# Compositional Construction of Abstractions for Infinite Networks of Switched Systems

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Abstract—We construct compositional continuous approximations for an interconnection of infinitely many discrete-time switched systems. An approximation (known as abstraction) is itself a continuous-space system, which can be used as a replacement of the original (known as concrete) system in a controller design process. Having synthesized a controller for the abstract system, the controller is refined to a more detailed controller for the concrete system. To quantify the mismatch between the output trajectory of the approximation and of that the original system, we use the notion of socalled simulation functions. In particular, each subsystem in the concrete network and its corresponding one in the abstract network is related through a local simulation function. We show that if the local simulation functions satisfy a certain smallgain type condition developed for a network of infinitely many subsystems, then the aggregation of the individual simulation functions provides an overall simulation function between the overall abstraction and the concrete network. For a network of linear switched systems, we systematically construct local abstractions and local simulation functions, where the required conditions are expressed in terms of linear matrix inequalities and can be efficiently computed. We illustrate the effectiveness of our approach through an application to frequency control in a power gird with a switched (i.e. time-varying) topology.

### I. INTRODUCTION

The high cost of incorrect configuration of a control system, on one hand, and safety concerns, on the other hand, call for automated and provably correct techniques for the verification and synthesis of modern control systems. In addition, emergent applications which consist of large-scale networked systems such as smart grids, connected automated vehicles, swarm robotics, etc. necessitate advanced control objectives going well beyond classic control problems such as regulation and tracking.

The complexity of control objectives, large and timevarying number of participating agents, and the complexity of the problem require methods on automated synthesis of provably correct controllers by joining forces from control theory and computer science. Particularly, a discrete abstraction (refereed to as symbolic model) provides automated synthesis of a correct-by-design controller for the original (referred to as concrete) system. In this approach, the controller synthesis problem can be algorithmically solved over a finite abstraction of the concrete system. Then, the constructed controller is refined back to the original system based on some behavioral relation between the original system and its finite abstraction such as approximate alternating simulation relations [1].

The applicability of finite abstractions is considerably limited due to the computational complexity of constructing discrete approximations of the concrete system. Therefore, a brute force approach to large-scale systems is not feasible. A way to reduce this computational complexity is to introduce a pre-processing step by constructing so-called continuous abstractions. In that way, a continuous-space system, but possibly with a *lower* dimension, is obtained for the concrete system [2]–[4]. To further manage the computational complexity, one may divide a possibly large-scale network into several smaller subsystems and then construct an abstraction for each subsystem individually. The methodology to achieve an abstraction for the overall network via the interconnection of the individual abstractions is called a compositional approach [5]-[7]. However, an efficient approach which is *independent* of the size of the network and potentially applicable to infinite-dimensional cases is still missing.

Motivated by the above discussion, this paper aims at providing a scale-free compositional approach for the construction of continuous abstractions for arbitrarily largescale networks of discrete-time switched systems. Inspired by the works in the literature regarding stability analysis of large-scale systems e.g. [8]-[12], to address the scalability issue, we over-approximate a finite-but-large network with a network composed of *infinitely* many subsystems, which we call it an infinite network. It is widely accepted that an infinite network captures the essence of its corresponding finite network; see, e.g., a vehicle platooning application in [13]. This treatment leads to an infinite-dimensional system and calls for a more rigorous and detailed setting. In particular, we adapt the notion of simulation functions [2] to the case of *infinite*-dimensional systems. The existence of a simulation function ensures that the error between the output trajectories of the abstract system and that of the concrete system is quantitatively bounded in a certain sense (cf. Definition III.1).

Following the compositionality approach, we assign to

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each subsystem an individual simulation function and construct each local abstraction accordingly. Then we aggregate them to compose an abstraction for the overall network. We show that the aggregation yields a continuous abstraction for the overall concrete network if a certain small-gain condition, which has been recently developed in [14], is satisfied. The effectiveness of our approach is verified by an application to frequency control in a power grid with a time-varying topology.

*Notation:* We write  $\mathbb{N}_0(\mathbb{N})$  for the set of nonnegative (positive) integers. For vector norms on finite- and infinitedimensional vector spaces, we write  $|\cdot|$ . By  $\ell^p$ ,  $p \in [1, \infty)$ , we denote the Banach space of all real sequences  $x = (x_i)_{i \in \mathbb{N}}$  with finite  $\ell^p$ -norm  $|x|_p < \infty$ , where  $|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$  for  $p < \infty$ . If X is a Banach space, we write r(T) for the spectral radius of a bounded linear operator  $T: X \to X$ . The identity function is denoted by id. We will consider  $\mathcal{K}$  and  $\mathcal{K}_{\infty}$  comparison functions, see [15, Chapter 4.4] for definitions.

#### **II. SYSTEM DESCRIPTION**

We study the interconnection of countably many switched systems, each given by a finite-dimensional difference equation. We define the switching signal functions  $\sigma_i : \mathbb{N}_0 \to S_i$ ,  $i \in \mathbb{N}$  for  $S_i \in \{1, 2, \ldots, r_i\}$  which is a finite index set with  $r_i \in \mathbb{N}$ . We denote the set of such switching signals by  $S_i$ . The *i*-th subsystem  $(i \in \mathbb{N})$  is written as

$$\Sigma_i: \begin{cases} \mathbf{x}_i(k+1) = f_{i,\sigma_i(k)}(\mathbf{x}_i(k), \mathbf{w}_i(k), \mathbf{u}_i(k)), \\ \mathbf{y}_i(k) = h_{i,\sigma_i(k)}(\mathbf{x}_i(k)), \end{cases}$$
(1)

where  $\mathbf{x}_i : \mathbb{N}_0 \to \mathbb{R}^{n_i}$ ,  $\mathbf{w}_i : \mathbb{N}_0 \to \mathbb{R}^{N_i}$ ,  $\mathbf{u}_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ , and  $\mathbf{y}_i : \mathbb{N}_0 \to \mathbb{R}^{q_i}$  are state signal, internal input signal, external input signal, and output signal, respectively.

The family  $(\Sigma_i)_{i\in\mathbb{N}}$  comes together with sequences  $(n_i)_{i\in\mathbb{N}}, (m_i)_{i\in\mathbb{N}}$  of positive integers and finite sets  $I_i \subset \mathbb{N}\setminus\{i\}$ , and  $\overline{I}_i \subset \mathbb{N}$  enumerate the neighbors of  $\Sigma_i$ , i.e., those systems  $\Sigma_j, j \in I_i, \Sigma_{j'}, j' \in \overline{I}_i$  that affect or are affected by  $\Sigma_i$ , respectively. By definition we require that  $i \notin I_i \cup \overline{I}_i, \forall i \in \mathbb{N}$ . We denote  $\mathbf{w}_i(k) = (\mathbf{w}_{ij}(k))_{j\in I_i} \in \mathbb{R}^{N_i}$  for  $N_i := \sum_{j\in I_i} n_j$  as the internal inputs to show the interconnections. The output functions  $h_{i,\sigma_i(k)}(\mathbf{x}_i(k)) = (h_{ij,\sigma_i(k)}(\mathbf{x}_i(k)))_{j\in(i\cup\overline{I}_i)}, \mathbf{y}_i(k) = (\mathbf{y}_{ij}(k))_{j\in(i\cup\overline{I}_i)}$  are elements of  $\mathbb{R}^{q_i}$ . Note that  $\mathbf{w}_i(k)$  and  $\mathbf{y}_i(k)$  are partitioned into sub-vectors and we aggregate all the subsystems  $\Sigma_i$  through the interconnection constraints given by  $\mathbf{w}_{ij}(k) = \mathbf{y}_{ji}(k)$  for all  $i \in \mathbb{N}$  and for all  $j \in I_i$ . In that way, the interconnection of  $\Sigma_i, i \in \mathbb{N}$ , is described by

$$\Sigma: \begin{cases} \mathbf{x}(k+1) = f_{\sigma(k)}(\mathbf{x}(k), \mathbf{u}(k)), \\ \mathbf{y}(k) = h_{\sigma(k)}(\mathbf{x}(k)), \end{cases}$$
(2)

where  $\mathbf{x}(k) = (\mathbf{x}_i(k))_{i \in \mathbb{N}}$ ,  $\mathbf{u}(k) = (\mathbf{u}_i(k))_{i \in \mathbb{N}}$ ,  $\mathbf{y}(k) = (\mathbf{y}_{ii}(k))_{i \in \mathbb{N}}$ ,  $\sigma(k) = (\sigma_i(k))_{i \in \mathbb{N}}$ ,  $f_{\sigma(k)}(\mathbf{x}(k), \mathbf{u}(k)) = (f_{i,\sigma_i(k)}(\mathbf{x}_i(k), \mathbf{w}_i(k), \mathbf{u}_i(k)))_{i \in \mathbb{N}}$ , and  $h_{\sigma(k)}(\mathbf{x}(k)) := (h_{ii,\sigma_i(k)}(\mathbf{x}_i(k)))_{i \in \mathbb{N}}$ .

Clearly, system (2) is an *infinite*-dimensional system, which asks for careful choice of the state and input spaces. We choose appropriate Banach spaces  $X \subset \prod_{i \in \mathbb{N}} \mathbb{R}^{n_i}$  and  $U \subset \prod_{i \in \mathbb{N}} \mathbb{R}^{m_i}$ , and restrict  $f_{\sigma(k)}$  to  $X \times U$ ,  $\sigma : \mathbb{N} \to S$ , for all  $k \in \mathbb{N}_0$ , where  $S = \prod_{i \in \mathbb{N}} S_i$ .

We model the state space X of  $\Sigma$  as a Banach space of sequences  $x = (x_i)_{i \in \mathbb{N}}$  with  $x_i \in \mathbb{R}^{n_i}$ . The most natural choice is an  $\ell^p$ -space. To define such a space, we first fix a norm on each  $\mathbb{R}^{n_i}$ . Then, for every  $p \in [1, \infty)$ , we put

$$\ell^p(\mathbb{N}, (n_i)) := \left\{ x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R}^{n_i}, \ \sum_{i \in \mathbb{N}} |x_i|^p < \infty \right\}$$

and equip this space with the norm  $|x|_p := (\sum_{i \in \mathbb{N}} |x_i|^p)^{1/p}$ .

As the state space of the system  $\Sigma$ , we consider  $X := \ell^p(\mathbb{N}, (n_i))$  for a fixed  $p \in [1, \infty)$ . Similarly, for a fixed  $q \in [1, \infty)$ , we consider the *external input space*  $U := \ell^q(\mathbb{N}, (m_i))$ , where we fix norms on  $\mathbb{R}^{m_i}$  that we simply denote by  $|\cdot|$  again. The space of admissible *external input functions*  $\mathbf{u}$  is defined by  $\mathcal{U} := \{\mathbf{u} : \mathbb{N}_0 \to U\}$ . We denote the corresponding solutions by  $\mathbf{x}(k, x, \sigma, \mathbf{u})$  for any  $k \in \mathbb{N}_0$ , any initial value  $x \in X$ , any switching signal  $\sigma \in S$ , and any control input  $\mathbf{u} \in \mathcal{U}$ .

We refer to system (2) as the *concrete* system, which is often hard to control. In order to synthesize these systems, using a simpler, though less precise system called an *abstract* system is beneficial. Adopting the same *notational convention* as those for  $\Sigma_i$  and  $\Sigma$ , but with the  $\hat{\cdot}$  sign on the top of the respective ones, we introduce the symbols for the abstract subsystems  $\hat{\Sigma}_i$  and the corresponding overall system and  $\hat{\Sigma}$ , respectively.

# III. ABSTRACTIONS FOR SWITCHED DISCRETE-TIME SYSTEMS

In this section, we introduce the notion of so-called simulation functions for the discrete-time switched systems with only external inputs. A simulation function of  $\hat{\Sigma}$  by  $\Sigma$  is a function over their state spaces which explain how a state trajectory of  $\hat{\Sigma}$  can be transformed into a state trajectory of  $\Sigma$  to make the distance between the associated output trajectories bounded. Formally, a simulation function is defined as follows:

**Definition III.1** Consider the systems  $\Sigma$  and  $\hat{\Sigma}$  with the same output spaces and fixed  $p,q \in [1,\infty)$ . Let  $V_s : X \times \hat{X} \to \mathbb{R}_+, s \in S$ , be a family of functions. Let there exist positive constants  $\alpha, b$ , such that for all  $s \in S$ ,  $x \in X$ ,  $\hat{x} \in \hat{X}$ ,

$$\alpha \left| h_s(x) - \hat{h}_s(\hat{x}) \right|_p^b \le V_s(x, \hat{x}), \tag{3}$$

and there exist a function  $\rho_{\text{ext}} \in \mathcal{K}$  and a positive constant  $\lambda < 1$ , such that for all consecutive  $s', s \in S$  (i.e.,  $s' = \sigma(k+1), s = \sigma(k)$  for  $k \in \mathbb{N}_0$ ), and all  $x \in X$ ,  $\hat{x} \in \hat{X}$  and  $\hat{u} \in \hat{U}$  there exist  $u \in U$  so that we have

$$V_{s'}(f_s(x, u), \hat{f}_s(\hat{x}, \hat{u})) - V_s(x, \hat{x}) \leq -\lambda V_s(x, \hat{x}) + \rho_{\text{ext}}(|\hat{u}|_q).$$
(4)

Then, the functions  $V_s$  are called the simulation functions from  $\hat{\Sigma}$  to  $\Sigma$ .

The following proposition shows the importance of the existence of a simulation function.

**Proposition III.2** Consider systems  $\Sigma$  and  $\hat{\Sigma}$ , the same output space, and fixed  $p, q \in [1, \infty)$ . Let a set of simulation functions  $V_s$ ,  $s \in S$ , from  $\hat{\Sigma}$  to  $\Sigma$  be given. Then there exist a function  $\gamma_{\text{ext}} \in \mathcal{K}$  and positive constants  $\vartheta$  and  $\beta < 1$ , such that for any  $\sigma \in S$ ,  $x \in X$ ,  $\hat{x} \in \hat{X}$ ,  $\hat{\mathbf{u}} \in \hat{\mathcal{U}}$ ,  $k \in \mathbb{N}_0$ , there exists  $\mathbf{u} \in \mathcal{U}$  so that we have

$$\begin{aligned} \left| \mathbf{y}(k, x, \sigma, \mathbf{u}) - \hat{\mathbf{y}}(k, \hat{x}, \sigma, \hat{\mathbf{u}}) \right|_{p} \\ &\leq \vartheta \beta^{k} (V_{\sigma(0)}(\xi, \hat{\xi}))^{\frac{1}{b}} + \gamma_{\text{ext}}(|\hat{\mathbf{u}}|_{q, \infty}), \end{aligned} \tag{5}$$

where  $|\hat{\mathbf{u}}|_{q,\infty} := \sup_{k \in \mathbb{N}_0} |\hat{\mathbf{u}}(k)|_q$  and b as in (3).

The proof is not presented due to space limitations. Basically it follows similar arguments as those in the proof of [16, Lemma 3.5].

**Remark III.3** If we are given an interface function  $\nu$  that maps every  $x, \hat{x}, \hat{u}$ , and s to an input  $u = \nu(x, \hat{x}, \hat{u}, s)$  so that (4) is satisfied, then, the input  $\mathbf{u}$  that realizes (5) is readily given by  $\mathbf{u}(k) = \nu(\mathbf{x}(k), \hat{\mathbf{x}}(k), \hat{\mathbf{u}}(k), \sigma(k))$ , see [17, Theorem 1].

## IV. COMPOSITIONAL CONSTRUCTION OF ABSTRACTIONS AND SIMULATION FUNCTIONS

In this section, we construct continuous compositional abstraction for an interconnection of countably many discretetime switched system and the corresponding simulation function from the abstractions of the subsystems and their corresponding simulation functions, respectively. Then, we focus on linear subsystems and provide conditions under which local quadratic simulation functions with their associated interface functions construct the abstractions.

We assume that subsystems  $\Sigma_i$  for  $i \in \mathbb{N}$ , given by (1), together with their abstractions  $\hat{\Sigma}_i$  and the there exist simulation functions  $V_{i,s_i}, s_i \in S_i$ , from  $\hat{\Sigma}_i$  to  $\Sigma_i$  satisfying the following assumption

**Assumption IV.1** Consider the subsystems  $\Sigma_i$  for  $i \in \mathbb{N}$ , together with their abstractions  $\hat{\Sigma}_i$ . For fixed  $p, q \in [1, \infty)$ , there exist functions  $V_{i,s_i} : \mathbb{R}^{n_i} \times \mathbb{R}^{\hat{n}_i} \to \mathbb{R}_+, s_i \in S_i$ , with the following properties.

 There are positive constants α<sub>i</sub> so that for all x<sub>i</sub> ∈ ℝ<sup>n<sub>i</sub></sup>, all x̂<sub>i</sub> ∈ ℝ<sup>n̂<sub>i</sub></sup>

$$\alpha_i \left| h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i) \right|^p \le V_{i,s_i}(x_i, \hat{x}_i).$$
 (6)

• There are positive constants  $\lambda_i < 1$ ,  $\rho_{i,\text{int}}$ ,  $\rho_{i,\text{ext}}$ , such that for all consecutive  $s'_i, s_i \in S_i, x_i \in \mathbb{R}^{n_i}, \hat{x}_i \in \mathbb{R}^{\hat{n}_i}$ ,  $\hat{u}_i \in \mathbb{R}^{\hat{m}_i}$ , there exist  $u_i \in \mathbb{R}^{m_i}$ , so that the following holds for all  $w_i \in \mathbb{R}^{N_i}$ ,  $\hat{w}_i \in \mathbb{R}^{\hat{N}_i}$ :

$$V_{i,s'_{i}}\left(f_{i,s_{i}}(x_{i},w_{i},u_{i}),\hat{f}_{i,s_{i}}(\hat{x}_{i},\hat{w}_{i},\hat{u}_{i})\right) - V_{i,s_{i}}(x_{i},\hat{x}_{i}) \\ \leq -\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q} + \rho_{i,\text{int}}|w_{i} - \hat{w}_{i}|^{p}.$$
(7)

We assume that the following uniformity conditions hold for the constants introduced above.

**Assumption IV.2** There are constants  $\underline{\alpha}, \underline{\lambda}, \overline{\rho}_{ext} > 0$  so that for all  $i \in \mathbb{N}$ , we have  $\underline{\alpha} \leq \alpha_i, \underline{\lambda} \leq \lambda_i, \rho_{i,ext} \leq \overline{\rho}_{ext}$ .

In order to formulate a small-gain condition, we further introduce the following matrices by utilizing the coefficients from (7):

$$\Lambda := \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \Gamma := (\gamma_{ij})_{i,j \in \mathbb{N}}, \qquad (8)$$

where

$$\gamma_{ij} := \begin{cases} \rho_{i,\text{int}} \bar{N}_i \frac{1}{\alpha_j}, & j \in I_i, \\ 0, & j \notin I_i, \end{cases}$$
(9)

for  $\bar{N}_i$  as the number of neighbors of subsystem *i*. Now, we define the following matrix by which we express our *small-gain* condition:

$$\Psi := \Lambda^{-1} \Gamma := (\psi_{ij})_{i,j \in \mathbb{N}}, \quad \psi_{ij} = \gamma_{ij} / \lambda_i.$$
(10)

We make the following spectral radius condition which provides a *quantitative* index on the strength of coupling between the subsystems.

# **Assumption IV.3** The spectral radius $r(\Psi) < 1$ .

We make an assumption on the boundedness of the operator  $\Gamma$ .

**Assumption IV.4** The operator  $\Gamma = (\gamma_{ij})_{i,j \in \mathbb{N}}$  satisfies  $\sup_{i \in \mathbb{N}} \sum_{i=1}^{\infty} \gamma_{ij} < \infty$ .

Note that the assumption above holds if each subsystem is interconnected to a finitely many subsystems.

The following theorem gives the *main result* of the paper, which is a compositional approach for construction of abstractions of infinite interconnected control systems and their corresponding simulation functions.

**Theorem IV.5** Consider the infinite networks  $\Sigma$  and  $\hat{\Sigma}$  with fixed  $p, q \in [1, \infty)$ . Suppose that Assumptions IV.1, IV.2, IV.3 and IV.4 hold. Then there exists a vector  $\mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^{\infty}$ satisfying  $\underline{\mu} \leq \mu_i \leq \overline{\mu}$  with some constants  $\underline{\mu}, \overline{\mu} > 0$ , such that the following is satisfied for a constant  $0 < \lambda_{\infty} < 1$ .

$$\frac{\mu^{\top}(-\Lambda+\Gamma)]_i}{\mu_i} \le -\lambda_{\infty} \quad \forall i \in \mathbb{N}, s \in S.$$
(11)

Moreover, for all  $s_i \in S_i$ ,  $s \in S$ ,

$$V_s(x,\hat{x}) = \sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i,\hat{x}_i), \quad V_s : X \times \hat{X} \to \mathbb{R}_+$$

are simulation functions of  $\hat{\Sigma}$  by  $\Sigma$  with  $b = p, \alpha = \underline{\mu}\underline{\alpha}$  as in (3) and  $\lambda = \lambda_{\infty}$  and  $\rho_{\text{ext}} : t \mapsto \overline{\mu} \ \overline{\rho}_{\text{ext}} t^q$  as in (4).

*Proof:* From [14, Lemma V.10], Assumption IV.3 (i.e.  $r(\Psi) < 1$ ) implies that there exists a vector  $\mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^{\infty}$  satisfying  $\underline{\mu} \leq \mu_i \leq \overline{\mu}$  such that (11) holds.

Now we show that V in (IV.5) satisfies (3) with  $\alpha = \underline{\mu}\underline{\alpha}$ . For any  $s \in S$ ,  $s_i \in S_i$ ,  $x \in X$ ,  $\hat{x} \in \hat{X}$ , and taking b = p, it follows from (6) and Assumption IV.2 that

$$\sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i, \hat{x}_i) \ge \sum_{i=1}^{\infty} \mu_i \alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p$$
$$\ge \underline{\mu} \underline{\alpha} \sum_{i=1}^{\infty} |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p$$
$$\ge \underline{\mu} \underline{\alpha} |h_s(x) - \hat{h}_s(\hat{x})|_p^p.$$

Now we show the inequality (4) holds as well. Considering (7) and (IV.5), we obtain the chain of inequalities in (12) for all  $s'_i, s_i \in S_i, s_j \in S_j, s', s \in S, i \in \mathbb{N}$ .

Letting  $V_{s_{vec}}(x, \hat{x}) := (V_{i,s_i}(x_i, \hat{x}_i))_{i \in \mathbb{N}}$  and using (12) and (11), we have that

$$\begin{aligned} V_{s'}(f_s(x,u), f_s(\hat{x}, \hat{u})) &- V_s(x, \hat{x}) \\ &\leq \left[ \mu^\top (-\Lambda + \Gamma) V_{s_{vec}}(x, \hat{x}) + \sum_{i=1}^\infty \mu_i \rho_{i, \text{ext}} |\hat{u}_i|^q \right] \\ &\leq -\lambda_\infty V_s(x, \hat{x}) + \rho_{\text{ext}}(|\hat{u}|_q), \end{aligned}$$
  
where  $\rho_{\text{ext}}(t) = \overline{\mu} \ \overline{\rho}_{\text{ext}} t^q$  for all  $t \geq 0.$ 

#### A. Abstractions for linear systems

In this section, we use the previous results to compute the compositional abstractions for a network of linear switched subsystems. We aim to construct abstractions with output trajectories close enough to those of the concrete system.

Consider the following network on interconnected linear switched systems:

$$\Sigma_{i}: \begin{cases} \mathbf{x}_{i}(k+1) = A_{i,\sigma_{i}(k)}\mathbf{x}_{i}(k) + D_{i,\sigma_{i}(k)}\mathbf{w}_{i}(k) \\ + B_{i,\sigma_{i}(k)}\mathbf{u}_{i}(k), \\ \mathbf{y}_{i}(k) = C_{i,\sigma_{i}(k)}\mathbf{x}_{i}(k), \end{cases}$$
(13)

where  $\sigma_i \in S_i$ ,  $A_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times n_i}$ ,  $B_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times m_i}$ ,  $C_{i,\sigma_i(k)} \in \mathbb{R}^{q_i \times n_i}$  and  $D_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times p_i}$  for  $i \in \mathbb{N}$ .

Choose  $X = \ell^2(\mathbb{N}, (n_i))$  and  $U = \ell^2(\mathbb{N}, (m_i))$ . We assume that we are given abstractions as  $\hat{\Sigma}_i$ , and then provide conditions under which  $V_{i,s_i}$  for  $s_i \in S_i$  are candidate simulation functions from  $\hat{\Sigma}_i$  to  $\Sigma_i$ . Assume that there exist a family of matrices  $K_{i,s_i}$ , positive definite matrices  $M_{i,s_i}$ and given  $0 < \kappa_i < 1$  for  $i \in \mathbb{N}$ , such that the following matrix inequalities hold for all consecutive  $s'_i, s_i \in S_i$  (i.e.,  $s'_i = \sigma_i(k+1), s_i = \sigma_i(k)$  for  $k \in \mathbb{N}_0$ ).

$$C_{i,s_{i}}^{\top}C_{i,s_{i}} \leq M_{i,s_{i}},$$
(14a)  

$$3(A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}})^{\top}M_{i,s_{i}'}(A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}}) - M_{i,s_{i}}$$
  

$$\leq -\kappa_{i}M_{i,s_{i}}.$$
(14b)

Note that (14b) could be transformed to a LMI using the Schur complement lemma (see [18, Remark 4.7]).

Take the following simulation function candidates for the mentioned chosen state space:

$$V_{i,s_i}(x_i, \hat{x}_i) = (x_i - P_i \hat{x}_i)^\top M_{i,s_i}(x_i - P_i \hat{x}_i).$$
(15)

The input is given by the interface function  $\nu_i$  as follows.

$$u_i = \nu_i(x_i, \hat{x}_i, \hat{u}_i, \hat{w}_i, s_i)$$
(16)

$$=K_{i,s_i}(x_i - P_i \hat{x}_i) + Q_{i,s_i} \hat{x}_i + R_{i,s_i} \hat{u}_i + T_{i,s_i} \hat{w}_i,$$

where  $P_i$ ,  $i \in \mathbb{N}$ , are appropriate dimension matrices. Assume that the following inequalities hold for some appropriate dimension matrices  $Q_{i,s_i}$ ,  $T_{i,s_i}$ .

$$A_{i,s_i}P_i = P_i\hat{A}_{i,s_i} - B_{i,s_i}Q_{i,\sigma_i},$$
(17a)

$$D_{i,s_i} = P_i \hat{D}_{i,s_i} - B_{i,s_i} T_{i,\sigma_i},$$
 (17b)

$$C_{i,s_i} P_i = \hat{C}_{i,s_i}.\tag{17c}$$

**Theorem IV.6** Consider two systems  $\Sigma_i = (A_{i,s_i}, B_{i,s_i}, C_{i,s_i}, D_{i,s_i})$  and  $\hat{\Sigma}_i = (\hat{A}_{i,s_i}, \hat{B}_{i,s_i}, \hat{C}_{i,s_i}, \hat{D}_{i,s_i})$  for  $i \in \mathbb{N}$ . Suppose that for all  $s_i \in S_i$  there exist appropriate matrices  $M_{i,s_i}$ ,  $P_i$ ,  $K_{i,s_i}$ ,  $Q_{i,s_i}$  and  $T_{i,s_i}$  which satisfy (14) and (17). Then, the functions defined in (15) are simulation functions from  $\hat{\Sigma}_i$  to  $\Sigma_i$  with inputs given by (16).

*Proof:* According to (17c), we have

$$\begin{aligned} |C_{i,s_i} x_i - \hat{C}_{i,s_i} \hat{x}_i| &= \\ \left( (x_i - P_i \hat{C}_{i,s_i})^\top C_{i,s_i}^\top C_{i,s_i} (x_i - P_i \hat{C}_{i,s_i}) \right)^{\frac{1}{2}}. \end{aligned}$$

Using (14a), it is clear that  $|C_{i,s_i}x_i - \hat{C}_{i,s_i}\hat{x}_i|^2 \leq V_{i,s_i}(x_i, \hat{x}_i)$  holds for all  $x_i \in \mathbb{R}^{n_i}$ ,  $\hat{x}_i \in \mathbb{R}^{\hat{n}_i}$ . Then, (6) is satisfied with  $\alpha_i = 1, i \in \mathbb{N}, p = 2$ .

Now, we proceed to show that (7) is satisfied, too.

By using (17a), (17b) and considering the  $u_i$  given by (16),  $A_{i,s_i}x_i + B_{i,s_i}u_i + D_{i,s_i}w_i - P_i(\hat{A}_{i,s_i}\hat{x}_i + \hat{B}_{i,s_i}\hat{u}_i + \hat{D}_{i,s_i}\hat{w}_i)$ is simplified to  $(A_{i,s_i} + B_{i,s_i}K_{i,s_i})(x_i - P_i\hat{x}_i) + D_{i,s_i}(w_i - \hat{w}_i) + (B_{i,s_i}R_{i,s_i} - P_i\hat{B}_{i,s_i})\hat{u}_i$ . Therefore, we obtain

$$\begin{aligned} V_{i,s'_{i}}\left(f_{i,s_{i}}(x_{i},w_{i},u_{i}),\hat{f}_{i,s_{i}}(\hat{x}_{i},\hat{w}_{i},\hat{u}_{i})\right) - V_{i,s_{i}}(x_{i},\hat{x}_{i}) \\ = & (x_{i} - P_{i}\hat{x}_{i})^{\top}[(A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}})^{\top}M_{i,s'_{i}} \\ \times & (A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}}) - M_{i,s_{i}}](x_{i} - P_{i}\hat{x}_{i}) \\ & + [2(x_{i} - P_{i}\hat{x}_{i})^{\top}(A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}})^{\top}]M_{i,s'_{i}} \\ \times & [D_{i,s_{i}}(w_{i} - \hat{w}_{i})] + [2(x_{i} - P_{i}\hat{x}_{i})^{\top} \\ & \times (A_{i,s_{i}} + B_{i,s_{i}}K_{i,s_{i}})^{\top}]M_{i,s'_{i}}[(B_{i,s_{i}}R_{i,s_{i}} - P_{i}\hat{B}_{i,s_{i}})\hat{u}_{i}] \\ & + [2(w_{i} - \hat{w}_{i})^{\top}D_{i,s_{i}}^{\top}]M_{i,s'_{i}}[(B_{i,s_{i}}R_{i,s_{i}} - P_{i}\hat{B}_{i,s_{i}})\hat{u}_{i}] \\ & + |\sqrt{M_{i,s'_{i}}}D_{i,s_{i}}(w_{i} - \hat{w}_{i})|^{2} \\ & + |\sqrt{M_{i,s'_{i}}}(B_{i,s_{i}}R_{i,s_{i}} - P_{i}\hat{B}_{i,s_{i}})\hat{u}_{i}|^{2}. \end{aligned}$$

Using Young's inequality and (14b), we can obtain the following:

$$\begin{aligned} V_{i,s'_{i}}\left(f_{i,s_{i}}(x_{i},w_{i},u_{i}),\hat{f}_{i,s_{i}}(\hat{x}_{i},\hat{w}_{i},\hat{u}_{i})\right) - V_{i,s_{i}}(x_{i},\hat{x}_{i}) \leq \\ &-\kappa_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + 3|\sqrt{M_{i,s'_{i}}}D_{i,s_{i}}|^{2}|w_{i} - \hat{w}_{i}|^{2} \\ &+ 3\left|\sqrt{M_{i,s'_{i}}}(B_{i,s_{i}}R_{i,s_{i}} - P_{i}\hat{B}_{i,s_{i}})\right|^{2}|\hat{u}_{i}|^{2}.\end{aligned}$$

Thus, (7) holds with p = q = 2,  $\lambda_i = \kappa_i$ ,  $\rho_{i,\text{ext}} = 3 \max_{s_i} \{ |\sqrt{M_{i,s'_i}} (B_{i,s_i} R_{i,s_i} - P_i \hat{B}_{i,s_i})|^2 \}$  and  $\rho_{i,\text{int}} = 3 \max_{s_i} \{ |\sqrt{M_{i,s'_i}} D_{i,s_i}|^2 \}$ .

Therefore, the functions of (15) are simulation functions from  $\hat{\Sigma}_i$  to  $\Sigma_i$ .

$$V_{s'}\left(f_{s}(x,u),\hat{f}_{s}(\hat{x},\hat{u})\right) - V_{s}(x,\hat{x}) = \sum_{i=1}^{\infty} \mu_{i}\left[V_{i,s'_{i}}\left(f_{i,s_{i}}(x_{i},w_{i},u_{i}),\hat{f}_{i,s_{i}}(\hat{x}_{i},\hat{w}_{i},\hat{u}_{i})\right) - V_{i,s_{i}}(x_{i},\hat{x}_{i})\right]$$

$$\leq \sum_{i=1}^{\infty} \mu_{i}(-\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \rho_{i,\text{int}}|w_{i} - \hat{w}_{i}|^{p} + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q})$$

$$\leq \sum_{i=1}^{\infty} \mu_{i}(-\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \sum_{j\in I_{i}} \rho_{i,\text{int}}\bar{N}_{i}|w_{ij} - \hat{w}_{ij}|^{p} + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q})$$

$$\leq \sum_{i=1}^{\infty} \mu_{i}(-\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \sum_{j\in I_{i}} \rho_{i,\text{int}}\bar{N}_{i}\left|h_{j,s_{j}}(x_{j}) - \hat{h}_{j,s_{j}}(\hat{x}_{j})\right|^{p} + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q})$$

$$\leq \sum_{i=1}^{\infty} \mu_{i}(-\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \sum_{j\in I_{i}} \rho_{i,\text{int}}\bar{N}_{i}\frac{1}{\alpha_{j}}V_{j,s_{j}}(x_{j},\hat{x}_{j}) + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q})$$

$$\stackrel{(9)}{\leq} \sum_{i=1}^{\infty} \mu_{i}\left(-\lambda_{i}V_{i,s_{i}}(x_{i},\hat{x}_{i}) + \sum_{j\in I_{i}} \gamma_{ij}V_{j,s_{j}}(x_{j},\hat{x}_{j}) + \rho_{i,\text{ext}}|\hat{u}_{i}|^{q}\right).$$

## V. EXAMPLE

We verify the effectiveness of our theoretical results by regulating the frequency deviations in a power network.

We consider a power network modeled by an interconnection of second-order systems, known as swing equation [19]. In particular, we consider two *circular* topologies: in the first one shown in Figure 1, subsystem *i* is fed by subsystem i-1; in the other configuration, subsystem *i* is fed by subsystem i+1, see Figure 2. We assume that the network topology switches between these two configurations at certain times. Let  $\sigma_i(k)$  be the switching signal which takes values in the set  $\{1, 2\}$ , where  $\sigma_i(k) = 1$  corresponds to the topology shown in Figure 1 and  $\sigma_i(k) = 2$  corresponds to that illustrated in Figure 2. To mathematically describe such a relation between the network topology and the switching signal, we define the function  $g_i$  by

$$g_i(s) = \begin{cases} i-1 & \text{if} \quad s=1, \\ i+1 & \text{if} \quad s=2. \end{cases}$$

In that way, each subsystem of the network is described by

where  $C_{ii} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ ,  $C_{ig_i(\sigma_i(k))} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ , and  $\delta_i$ ,  $\omega_i$ ,  $m_i$ ,  $d_i$ , and  $\mathbf{u}_i$  are the phase angle, frequency, inertia, damping coefficient, and the mechanical input power of bus i, respectively. The coefficient  $l_{ig_i(s_i)} = |v_i||v_{g_i(s_i)}|b_{ig_i(s_i)}$ , where  $|v_i|$  is the absolute value of the voltage of bus i, and  $b_{is_i}$  is the susceptance of the line  $(i, g_i(s_i))$  for  $s_i \in \{1, 2\}$ .



Fig. 1. The interconnected system  $\Sigma$  for  $s_i = 1$ .

$\mathbf{y}_{33}$ $\mathbf{u}_3$ $\mathbf{u}_3$ $\mathbf{y}_{32}$ $\Sigma_3$ $\mathbf{y}_{34}$	$\Sigma_4$ $\Sigma_4$ $\cdots$	$\underbrace{\overset{\mathbf{y}_{(i-1)(i-1)}}{\underbrace{\sum_{i-1}}} \mathbf{u}_{i-1}}_{\sum_{i-1}}$
$\mathbf{u}_2$ $\mathbf{y}_{22}$ $\mathbf{y}_{22}$ $\mathbf{y}_{21}$	$\mathbf{u}_1$ $\mathbf{y}_{11}$ $\mathbf{y}_{11}$	$\cdot$ $\underbrace{\mathbf{u}_i}_{\sum_i}$ $\underbrace{\mathbf{y}_{ii}}_{\mathbf{y}_{i(i-1)}}$

Fig. 2. The interconnected system  $\Sigma$  for  $s_i = 2$ .

The interconnection structure switches between two *circular* topologies shown in Figures 1-2.

To construct abstractions for  $\Sigma$ , we construct an abstraction for  $\Sigma_i$ ,  $i \in \mathbb{N}$ , for both communication topologies, i.e. for both  $s_i = 1, 2$ . Taking  $K_{i,s_i} = \begin{bmatrix} l_{ig_i(s_i)} - \frac{9}{16}m_i & d_i - 1.5m_i \end{bmatrix}$ , for both  $s_i = 1, 2$ , and  $\kappa_i = 0.2$ , we compute  $M_{i,s'_i} = M_{i,s_i} = \begin{bmatrix} 11.20 & 12.50 \\ 12.50 & 17.83 \end{bmatrix}$ , for  $s'_i, s_i \in \{1, 2\}$ , which satisfies (14). Now we proceed to compute other matrices so that (17) holds. Using (17a), we take  $Q_{i,s_i} = l_{ig_i(s_i)}$  for  $s_i = 1, 2$ . We obtain  $\hat{A}_{i,s_i} = c_i, s_i = 1, 2$ , and  $P_i = [1; c_i - 1]$  for constant  $c_i$  which is determined by solving the equation  $c_i^2 + c_i(\frac{d_i}{m_i} - 2) + 1 = 0$ . Therefore,  $c_i = 1 - \frac{d_i}{2m_i}$ . Moreover, using (17b), we get  $\hat{D}_{i,s_i} = 0$  and  $T_{i,s_i} = -l_{ig_i(s)}$ . Accordingly,  $\hat{C}_{i,s_i} = C_{i,s_i} \begin{bmatrix} 1 \\ c_i - 1 \end{bmatrix}$ . We also choose  $\hat{B}_{i,s_i} = \frac{d_i}{2m_i} - 0.6$  and  $R_{i,s_i} = (B_{i,s_i}^\top M_{i,s_i} B_{i,s_i})^{-1} B_{i,s_i}^\top M_{i,s_i} P_i \hat{B}_{i,s_i}$  to minimize  $\rho_{i,ext}$  as suggested in [2].

With the choice of  $V_i$ , Assumptions IV.1 and IV.2 hold with  $\alpha_i = 1$ ,  $\lambda_i = 0.2$ ,  $\rho_{i,\text{int}} = 0.1455$ ,  $\rho_{i,\text{ext}} = 8.1487 \times 10^{-11}$ . Recalling the circular interconnection topologies, each subsystem is directly fed by either subsystem i - 1 or i + 1 at each time instant. So (9) gives  $\gamma_{ij} = 3 \max_{s_i} \{ |\sqrt{M_{i,s'_i}} D_{i,s_i}|^2 \} \bar{N}_i \frac{1}{\alpha_j} = 0.1455$  for  $j \in I_i$  and



Fig. 3. The error norm between the output trajectories of  $\Sigma$  and  $\hat{\Sigma}$ , consisting of 1000 subsystems.



Fig. 4. The external outputs  $\mathbf{y}_{ii}$  (i.e. frequency deviations) for  $i = 1, \ldots, 1000$ .

 $\gamma_{ij} = 0$  for  $j \notin I_i$ . Then we get

$$r(\Psi) < \sup_{j \in \mathbb{N}} \sum_{i=1}^{\infty} \psi_{ij} < \frac{0.1455}{0.2} < 1.$$

which implies that the spectral radius condition IV.3 is fulfilled. Therefore all the hypotheses of Theorem IV.5 are satisfied.

For the sake of simulations, we consider a network of 1000 subsystems. The parameter values are set as  $m_i = 10^5 \text{kgm}^2$ ,  $d_i = 1s^{-1}, \ l_{ii} = 4 \times 10^3 \text{ for all } i \in \{1, \cdots, 1000\}.$ Additionally for  $s_i = 1$  and  $s_i = 2$ , we have  $l_{i(i-1)} = 4 \times 10^3$ and  $l_{i(i+1)} = 4 \times 10^3$ , respectively. Recalling the computed matrices  $\hat{A}_{i,s_i}$  and  $\hat{B}_{i,s_i}$ , by taking each local controller of the abstract subsystem as  $\hat{u}_i = \hat{x}_i$  the network  $\hat{\Sigma}$  gets stabilized at the origin. The switching between  $s_i = 1$  and  $s_i = 2$  occurs at  $k = 5n, n \in \mathbb{N}$  time instants. The norm of the overall error between the output trajectories of the abstract and concrete systems and the closed-loop output trajectories of the concrete subsystems are, respectively, depicted by Figures 3 and 4. From the choice of  $\hat{u}$  and stabilizability of  $\Sigma$  at the origin,  $\lim_{k\to\infty} |\hat{\mathbf{u}}(k)|_2 \to 0$ . This together with (4) implies that the mismatch between output trajectories converges to zero, which is illustrated by Fig. 3.

#### VI. CONCLUSIONS

We constructed continuous abstractions compositionally for an infinite network of switched discrete-time systems with arbitrary switching signals. To do this, we extended the notion of simulation functions to infinite-dimensional systems (networks of infinitely many finite-dimensional systems). Following the compositionality approach, we assigned to each subsystem an individual simulation function and constructed each local abstraction accordingly. Finally we composed the local abstractions to provide an abstraction of the overall network. We showed that the aggregation yields a continuous abstraction of the overall concrete network if the small-gain condition, expressed in terms of a spectral radius criterion, is satisfied. For linear systems, our approach boils down to linear matrix inequality conditions which can be computed efficiently. We applied our result to a power network with a switched topology.

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