Fixed-time convergent consensus algorithm of networked nonholonomic multi-agent systems

Michael Defoort, Thierry Floquet, Wilfrid Perruquetti

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Abstract— **In this paper, the problem of fixed-time leader-follower consensus problem of nonholonomic multi-agent systems is under study. Using the "desingularisation method" introduced in the seminal paper by J.M. Coron [5], new fixed-time controllers/observers for the double integrator system are designed. Following those results, a switching consensus protocol which guarantees the tracking errors stabilization in fixed-time which does not depend on the initial conditions of the multi-agent system is provided. Simulation results on a fleet of wheeled mobile robots show the effectiveness of the proposed scheme.**

I. Introduction

Stabilization and tracking problems of nonholonomic systems, e.g. systems subject to nonintegrable constraints with respect to velocity, have received a lot of attention in the literature (see [14] for an extended survey). Indeed, there are significant technical challenges due to the Brockett's necessary condition [2]. Hence, several time-varying or discontinuous controllers have been investigated such as sinusoidal and polynomial controls [17], smooth time-varying feedbacks [22], backstepping based schemes [25] and hybrid schemes [15]. Nowadays, there are lot of applications in autonomous vehicles, wheeled mobile robots [8], under-actuated ships [10], etc.

During the last decades, distributed control of multi-agent systems (MAS) has attracted a lot of attention due to its wide range of applications in different areas such as flocking [27], swarming, target tracking, etc. Among them, the consensus problem is one of the most fundamental problems for cooperative control of MAS. Its objective is to design distributed controllers so that each agent reaches an agreement regarding to a given quantity of interest using only local information [20].

The consensus problem has been widely investigated for linear systems [24], [16], [23]. Nevertheless, many mechanical systems (e.g. wheeled mobile robots, UAVs, or manipulators) cannot be described by quasi-linear systems since they present nonintegrable constraints on velocity. Hence, consensus protocols have been derived for nonholonomic systems [11], [12], [4]. However, these controllers only guarantee asymptotic consensus or finite-time consensus (i.e. the settling time depends on the initial conditions of the multi-agent system).

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The concept of fixed-time stability has been proposed to derive controllers such that the settling time is bounded independently of the initial conditions [6], [19]. Based on this concept, fixed-time consensus protocols have been proposed for first and second integrator MAS [24], [16], [9]. Recently, the fixed-time consensus problem for nonholonomic MAS has been investigated [8], [21], [1], [18]. In [8], only the case of static leader is studied. This work has been extended in [21], [1], [18] to the dynamic leader case. In [1], a specific sliding surface is designed based on results from [19] for second order systems. However, it requires a careful tuning of four control parameters and the settling time is highly over-estimated. The methodology given for instance in [21], [18] is based on the power integrator method (which relies on [5]) that requires the design of a sufficiently small positive constant. Furthermore, the resulting controller is relatively complex with many parameters to be carefully tuned.

In this paper, a new fixed-time leader-follower consensus controller for nonholonomic MAS are derived. The main features of the present work are as follows:

i) A new fixed-time leader-follower consensus controller for nonholonomic MAS is provided. The settling time is upper bounded by a positive constant which only depends on the controller parameters.
ii) A detailed Lyapunov analysis is provided to show the fixed-time stabilization of the tracking errors for the double integrator system.
iii) Compared to [21], [1], [18], the controller, based on the "desingularisation method" (see [5]), uses exponential functions and shows good robustness properties.

This paper is structured as follows. The consensus problem for nonholonomic MAS is stated in Section II. The next section III is divided as follows: new fixed-time controllers/observers are derived for the double integrator system (see Subsection III-A/III-B). Then, based on these results, a switching consensus protocol which guarantees the fixed-time stabilization of the tracking errors of the multi-agent system is derived in Subsection III-C.2. Finally, simulation results for a fleet of wheeled mobile robots are shown in Section IV.

Notations: $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Γ denotes the Gamma function. For any real number $a \geq 0$ and for all $x \in \mathbb{R}$, the signed power a of x is defined by $\{x\}^a = \operatorname{sign}(x)|x|^a$. Clearly we have: $\{x\}^0 = \operatorname{sign}(x), \{\{x\}^a\}^b = \{x\}^{ab}, \{x\}^a \{x\}^b = |x|^{a+b}$, $\{x\}^a |x|^b = \{x\}^{a+b}$, and for $a \geq 1$, $\frac{d\{x\}^a}{dx} = a\,|x|^{a-1}$ and $\frac{d|x|^a}{dx} = a\,\{x\}^{a-1}$. Let us define the two following functions

$\forall (x,y) \in \mathbb{R}_+^2, \forall \mu, \nu \in \mathbb{R}_+$:

$$\varphi_\mu(x,y) = \{y\}^\mu - \{x\}^\mu, \quad (1)$$

$$\Phi_\mu^\nu(x,y) = \int_x^y \{\varphi_\mu(x,s)\}^\nu \, ds \geq 0. \quad (2)$$

$\mathcal{C}^k(X,Y)$ is the set of functions $f : X \to Y$ which are k times continuously differentiable (noted as $\mathcal{C}^k$ when the sets X, Y are obvious from the context). $\mathcal{CL}(X,Y)$ is the set of continuous functions (including at 0) $f : X \to Y$ which are locally Lipschitz everywhere except at 0. A continuous function $\alpha : [0, a[\subset \mathbb{R}_+ \to \mathbb{R}_+, r \mapsto \alpha(r)$, is said to be a *class-$\mathcal{K}$ function* if it is strictly increasing with $\alpha(0) = 0$. α is a *class-$\mathcal{K}_\infty$ function* if it is a class-$\mathcal{K}$ function with $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, +\infty[\subset \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+, (r,t) \mapsto \beta(r,t)$, belongs to class-$\mathcal{KL}$ if for each fixed t, the mapping $r \mapsto \beta(r,t)$ belongs to class $\mathcal{K}_\infty$ with respect to r and for each fixed $r \in \mathbb{R}_+$, the mapping $t \mapsto \beta(r,t)$ is decreasing with respect to t and $\lim_{t \to +\infty} \beta(r,t) = 0$.

In this paper, all the proofs are given in the Appendix.

II. Problem statement and preliminaries

A. Problem statement

Consider a set of $N+1$ homogeneous nonholonomic agents (i.e. one leader that can be virtual and N followers). Each agent i is described by the following kinematic equations:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & 0 \\ \sin(\theta_i) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix}, i \in \{0, \ldots, N\}, \quad (3)$$

where $v_i(t)$ and $\omega_i(t)$ are linear and angular velocities, respectively. This model can be used in many applications, as for example when the agents are unicycle mobile robots or a $(2,0)-$type mobile robots [3]. The leader state is $q_0 = (x_0, y_0, \theta_0)^\top$ and the leader control input is $u_0 = (v_0, \omega_0)^\top$. The state of agent i is $q_i = (x_i, y_i, \theta_i)^\top$ and the control input of agent i is $u_i = (v_i, \omega_i)^\top$.

The communication topology among the N follower agents can be represented by a fixed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} = \{1, \ldots, N\}$ is the node set and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ defines the edge set. There is a link between two agents i and j, i.e. $(j,i) \in \mathcal{E}$, with $i \neq j$, if agent i receives information from agent j. The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ is defined as $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The corresponding Laplacian matrix is given by $L = (l_{ij}) \in \mathbb{R}^{N \times N}$ with $l_{ii} = \sum_{j=1,\, j \neq i}^N a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. Since the leader state is available to only a portion of the N followers, the communication topology among the followers and the leader is given by the matrix $D = \mathrm{diag}(d_1, \ldots, d_N)$ where $d_i > 0$ if the leader state is available to agent i and where $d_i = 0$ otherwise.

In this paper, the control objective is to design a distributed controller u_i for each follower agent ($i = 1, \ldots, N$), based on available information, such that the **leader-follower consensus problem is solved in fixed time**. This means that there is a positive constant T (independent of the initial conditions) such that $\forall q_i(0) \in \mathbb{R}^3$, $\forall i = 1, \ldots, N$,

$$\begin{array}{l} \lim_{t \to T} \|q_i(t) - q_0(t)\| = 0, \\ q_i(t) = q_0(t), \ \forall t \geq T. \end{array} \quad (4)$$

Before designing the consensus controller, let us consider the so-called "chained form transformation". For any $i \in \{0, \cdots, N\}$, with $\theta_i \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, the mapping $\phi_i : (\theta_i, v_i, \omega_i) \mapsto (\Theta_i = \tan(\theta_i), u_{1,i} = v_i \cos(\theta_i), u_{2,i} = \omega_i(1 + \tan^2(\theta_i))$) is a diffeomorphism under which system (3) can be rewritten into the so-called chained form as follows

$$\left\{ \begin{array}{lcl} \dot{x}_i & = & u_{1,i} \\ \dot{y}_i & = & u_{1,i}\Theta_i \\ \dot{\Theta}_i & = & u_{2,i} \end{array} \right., \qquad i \in \{0, \cdots, N\}. \quad (5)$$

In order to solve the leader-follower consensus problem, the following assumptions are needed.

Assumption 1: The graph $\mathcal{G}$ is undirected and connected. The leader state is available at least to one agent i, i.e. $\exists i, \ d_i > 0$.

Assumption 2: The control input of the leader is bounded as follows

$$\left\{ \begin{array}{l} u_{1,0}^{\min} < u_{1,0}(t) < u_{1,0}^{\max}, \\ |\dot{u}_{1,0}(t)| < \tau, \\ |u_{2,0}(t)| < u_{2,0}^{\max}, \end{array} \right. \quad (6)$$

with $u_{1,0}^{\min}, u_{1,0}^{\max}, \tau, u_{2,0}^{\max} \in \mathbb{R}_+$.

Assumption 3: All the followers do not know the leader input. However, its neighboring agents know its upper bounds.

Remark 1: It should be highlighted that system (5) can be seen as two coupled subsystems: a single integrator and a second-order subsystems. Hence, it appears to be important to derive a fixed-time controller for single and double integrator systems before designing the fixed-time leader-follower consensus protocol.

B. Non asymptotic concepts

Let us consider

$$\dot{z}(t) = f(z), z \in \mathbb{R}^n, \quad (7)$$

where the function f is assumed to be of $\mathcal{CL}-$class (continuous everywhere and locally Lipschitz in z except at the origin). Let us denote $\Phi^t(z(t_0))$ the solution of system (7) starting from $z(t_0)$. Using class-$\mathcal{KL}$ functions, the stability properties are given as follows:

Definition 1: At equilibrium $z = 0$ the system (7) is said to be

- *Uniformly finite-time stable (in short $0-$UFTS)* if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that $\forall t \geq t_0 \geq 0, \forall \|z(t_0)\| < c$:
$$\|\Phi^t(z(t_0))\| \leq \beta(\|z(t_0)\|, t),$$
with $\beta(\|z(t_0)\|, t) = 0, \forall t \geq T(z(t_0))$,
- *Uniformly fixed-time stable (in short $0-$UFxTS)* if it is finite-time stable with $\sup_{\|z_0\| < c} T(z_0) < +\infty$.

These notions have their "global" version when $c = \infty$ and are denoted as $0-$**UGFTS** and $0-$**UGFxTS**.

C. Preliminary results

Lemma 1: System (7) with $f(z) = -r(z), r(0) = 0, n = 1$ is $0-$UGFxTS iff $xr(x) > 0, \forall x \in \mathbb{R} \setminus \{0\}$ and $\sup_{x_0 \in \mathbb{R}} \int_0^{|x_0|} \frac{dx}{r(x)} < \infty$.
From which one deduces the following result:

Theorem 1: For $0 \leq \alpha < 1$ and $k > 0, b > 0$, system $\dot{x} = -k\{x\}^\alpha \exp(b|x|), x \in \mathbb{R}$ is $0-$UGFxTS with settling time bounded as follows $T(x_0) \leq T_{\max} = \frac{\Gamma(1-\alpha)}{kb^{(1-\alpha)}}$. Moreover, any differentiable positive scalar function $V(t)$ satisfying the following differential inequality $\dot{V} \leq -kV^\alpha \exp(b|V|)$, decreases to zero in fixed-time less than $T_{\max}$.

III. Main results

Before designing the fixed-time leader-follower consensus protocol, let us derive new fixed-time controllers and observers for the double integrator system.

A. Fixed-time stabilization of a double integrator system

Let us consider the double integrator system:

$$\dot{x}_1 = x_2 + \pi_1(x),\ \dot{x}_2 = u + \pi_2(x), \tag{8}$$

where the state is $x = (x_1, x_2)^\top \in \mathbb{R}^2$, the control is $u \in \mathbb{R}$ and $\pi_i, i = 1, 2$ represents disturbances.

Theorem 2: Assume that $\pi_i, i = 1, 2$ satisfy:

$$|\pi_1(x)| \leq |x_1|\Pi_1(x), \tag{9}$$
$$|\pi_2(x)| \leq \Pi_2(x), \tag{10}$$

where Π_1, Π_2 are known smooth positive functions of their arguments and Π_1 is assumed to be $\mathcal{C}^1$. Then, system (8) is $0-$**UGFxTS** under the following feedback control:

$$u(x) = -\{\zeta(x)\}^{4\alpha-3}\gamma_2(x) - \operatorname{sign}(\zeta(x))\Pi_2(x), \tag{11}$$
$$\zeta(x) = \varphi_{\frac{1}{2\alpha-1}}(x_2^\star(x_1), x_2), \tag{12}$$
$$x_2^\star(x_1) = -\{x_1\}^{2\alpha-1}\gamma_1(x), \tag{13}$$
$$\gamma_1(x) = 2 + \frac{k}{2}\exp(bx_1^2) + (1 + |x_1|^{2(1-\alpha)})\Pi_1(x), \tag{14}$$
$$\gamma_2(x) = 2 + A(x) + B(x) + \frac{1}{2}(A(x)\gamma_1(x))^{2\alpha} + \frac{|x_1\zeta(x)|^{2(1-\alpha)}}{4}A^2(x)\Pi_1(x), \tag{15}$$
$$A(x) = 6|x_2^\star|^{\frac{2(1-\alpha)}{2\alpha-1}}\left|\frac{\partial x_2^\star}{\partial x_1}\right|, \tag{16}$$
$$B(x) = k2^{2\alpha-1}\exp\left(4b\zeta^2(x)\right), \tag{17}$$

where u is discontinuous and $x_2^\star$ is a $\mathcal{CL}-$class function, with parameters tuned as follows:

- $\frac{3}{4} < \alpha < 1$,
- k, b are positive parameters to be tuned for selecting the settling-time bound $T_{\max} = \frac{\Gamma(1-\alpha)}{kb^{(1-\alpha)}}$.

Remark 2: Let us stress that γ_1 is $\mathcal{C}^1$ (since Π_1 is assumed to be $\mathcal{C}^1$), thus $x_2^\star$ is a $\mathcal{CL}-$class function. Moreover, $A : \mathbb{R} \to \mathbb{R}_+, x \mapsto A(x_1)$ (given by (16) is continuous with respect to x_1 and $B : \mathbb{R}^2 \to \mathbb{R}_+, x \mapsto B(x)$ (given by (17)) is $\mathcal{C}^\infty$, thus the applied control u has two parts: a continuous one $-\{\zeta(x)\}^{4\alpha-3}\gamma_2(x)$ and a discontinuous one (due to the signum function).

B. Fixed-time observation of a double integrator system

Let us consider system (8). Here, the objective is to estimate the state x in fixed time using only the measurement $y = x_1$. The control input u is assumed to be known. The observer is designed as

$$\dot{\hat{x}}_1 = \hat{x}_2 + O_1(e_1),\ \dot{\hat{x}}_2 = u + O_2(e_1). \tag{18}$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2)^T \in \mathbb{R}^2$ is the estimated state and the output injection terms $O_1(\cdot)$ and $O_2(\cdot)$ are given hereafter.

The observation error dynamics $e_i = x_i - \hat{x}_i, i = 1, 2$ are as follows

$$\dot{e}_1 = e_2 - O_1(e_1),\ \dot{e}_2 = \pi_2 - O_2(e_1). \tag{19}$$

Theorem 3: Assume that $\pi_i, i = 1, 2$ satisfy:

$$\pi_1(x_1) = 0, \tag{20}$$
$$\pi_2(x) \leq \Pi_2, \tag{21}$$

where Π_2 is a known constant. System (19) is $0-$**UGFxTS** under the following output injection terms

$$O_1(e_1) = -l_1\{e_1\}^{\alpha_1}\exp(b_1e_1^2) \tag{22}$$
$$O_2(e_1) = -l_2\{e_1\}^{\alpha_2}\exp(b_2e_1^2) - \Pi_2\operatorname{sign}(e_1) \tag{23}$$

where the parameters are selected as follows:

- $l_1 > 0, l_2 > 0, b_1 > 0, b_2 > 0$ (free parameters),
- $0 < \alpha_1 < 1, \alpha_2 = 2\alpha_1 - 1$.

C. Fixed-time leader-follower consensus protocol

To avoid the communication loop problem due to the dependence of the control inputs of the followers on the inputs of its neighbors in [8], distributed fixed-time observers are designed to estimate the state of the leader. Then, based on this estimate, an observer-based consensus protocol is designed to achieve the objective stated in (4).

Assumption 4: All the agents can measure its whole state q_i. The leader transmits its state only to its neighbors. All the followers $i = 1, \ldots, N$ transmit their estimate of the leader denoted $\hat{\xi}_i$ computed according to the next subsection.

Remark 3: One should highlight that the observer, derived hereafter, is not designed to reconstruct the system state from incomplete measurements. Its objective is to provide, in a distributed way, an accurate estimate of the leader state for each follower. Indeed, the leader state is only transmitted to a portion of agents. According to Assumption 4, the estimate $\hat{\xi}_i$ is sent from agent i to its neighbors instead of q_0 to avoid safety and robustness issues due to centralization.

1) Distributed fixed-time observers: Now setting, $\xi_{1,i} = x_i$, $\xi_{2,i} = u_{1,i}, \dot{\xi}_{2,i} = \dot{u}_{1,i}$, $\xi_{3,i} = y_i$, $\xi_{4,i} = \Theta_i$, system (5) can be rewritten as

$$\left\{\begin{array}{rcl} \dot{\xi}_{1,i} & = & \xi_{2,i} \\ \dot{\xi}_{2,i} & = & \dot{u}_{1,i} \\ \dot{\xi}_{3,i} & = & \xi_{4,i}\xi_{2,i} \\ \dot{\xi}_{4,i} & = & u_{2,i} \end{array}\right., \quad i \in \{0, \cdots, N\}. \tag{24}$$

For each follower, let us denote the disagreement: $\forall i = \{1,\ldots,N\}$, $\forall k = \{1,\ldots,4\}$,

$$\Gamma_{k,i} = \sum_{j=1}^{N} a_{ij}(\hat{\xi}_{k,j} - \hat{\xi}_{k,i}) + d_i(\xi_{k,0} - \hat{\xi}_{k,i}). \tag{25}$$

Based on [1] and using Theorem 1, the following distributed observers are applied

$$\begin{cases} \dot{\hat{\xi}}_{1,i} &= \hat{\xi}_{2,i} + k_1\{\Gamma_{1,i}\}^{\alpha'}\exp(k_2|\Gamma_{1,i}|), \\ \dot{\hat{\xi}}_{2,i} &= k_1\{\Gamma_{2,i}\}^{\alpha'}\exp(k_2|\Gamma_{2,i}|) + \tau\mathrm{sign}(\Gamma_{2,i}), \\ \dot{\hat{\xi}}_{3,i} &= \hat{\xi}_{4,i}\hat{\xi}_{2,i} + k_1\{\Gamma_{3,i}\}^{\alpha'}\exp(k_2|\Gamma_{3,i}|), \\ \dot{\hat{\xi}}_{4,i} &= k_1\{\Gamma_{4,i}\}^{\alpha'}\exp(k_2|\Gamma_{4,i}|) + u_{2,0}^{\max}\mathrm{sign}(\Gamma_{4,i}), \end{cases} \tag{26}$$

where $\hat{\xi}_i = \left(\hat{\xi}_{1,i}, \hat{\xi}_{2,i}, \hat{\xi}_{3,i}, \hat{\xi}_{4,i}\right)^T$ $(i = \{1,\ldots,N\})$ is the estimation of the state of the leader ξ_0 for the ith follower. k_1, k_2 are positive constants and $0 \leq \alpha' < 1$. Let us define the estimation errors

$$\tilde{\xi}_{k,i} = \hat{\xi}_{k,i} - \xi_{k,0}, \qquad (i = \{1,\ldots,N\},\ k = \{1,\ldots,4\}). \tag{27}$$

Hence, the disagreement (25) can be expressed as $\forall i = \{1,\ldots,N\}$, $\forall k = \{1,\ldots,4\}$,

$$\Gamma_{k,i} = \sum_{j=1}^{N} a_{ij}(\tilde{\xi}_{k,j} - \tilde{\xi}_{k,i}) - d_i\tilde{\xi}_{k,i}. \tag{28}$$

Denoting $\Gamma_k = (\Gamma_{k,1},\ldots,\Gamma_{k,N})^T$ and $\tilde{\xi}_k = \left(\tilde{\xi}_{k,1},\ldots,\tilde{\xi}_{k,N}\right)^T$, Eq. (28) can be written in a compact form as

$$\Gamma_k = -(L+D)\tilde{\xi}_k. \tag{29}$$

Note that matrix $(L+D)$ is symmetric positive definite under Assumption 1 (see [26]). λ_m denotes the smallest eigenvalues of $(L+D)$.

Theorem 4: Suppose that Assumptions 1-4 hold. Using the distributed observer (26), the estimation errors (27) converge to zero in a fixed time bounded by

$$T_o = 2\frac{N\Gamma(1-\alpha')}{k_1\lambda_m k_2^{1-\alpha'}}. \tag{30}$$

2) Observer-based fixed-time consensus: Using the proposed distributed observers, each follower i can estimate the leader state after time T_o and uses the estimate in the consensus protocol. For each agent, after T_o, the tracking errors are defined as $\forall i = \{1,\ldots,N\}$, $\forall k = \{1,3,4\}$,

$$e_{k,i} = \xi_{k,i} - \hat{\xi}_{k,i}. \tag{31}$$

From Theorem 4, for each follower $i = \{1,\ldots,N\}$ and after time T_o, the tracking error dynamics reduces to

$$\begin{array}{lrcl} (\Sigma_1) & \dot{e}_{1,i} &=& u_{1,i} - u_{1,0}, \\ & & & \\ (\Sigma_2) & \dot{e}_{3,i} &=& e_{4,i}u_{1,0} + (e_{4,i} + \xi_{4,0})(u_{1,i} - u_{1,0}), \\ & \dot{e}_{4,i} &=& u_{2,i} - u_{2,0}. \end{array} \tag{32}$$

One can note that system (32) is divided into two coupled subsystems (i.e. Σ_1 and Σ_2).

To deal with the consensus tracking problem, for each follower $i = \{1,\ldots,N\}$, the following two steps are proposed:

- The controller $u_{1,i}$ is designed such that the origin of Σ_1 is fixed-time stable with the settling time estimate T_s.
- For $t \geq T_s$, the controller $u_{2,i}$ is designed such that the origin of Σ_2 is fixed-time stable with the settling time estimate T_{max}. It could be highlighted that for $t \geq T_s$, Σ_2 reduces to

$$\begin{array}{rcl} \dot{e}_{3,i} &=& e_{4,i}u_{1,0}, \\ \dot{e}_{4,i} &=& u_{2,i} - u_{2,0}. \end{array} \tag{33}$$

Using results of the previous subsection, the following theorem can be derived.

Theorem 5: Suppose that Assumptions 1-4 hold. The fixed-time leader follower consensus problem is achieved under the distributed controller

$$u_{1,i} = \begin{cases} 0, & \forall t < T_o \\ -k_3\{e_{1,i}\}^{\beta}\exp(k_4|e_{1,i}|) - u_{1,0}^{\max}\mathrm{sign}(e_{1,i}), & \forall t \geq T_o \end{cases} \tag{34}$$

$$u_{2,i} = \begin{cases} 0, & \forall t < T_s \\ \frac{1}{\hat{\xi}_{2,i}}u(e_{3,i}, \hat{\xi}_{2,i}e_{4,i}), & \forall t > T_s \end{cases} \tag{35}$$

where $0 \leq \beta < 1$, $k_3 > 0$, $k_4 > 0$, $u(e_{3,i}, \hat{\xi}_{2,i}e_{4,i})$ is given in Theorem 2 with $\Pi_1 = 0$ and $\Pi_2 = u_{2,0}^{\max}$, the switching time is defined as

$$T_s = T_o + \frac{\Gamma(1-\beta)}{k_3 k_4^{(1-\beta)}}, \tag{36}$$

with T_o given by (30). The settling time bound is given by $T_{max} = T_s + \frac{\Gamma(1-\alpha)}{kb^{(1-\alpha)}}$.

IV. Simulations

Let us consider a MAS which consists of $N = 6$ followers labeled by $1-6$ and one leader labeled by 0. Each agent is described by Eq. (3). The communication topology between agents is given in Figure 1. One can easily check that Assumption 1 is fulfilled.

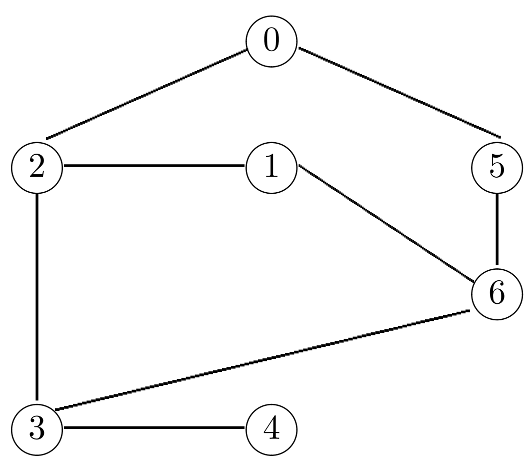

Fig. 1. Communication topology for the nonholonomic MAS.

In the following simulation, the leader control input is set to $u_{1,0} = 1 - 0.1\cos(t)$, $u_{2,0} = -0.2\cos(0.1t)$. Hence, Assumption 2 is verified with $u_{1,0}^{\min} = 0.9$, $u_{1,0}^{\max} = 1.1$, $\tau = 0.1$ and $u_{2,0}^{\max} = 0.2$. The observer (26) is implemented with $k_1 = 20$, $k_2 = 0.5$ and $\alpha' = \frac{1}{3}$. The control input (34)-(35) is implemented with $k_3 = 2$, $k_4 = 0.1$, $\beta = \frac{1}{3}$, $k = 0.01$, $b = 0.001$, $\alpha = \frac{7}{8}$. The initial state of the leader is $q_0(0) = [3, 2, 0.46]^T$ and the initial states of the followers are $q_1(0) = [5, 3, -0.78]^T$, $q_2(0) = [2, -1, 0]^T$, $q_3(0) = [7, 9, -1.1]^T$, $q_4(0) = [5, 3, -0.78]^T$, $q_5(0) = [7, -3, 0.78]^T$, $q_6(0) = [3, -0.5, 0.2]^T$. The upper bound of the settling time for the observer part can be estimated as $T_o = 1s$.

From Theorem 4, one can show that the decentralized observer (26) ensures the fixed-time stabilization of the estimation errors toward the origin. This can be seen in Fig. 2. The switching time in (35) is equal to $T_s = 5s$. One can see in Fig. 3 that the tracking errors converge to zero in a fixed time. The trajectories of the nonholonomic robots are depicted in Fig. 4. One can see that the proposed observer-based protocol ensure the fixed-time leader-follower consensus in fixed-time.

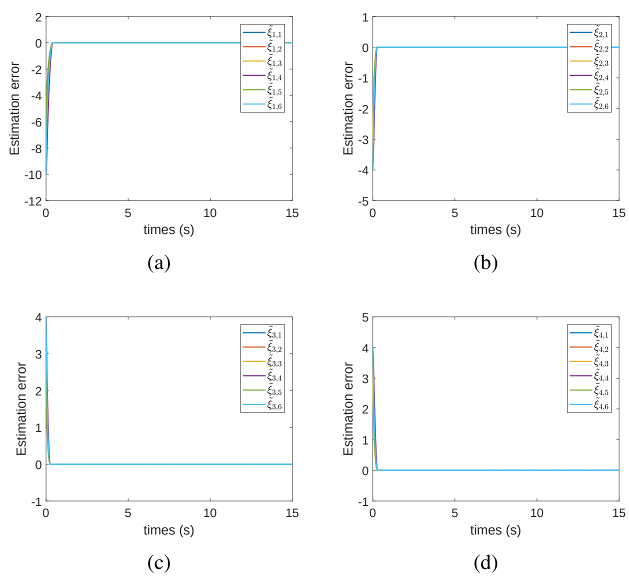

Fig. 2. Evolution of the estimation errors for each follower.

V. Conclusion

In this paper, we have studied the fixed-time leader-follower consensus problem of nonholonomic multi-agent systems. Using the "desingularisation method", new fixed-time controllers for the double integrator system are designed. Following those results, a switching consensus protocol which guarantees the stabilization of the tracking errors in fixed-time which does not depend on the initial conditions of the multi-agent system is provided. Simulation results on a fleet of wheeled mobile robots have shown the effectiveness of the proposed scheme.

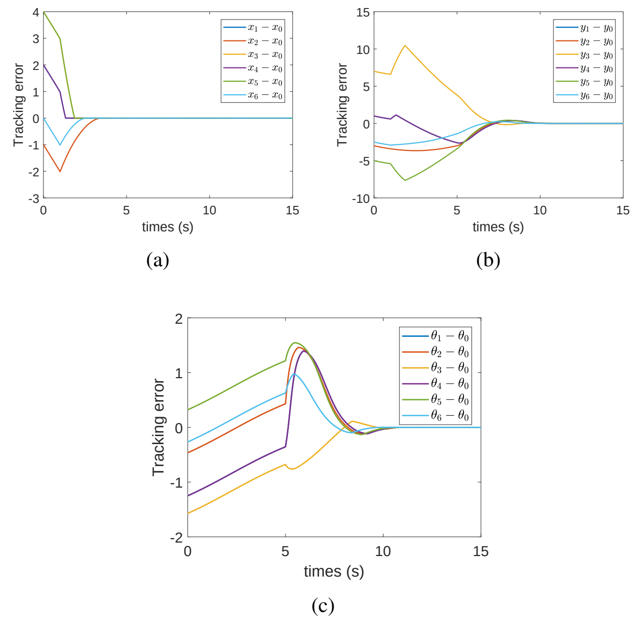

Fig. 3. Evolution of the tracking errors between each agent and the leader.

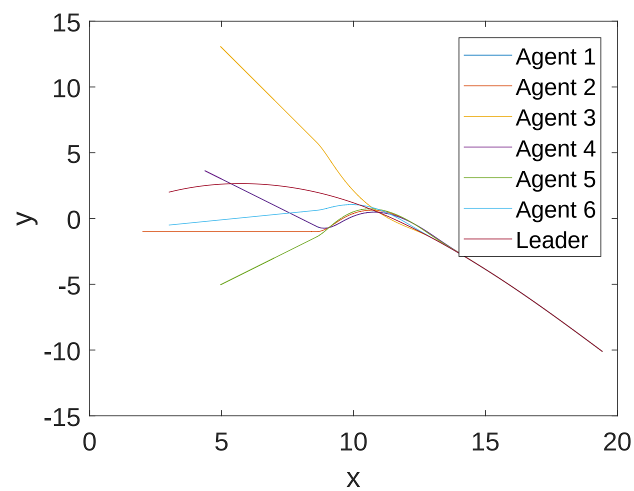

Fig. 4. Trajectory for each agent.

Appendix

Jensen's inequality, leads to

Lemma 2: For any $(V_1, V_2) > 0, 0 < \alpha < 1, b > 0$ the following inequality holds ($V = \sum_{i=1}^{2} V_i$):

$$-2^{\alpha-1} \sum_{i=1}^{2} V_i^{\alpha} \exp(2bV_i) \leq -V^{\alpha} \exp(bV). \tag{37}$$

Due to space limitation, we only provide for Theorem 2 some sketch of the proof (which is inspired from papers [13], [7]):

Proof: Let us define $V_1 = \frac{1}{2}x_1^2$, we get

$$\begin{aligned} \dot{V}_1 &= x_1 x_2^\star + x_1(x_2 - x_2^\star + \pi_1), \\ &\leq x_1 x_2^\star + x_1(x_2 - x_2^\star) + |x_1|^{2\alpha}|x_1|^{2(1-\alpha)}\Pi_1(x). \end{aligned}$$

In the sequel we use $\zeta(x) = \varphi_{\frac{1}{2\alpha-1}}(x_2^\star(x_1), x_2) = \{x_2\}^{\frac{1}{2\alpha-1}} - \{x_2^\star(x_1\}^{\frac{1}{2\alpha-1}}$ for which $(x_2^\star(x_1), x_2)$ the arguments of $\varphi_{\frac{1}{2\alpha-1}}$ will be omitted for sake of brevity. We have $|x_1(x_2 - x_2^\star)| \leq 2|x_1||\varphi_{\frac{1}{2\alpha-1}}|^{2\alpha-1}$ which gives:

$$|x_1(x_2 - x_2^\star)| \leq |x_1|^{2\alpha} + \frac{2\alpha-1}{\alpha^{\frac{2\alpha}{2\alpha-1}}}|\zeta(x)|^{2\alpha}.$$

Since $\frac{3}{4} < \alpha < 1$ we have $\frac{2\alpha-1}{\alpha^{\frac{2\alpha}{2\alpha-1}}} < \frac{4}{3} < 2$. Thus, using $x_2^\star(x_1)$ given by (13), we obtain:

$$\begin{aligned} \dot{V}_1 &\leq -W_1(x_1) + 2|\zeta(x)|^{2\alpha} - (1+\Pi_1(x))|x_1|^{2\alpha}, \\ W_1(x_1) &= k2^{\alpha-1}V_1^\alpha \exp(2bV_1). \end{aligned} \tag{38}$$

Note that when $x_2 = x_2^\star$ (thus $\zeta(x) = 0$), we have $\dot{V}_1 \leq -W_1(x_1)$. Using (2), let us define $V_2(x) = \Phi^{3-2\alpha}_{\frac{1}{2\alpha-1}}(x_2^\star(x_1), x_2)$ which is clearly $\mathcal{C}^1$ due to the fact that $\frac{3}{4} < \alpha < 1$ thus $(3-2\alpha) > 1$. Take $V(x) = V_1(x_1) + V_2(x)$. Since $V_2(x) = 0 \Leftrightarrow x_2^\star(x_1) = x_2$ thus $V(x) \geq 0$ (positive definite). We have $V_2(x) \leq 2\varphi^2_{\frac{1}{2\alpha-1}}(x_2^\star(x_1), x_2) = 2\zeta^2(x)$ and

$$\begin{aligned} \dot{V}_2 &= \frac{\partial V_2}{\partial x_1}(x_2 + \pi_1) + \frac{\partial V_2}{\partial x_2}(u + \pi_2), \\ \frac{\partial V_2}{\partial x_2} &= \left\{\varphi_{\frac{1}{2\alpha-1}}(x_2^\star(x_1), x_2)\right\}^{3-2\alpha}. \end{aligned}$$

After some lengthly computations we finally get

$$\begin{aligned} \dot{V} \leq\ & -W_1(x_1) - (1+\Pi_1(x))|x_1|^{2\alpha} \\ & +(1+\Pi_1(x))|x_1|^{2\alpha} + \{\zeta(x)\}^{3-2\alpha}u + |\zeta(x)|^{3-2\alpha}\Pi_2(x) \\ & + \left(2 + A(x) + \frac{1}{2}(A(x)\gamma_1(x))^{2\alpha}\right)|\zeta(x)|^{2\alpha} \\ & + \frac{|x_1\zeta(x)|^{2(1-\alpha)}}{4}A^2(x)\Pi_1(x)\,|\zeta(x)|^{2\alpha}. \end{aligned}$$

Finally, using control u given by (11)-(17), we have

$$\dot{V} \leq -W_1(x_1) - W_2(\zeta(x)), \tag{39}$$

$$W_2(\zeta(x)) = k2^{3\alpha-1}|\zeta(x)|^{2\alpha}\exp\left(4b\zeta^2(x)\right). \tag{40}$$

Since $V_2(x) \leq 2\zeta^2(x)$ one gets $-W_2(\zeta(x)) \leq -k2^{\alpha-1}V_2^\alpha \exp(2bV_2)$. Thus, we obtain:

$$\dot{V} \leq -k2^{\alpha-1}(V_1^\alpha \exp(2bV_1) + V_2^\alpha \exp(2bV_2)).$$

Using (37) we finally get:

$$\dot{V} \leq -kV^\alpha \exp(bV),$$

from which one concludes using Theorem 1. ■

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