w(t) \in \mathbb{R}^{n_p}$ is a disturbance input, and suppose that there exists a continuous algebraic solution P to the algebraic inequality (36) such that

(i) $(x^\top P(x) + \delta(x, \xi)^\top) \mu_{i,j}(x) = 0$, for any mapping $\mu_{i,j}$ belonging to μ^o;

(ii) there exists an open set $\mathcal{U} \subset \mathbb{R}^n$, containing the origin, in which

$$0 > \Big(\delta(x, \xi)^\top \bar{f}(x) - x^\top \Sigma(x) x$$
$$+ \mathcal{R}_r(P(x)x + \delta(x, \xi), \mu^d)$$
$$+ \frac{1}{2} \delta(x, \xi)^\top (p(x)p(x)^\top - l(x)l(x)^\top)(2P(x)x$$
$$+ \delta(x, \xi)) \Big)|_{\xi = \pi(x)}.$$

Then, there exist $\hat{\mathcal{U}} \subseteq \mathcal{U}$ and $\kappa^\star > 0$ such that the dynamic state-feedback

$$\begin{aligned} \dot{\xi} &= -\kappa V_\xi^\top, \\ u &= -g(x)^\top (P(\xi)x + R(x - \xi)) \end{aligned} \qquad (39)$$

solves Problem 2 with $\nu = n$ for any $x \in \hat{\mathcal{U}}$ and $\kappa \in (\kappa^\star, \infty)$. $\diamond$

VI. A SIMPLE EXAMPLE

The theory is corroborated in this section by discussing a simple numerical example. To this end, consider the input-quadratic system described by the equation

$$\dot{x} = u_1 + u_1 u_2, \qquad (40)$$

which may be expressed as (4) with $f(x) = 0$, $g(x) = [1, 0]$ and

$$h(x) = h_1(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad (41)$$

for any $x \in \mathbb{R}$. According to the notation introduced above, $\mu_{1,1}(x) = \mu_{2,2}(x) = 0$ and $\mu_{1,2}(x) = 1$, for any x, and hence, consequently $\det(\mathcal{M}(x, V_x)) = 1 - V_x^2$ and

$$\mathcal{M}^{\mathrm{ad}}(x, V_x) = \begin{bmatrix} 1 & -V_x \\ -V_x & 1 \end{bmatrix}.$$

Therefore, the partial differential equation associated to the auxiliary disturbance attenuation problem is

$$0 = \frac{1}{2} x^2 - \frac{1}{2}(1 + x^2)V_x^2, \qquad (42)$$

hence with $\mathcal{R}(V_x, \mu^d, \mu^o)$ in (22) equal to zero for all x, the solution of which is given by

$$V = \sqrt{x^2 + 1} - 1. \qquad (43)$$

The latter function then yields the optimal control input

$$u^\star(x) = \begin{bmatrix} -(\sqrt{x^2 + 1})x \\ x^2 \end{bmatrix}. \qquad (44)$$

VII. CONCLUSIONS

In this paper we have studied the infinite-horizon optimal control problem for multi-input input-quadratic nonlinear systems. It has been shown that the property of optimality of the closed-loop system is in fact strongly related to the property that an auxiliary input-affine system possesses a $\mathcal{L}_2$-gain smaller or equal to one and provided an additional technical condition is verified. Such conditions have then been relaxed to a pair of algebraic inequalities that are combined with a dynamic extension to yield an approximate solution to the optimal control problem.

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