State Variance Estimation in Large-Scale Network Systems

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***Abstract*— State variance of a network system is a nonlinear functional computed as the squared deviation of the network's state vector. Such a quantity is useful to monitor how much the states of network nodes are spread around their average mean. Estimating state variance is crucial when the full state estimation of a network system is not possible due to limited computational and sensing resources. We propose a novel methodology to estimate the state variance in a computationally efficient way. First, clusters are identified in the network such that the state variance can be approximated from the average states of the clusters. Then, the approximated state variance is estimated from the average state observer.**

I. Introduction

Large-scale network systems are ubiquitous in modern engineering applications such as traffic networks, building thermal systems, and distributed sensor networks. Complete monitoring of such large-scale systems is usually not possible due to limited computational and sensing resources. Limited computational resources can make the real-time state estimation task infeasible, whereas limited number of sensors may render the system unobservable. It is reasonable, therefore, to monitor the network system by estimating the aggregated state profiles such as the state average and variance. In particular, our goal is to estimate the aggregated state profiles of unmeasured nodes from the knowledge of the state measurements at few gateway (or measured) nodes.

The state average is defined as the mean of the unmeasured states, whereas the state variance is the squared deviation of the unmeasured states from the state average. Both are state functionals that evolve with time and are meaningful in several applications. For instance, estimating the traffic density of every road in an urban traffic network is often not possible [1]. However, the aggregated quantities like average and variance of traffic densities over a sector of the traffic network are considered to be quite suitable for monitoring traffic. In building thermal systems [2], the average (mean operative temperature) and the variance inside a room provide a good measure for thermal comfort [3]. Finally, in distributed sensor networks [4], estimating the state average and variance allows to monitor the consensus value and consensus formation, respectively.

State average estimation has been studied in [5], which rely on the theory of minimum-order linear functional observers [6]–[8]. However, to the best of our knowledge, the problem of state variance estimation is novel and has not been studied in the literature.

The fundamental concepts of nonlinear functional observers are presented in [9], which can be employed to estimate the state variance. However, the fundamental limit with regards to the state variance is that a nonlinear functional observer should have an order of at least the number of unmeasured nodes minus one. That is, even if the observer is of minimum order, it is equivalent to estimate all but one states of the unmeasured nodes in order to asymptotically estimate the state variance. This makes the problem trivial because it can be argued that, in large-scale systems, estimating all but one states is almost same as estimating all the states. Thus, if we can estimate all the states, there is no need to estimate the state variance. However, in this paper, we assume that the system need not be observable.

The infeasibility of designing a nonlinear functional observer directs us towards the estimation of state variance in an approximate sense. That is, we first approximate the state variance by projecting the network system on a lower dimensional state space by clustering. The state vector of the projected system include the states of the measured nodes and the average states of the clusters. Then, by estimating the average states of the clusters, we can estimate the approximated state variance.

II. Problem Statement

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \cdots, \ell\}$ is the set of nodes, $\ell = m + n$ with m measured nodes and n unmeasured nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges. Let $A \in \mathbb{R}^{\ell \times \ell}$ be a Metzler matrix such that, for $i \neq j$, $[A]_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $[A]_{ij} = 0$ otherwise; also, $[A]_{ii} \leq 0$ for all $i \in \mathcal{V}$. In particular, we consider $A = \Lambda - \mathcal{L}(\mathcal{G})$, where $\mathcal{L}(\mathcal{G})$ is the Laplacian matrix of $\mathcal{G}$ and $\Lambda = \texttt{diag}([\, a_{11} \ \cdots \ a_{\ell\ell} \,])$ with $a_{ii} \leq 0$ for all $i \in \mathcal{V}$. Thus, the spectrum $\texttt{eig}(A) \subset \mathbb{C}_{<0} \cup \{0\}$, where $\mathbb{C}_{<0}$ is the open left-half complex plane.

Each node i has a state $x_i(t) \in \mathcal{X}_i \subset \mathbb{R}$ for $t \in \mathbb{R}_{\geq 0}$, where $\mathcal{X}_i$ is a bounded interval. A network system with a structure given by the graph $\mathcal{G}$ is represented as

$$\boldsymbol{\Sigma} : \begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

where $\mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^{m+n}$ is the network state vector, $\mathbf{u}(t) \in \mathbb{R}^p$ is the input vector, and $\mathbf{y}(t) \in \mathbb{R}^m$ is the measurement vector. Without loss of generality, we suppose $\mathcal{V}_1 = \{1, \cdots, m\}$ to be the set of measured nodes and $\mathcal{V}_2 = \{m+1, \cdots, m+n\}$ to be the set of unmeasured nodes. Then, we can write $\mathbf{x}(t) = [\, \mathbf{x}_1^{\mathrm{T}}(t) \ \mathbf{x}_2^{\mathrm{T}}(t) \,]^{\mathrm{T}}$, where

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$\mathbf{x}_1(t) = \mathbf{y}(t) = [\, x_1(t) \cdots x_m(t) \,]^{\mathrm{T}}$ is the state vector of $\mathcal{V}_1$ and $\mathbf{x}_2(t) = [\, x_{m+1}(t) \cdots x_{m+n}(t) \,]^{\mathrm{T}}$ is the state vector of $\mathcal{V}_2$. Correspondingly, we obtain the following partition of system matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; C = \begin{bmatrix} I_m & 0_{m\times n} \end{bmatrix}.$$

Let the *state average* be $x_{\mathrm{a}}(t) = \frac{1}{n}\sum_{j\in\mathcal{V}_2} x_j(t)$. Then, the *state variance* is defined as

$$x_{\mathrm{v}}(t) = \frac{1}{n}\sum_{j\in\mathcal{V}_2} \big(x_j(t) - x_{\mathrm{a}}(t)\big)^2 = \frac{1}{n}\mathbf{x}_2(t)^{\mathrm{T}} J_n \mathbf{x}_2(t) \quad (1)$$

where the matrix $J_n = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$ is symmetric ($J_n = J_n^{\mathrm{T}}$), idempotent ($J_n^2 = J_n$) and positive semi-definite with I_n the identity matrix of size $n\times n$ and $\mathbf{1}_n$ the vector of ones of size $n \times 1$. Then, the goal is to estimate the state variance $x_{\mathrm{v}}(t)$ when the measurement vector $\mathbf{y}(t)$ and the input vector $\mathbf{u}(t)$ are known.

III. Review of Functional Observers

In this section, we briefly review the fundamental concepts of functional observers [9].

Given a nonlinear functional $x_{\mathrm{v}}(t)$, a functional observer of order k is a system

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \mathbf{f}(\mathbf{w}(t), \mathbf{y}(t), \mathbf{u}(t)) \\ \hat{x}_{\mathrm{v}}(t) &= h(\mathbf{w}(t), \mathbf{y}(t)) \end{aligned} \quad (2)$$

with $\mathbf{f} : \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^k$ and $h : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}$ designed such that the error $x_{\mathrm{v}}(t) - \hat{x}_{\mathrm{v}}(t)$ converges to zero exponentially. Equivalently, there exists an invariant manifold $\mathbf{w} = \mathbf{g}(\mathbf{x})$ such that

$$\begin{aligned} \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\big(A\mathbf{x} + B\mathbf{u}\big) &= \mathbf{f}(\mathbf{g}(\mathbf{x}), \mathbf{y}, \mathbf{u}) \\ x_{\mathrm{v}} &= h(\mathbf{g}(\mathbf{x}), \mathbf{y}). \end{aligned} \quad (3)$$

For linear systems, we need to find a linear map $\mathbf{g}(\mathbf{x}) = P^{\mathrm{T}}\mathbf{x}$, where $P \in \mathbb{R}^{(m+n)\times k}$, in order to satisfy the condition (3). This can be stated as follows:

Proposition 1: Consider an observer (2) with

$$\begin{aligned} \mathbf{f}(\mathbf{w}(t), \mathbf{y}(t), \mathbf{u}(t)) &= M\mathbf{w}(t) + K\mathbf{y}(t) + N\mathbf{u}(t) \\ h(\mathbf{w}(t), \mathbf{y}(t)) &= \mathbf{w}^{\mathrm{T}}(t)D\mathbf{w}(t) + \mathbf{y}^{\mathrm{T}}(t)L\mathbf{y}(t) \end{aligned} \quad (4)$$

where $M \in \mathbb{R}^{k\times k}$ is a Hurwitz matrix and all the other matrices K, N, D, L are of appropriate dimensions. Then, the error $x_{\mathrm{v}}(t) - \hat{x}_{\mathrm{v}}(t)$ converges to zero exponentially if and only if there exists a matrix $P \in \mathbb{R}^{(m+n)\times k}$ such that

$$P^{\mathrm{T}}A - MP^{\mathrm{T}} = KC \quad (5a)$$

$$PDP^{\mathrm{T}} = T \quad (5b)$$

where $T = \begin{bmatrix} -L & 0_{m\times n} \\ 0_{n\times m} & \frac{1}{n}J_n \end{bmatrix}$. ⌟

The proof is a straightforward consequence of (3).

We remark that, in order to design an observer of the form (4) under the constraint that M is Hurwitz, one has to determine the order k and find $P \in \mathbb{R}^{(n+m)\times k}$ that satisfies (5). Finding a minimal order k such that the Sylvester equation (5a) is solvable is known to be quite challenging, [7], [8]. Moreover, in order to solve (5b), we see that the order k of the observer must be at least $n-1$. To elucidate this fact, we suppose $P^{\mathrm{T}} = [\, P_1^{\mathrm{T}} \;\; P_2^{\mathrm{T}} \,]$ with $P_1 \in \mathbb{R}^{m\times k}$ and $P_2 \in \mathbb{R}^{n\times k}$, then (5b) can be written as

$$\begin{bmatrix} P_1DP_1^{\mathrm{T}} & P_1DP_2^{\mathrm{T}} \\ P_2DP_1^{\mathrm{T}} & P_2DP_2^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -L & 0_{m\times n} \\ 0_{n\times m} & \frac{1}{n}J_n \end{bmatrix}.$$

Apart from $P_1DP_1^{\mathrm{T}} = -L$, $P_1DP_2^{\mathrm{T}} = 0$ and $P_2DP_1^{\mathrm{T}} = 0$, we also need to satisfy $P_2DP_2^{\mathrm{T}} = \frac{1}{n}J_n$, which implies that $\mathtt{rank}(P_2) \geq n-1$ because $\mathtt{rank}(J_n) = n-1$. Hence, it is necessary that $k \geq n-1$, which is a lower bound on the order of the functional observer (4).

Even if the functional observer is of minimum order, i.e., $k = n-1$, the estimation is still not feasible because n can be very large. Such an observer estimates all but one states of the unmeasured nodes to compute the state variance. This is because $\mathtt{rank}(J_n) = n-1$ and $\mathbf{1}_n^{\mathrm{T}}J_n = 0$, which means that if we estimate $n-1$ elements of the vector $J_n\mathbf{x}_2(t) = \mathbf{x}_2(t) - \mathbf{1}_n x_{\mathrm{a}}(t)$, the n-th element equals the negative sum of the estimated $n-1$ elements. The problem of interest, however, is to estimate the variance $x_{\mathrm{v}}(t)$ without estimating the whole vector $J_n\mathbf{x}_2(t)$, which is not possible due to the limitation on the order of the observer. Therefore, instead of the asymptotic estimation, i.e., $x_{\mathrm{v}}(t) - \hat{x}_{\mathrm{v}}(t) \to 0$ exponentially as $t \to \infty$, we would like to find an optimal approximate estimation solution, where the order k is chosen according to the available computational capability.

IV. State Variance Approximation by Network Clustering

To approximate the state variance, we approximate the state trajectory $\mathbf{x}_2(t)$ of unmeasured nodes $\mathcal{V}_2$ by projecting it on a lower dimensional state space. That is, we partition $\mathcal{V}_2$ into k clusters such that the states of nodes in each cluster can be approximated by its average state, which is similar to a k-means clustering problem [10]. Finally, the approximated state variance is then computed from the average states of the clusters.

A. Clustering problem

Let $k < n$ to be a given number of clusters and $\mathcal{Q} = \{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$ to be a clustering (or partition) of the unmeasured nodes $\mathcal{V}_2 = \{m+1, \cdots, m+n\}$, where $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k = \mathcal{V}_2$ and $\mathcal{C}_\alpha \cap \mathcal{C}_\beta = \emptyset$, for $\alpha \neq \beta$ and $\alpha, \beta = 1, \cdots, k$. The characteristic matrix $Q \in \mathbb{B}^{n\times k}$ of the clustering $\mathcal{Q}$, for $i \in \{1, \cdots, n\}$ and $\alpha \in \{1, \cdots, k\}$, is defined as

$$[Q]_{i\alpha} = \begin{cases} 1, & \text{if } m+i \in \mathcal{C}_\alpha \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbb{B}^{n\times k} = \{X \in \{0,1\}^{n\times k} : X\mathbf{1}_k = \mathbf{1}_n\}$ is a set that ensures that each cluster is nonempty and each node belongs to at most one cluster.

The rationale for an approximated state variance is as follows: If $Q \in \mathbb{B}^{n\times k}$ is such that $\mathbf{x}_2 \approx Q\mathbf{z}_{\mathrm{a}}$, where $\mathbf{z}_{\mathrm{a}} = Q^{+}\mathbf{x}_2$ and $Q^{+} = (Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}}$, then $x_{\mathrm{v}} \approx \frac{1}{n}\mathbf{z}_{\mathrm{a}}^{\mathrm{T}}Q^{\mathrm{T}}J_nQ\mathbf{z}_{\mathrm{a}}$. That is, if a clustering is such that the states of all nodes in a cluster $\mathcal{C}_\alpha$, for $\alpha = 1, \cdots, k$, can be approximated by its average $x_{\mathrm{a},\alpha} = \frac{1}{n_\alpha}\sum_{j\in\mathcal{C}_\alpha} x_j$, then the state variance can be

approximated as

$$x_{\rm v} \approx \frac{1}{n}\mathbf{z}_{\rm a}^{\rm T}Q^{\rm T}J_nQ\mathbf{z}_{\rm a} = \frac{1}{n}\sum_{\alpha=1}^{k} n_\alpha x_{{\rm a},\alpha}^2(t) - \left(\frac{1}{n}\sum_{\alpha=1}^{k} n_\alpha x_{{\rm a},\alpha}(t)\right)^2 \quad (6)$$

where $\mathbf{z}_{\rm a}(t) = [\, x_{{\rm a},1}(t) \ \cdots \ x_{{\rm a},k}(t)\,]^{\rm T} \in \mathbb{R}^k$ and $n_\alpha = |\mathcal{C}_\alpha|$ with $\sum_{\alpha=1}^k n_\alpha = n$.

Let the multi-cluster deviation vector to be $\boldsymbol{\sigma} = \mathbf{x}_2 - Q\mathbf{z}_{\rm a}$ with i-th entry, for $i = 1, \cdots, n$, given by $\sigma_i = x_j - x_{{\rm a},\alpha}$, where $j = m + i$, $j \in \mathcal{C}_\alpha$, and $\alpha \in \{1, \cdots, k\}$. That is, the entries of $\boldsymbol{\sigma}$ are the differences between the states of unmeasured nodes and the average states of the corresponding clusters. We can write $\boldsymbol{\sigma} = D_Q\mathbf{x}$, where $D_Q = [\ 0_{n\times m} \quad I_n - QQ^+\ ]$. Then, the transfer function from $\mathbf{u}$ to $\boldsymbol{\sigma}$ is given by $\mathbf{G}_\sigma(s) = D_Q(sI - A)^{-1}B$ with the $\mathcal{H}_2(\tau)$-norm defined as, see [11], $\|\mathbf{G}_\sigma\|_{\mathcal{H}_2(\tau)}^2 = \mathtt{tr}(D_Q\Gamma_\tau D_Q^{\rm T})$, where, for some $\tau \in \mathbb{R}_{>0}$, $\Gamma_\tau = \int_0^\tau \exp(At)BB^{\rm T}\exp(A^{\rm T}t)dt$ is the finite-horizon controllability grammian of the network system $\boldsymbol{\Sigma}$. If the state matrix A is Hurwitz, then the standard $\mathcal{H}_2$-norm can also be considered, which can be computed by using the infinite-horizon controllability grammian, [11].

The network clustering problem is defined as follows: Find $Q \in \mathbb{B}^{n\times k}$ such that

$$\min_{Q\in\mathbb{B}^{n\times k}} \ \mathtt{tr}(D_Q\Gamma_\tau D_Q^{\rm T}) \quad (7)$$

where $\mathbb{B}^{n\times k} = \{X \in \{0,1\}^{n\times k} : X\mathbf{1}_k = \mathbf{1}_n\}$. The clustering problem (7) is a non-convex, mixed-integer NP-hard optimization problem.

B. Clustering algorithm

In this subsection, we provide a heuristic algorithm to find a suboptimal solution to (7) in polynomial time. Let $\psi = \mathtt{tr}(D_Q\Gamma_\tau D_Q^{\rm T})$ be the cost of (7) for some $Q \in \mathbb{B}^{n\times k}$, which is the characteristic matrix of the clustering $\mathcal{Q}$. Similarly, let $\psi_0 = \mathtt{tr}(D_{Q_0}\Gamma_\tau D_{Q_0}^{\rm T})$ be the cost of a randomly initialized clustering $\mathcal{Q}_0$.

Algorithm 1: The algorithm requires the number of clusters k, the number of measured nodes m, the number of unmeasured nodes n, the controllability grammian Γ_τ, an initial clustering $\mathcal{Q}_0$, and a stopping criterion $\delta > 0$ which is a very small number (e.g., 10^{-6}). It outputs a suboptimal clustering $\mathcal{Q} = \{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$ with $\psi = \mathtt{tr}(D_Q\Gamma_\tau D_Q^{\rm T})$.

Compute ψ_0 and assign $\mathcal{Q}_1 \leftarrow \mathcal{Q}_0$.
repeat
  Assign $\psi_1 \leftarrow \psi_0$.
  for $i = 1, 2, \cdots, n$ **do**
    Assign $\mathcal{Q}_2 \leftarrow \mathcal{Q}_1$.
    Let $\beta \in \{1, \cdots, k\}$ be such that $m + i \in \mathcal{C}_\beta$.
    if $\mathcal{C}_\beta$ has more than one node, i.e., $|\mathcal{C}_\beta| > 1$, **then**
      for $\alpha = 1, 2, \cdots, k$ and $\alpha \neq \beta$ **do**
        Move $m + i$ from its cluster to $\mathcal{C}_\alpha$.
        Update $\mathcal{Q}_2$ accordingly and compute ψ_2.
        if $\psi_2 < \psi_0$ **then**
          Update $\psi_0 \leftarrow \psi_2$ and $\mathcal{Q}_1 \leftarrow \mathcal{Q}_2$.
        end if
      end for
    end if
  end for
until $\psi_1 - \psi_0 > \delta$
Assign $\mathcal{Q} \leftarrow \mathcal{Q}_1$.
return $\mathcal{Q} = \{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$, $\psi = \mathtt{tr}(D_Q\Gamma_\tau D_Q^{\rm T})$. ⌟

The suboptimal solution obtain from Algorithm 1 depends on the initial clustering $\mathcal{Q}_0$. Therefore, to obtain a better solution, the following algorithm repeatedly runs Algorithm 1, where at every iteration the clustering is initialized randomly.

Algorithm 2: The algorithm requires a positive integer $c > 0$ that is the maximum value of a counter and the same inputs $k, m, n, \Gamma_\tau, \mathcal{Q}_0, \delta$ as those of Algorithm 1. It outputs a suboptimal clustering $\mathcal{Q}^* = \{\mathcal{C}_1^*, \cdots, \mathcal{C}_k^*\}$ with a cost $\psi^* = \mathtt{tr}(D_{Q^*}\Gamma_\tau D_{Q^*}^{\rm T})$ such that $\psi^* \leq \psi$, where ψ is obtained by running Algorithm 1 once.

Assign $a \leftarrow 0$ and $b \leftarrow 0$.
repeat
  Compute $\psi_0 = \mathtt{tr}(D_{Q_0}\Gamma_\tau D_{Q_0}^{\rm T})$
  Run Algorithm 1 and store Q and ψ.
  Assign $a \leftarrow a + 1$.
  if $a = 1$ **then**
    Assign $\psi^* \leftarrow \psi$ and $\mathcal{Q}^* \leftarrow \mathcal{Q}$.
    Randomly initialize $\mathcal{Q}_0$ and compute D_{Q_0}.
  else
    if $\psi < \psi^*$ **then**
      Assign $\psi^* \leftarrow \psi$ and $\mathcal{Q}^* \leftarrow \mathcal{Q}$.
      Randomly initialize $\mathcal{Q}_0$ and compute D_{Q_0}.
    else
      Assign $b \leftarrow b + 1$.
    end if
  end if
until $b \leq c$
return $\mathcal{Q}^* = \{\mathcal{C}_1^*, \cdots, \mathcal{C}_k^*\}$, $\psi^* = \mathtt{tr}(D_{Q^*}\Gamma_\tau D_{Q^*}^{\rm T})$. ⌟

C. Illustrative example

Consider a network system shown in Fig. 1, where the input $\mathbf{u} = [\, u_1 \ u_2\,]^{\rm T}$ and the output $\mathbf{y} = [\, y_1 \ y_2\,]^{\rm T}$. The measured nodes $\mathcal{V}_1 = \{1, 2\}$ and the unmeasured nodes $\mathcal{V}_2 = \{3, 4, \cdots, 10\}$. Let the number of clusters be $k = 3$. Then, the clustering $\mathcal{Q}^* = \{\mathcal{C}_1^*, \mathcal{C}_2^*, \mathcal{C}_3^*\}$ is obtained from Algorithm 2, where $\mathcal{C}_1^* = \{3, 6, 9\}$, $\mathcal{C}_2^* = \{4, 7\}$, and $\mathcal{C}_3^* = \{5, 8, 10\}$, which are specified by the dashed lines in Fig. 1.

The clustering obtained by Algorithm 2 ensures that the state trajectories of each cluster stay closer to each other as time progresses, shown in Fig. 2(a). For instance, initially the states $x_5(0), x_8(0), x_{10}(0)$ are not close to each other, however, as $t > 1$, we see that their trajectories converge closer to each other. Consequently, the state variance can be approximated as shown in Fig. 2(b).

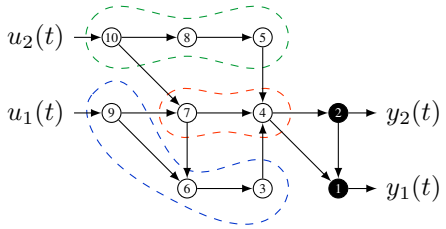
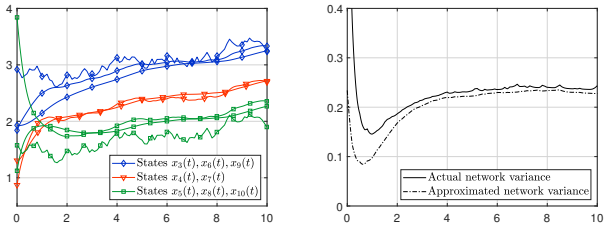


Fig. 1: Three clusters (enclosed by the dashed lines) identified by Algorithm 2 for the example network.



(a) (b)

Fig. 2: (a) State trajectories of clustered unmeasured nodes of the network system shown in Fig. 1. (b) The plots of actual state variance $x_{\rm v}(t)$ computed by (1) from the states of unmeasured nodes and approximated state variance computed by (6) from the average mean values of the identified clusters.

V. State Variance Estimation by Average State Observer

The average state $\mathbf{z}_{\rm a}(t) = Q^{+}\mathbf{x}_2(t) \in \mathbb{R}^k$ is the vector of average mean states of clusters that are obtained from Algorithm 2, where $Q \in \mathbb{B}^{n\times k}$ is the characteristic matrix of the clustering. In this section, we present an observer design procedure to estimate $\mathbf{z}_{\rm a}(t)$, from which the state variance will be estimated.

The network system $\mathbf{\Sigma}$ whose nodes are clustered according to $\mathcal{Q}$ is called a clustered network system. Since $\mathbf{x}_2 = Q\mathbf{z}_{\rm a} + \boldsymbol{\sigma}$, we examine the dynamics of $\mathbf{z}_{\rm a}(t)$ from a projected network system, [5],

$$\mathring{\mathbf{\Sigma}}: \begin{cases} \dot{\mathbf{z}}(t) = E\mathbf{z}(t) + F\boldsymbol{\sigma}(t) + G\mathbf{u}(t) \\ \mathbf{y}(t) = H\mathbf{z}(t) \\ 0 \;\; = Q^{+}\boldsymbol{\sigma} \end{cases}$$

where $\mathbf{z}(t) = [\,\mathbf{x}_1^{\mathsf{T}}(t)\;\; \mathbf{z}_{\rm a}^{\mathsf{T}}(t)\,]^{\mathsf{T}} \in \mathbb{R}^{m+k}$, $\boldsymbol{\sigma}(t) \in \mathbb{R}^n$, and

$$E = \begin{bmatrix} A_{11} & A_{12}Q \\ Q^{+}A_{21} & Q^{+}A_{22}Q \end{bmatrix}, \quad F = \begin{bmatrix} A_{12} \\ Q^{+}A_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} I_m & 0_{m\times k} \end{bmatrix}, \quad G = \begin{bmatrix} B_1 \\ Q^{+}B_2 \end{bmatrix}.$$

The projected network system $\mathring{\mathbf{\Sigma}}$ is an aggregated model where each cluster is considered as a single node, and whose state vector contains the states of measured nodes and the average states of the clusters of unmeasured nodes. We use this model to design an average state observer of order k that estimates the average states of clusters.

A. Average state observer

To estimate the average state $\mathbf{z}_{\rm a}(t) \in \mathbb{R}^k$, we consider a minimum order average state observer (ASO), presented in [5],

$$\mathbf{\Omega}_{\rm a}: \begin{cases} \dot{\mathbf{w}}(t) \;=\; M\mathbf{w}(t) + K\mathbf{y}(t) + N\mathbf{u}(t) \\ \hat{\mathbf{z}}_{\rm a}(t) \;=\; \mathbf{w}(t) + L\mathbf{y}(t) \end{cases}$$

with $\mathbf{w}(t) \in \mathbb{R}^k$ the state of the observer, $\hat{\mathbf{z}}_{\rm a}(t)$ the estimated average state of clusters, and

$$\begin{aligned} M &= Q^{+}A_{22}Q - LA_{12}Q \\ N &= Q^{+}B_2 - LB_1 \\ K &= ML - LA_{11} + Q^{+}A_{21} \\ L &= (Q^{+}A_{22} - VQ^{+})A_{12}^{+} \end{aligned} \tag{8}$$

where $V = \alpha Q^{+}A_{22}Q$ is a Hurwitz matrix with $\alpha > 0$ the tuning parameter to be chosen such that M is Hurwitz and the average state estimation (ASE) error $\boldsymbol{\zeta}_{\rm a}(t) = \mathbf{z}_{\rm a}(t) - \hat{\mathbf{z}}_{\rm a}(t)$ is minimum as $t \to \infty$. The following results are presented in [5] with slight modifications.

Theorem 2: The following statements are true:

(i) Let $b, c \in \mathbb{R}_{>0}$ be some positive constants. Then, the ASE error satisfies $\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 \le c\, e^{-\frac{\alpha}{b}t}$, where $\alpha > 0$ is the tuning parameter, if and only if

$$\mathtt{rank}\left(\begin{bmatrix} A_{12} \\ Q^{+} \\ Q^{+}A_{22} \end{bmatrix}\right) = \mathtt{rank}\,(A_{12})\,.$$

(ii) Let $b, c_2 \in \mathbb{R}_{>0}$ be some positive constants and $c_1 \in \mathbb{R}_{\ge 0}$ be some non-negative constant. Then, the ASE error satisfies

$$\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 \le \frac{c_1}{\alpha} + c_2\, e^{-\frac{\alpha}{b}t}, \tag{9}$$

where $\alpha > 0$ is the tuning parameter, if and only if

$$\mathtt{rank}\left(\begin{bmatrix} A_{12} \\ Q^{+} \end{bmatrix}\right) = \mathtt{rank}\,(A_{12})\,.$$

(iii) Let $b, c \in \mathbb{R}_{>0}$ be some positive constants. Then, the ASE error satisfies $\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 \le c\, e^{-\frac{1}{b}t}$ if and only if $\mathtt{ker}(Q^{+}) \subseteq \mathtt{ker}(Q^{+}A_{22})$. Notice that the inequality is without a tuning parameter α. ⌟

The proofs of the theorem statements given above are similar to the ones provided in [5] for average observability and average detectability. Theorem 2(i) corresponds to the case where $\lim_{t\to\infty}\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 = 0$ for any tuning parameter $\alpha > 0$, i.e., this is the average observability case and the ASO is a tunable observer. Theorem 2(iii) is the case where $\lim_{t\to\infty}\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 = 0$ but without any tuning parameter α, i.e., this is the average detectability case and is closely related to the notion of lumpability [12]. Theorem 2(ii) implies that $\lim_{t\to\infty}\|\boldsymbol{\zeta}_{\rm a}(t)\|_2 = c_1/\alpha$, which can be made arbitrarily small by choosing α to be very large. Notice that if the condition in Theorem 2(iii) is satisfied, then the conditions in Theorem 2(i) and (ii) are equivalent and $c_1 = 0$ in (9).

The network interpretation of the conditions in Theorem 2(i) and (ii) is as follows: It is necessary that the measured nodes span all the unmeasured nodes. In other words, for every unmeasured node $j \in \mathcal{V}_2$, there exists at least one measured node $i \in \mathcal{V}_1$ to which j is connected,

i.e., $(i,j) \in \mathcal{E}$ or $[A]_{ij} > 0$. This condition is not sufficient because we also require that the weights of the edges between measured and unmeasured nodes are balanced is such a way that Q^+ is in the rowspace of A_{12}.

The network interpretation of the condition in Theorem 2(iii) is that the characteristic matrix Q corresponds to an equitable partition, which means the following. First, the induced subgraphs formed by the nodes of each cluster are weighted out-regular, i.e., the weighted out-degrees of nodes with respect to their own cluster is equal. Second, the induced bipartite subgraphs formed by the nodes of every pair of clusters must also be weighted out-regular, i.e., the weighted out-degrees of nodes of one cluster with respect to another cluster is equal.

B. Minimizing the ASE error

The conditions given in Theorem 2 require strict structural constraints on the clustered network system ($\boldsymbol{\Sigma}$ clustered according to $\mathcal{Q}$). Since Algorithm 1 and 2 do not take these constraints into account, therefore these conditions are not satisfied in general, which implies that the ASE error doesn't converge to zero. In this subsection, we present a methodology to deal with general clustered network systems. The problem is formulated as a convex optimization in order to minimize the ASE error $\|\boldsymbol{\zeta}_{\mathrm{a}}(t)\|_2$.

Let $R_\alpha = Q^+A_{22} - LA_{12}$, then, from (8), we obtain

$$R_\alpha = Q^+A_{22}\left(I_n - (I_n - \alpha QQ^+)A_{12}^+A_{12}\right).$$

The matrix $M = R_\alpha Q$ and the dynamics of the ASE error is given by $\dot{\boldsymbol{\zeta}}_{\mathrm{a}}(t) = R_\alpha Q\boldsymbol{\zeta}_{\mathrm{a}}(t) + R_\alpha\boldsymbol{\sigma}(t)$, where $\boldsymbol{\sigma}(t) = D_Q\mathbf{x}(t)$. Suppose $\check{\alpha} > 0$ be such that $R_\alpha Q$ is Hurwitz for $\alpha > \check{\alpha}$. The transfer function from $\mathbf{x}$ to $\boldsymbol{\zeta}_{\mathrm{a}}$ is given by $\mathbf{G}_\zeta(s) = (sI - R_\alpha Q)^{-1}R_\alpha D_Q$ with the $\mathcal{H}_2(\tau)$-norm defined as $\|\mathbf{G}_\zeta\|^2_{\mathcal{H}_2(\tau)} = \mathtt{tr}(\Xi_\tau(\alpha))$, where $\Xi_\tau(\alpha) = \int_0^\tau \exp(R_\alpha Qt)R_\alpha D_Q D_Q^{\mathrm{T}}R_\alpha^{\mathrm{T}}\exp(Q^{\mathrm{T}}R_\alpha^{\mathrm{T}}t)dt$. The ASE error minimization problem is as follows: Find $\alpha > \check{\alpha}$ such that

$$\min_{\alpha>\check{\alpha}} \quad \mathtt{tr}(\Xi_\tau(\alpha)). \tag{10}$$

For some $\alpha > \check{\alpha}$, let $\gamma_\alpha = \mathtt{tr}(\Xi_\tau(\alpha))$ to be the cost of (10).

Algorithm 3: The algorithm requires a small real number $\varepsilon > 0$, a large positive integer η, and all the matrices required to compute $\Xi_\tau(\alpha)$ for a given $\alpha > \check{\alpha}$. It outputs α^* the suboptimal solution of (10).

1: Initialize $\alpha \leftarrow 0$.
2: **repeat**
3: Assign $\alpha \leftarrow \alpha + \varepsilon$ and compute $R_\alpha Q$.
4: **until** $R_\alpha Q$ is Hurwitz.
5: Assign $\alpha \leftarrow \alpha + \varepsilon$ and compute $\gamma_\alpha = \mathtt{tr}(\Xi_\tau(\alpha))$.
6: **repeat**
7: Assign $\gamma \leftarrow \gamma_\alpha$ and $\alpha \leftarrow \alpha + \varepsilon$.
8: Compute $\gamma_\alpha = \mathtt{tr}(\Xi_\tau(\alpha))$.
9: **until** $\gamma_\alpha > \gamma$
10: Assign $\alpha_1 \leftarrow \alpha - 2\varepsilon$ and $\alpha_2 \leftarrow \alpha$.
11: Let $\mathcal{I}$ be the set of η equidistant points in $[\alpha_1, \alpha_2]$.
12: Compute $\alpha^* = \arg\min_{\alpha\in\mathcal{I}} \mathtt{tr}(\Xi_\tau(\alpha))$.
13: **return** α^* ⌟

C. State variance estimation error

In this subsection, we derive an expression for the state variance estimation (SVE) error as the sum of the variance approximation error and a quantity proportional to the ASE error. This is to justify that if the variance approximation error and the ASE error are small, then the SVE error is also small. Moreover, if the conditions of Theorem 2 are satisfied, then the asymptotic SVE error only depends on the asymptotic variance approximation error.

The estimated state variance computed from the estimated average states is given by

$$\hat{x}_{\mathrm{v}}(t) = \frac{1}{n}\hat{\mathbf{z}}_{\mathrm{a}}^{\mathrm{T}}(t)Q^{\mathrm{T}}J_nQ\hat{\mathbf{z}}_{\mathrm{a}}(t). \tag{11}$$

Since the ASE error $\boldsymbol{\zeta}_{\mathrm{a}} = \mathbf{z}_{\mathrm{a}} - \hat{\mathbf{z}}_{\mathrm{a}}$, we have

$$\hat{x}_{\mathrm{v}} \;=\; \frac{1}{n}\left(\mathbf{z}_{\mathrm{a}}^{\mathrm{T}}Q^{\mathrm{T}}J_nQ\mathbf{z}_{\mathrm{a}} + \boldsymbol{\zeta}_{\mathrm{a}}^{\mathrm{T}}Q^{\mathrm{T}}J_nQ\boldsymbol{\zeta}_{\mathrm{a}} - 2\mathbf{z}_{\mathrm{a}}Q^{\mathrm{T}}J_nQ\boldsymbol{\zeta}_{\mathrm{a}}\right)$$

and since $\mathbf{x}_2 = \boldsymbol{\sigma} + Q\mathbf{z}_{\mathrm{a}}$, we have

$$x_{\mathrm{v}} \;=\; \frac{1}{n}\left(\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma} + \mathbf{z}_{\mathrm{a}}^{\mathrm{T}}Q^{\mathrm{T}}J_nQ\mathbf{z}_{\mathrm{a}}\right)$$

where we used the facts: $\boldsymbol{\sigma}^{\mathrm{T}}J_n\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\mathrm{T}}J_nQ\mathbf{z}_{\mathrm{a}} = 0$. Therefore, the SVE error $\xi_{\mathrm{v}} = x_{\mathrm{v}} - \hat{x}_{\mathrm{v}}$ can be written as

$$\xi_{\mathrm{v}} = \frac{1}{n}\left(\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma} + (2\mathbf{z}_{\mathrm{a}} - \boldsymbol{\zeta}_{\mathrm{a}})^{\mathrm{T}}Q^{\mathrm{T}}J_nQ\boldsymbol{\zeta}_{\mathrm{a}}\right). \tag{12}$$

The above expressions contains two summands, first is the square of the norm of state variance approximation error $\|\boldsymbol{\sigma}(t)\|_2^2$ and the second is proportional to the ASE error $\boldsymbol{\zeta}_{\mathrm{a}}(t)$. If the optimization problems (7) and (10) admit a solution that yields a small cost, then the state variance approximation error and the ASE error will also be small. Consequently, the SVE error will be small.

VI. Simulation Example

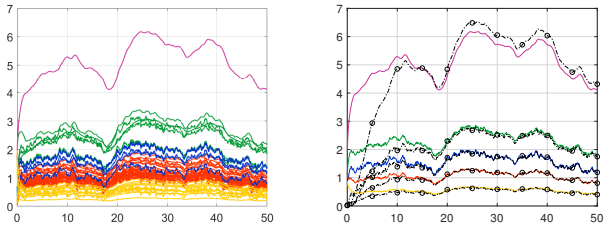
We consider a linear compartmental system, [13], where each compartment is a node with a state $x_i(t) \in \mathbb{R}_{\geq 0}$ that represents some physical quantity. The nodes $\mathcal{V}$ are connected via an underlying graph $\mathcal{G}$ and the rate of change of node i's state equals the difference between the inflow to i and the outflow from i, i.e., $\dot{x}_i(t) = f_i^{\mathrm{in}}(t) - f_i^{\mathrm{out}}(t)$ where $f_i^{\mathrm{in}}(t)$ and $f_i^{\mathrm{out}}(t)$ represent the flow-in and the flow-out, respectively, and are given by

$$f_i^{\mathrm{in}} \;=\; \textstyle\sum_{j\in\mathcal{N}_i^\downarrow} a_{ij}x_j(t) + b_{il}^+u_l^+(t),$$

$$f_i^{\mathrm{out}} \;=\; \textstyle\sum_{h\in\mathcal{N}_i^\uparrow} a_{hi}x_i(t) + b_{il}^-u_l^-(t)$$

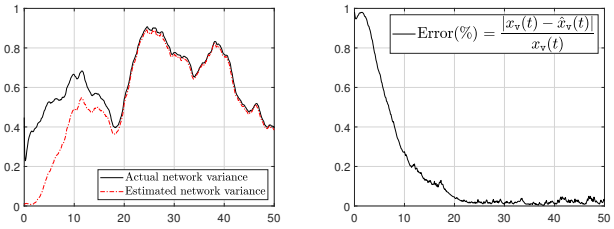
with $\mathcal{N}_i^\downarrow$ and $\mathcal{N}_i^\uparrow$ the in-neighbors and the out-neighbors of node i, respectively; $u_l^+(t) \in \mathbb{R}_{\geq 0}$, $u_l^-(t) \in \mathbb{R}_{\leq 0}$ are the positive and the negative inputs, respectively, $b_{il}^+, b_{il}^- \in \{0,1\}$ are the scalars that determine if the input-l is applied at node i.

The state matrix $A = \mathcal{A}(\mathcal{G}) - \mathcal{D}^\uparrow(\mathcal{G})$, where $\mathcal{A}(\mathcal{G})$ and $\mathcal{D}^\uparrow(\mathcal{G})$ are the weighted adjacency and out-degree matrices of the graph $\mathcal{G}$, respectively. We generate a graph $\mathcal{G}$ of 55 nodes by an Erdos-Renyi model with a probability of a directed edge between any pair of nodes equal to 0.15. The number of measured nodes $m = 5$ and the number of unmeasured nodes $n = 50$. We choose the number of clusters to be $k = 5$. We consider the input vector to be $\mathbf{u}(t) = [\, u_1(t)\ u_2(t)\,]^{\mathrm{T}}$,



(a) (b)

Fig. 3: (a). The state trajectories $\mathbf{x}_2(t)$ of the 50 unmeasured nodes in the example network system. The colors of the trajectories correspond to the 5 clusters identified by Algorithm 2. (b) Average state estimation of the identified clusters. The colored solid trajectories show the average states of the clusters $\mathbf{z}_{\mathrm{a}}(t)$ and black dashed trajectories show the estimated average states $\hat{\mathbf{z}}_{\mathrm{a}}(t)$.



(a) (b)

Fig. 4: (a) The actual state variance $x_{\mathrm{v}}(t)$ plotted with black solid line vs. the estimated state variance $\hat{x}_{\mathrm{v}}(t)$ plotted with red dotted line. (b) Percentage variance estimation error.

where each input $u_l(t) = u_l^+(t) - u_l^-(t)$ with $u_l^+(t), u_l^-(t) \in [0, 1]$ represent a random, discontinuous signal. The inputs are directly applied to 10% of nodes chosen in a uniformly random way.

The state trajectories of the system are shown in Fig. 3(a), where the trajectories of the same color correspond to the clusters identified by Algorithm 2. We obtain the optimal tuning parameter $\alpha^* = 1.8879$ from Algorithm 3, where Ξ_τ is computed at every iteration with $\tau = 10$. The estimation of the average states are shown in Fig. 3(b). Notice that the estimation of the four average states shown as green, blue, red, and yellow is very accurate, whereas the estimation of the average state shown as magenta is not very accurate. This can be due to the fact that the cluster represented by magenta has only one node, which is an outlier node since its trajectory is far from the states of other nodes. The average state observer is designed to optimally estimate the average states of a cluster of several nodes, and not the states of individual nodes. Another reason can be the fact that we optimize a single parameter α in problem (10), which changes the eigenvalues of $R_\alpha Q$ with an equal proportion. After obtaining an optimal α^*, if the spectrum $\mathtt{eig}(R_\alpha Q)$ contains a very small eigenvalue, then the corresponding estimated average state trajectory will not be accurate, as shown in the figure.

The plot of the actual and the estimated state variance is shown in Fig. 4(a) and the percentage state variance estimation error in Fig 4(b). We see that the state variance estimation is very accurate, which is due to the following reasons: (i) The variance approximation is very accurate because of the state trajectories of the identified clusters are very close to each other. (ii) The average state estimation is very accurate because of the optimal tuning parameter α^*. From the discussion that follows after the SVE error equation (12), we conclude that the accuracy of state variance approximation and average state estimation results in the accuracy of state variance estimation.

VII. Concluding Remark

We presented a methodology to estimate the state variance, which is a nonlinear functional of the state vector that provides a measure of the square deviation of state trajectories around their average mean. The methodology comprises two steps: (i) Offline: Run Algorithm 2 to identify k clusters of network nodes such that the gain from the input to the average deviation vector is minimized. Then, run Algorithm 3 to obtain an optimal tuning parameter for an average state observer that estimates the average states of the clusters. (ii) Online: Compute the estimated state variance from the output of the average state observer. Our proposed methodology is computationally tractable for large-scale systems and can be generalized to estimate any nonlinear functional or higher moments (e.g., skewness) of the state vector.

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