

Compositional Verification of Initial-State Opacity for Switched Systems

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Abstract—In this work, we propose a compositional framework for the verification of approximate initial-state opacity for networks of discrete-time switched systems. The proposed approach is based on a notion of approximate initial-state opacity-preserving simulation functions (InitSOPSFs), which characterize how close concrete networks and their finite abstractions are in terms of the satisfaction of approximate initial-state opacity. We show that such InitSOPSFs can be obtained compositionally by assuming some small-gain type conditions and composing so-called local InitSOPSFs constructed for each subsystem separately. Additionally, for switched systems satisfying certain stability property, we provide an approach to construct their finite abstractions together with the corresponding local InitSOPSFs. Finally, the effectiveness of our results is illustrated through an example.

I. INTRODUCTION

In recent decades, CPSs have become ubiquitous in critical infrastructures and industrial control systems, including power plants, medical devices and smart communities [1]. While the increased connectivity between cyber and physical components brings in the benefit of improving systems functionalities, it also exposes CPSs to more vulnerabilities and security challenges. More recently, the world has witnessed numerous cyber-attacks which have led to great losses in people’s livelihoods [2]. Therefore, ensuring the security of CPSs has become significantly more important.

In this work, we focus on an information-flow security property, called *opacity*, which characterizes the ability that a system forbids leaking its secret information to a malicious intruder outside the system. Opacity was firstly introduced in [3] to analyze cryptographic protocols. Later, opacity was widely studied in the domain of Discrete Event Systems (DESs), see [4] and the references therein. In this context, existing works on the analysis of various notions of opacity mostly apply to systems modeled by finite state automata, which are more suitable for the cyber-layers of CPSs. However, for the physical components, system dynamics are in general hybrid with uncountable number of states.

There have been some recent attempts to extend the notions of opacity to continuous-space dynamical systems [5], [6], [7], [8]. In [5], a framework for opacity was introduced for the class of discrete-time linear systems, where the notion

of opacity was formulated as an output reachability property rather than an information-flow one. The results in [6] presented a formulation of opacity-preserving (bi)simulation relations between transition systems, which allows one to verify opacity of an infinite-state transition system by leveraging its associated finite quotient one. However, the notion of opacity proposed in this work assumes that the outputs of systems are symbols and are exactly distinguishable from each other, thus, is only suitable for systems with purely logical output sets. In a more recent paper [7], a new notion of *approximate opacity* was proposed to accommodate imperfect measurement precision of physical systems. Based on this, the authors proposed a notion of so-called approximate opacity-preserving simulation relation to capture the closeness between continuous-space systems and their finite abstractions (a.k.a symbolic models) in terms of preservation of approximate opacity.

The recent results in [8] investigated opacity for discrete-time stochastic control systems using a notion of so-called initial-state opacity-preserving stochastic simulation functions between stochastic control systems and their finite abstractions (a.k.a. finite Markov decision processes). Though promising, the computational complexity of the construction of finite abstractions grows exponentially with respect to the dimension of the state set, and, hence, the existing approaches [6], [7], [8] will become computationally intractable when dealing with large-scale systems.

Motivated by those abstraction-based techniques in [6], [7], [8] and their limitations, this work proposes an approach to analyze approximate initial-state opacity for networks of switched systems by constructing their opacity-preserving finite abstractions compositionally. There have been some recent results proposing compositional techniques for constructing finite abstractions for networks of systems (see the results in [9], [10], [11], [12] for more details). However, the aforementioned compositional schemes are proposed for the sake of controller synthesis for temporal logic properties, and none of them are applicable to deal with security properties.

In this paper, we provide a compositional approach to analyze approximate initial-state opacity of a network of switched systems using their finite abstractions. We first define a notion of so-called local approximate initial-state opacity-preserving simulation functions (InitSOPSFs) to relate each switched system and its finite abstraction. Then, by leveraging some small-gain type conditions, we construct an InitSOPSF as a relation between the network of switched systems and that of their finite abstractions using local InitSOPSFs. This InitSOPSF characterizes the closeness between the two networks in terms of the preservation of approximate initial-state opacity. Moreover, under some

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assumptions ensuring incremental input-to-state stability of discrete-time switched systems, we provide an approach to construct their finite abstractions together with their local InitSOPSFs. Finally, an illustrative example is presented to show how one can leverage our compositionality results for the verification of opacity for a network of switched systems.

Due to lack of space, we provide the proofs of all statements in an arXiv version of the paper [13].

II. NOTATION AND PRELIMINARIES

Notation: We denote by \mathbb{R} and \mathbb{N} the set of real numbers and non-negative integers, respectively. These symbols are annotated with subscripts to restrict them in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote the closed, open, and half-open intervals in \mathbb{R} by $[a, b]$, (a, b) , $[a, b)$, and $(a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b]$, $(a; b)$, $[a; b)$, and $(a; b]$ to denote the corresponding intervals in \mathbb{N} . Given any $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a . Given $N \in \mathbb{N}_{\geq 1}$ vectors $\nu_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in [1; N]$, we use $\nu = [\nu_1; \dots; \nu_N]$ to denote the vector in \mathbb{R}^n with $n = \sum_i n_i$ consisting of the concatenation of vectors ν_i . Moreover, $\|\nu\|$ denotes the infinity norm of ν . The individual elements in a matrix $A \in \mathbb{R}^{m \times n}$, are denoted by $\{A\}_{i,j}$, where $i \in [1; m]$ and $j \in [1; n]$. We denote the zero matrix in $\mathbb{R}^{n \times n}$ by 0_n . We denote by $\text{card}(\cdot)$ the cardinality of a given set and by \emptyset the empty set. For any set $S \subseteq \mathbb{R}^n$ of the form of finite union of boxes, e.g., $S = \bigcup_{j=1}^M S_j$ for some $M \in \mathbb{N}$, where $S_j = \prod_{i=1}^n [c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$, we define $\text{span}(S) = \min_{j=1, \dots, M} \eta_{S_j}$ and $\eta_{S_j} = \min\{|d_1^j - c_1^j|, \dots, |d_n^j - c_n^j|\}$. Moreover, for a set in the form of $X = \prod_{i=1}^N X_i$, where $X_i \subseteq \mathbb{R}^{n_i}$, $\forall i \in [1; N]$, are of the form of finite union of boxes, and any positive (component-wise) vector $\phi = [\phi_1; \dots; \phi_N]$ with $\phi_i \leq \text{span}(X_i)$, $\forall i \in [1; N]$, we define $[X]_\phi = \prod_{i=1}^N [X_i]_{\phi_i}$, where $[X_i]_{\phi_i} = [\mathbb{R}^{n_i}]_{\phi_i} \cap X_i$ and $[\mathbb{R}^{n_i}]_{\phi_i} = \{a \in \mathbb{R}^{n_i} \mid a_j = k_j \phi_j, k_j \in \mathbb{Z}, j = 1, \dots, n_i\}$. Note that if $\phi = [\eta; \dots; \eta]$, where $0 < \eta \leq \text{span}(S)$, we simply use notation $[S]_\eta$ rather than $[S]_\phi$. We use notations \mathcal{K} and \mathcal{K}_∞ to denote different classes of comparison functions, as follows: $\mathcal{K} = \{\alpha: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \mid \alpha \text{ is continuous, strictly increasing, and } \alpha(0) = 0\}$; $\mathcal{K}_\infty = \{\alpha \in \mathcal{K} \mid \lim_{r \rightarrow \infty} \alpha(r) = \infty\}$. For $\alpha, \gamma \in \mathcal{K}_\infty$ we write $\alpha \leq \gamma$ if $\alpha(r) \leq \gamma(r)$, and, with abuse of the notation, $\alpha = c$ if $\alpha(r) = cr$ for all $c, r \geq 0$. Finally, we denote by \mathcal{I}_d the identity function over $\mathbb{R}_{\geq 0}$, that is $\mathcal{I}_d(r) = r, \forall r \in \mathbb{R}_{\geq 0}$. Given sets X and Y with $X \subset Y$, the complement of X with respect to Y is defined as $Y \setminus X = \{x : x \in Y, x \notin X\}$.

A. Discrete-Time Switched Systems

In this work we study discrete-time switched systems of the following form.

Definition 1: A discrete-time switched system (dt-SS) Σ is defined by the tuple $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$, where

- $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{W} \subseteq \mathbb{R}^m$, and $\mathbb{Y} \subseteq \mathbb{R}^q$ are the state set, internal input set, and output set, respectively;
- $P = \{1, \dots, m\}$ is the finite set of modes;
- $F = \{f_1, \dots, f_m\}$ is a collection of set-valued maps $f_p : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{X}$ for all $p \in P$;
- $h : \mathbb{X} \rightarrow \mathbb{Y}$ is the output map.

The dt-SS Σ is described by difference inclusions of the form

$$\Sigma : \begin{cases} \mathbf{x}(k+1) \in f_{p(k)}(\mathbf{x}(k), \omega(k)), \\ \mathbf{y}(k) = h(\mathbf{x}(k)), \end{cases} \quad (1)$$

where $\mathbf{x} : \mathbb{N} \rightarrow \mathbb{X}$, $\mathbf{y} : \mathbb{N} \rightarrow \mathbb{Y}$, $p : \mathbb{N} \rightarrow P$, and $\omega : \mathbb{N} \rightarrow \mathbb{W}$ are the state, output, switching, and internal input signal, respectively. Let $\varphi_k, k \in \mathbb{N}_{\geq 1}$, denote the time when the k -th switching instant occurs. We assume that signal p satisfies a dwell-time condition [14] (i.e. there exists $k_d \in \mathbb{N}_{\geq 1}$, called the dwell-time, such that for all consecutive switching time instants $\varphi_k, \varphi_{k+1}, \varphi_{k+1} - \varphi_k \geq k_d$). If for all $x \in \mathbb{X}, p \in P, w \in \mathbb{W}$, $\text{card}(f_p(x, w)) \leq 1$ we say the system Σ is deterministic, and non-deterministic otherwise. System Σ is called finite if \mathbb{X}, \mathbb{W} are finite sets and infinite otherwise. Furthermore, if for all $x \in \mathbb{X}$ there exist $p \in P$ and $w \in \mathbb{W}$ such that $\text{card}(f_p(x, w)) \neq 0$ we say the system is non-blocking. In this paper, we assume that all systems are non-blocking.

Note that in this work, we consider switched systems with some secret states. Hereafter, we slightly modify the formulation in Definition 1 to accommodate for initial and secret states, as $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, \mathbb{W}, F, \mathbb{Y}, h)$, where $\mathbb{X}_0, \mathbb{X}_s \subseteq \mathbb{X}$ are the sets of initial and secret states, respectively.

B. Transition Systems

In this section, we employ the notion of transition systems, introduced in [15], to provide an alternative description of switched systems that can be later directly related to their finite abstractions.

Definition 2: Given a dt-SS $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, \mathbb{W}, F, \mathbb{Y}, h)$, we define the associated transition system $T(\Sigma) = (X, X_0, X_s, U, W, \mathcal{F}, Y, \mathcal{H})$ where:

- $X = \mathbb{X} \times P \times \{0, \dots, k_d - 1\}$ is the state set; $X_0 = \mathbb{X}_0 \times P \times \{0\}$ is the initial state set; $X_s = \mathbb{X}_s \times P \times \{0, \dots, k_d - 1\}$ is the secret state set;
- $U = P$ is the external input set; $W = \mathbb{W}$ is the internal input set; $Y = \mathbb{Y}$ is the output set; $\mathcal{H} : X \rightarrow Y$ is the output map defined as $\mathcal{H}(x, p, l) = h(x)$;
- \mathcal{F} is the transition function given by $(x^+, p^+, l^+) \in \mathcal{F}((x, p, l), u, w)$ if and only if $x^+ \in f_p(x, w), u = p$ and the following scenarios hold:
 - $l < k_d - 1, p^+ = p$ and $l^+ = l + 1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;
 - $l = k_d - 1, p^+ = p$ and $l^+ = k_d - 1$: switching is allowed but no switch occurs;
 - $l = k_d - 1, p^+ \neq p$ and $l^+ = 0$: switching is allowed and a switch occurs.

The following proposition is borrowed from [12] showing that the output runs of a dt-SS Σ and its associated transition system $T(\Sigma)$ are equivalent so that one can use Σ and $T(\Sigma)$ interchangeably.

Proposition 3: Consider a transition system $T(\Sigma)$ in Definition 2 associated to Σ as defined in (1). Any output trajectory of Σ can be uniquely mapped to an output trajectory of $T(\Sigma)$ and vice versa.

Next, let us provide a formal definition of networks of dt-SS (or equivalently, networks of transition systems).

C. Networks of Systems

Consider $N \in \mathbb{N}_{\geq 1}$ dt-SS $\Sigma_i = (\mathbb{X}_i, \mathbb{X}_{0_i}, \mathbb{X}_{s_i}, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i, h_i)$, $i \in [1; N]$, with partitioned internal inputs and outputs as

$$w_i = [w_{i1}; \dots; w_{i(i-1)}; w_{i(i+1)}; \dots; w_{iN}], \quad (2)$$

$$h_i(x_i) = [h_{i1}(x_i); \dots; h_{iN}(x_i)], \quad (3)$$

with $\mathbb{W}_i = \prod_{j=1, j \neq i}^N \mathbb{W}_{ij}$, $\mathbb{Y}_i = \prod_{j=1}^N \mathbb{Y}_{ij}$, $w_{ij} \in \mathbb{W}_{ij}$, $y_{ij} = h_{ij}(x_i) \in \mathbb{Y}_{ij}$. The outputs y_{ii} are considered as external ones, whereas y_{ij} , with $i \neq j$, are interpreted as internal ones which are used to construct interconnections between systems. In particular, we assume that w_{ij} equals to y_{ji} if there is connection from system Σ_j to Σ_i , otherwise we set $h_{ji} \equiv 0$. In the sequel, we denote by $\mathcal{N}_i = \{j \in [1; N], j \neq i | h_{ji} \neq 0\}$ the collection of neighboring systems $\Sigma_j, j \in \mathcal{N}_i$, that provide internal inputs to system Σ_i .

Now, we are ready to provide a formal definition of the network consisting of $N \in \mathbb{N}_{\geq 1}$ dt-SS.

Definition 4: Consider $N \in \mathbb{N}_{\geq 1}$ dt-SS $\Sigma_i = (\mathbb{X}_i, \mathbb{X}_{0_i}, \mathbb{X}_{s_i}, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i, h_i)$, $i \in [1; N]$, with the input-output structure given by (2) and (3). The network, representing the interconnection of N dt-SS Σ_i , is a tuple $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, F, \mathbb{Y}, h)$, denoted by $\mathcal{I}_{\mathcal{M}}(\Sigma_1, \dots, \Sigma_N)$, where $\mathbb{X} = \prod_{i=1}^N \mathbb{X}_i$, $\mathbb{X}_0 = \prod_{i=1}^N \mathbb{X}_{0_i}$, $\mathbb{X}_s = \prod_{i=1}^N \mathbb{X}_{s_i}$, $P = \prod_{i=1}^N P_i$, $F = \prod_{i=1}^N F_i$, $\mathbb{Y} = \prod_{i=1}^N \mathbb{Y}_{ii}$, $h = \prod_{i=1}^N h_{ii}$, and $\mathcal{M} \in \mathbb{R}^{N \times N}$ is a matrix with elements $\{\mathcal{M}\}_{ii} = 0$, $\{\mathcal{M}\}_{ij} = \phi_{ij}, \forall i, j \in [1; N], i \neq j$, $0 \leq \phi_{ij} \leq \text{span}(\mathbb{Y}_{ji})$, subject to the constraint:

$$\|y_{ji} - w_{ij}\| \leq \phi_{ij}, [\mathbb{Y}_{ji}]_{\phi_{ij}} \subseteq \mathbb{W}_{ij}, \forall i \in [1; N], j \in \mathcal{N}_i. \quad (4)$$

Remark 5: In this paper, when we are talking about the network of concrete switched systems, y_{ji} is always equal to w_{ij} , which naturally implies $\phi_{ij} = 0$ and $\mathcal{M} = 0_N$. However, for the network of finite abstractions, due to possibly different granularities of the internal input and output sets, the designed parameters ϕ_{ij} are not necessarily zero. Note that whenever $\phi_{ij} \neq 0$, the sets $\mathbb{Y}_{ji}, \forall i, j \in [1; N], i \neq j$, are assumed to be finite unions of boxes.

Similarly, given transition systems $T(\Sigma_i)$, one can also define a network of transition systems $\mathcal{I}_{\mathcal{M}}(T(\Sigma_1), \dots, T(\Sigma_N))$.

III. OPACITY-PRESERVING SIMULATION FUNCTIONS

In this section, we start by defining approximate initial-state opacity property [7] for networks of transition systems. This property is, in general, hard to check for a concrete network as its state set is infinite and so far there is no systematic way in the literature to verify opacity of such systems. On the other hand, existing tool DESUMA¹ and algorithms [16],[17],[6, Sec. IV] in DESs literature can be leveraged to check opacity for systems with finite state sets. Therefore, it would be feasible to check opacity for finite networks (i.e, networks consisting of finite abstractions) and then carry back the reasoning to concrete ones, as long as there is a formal relation between those networks. To this purpose, we introduce a new notion of approximate initial-state opacity preserving simulation functions (InitSOPSF)

which formally relate two networks of transition systems and their approximate initial-state opacity properties.

Before defining the notion of approximate initial-state opacity for networks of transition systems, we introduce some notations as follows. Consider network $T(\Sigma)$. We use z^k to denote the state of $T(\Sigma)$ reached at time $k \in \mathbb{N}$ from the initial state z^0 under the input sequence \bar{u} with length k , and denote by $\{z^0, z^1, \dots, z^k\}$ a finite state run of $T(\Sigma)$ with length $k \in \mathbb{N}$.

Definition 6: Consider network $T(\Sigma) = (X, X_0, X_s, U, \mathcal{F}, Y, \mathcal{H})$ and a constant $\delta \geq 0$. Network $T(\Sigma)$ is said to be δ -approximate initial-state opaque if for any $z^0 \in X_0 \cap X_s$ and finite state run $\{z^0, z^1, \dots, z^k\}$, there exist $\bar{z}^0 \in X_0 \setminus X_s$ and a finite state run $\{\bar{z}^0, \bar{z}^1, \dots, \bar{z}^k\}$ such that

$$\max_{t \in [0; k]} \|\mathcal{H}(z^t) - \mathcal{H}(\bar{z}^t)\| \leq \delta.$$

Now, we can introduce a notion of approximate InitSOPSF to quantitatively relates two networks of transition systems in terms of preservation of approximate opacity as defined above.

Definition 7: Consider $T(\Sigma) = (X, X_0, X_s, U, \mathcal{F}, Y, \mathcal{H})$ and $T(\hat{\Sigma}) = (\hat{X}, \hat{X}_0, \hat{X}_s, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ with $\hat{Y} \subseteq Y$. For $\varepsilon \in \mathbb{R}_{\geq 0}$, a function $\mathcal{S}: X \times \hat{X} \rightarrow \mathbb{R}_{\geq 0}$ is called an ε -approximate initial-state opacity-preserving simulation function (ε -InitSOPSF) from $T(\Sigma)$ to $T(\hat{\Sigma})$ if there exists $\alpha \in \mathcal{K}_{\infty}$ such that

- 1 (a) $\forall z^0 \in X_0 \cap X_s, \exists \hat{z}^0 \in \hat{X}_0 \cap \hat{X}_s$, s.t. $\mathcal{S}(z^0, \hat{z}^0) \leq \varepsilon$;
- (b) $\forall \hat{z}^0 \in \hat{X}_0 \setminus \hat{X}_s, \exists z^0 \in X_0 \setminus X_s$, s.t. $\mathcal{S}(z^0, \hat{z}^0) \leq \varepsilon$;
- 2 $\forall z \in X, \forall \hat{z} \in \hat{X}, \alpha(\|\mathcal{H}(z) - \mathcal{H}(\hat{z})\|) \leq \mathcal{S}(z, \hat{z})$;
- 3 $\forall z \in X, \forall \hat{z} \in \hat{X}$ s.t. $\mathcal{S}(z, \hat{z}) \leq \varepsilon$, one has:
 - (a) $\forall u \in U, \forall z^+ \in \mathcal{F}(z, u), \exists \hat{u} \in \hat{U}, \exists \hat{z}^+ \in \hat{\mathcal{F}}(\hat{z}, \hat{u})$, s.t. $\mathcal{S}(z^+, \hat{z}^+) \leq \varepsilon$;
 - (b) $\forall \hat{u} \in \hat{U}, \forall \hat{z}^+ \in \hat{\mathcal{F}}(\hat{z}, \hat{u}), \exists u \in U, \exists z^+ \in \mathcal{F}(z, u)$, s.t. $\mathcal{S}(z^+, \hat{z}^+) \leq \varepsilon$.

Although Definition 7 is general in the sense that networks $T(\Sigma)$ and $T(\hat{\Sigma})$ can be either infinite or finite, practically, network $T(\hat{\Sigma})$ potentially consists of $N \in \mathbb{N}_{\geq 1}$ finite abstractions. Hence, checking approximate initial-state opacity for this network is decidable in comparison to network $T(\Sigma)$.

Before showing the next result, let us recall the definition of approximate initial-state opacity-preserving simulation relation which was originally proposed in [7].

Definition 8: Consider networks $T(\Sigma) = (X, X_0, X_s, U, \mathcal{F}, Y, \mathcal{H})$ and $T(\hat{\Sigma}) = (\hat{X}, \hat{X}_0, \hat{X}_s, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{Y} \subseteq Y$. For $\hat{\varepsilon} \in \mathbb{R}_{\geq 0}$, a relation $R \subseteq X \times \hat{X}$ is called an $\hat{\varepsilon}$ -approximate initial-state opacity-preserving simulation relation ($\hat{\varepsilon}$ -InitSOP simulation relation) from $T(\Sigma)$ to $T(\hat{\Sigma})$ if

- 1 (a) $\forall z^0 \in X_0 \cap X_s, \exists \hat{z}^0 \in \hat{X}_0 \cap \hat{X}_s$, s.t. $(z^0, \hat{z}^0) \in R$;
- (b) $\forall \hat{z}^0 \in \hat{X}_0 \setminus \hat{X}_s, \exists z^0 \in X_0 \setminus X_s$, s.t. $(z^0, \hat{z}^0) \in R$;
- 2 $\forall (z, \hat{z}) \in R, \|\mathcal{H}(z) - \hat{\mathcal{H}}(\hat{z})\| \leq \hat{\varepsilon}$;
- 3 For any $(z, \hat{z}) \in R$, we have
 - (a) $\forall u \in U, \forall z^+ \in \mathcal{F}(z, u), \exists \hat{u} \in \hat{U}, \exists \hat{z}^+ \in \hat{\mathcal{F}}(\hat{z}, \hat{u})$, s.t. $(z^+, \hat{z}^+) \in R$;
 - (b) $\forall \hat{u} \in \hat{U}, \forall \hat{z}^+ \in \hat{\mathcal{F}}(\hat{z}, \hat{u}), \exists u \in U, \exists z^+ \in \mathcal{F}(z, u)$, s.t. $(z^+, \hat{z}^+) \in R$.

The following corollary borrowed from [7] shows the usefulness of Definition 8 in terms of preservation of approximate opacity across related networks.

¹Available at URL <http://www.eecs.umich.edu/umdes/toolboxes.html>.

Corollary 9: Consider networks $T(\Sigma) = (X, X_0, X_s, U, \mathcal{F}, Y, \mathcal{H})$ and $T(\hat{\Sigma}) = (\hat{X}, \hat{X}_0, \hat{X}_s, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{Y} \subseteq Y$. Let $\hat{\varepsilon}, \delta \in \mathbb{R}_{\geq 0}$. If there exists an $\hat{\varepsilon}$ -InitSOP simulation relation from $T(\Sigma)$ to $T(\hat{\Sigma})$ as in Definition 8 and $\hat{\varepsilon} \leq \frac{\delta}{2}$, then the following implication holds

$$\begin{aligned} T(\hat{\Sigma}) \text{ is } (\delta - 2\hat{\varepsilon})\text{-approximate initial-state opaque} \\ \Rightarrow T(\Sigma) \text{ is } \delta\text{-approximate initial-state opaque.} \end{aligned}$$

The next result shows that the existence of an ε -InitSOPSF for networks of transition systems implies the existence of an $\hat{\varepsilon}$ -InitSOP simulation relation between them.

Proposition 10: Consider networks $T(\Sigma) = (X, X_0, X_s, U, \mathcal{F}, Y, \mathcal{H})$ and $T(\hat{\Sigma}) = (\hat{X}, \hat{X}_0, \hat{X}_s, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{Y} \subseteq Y$. Assume \mathcal{S} is an ε -InitSOPSF from $T(\Sigma)$ to $T(\hat{\Sigma})$ as in Definition 7. Then, relation $R \subseteq X \times \hat{X}$ defined by

$$R = \left\{ (z, \hat{z}) \in X \times \hat{X} \mid \mathcal{S}(z, \hat{z}) \leq \varepsilon \right\}, \quad (5)$$

is an $\hat{\varepsilon}$ -InitSOP simulation relation from $T(\Sigma)$ to $T(\hat{\Sigma})$ with

$$\hat{\varepsilon} = \alpha^{-1}(\varepsilon). \quad (6)$$

Given the results of Corollary 9 and Proposition 10, one can readily see that if there exists an ε -InitSOPSF from $T(\Sigma)$ to $T(\hat{\Sigma})$ as in Definition 7 and $T(\hat{\Sigma})$ is $(\delta - 2\varepsilon)$ -approximate initial-state opaque, then $T(\Sigma)$ is δ -approximate initial-state opaque, where $\hat{\varepsilon} = \alpha^{-1}(\varepsilon) \leq \frac{\delta}{2}$, $\delta \in \mathbb{R}_{\geq 0}$.

IV. COMPOSITIONALITY RESULT

We saw in the previous section that ε -InitSOPSFs are very useful for checking approximate initial-state opacity of concrete networks based on checking that of their finite abstractions. However, constructing such a function for networks consisting of a large number of systems is not feasible in general. Hence, in this section, we introduce a compositional technique based on which one can construct an ε -InitSOPSF from the concrete network to a network of finite abstractions by using so-called local ε_i -InitSOPSFs between subsystems and their abstractions.

A. Compositional Construction of ε -InitSOPSF

Suppose that we are given N dt-SS Σ_i , or equivalently, $T(\Sigma_i)$. Moreover, we assume that systems $T(\Sigma_i)$ and $T(\hat{\Sigma}_i)$ admit a local ε_i -InitSOPSF as defined next.

Definition 11: Consider $T(\Sigma_i) = (X_i, X_{0i}, X_{si}, U_i, W_i, \mathcal{F}_i, Y_i, \mathcal{H}_i)$ and $T(\hat{\Sigma}_i) = (\hat{X}_i, \hat{X}_{0i}, \hat{X}_{si}, \hat{U}_i, \hat{W}_i, \hat{\mathcal{F}}_i, \hat{Y}_i, \hat{\mathcal{H}}_i)$, $\forall i \in [1; N]$, where $\hat{W}_i \subseteq W_i$ and $\hat{Y}_i \subseteq Y_i$. For $\varepsilon_i \in \mathbb{R}_{\geq 0}$, a function $\mathcal{S}_i: X_i \times \hat{X}_i \rightarrow \mathbb{R}_{\geq 0}$ is called a local ε_i -InitSOPSF from $T(\Sigma_i)$ to $T(\hat{\Sigma}_i)$ if there exist $\vartheta_i \in \mathbb{R}_{\geq 0}$ and $\alpha_i \in \mathcal{K}_{\infty}$ such that

- 1 (a) $\forall z_i^0 \in X_{0i} \cap X_{si}, \exists \hat{z}_i^0 \in \hat{X}_{0i} \cap \hat{X}_{si}$, s.t. $\mathcal{S}_i(z_i^0, \hat{z}_i^0) \leq \varepsilon_i$;
- (b) $\forall \hat{z}_i^0 \in \hat{X}_{0i} \setminus \hat{X}_{si}, \exists z_i^0 \in X_{0i} \setminus X_{si}$, s.t. $\mathcal{S}_i(z_i^0, \hat{z}_i^0) \leq \varepsilon_i$;
- 2 $\forall z_i \in X_i, \forall \hat{z}_i \in \hat{X}_i$, $\alpha_i(\|\mathcal{H}_i(z_i) - \hat{\mathcal{H}}_i(\hat{z}_i)\|) \leq \mathcal{S}_i(z_i, \hat{z}_i)$;
- 3 $\forall z_i \in X_i, \forall \hat{z}_i \in \hat{X}_i$ s.t. $\mathcal{S}_i(z_i, \hat{z}_i) \leq \varepsilon_i, \forall w_i \in W_i, \forall \hat{w}_i \in \hat{W}_i$ s.t. $\|w_i - \hat{w}_i\| \leq \vartheta_i$, the following conditions hold:
 - (a) $\forall u_i \in U_i, \forall z_i^+ \in \mathcal{F}_i(z_i, u_i, w_i), \exists \hat{u}_i \in \hat{U}_i, \exists \hat{z}_i^+ \in \hat{\mathcal{F}}_i(\hat{z}_i, \hat{u}_i, \hat{w}_i)$, s.t. $\mathcal{S}_i(z_i^+, \hat{z}_i^+) \leq \varepsilon_i$;
 - (b) $\forall \hat{u}_i \in \hat{U}_i, \forall \hat{z}_i^+ \in \hat{\mathcal{F}}_i(\hat{z}_i, \hat{u}_i, \hat{w}_i), \exists u_i \in U_i, \exists z_i^+ \in \mathcal{F}_i(z_i, u_i, w_i)$, s.t. $\mathcal{S}_i(z_i^+, \hat{z}_i^+) \leq \varepsilon_i$.

If there exists a local ε_i -InitSOPSF from $T(\Sigma_i)$ to $T(\hat{\Sigma}_i)$, and $T(\hat{\Sigma}_i)$ is finite (\hat{X}_i and \hat{W}_i are finite sets), $T(\hat{\Sigma}_i)$ is

called a finite abstraction (or symbolic model) of $T(\Sigma_i)$, which is constructed later in Definition 15. Note that local ε_i -InitSOPSFs are mainly for constructing an ε -InitSOPSF for the networks and they are not directly used for deducing approximate initial-state opacity-preserving simulation relation.

The next theorem provides a compositional approach to construct an ε -InitSOPSF from $T(\Sigma)$ to $T(\hat{\Sigma})$ via local ε_i -InitSOPSFs from $T(\Sigma_i)$ to $T(\hat{\Sigma}_i)$.

Theorem 12: Consider network $T(\Sigma) = \mathcal{I}_{0_N}(T(\Sigma_1), \dots, T(\Sigma_N))$. Assume that there exists a local ε_i -InitSOPSF \mathcal{S}_i from $T(\Sigma_i)$ to $T(\hat{\Sigma}_i)$, $\forall i \in [1; N]$, as in Definition 11. Let $\varepsilon = \max \varepsilon_i$, and $\hat{\mathcal{M}} \in \mathbb{R}^{N \times N}$ be a matrix with elements $\{\hat{\mathcal{M}}\}_{ii} = 0, \{\hat{\mathcal{M}}\}_{ij} = \phi_{ij}, \forall i, j \in [1; N], i \neq j, 0 \leq \phi_{ij} \leq \text{span}(\hat{Y}_{ji})$. If $\forall i \in [1; N], \forall j \in \mathcal{N}_i$,

$$\alpha_j^{-1}(\varepsilon_j) + \phi_{ij} \leq \vartheta_i, \quad (7)$$

then, function $\mathcal{S}: X \times \hat{X} \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$\mathcal{S}(z, \hat{z}) := \max_i \left\{ \frac{\varepsilon}{\varepsilon_i} \mathcal{S}_i(z_i, \hat{z}_i) \right\}, \quad (8)$$

is an ε -InitSOPSF from $T(\Sigma) = \mathcal{I}_{0_N}(T(\Sigma_1), \dots, T(\Sigma_N))$ to $T(\hat{\Sigma}) = \mathcal{I}_{\hat{\mathcal{M}}}(T(\hat{\Sigma}_1), \dots, T(\hat{\Sigma}_N))$.

Remark 13: Let $\phi_i = [\phi_{i1}; \dots; \phi_{iN}]$. Vectors ϕ_i serve later as internal input quantization parameters for the construction of symbolic models for $T(\Sigma_i)$ (see Definition 15). Moreover, the values of ϕ_i will be designed later in Theorem 20.

V. CONSTRUCTION OF SYMBOLIC MODELS

In this section, we consider $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, \mathbb{W}, F, \mathbb{Y}, h)$ as an infinite, deterministic dt-SS. Note that throughout this section, we are mainly talking about switched subsystems rather than the overall network. However, for the sake of better readability, we often omit index i of subsystems throughout the text in this section. We assume the output map h satisfies the following general Lipschitz assumption: there exists an $\ell \in \mathcal{K}_{\infty}$ such that: $\|h(x) - h(y)\| \leq \ell(\|x - y\|)$ for all $x, y \in \mathbb{X}$. Here, we also use Σ_p to denote a dt-SS Σ in (1) with constant switching signal $p(k) = p, \forall k \in \mathbb{N}$.

Here, we establish an ε -InitSOPSF between $T(\Sigma)$ and its symbolic model by assuming that, for all $p \in P, \Sigma_p$ is incrementally input-to-state stable (δ -ISS) [18] as defined next.

Definition 14: System Σ_p is δ -ISS if there exist functions $V_p: \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}, \underline{\alpha}_p, \bar{\alpha}_p, \rho_p \in \mathcal{K}_{\infty}$, and constant $0 < \kappa_p < 1$, such that for all $x, \hat{x} \in \mathbb{X}$, and for all $w, \hat{w} \in \mathbb{W}$

$$\underline{\alpha}_p(\|x - \hat{x}\|) \leq V_p(x, \hat{x}) \leq \bar{\alpha}_p(\|x - \hat{x}\|), \quad (9)$$

$V_p(f_p(x, w), f_p(\hat{x}, \hat{w})) \leq \kappa_p V_p(x, \hat{x}) + \rho_p(\|w - \hat{w}\|)$. (10) We say that $V_p, \forall p \in P$, are multiple δ -ISS Lyapunov functions for system Σ if it satisfies (9) and (10). Moreover, if $V_p = V_q, \forall p, q \in P$, we omit the index p in (9)-(10), and say that V is a common δ -ISS Lyapunov function for system Σ .

Now, we show how to construct a symbolic model $T(\hat{\Sigma})$ of $T(\Sigma)$ associated with the dt-SS Σ .

Definition 15: Consider a transition system $T(\Sigma) = (X, X_0, X_s, U, W, \mathcal{F}, Y, \mathcal{H})$, associated with the switched system $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, \mathbb{W}, F, \mathbb{Y}, h)$, where \mathbb{X}, \mathbb{W} are assumed to be finite unions of boxes. Let $\Sigma_p, \forall p \in P$, be δ -ISS as in Definition 14. Then one can construct a symbolic model $T(\hat{\Sigma}) = (\hat{X}, \hat{X}_0, \hat{X}_s, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where:

- $\hat{X} = \hat{\mathbb{X}} \times P \times \{0, \dots, k_d - 1\}$, where $\hat{\mathbb{X}} = [\mathbb{X}]_\eta$ and $0 < \eta \leq \min\{\text{span}(\mathbb{X}_s), \text{span}(\mathbb{X} \setminus \mathbb{X}_s)\}$ is the state set quantization parameter; $\hat{X}_0 = \hat{\mathbb{X}}_0 \times P \times \{0\}$, where $\hat{\mathbb{X}}_0 = [\mathbb{X}]_\eta$; $\hat{X}_s = \hat{\mathbb{X}}_s \times P \times \{0, \dots, k_d - 1\}$, where $\hat{\mathbb{X}}_s = [\mathbb{X}_s]_\eta$;
- $\hat{U} = U = P$; $\hat{Y} = Y$; $\hat{\mathcal{H}}: \hat{X} \rightarrow \hat{Y}$, defined as $\hat{\mathcal{H}}(\hat{x}, p, l) = \mathcal{H}(\hat{x}, p, l) = h(\hat{x})$; $\hat{W} = [\mathbb{W}]_\phi$, where ϕ , satisfying $0 < \|\phi\| \leq \text{span}(\mathbb{W})$, is the internal input set quantization parameter;
- $(\hat{x}^+, p^+, l^+) \in \hat{\mathcal{F}}(\hat{x}, p, l, \hat{u}, \hat{w})$ if and only if $\|f_p(\hat{x}, \hat{w}) - \hat{x}^+\| \leq \eta$, $\hat{u} = p$ and the following scenarios hold:
 - $l < k_d - 1$, $p^+ = p$ and $l^+ = l + 1$;
 - $l = k_d - 1$, $p^+ = p$ and $l^+ = k_d - 1$;
 - $l = k_d - 1$, $p^+ \neq p$ and $l^+ = 0$;

In order to construct a local ε -InitSOPSF from $T(\Sigma)$ to $T(\hat{\Sigma})$, we raise the following assumptions on functions V_p appeared in Definition 14.

Assumption 16: There exists $\mu \geq 1$ such that

$$\forall x, y \in \mathbb{X}, \quad \forall p, q \in P, \quad V_p(x, y) \leq \mu V_q(x, y). \quad (11)$$

Assumption 16 is an incremental version of a similar assumption in [19] that is used to prove input-to-state stability of switched systems under constrained switching signals.

Assumption 17: Assume $\exists \gamma_p \in \mathcal{K}_\infty, \forall p \in P$, such that

$$\forall x, y, z \in \mathbb{X}, \quad V_p(x, y) \leq V_p(x, z) + \gamma_p(\|y - z\|). \quad (12)$$

Assumption 17 is non-restrictive as shown in [20] provided that one is interested to work on a compact subset of \mathbb{X} .

Now, we establish the relation between $T(\Sigma)$ and $T(\hat{\Sigma})$ via the notion of local ε -InitSOPSF as in Definition 11.

Theorem 18: Consider a dt-SS $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, \mathbb{W}, F, \mathbb{Y}, h)$ with its equivalent transition system $T(\Sigma) = (X, X_0, X_s, U, W, \mathcal{F}, Y, \mathcal{H})$. Suppose $\Sigma_p, \forall p \in P$, is δ -ISS as in Definition 14, with functions $V_p, \underline{\alpha}_p, \bar{\alpha}_p, \rho_p$ and constant κ_p , and assume Assumptions 16 and 17 hold. Let $\varepsilon > 1$. For any design parameters $\varepsilon, \vartheta \in \mathbb{R}_{\geq 0}$, let $T(\hat{\Sigma})$ be a symbolic model of $T(\Sigma)$ constructed as in Definition 15 with any quantization parameter η satisfying

$$\eta \leq \min\{\hat{\gamma}^{-1}((1 - \kappa)\varepsilon - \rho(\vartheta)), \bar{\alpha}^{-1}(\varepsilon)\}, \quad (13)$$

where $\kappa = \max_{p \in P} \left\{ \kappa_p^{\frac{\varepsilon-1}{\varepsilon}} \right\}$, $\rho = \max_{p \in P} \left\{ \kappa_p^{-\frac{k_d}{\varepsilon}} \rho_p \right\}$, $\hat{\gamma} = \max_{p \in P} \left\{ \kappa_p^{-\frac{k_d}{\varepsilon}} \gamma_p \right\}$, and $\bar{\alpha} = \max_{p \in P} \left\{ \kappa_p^{-\frac{l}{\varepsilon}} \bar{\alpha}_p \right\}$. If, $\forall p \in P$, $k_d \geq \varepsilon \frac{\ln(\mu)}{\ln(\frac{1}{\kappa_p})} + 1$, then function \mathcal{V} defined as

$$\mathcal{V}((x, p, l), (\hat{x}, p, l)) := V_p(x, \hat{x}) \kappa_p^{\frac{l}{\varepsilon}}, \quad (14)$$

is a local ε -InitSOPSF from $T(\Sigma)$ to $T(\hat{\Sigma})$.

Given the results of Theorems 12 and 18, one can see that conditions (7) and (13) may not hold simultaneously. Therefore, we raise the following assumption which provides

a small-gain type condition such that one can verify if conditions (7) and (13) can be satisfied simultaneously.

Assumption 19: Consider network $\mathcal{I}_{0,N}(T(\Sigma_1), \dots, T(\Sigma_N))$ induced by $N \in \mathbb{N}_{\geq 1}$ transition systems $T(\Sigma_i)$. Assume that each $T(\Sigma_i)$ and its symbolic model $T(\hat{\Sigma}_i)$ admit a local ε_i -InitSOPSF \mathcal{V}_i as in (14), associated with functions and constants κ_i, α_i , and ρ_i appeared in Theorem 18. Define

$$\gamma_{ij} := \begin{cases} (1 - \kappa_i)^{-1} \rho_i \circ \alpha_j^{-1} & \text{if } j \in \mathcal{N}_i, \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

for all $i, j \in [1; N]$, and assume that functions γ_{ij} defined in (15) satisfy

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \dots \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathcal{I}_d, \quad (16)$$

$\forall (i_1, \dots, i_r) \in \{1, \dots, N\}^r$, where $r \in \{1, \dots, N\}$.

The next theorem is main result of the paper. We show that under the above small-gain assumption, one can always compositionally design local quantization parameters to satisfy conditions (7) and (13) simultaneously.

Theorem 20: Suppose that Assumption 19 holds. Then, there always exist local quantization parameters η_i and $\phi_{ij}, \forall i, j \in [1; N]$, such that (7) and (13) can be satisfied simultaneously.

VI. CASE STUDY

Consider a network of discrete-time switched system $\Sigma = (\mathbb{X}, \mathbb{X}_0, \mathbb{X}_s, P, F, \mathbb{Y}, h)$ as in Definition 4, consisting of n systems Σ_i each described by:

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = a_{i p_i(k)} \mathbf{x}_i(k) + d_i \omega_i(k) + b_{i p_i(k)}, \\ \mathbf{y}_i(k) = c_i \mathbf{x}_i(k), \end{cases} \quad (17)$$

where $p_i(k) \in P_i = \{1, 2\}$, $k \in \mathbb{N}$, denote the modes of each system Σ_i . The switching signal is set to be $p_i(k) = 1$ if k is odd and $p_i(k) = 2$ if k is even, $\forall k \in \mathbb{N}$. The other parameters are as the following: $a_{i1} = 0.05$, $a_{i2} = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 0.15$, $d_i = 0.05$, $c_i = [c_{i1}; \dots; c_{in}]$ with $c_{i(i+1)} = 1$, $c_{ij} = 0, \forall i \in [1; n-1], \forall j \neq i+1$, $c_{n1} = c_{nn} = 1$, $c_{nj} = 0, \forall j \in [2; n-1]$. The internal inputs are subjected to the constraints $\omega_1(k) = c_{n1} \mathbf{x}_n(k)$ and $\omega_i(k) = c_{(i-1)i} \mathbf{x}_{(i-1)}(k), \forall i \in [2; n]$. For each switched system, the state sets are $\mathbb{X}_i = \mathbb{X}_{0i} = (0, 0.6), \forall i \in [1; n]$, the secret sets are $\mathbb{X}_{s1} = (0, 0.2)$, $\mathbb{X}_{s2} = [0.4, 0.6)$, $\mathbb{X}_{si} = (0, 0.6), \forall i \in [3; n]$, the output sets are $\mathbb{Y}_i = \prod_{j=1}^n \mathbb{Y}_{ij}$ where $\mathbb{Y}_{i(i+1)} = (0, 0.6)$, $\mathbb{Y}_{ii} = \mathbb{Y}_{ij} = \{0\}, \forall i \in [1; n-1], \forall j \neq i+1$, $\mathbb{Y}_{nn} = \mathbb{Y}_{n1} = (0, 0.6)$, $\mathbb{Y}_{nj} = \{0\}, \forall j \in [2; n-1]$, and internal input sets are $\mathbb{W}_1 = \mathbb{Y}_{ni}, \mathbb{W}_i = \mathbb{Y}_{(i-1)i}, \forall i \in [2; n]$. Intuitively, the output of the network is the external output of the last system Σ_n . The main goal of this example is to check approximate initial-state opacity of the concrete network using its symbolic model. Now, let us construct a symbolic model of Σ compositionally with accuracy $\hat{\varepsilon} = 0.25$ as defined in (6). We use our compositional approach to achieve this goal.

Consider functions $V_{i p_i} = |x_i - \hat{x}_i|, \forall i \in [1; n]$. It can be readily verified that (9) and (10) are satisfied with $\underline{\alpha}_{i p_i} = \bar{\alpha}_{i p_i} = \mathcal{I}_d, \rho_{i p_i} = 0.05, \forall p_i \in P_i, \kappa_{i1} = a_{i1} = 0.05, \kappa_{i2} = a_{i2} = 0.1$. Condition (12) is satisfied with $\gamma_{i p_i} = \mathcal{I}_d, \forall p_i \in P_i$. Moreover, since $V_{i p_i} = V_{i q_i}, \forall p_i, q_i \in P_i, V_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$

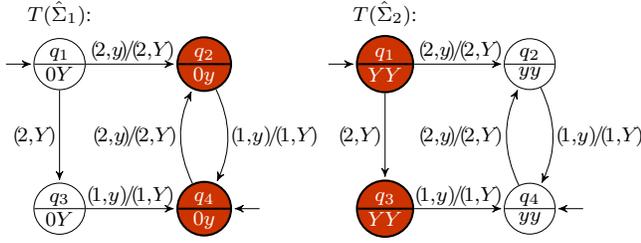


Fig. 1. Local symbolic models of transition systems.
 $\mathcal{I}(T(\hat{\Sigma}_1), T(\hat{\Sigma}_2))$:

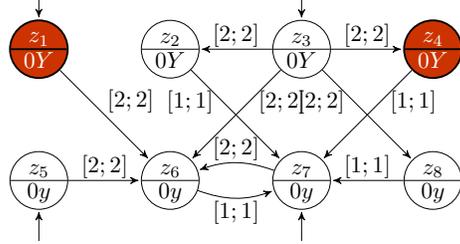


Fig. 2. Symbolic model of a network of 2 transition systems.

is a common δ -ISS Lyapunov function for system Σ_i . Next, given functions $\kappa_i = 0.1$, $\rho_i = 0.06\mathcal{I}_d$, $\alpha_i = \mathcal{I}_d$, $\hat{\gamma}_i = 1.05\mathcal{I}_d$, and $\bar{\alpha}_i = \mathcal{I}_d$ as appeared in Theorem 18, we have $\gamma_{ij} < \mathcal{I}_d$ by (15), $\forall i, j \in [1; n]$. Hence, the small-gain condition (16) is satisfied. Then, by applying Theorem 20, we obtain proper pairs of local parameters $(\varepsilon_i, \vartheta_i) = (0.25, 0.25)$ for all of the transition systems. Accordingly, we provide a suitable choice of local quantization parameters as $\eta_i = 0.2$, $\forall i \in [1; n]$, such that inequality (13) for each transition system $T(\hat{\Sigma}_i)$ is satisfied. Then, we construct local symbolic models $T(\hat{\Sigma}_i) = (\hat{X}_i, \hat{X}_{0_i}, \hat{X}_{s_i}, \hat{U}_i, \hat{W}_i, \hat{F}_i, \hat{Y}_i, \hat{H}_i)$ as defined in Definition 15, where $\hat{X}_i = \hat{X}_{0_i} = \{0.2, 0.4\} \times \{1, 2\} \times \{0\}$, $\hat{X}_{s_1} = \{0.2\} \times \{1, 2\} \times \{0\}$, $\hat{X}_{s_2} = \{0.4\} \times \{1, 2\} \times \{0\}$, $\hat{X}_{s_i} = \{0.2, 0.4\} \times \{1, 2\} \times \{0\}$, $\forall i \in [3; n]$, $\hat{Y}_i = \prod_{j=1}^i \{0\} \times \{0.2, 0.4\} \times \prod_{j=i+2}^n \{0\}$, $\forall i \in [1; n-1]$, $\hat{Y}_n = \{0.2, 0.4\} \times \prod_{j=2}^{n-1} \{0\} \times \{0.2, 0.4\}$, $\hat{W}_i = \{0.2, 0.4\}$, $\forall i \in [1; n]$. Now, using the result in Theorem 18, one can verify that $V_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = |x_i - \hat{x}_i|$ is a local ε_i -InitSOPSF from each $T(\Sigma_i)$ to its symbolic model $T(\hat{\Sigma}_i)$. Furthermore, by the compositionality result in Theorem 12, we obtain that $V = \max_i \{V_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i))\} = \max_i \{|x_i - \hat{x}_i|\}$ is an ε -InitSOPSF from $T(\Sigma) = \mathcal{I}_{0_N}(T(\Sigma_1), \dots, T(\Sigma_N))$ to $T(\hat{\Sigma}) = \mathcal{I}_{0_N}(T(\hat{\Sigma}_1), \dots, T(\hat{\Sigma}_N))$ with $\varepsilon = \max_i \varepsilon_i = 0.25$.

Now, let us verify approximate initial-state opacity for $T(\Sigma)$ using the network of symbolic models $T(\hat{\Sigma})$. To do this, we first show an example of a network consisting of 2 transition systems, as shown in Figures 1 and 2. The two automata in Figure 1 represent the symbolic models of the local transition systems, and the one in Figure 2 is the network of symbolic models. Each circle is labeled by the state (top half) and the corresponding output (bottom half). Initial states are distinguished by being the target of a sourceless arrow. The symbols on the edges show the switching signals $p_i(k) \in \{1, 2\}$ and internal inputs coming from other local transition systems. For simplicity of demonstration, we use symbols to represent the state and output vectors, where $q_1 = [0.4, 2, 0]$, $q_2 = [0.2, 1, 0]$,

$q_3 = [0.4, 1, 0]$, $q_4 = [0.2, 2, 0]$, $z_1 = [q_4; q_1]$, $z_2 = [q_3; q_3]$, $z_3 = [q_1; q_1]$, $z_4 = [q_2; q_3]$, $z_5 = [q_1; q_4]$, $z_6 = [q_2; q_2]$, $z_7 = [q_4; q_4]$, $z_8 = [q_3; q_2]$, $y = 0.2$, $Y = 0.4$, $0y = [0; 0.2]$, $0Y = [0; 0.4]$, $yy = [0.2; 0.2]$, $YY = [0.4; 0.4]$. One can easily see that $\mathcal{I}_{0_N}(T(\hat{\Sigma}_1), T(\hat{\Sigma}_2))$ is 0-approximate initial-state opaque, since for any run starting from any secret state, i.e. z_1 and z_4 , there exists a run starting from a non-secret state, i.e. z_2 and z_3 , such that the output trajectories are exactly the same. One can readily verify that the symbolic network has this property regardless of the number of systems (i.e. n), due to the homogeneity of systems Σ_i and the symmetry of the circular network topology. Thus, one can conclude that $T(\hat{\Sigma}) = \mathcal{I}_{0_N}(T(\hat{\Sigma}_1), \dots, T(\hat{\Sigma}_n))$ is 0-approximate initial-state opaque. Therefore, by Corollary 9, we obtain that the original network $T(\Sigma) = \mathcal{I}_{0_N}(T(\Sigma_1), \dots, T(\Sigma_n))$ is 0.5-approximate initial-state opaque.

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