The Conditional Poincaré Inequality for Filter Stability

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Abstract— **This paper is concerned with the problem of nonlinear filter stability of ergodic Markov processes. The main contribution is the conditional Poincaré inequality (PI), which is shown to yield filter stability. The proof is based upon a recently discovered duality which is used to transform the nonlinear filtering problem into a stochastic optimal control problem for a backward stochastic differential equation (BSDE). Based on these dual formalisms, a comparison is drawn between the stochastic stability of a Markov process and the filter stability. The latter relies on the conditional PI described in this paper, whereas the former relies on the standard form of PI.**

I. Introduction

The Poincaré (or spectral gap) inequality (PI) is central to the subject of stochastic stability of Markov processes [1, Ch. 4]. The PI is the simplest condition which quantifies ergodicity and convergence to stationarity: The Poincaré constant gives the rate of exponential decay. Apart from stochastic stability, the PI has a rich history. It is the fundamental inequality in the study of the elliptic PDEs.

The goal of this paper is to propose a generalization of the PI for the purpose of nonlinear filter stability analysis. Specifically, a continuous-time filtering model is considered where (a) the Markov process is ergodic, and (b) the observations are corrupted by additive white noise. In the study of the Wonham filter, this model is referred to as the *ergodic signal* case. A companion paper, also published in the proceedings of this conference, tackles the more general non-ergodic signal case [2].

For the case of ergodic signal, a pioneering early contribution is [3]. Early work is based on contraction analysis of the random matrix products arising from recursive application of the Bayes' formula [4] (see also [5, Ch. 4.3]). Using related techniques, the analysis of the Zakai equation leads to useful formulae for the Lyapunov exponents under assumptions on model parameters and noise limits [6], [7]. For the ergodic signal case, a comprehensive account appears in [8] and the first complete solution is given in [9]. A necessary and sufficient characterization of the asymptotic properties of the Wonham filter, for both ergodic and non-ergodic cases, appears in [10]. For an accessible account of the problem and the solution methods, see the review papers [11], [12] and references therein.

Notwithstanding the fundamental importance of the PI for stochastic stability of Markov processes, it is *not* a major theme in the filter stability literature. The closest is the appearance of the Brascamp-Lieb (B-L) inequality in [13, Chapter 4] and [14]. In both these references, the B-L inequality is employed for the filter stability analysis of the Itô diffusions assuming linear observations. The main point is that the B-L inequality is *not* central to the convergence analysis. In particular, the convergence rate relies on the uniform convexity of the optimal value function in [13] and (equivalently) the log concavity of the posterior in [14] (and this is primarily a result of the linear observation model). The value function and the posterior are related through the log transformation which is how duality in the nonlinear filters is historically understood [15], [16], [17]. Apart from these works, it is known that in the large noise limit, the top Lyapunov exponent converges to the spectral gap of the ergodic Markov process [18].

This paper has a single contribution: generalization of the PI for Markov processes to the proposed conditional PI for the nonlinear filter. The conditional PI is shown to play the same role for filter stability as the standard PI for stochastic stability. The proof technique relies on a recently discovered duality result whereby the nonlinear filtering problem is cast as a BSDE-constrained stochastic optimal control problem [19]. Using these methods, we are able to derive *all* the prior results where explicit convergence rates are obtained. These are pointed in the paper.

The outline of the remainder of this paper is as follows: The problem formulation appears in Sec. II. The PI and the conditional PI are introduced in Sec. III. The filter stability results appear in Sec. IV. The Appendix contains the proofs.

II. Problem formulation

Notation: The state-space $\mathbb{S} := \{1,2,\cdots\}$ is either finite (with d elements) or is countable. The set of probability vectors on $\mathbb{S}$ is denoted by $\mathcal{P}(\mathbb{S})$: $\mu \in \mathcal{P}(\mathbb{S})$ if $\mu(x) \geq 0$ and $\sum_{x\in\mathbb{S}} \mu(x) = 1$. The space of functions on $\mathbb{S}$ is denoted $C(\mathbb{S})$ and the space bounded functions on $\mathbb{S}$ is denoted as $C_b(\mathbb{S})$: $f \in C_b(\mathbb{S})$ if $|f(x)| < C$ for some constant C for all $x \in \mathbb{S}$. When the cardinality of $\mathbb{S}$ is finite (d), the space of functions on $\mathbb{S}$ is identified with $\mathbb{R}^d$. For a measure $\mu \in \mathcal{P}(\mathbb{S})$ and a function $f \in C_b(\mathbb{S})$, $\mu(f) := \sum_{x\in\mathbb{S}} \mu(x) f(x)$. For two functions $f,h \in C_b(\mathbb{S})$, fh is the function obtained through element-wise product: $(fh)(x) = f(x)h(x)$ for all $x \in \mathbb{S}$ and similarly $f^2 = ff$ is the square of the function. The functions of all ones is denoted as 1, i.e., $1(x) = 1$ for all $x \in \mathbb{S}$.

A. Filtering model

Consider a pair of continuous-time stochastic processes (X,Z) defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The state

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process $X := \{X_t \in \mathbb{S} : t \ge 0\}$ is a stationary Markov process with generator A and an everywhere positive invariant measure $\bar{\mu} \in \mathcal{P}(\mathbb{S})$: $\bar{\mu}(x) > 0$ for all $x \in \mathbb{S}$ and $\bar{\mu}(Af) = 0$ for all $f \in C_b(\mathbb{S})$ (this is the case for any irreducible positive recurrent Markov process). The observation process $Z = \{Z_t \in \mathbb{R}^m : t \ge 0\}$ is defined according to the following model:

$$Z_t := \int_0^t h(X_s)\,\mathrm{d}s + W_t \tag{1}$$

where $h : \mathbb{S} \to \mathbb{R}^m$ is the observation function and $W = \{W_t \in \mathbb{R}^m : t \ge 0\}$ is a Wiener process (w.p.) that is assumed to be independent of X. The covariance of W is denoted R which is assumed to be strictly positive-definite. The filtration generated by Z is denoted $\mathcal{Z} := \{\mathcal{Z}_t : 0 \le t \le T\}$ where $\mathcal{Z}_t = \sigma(\{Z_s : 0 \le s \le t\})$.

Function spaces: To stress the choice of the initial stationary prior $\bar{\mu}$, we write the probability measure P as $\mathsf{P}^{\bar{\mu}}$, the expectation is denoted $\mathsf{E}^{\bar{\mu}}(\cdot)$, and $L^p(\bar{\mu})$ is the space of random variables Y with $\mathsf{E}^{\bar{\mu}}(|Y|^p) < \infty$. The space of square-integrable deterministic functions on $\mathbb{S}$ is denoted as $L^2(\mathbb{S})$: $f \in L^2(\mathbb{S})$ if $\mathsf{E}^{\bar{\mu}}(|f(X_T)|^2) = \sum_x |f(x)|^2 \bar{\mu}(x) < \infty$. The space of square-integrable $\mathcal{Z}_T$-measurable random functions on $\mathbb{S}$ is denoted as $L^2_{\mathcal{Z}_T}(\mathbb{S})$: $F \in L^2_{\mathcal{Z}_T}(\mathbb{S})$ if F is $\mathcal{Z}_T$-measurable and $\mathsf{E}^{\bar{\mu}}(|F(X_T)|^2) < \infty$. Likewise, the space of $\mathcal{Z}$-adapted square-integrable S-valued stochastic processes is denoted $L^2_{\mathcal{Z}}([0,T];S)$. Examples are $S = \mathbb{R}^m$ for vector-valued and $S = C(\mathbb{S})$ for function-valued stochastic processes.

The filtering problem is to compute the conditional distribution (posterior) of the state X_t given $\mathcal{Z}_t$. The posterior distribution at time t is denoted $\pi_t^{\bar{\mu}} \in \mathcal{P}(\mathbb{S})$. For $f \in C_b(\mathbb{S})$

$$\pi_t^{\bar{\mu}}(f) := \mathsf{E}^{\bar{\mu}}(f(X_t)|\mathcal{Z}_t)$$

B. Definition of filter stability

For each $f \in C_b(S)$, the Wonham filter is given by the stochastic differential equation:

$$\mathrm{d}\pi_t(f) = \pi_t(Af)\,\mathrm{d}t + (\pi_t(hf) - \pi_t(h)\pi_t(f))R^{-1}(\mathrm{d}Z_t - \pi_t(h)\,\mathrm{d}t) \tag{2}$$

With an initialization $\pi_0 = \mu \in \mathcal{P}(\mathbb{S})$, the solution of the Wonham filter is denoted as $\pi^{\mu} := \{\pi_t^{\mu} \in \mathcal{P}(\mathbb{S}) : t \ge 0\}$. The posterior $\pi^{\bar{\mu}}$ results from the choice of the initial condition $\pi_0 = \bar{\mu}$.

Remark 1: (see also the discussion in [12, Sec. 1].) Suppose $\mu \in \mathcal{P}(\mathbb{S})$ then (because $\bar{\mu}$ is everywhere positive) $\mu \ll \bar{\mu}$ and setting the Radon-Nikodym (R-N) derivative

$$\frac{\mathrm{d}\mathsf{P}^{\mu}}{\mathrm{d}\mathsf{P}^{\bar{\mu}}}(\omega) = \sum_x \frac{\mu(x)}{\bar{\mu}(x)} 1_{[X_0 = x]}(\omega)$$

yields a new probability measure P^{μ} on the common measurable space $(\Omega, \mathcal{F})$. With respect to P^{μ}, the expectation operator is denoted $\mathsf{E}^{\mu}(\cdot)$ and $L^p(\mu)$ is the space of random variables Y with $\mathsf{E}^{\mu}(|Y|^p) < \infty$. The solution of the Wonham filter $\pi_t^{\mu}(f) = \mathsf{E}^{\mu}(f(X_t)|\mathcal{Z}_t)$.

Definition 1: The Wonham filter is *stable* if for each $f \in C_b(\mathbb{S})$, $\pi_T^{\mu}(f) \xrightarrow{L^1(\bar{\mu})} \pi_T^{\bar{\mu}}(f)$, i.e.,

$$\mathsf{E}^{\bar{\mu}}\big(|\pi_T^{\mu}(f) - \pi_T^{\bar{\mu}}(f)|\big) \longrightarrow 0 \quad \text{as } T \to \infty \tag{3}$$

for all $\mu \in \mathcal{P}_0 \subset \mathcal{P}(\mathbb{S})$.

Our goals are as follows: (i) characterize the subset $\mathcal{P}_0 \subset \mathcal{P}(\mathbb{S})$ for which the filter stability holds (ideally $\mathcal{P}_0 = \mathcal{P}(\mathbb{S})$); and (ii) obtain explicit estimates on the rate of convergence.

Remark 2: Suppose $f \in C_b(\mathbb{S})$, $\mu, \nu \in \mathcal{P}_0$ such that (3) holds. Then

1) Because $\pi_T^{\mu}(f) \xrightarrow{L^1(\bar{\mu})} \pi_T^{\bar{\mu}}(f)$ and $\pi_T^{\nu}(f) \xrightarrow{L^1(\bar{\mu})} \pi_T^{\bar{\mu}}(f)$, by the use of triangle inequality, it also follows that $\pi_T^{\mu}(f) \xrightarrow{L^1(\bar{\mu})} \pi_T^{\nu}(f)$.
2) (see [3, Remark 3.3]) Also $\pi_T^{\mu}(f) \xrightarrow{L^p(\bar{\mu})} \pi_T^{\nu}(f)$ for all $p \ge 1$. This is because, e.g., with $p = 2$,
$$\mathsf{E}^{\bar{\mu}}(|\pi_T^{\mu}(f) - \pi_T^{\nu}(f)|^2) \le 2\|f\|_{\infty}\,\mathsf{E}^{\bar{\mu}}(|\pi_T^{\mu}(f) - \pi_T^{\nu}(f)|)$$
3) If $\pi_T^{\mu}(f) \xrightarrow{L^1(\bar{\mu})} \pi_T^{\nu}(f)$ then (owing to Remark 1) it also follows $\pi_T^{\mu}(f) \xrightarrow{L^1(\nu)} \pi_T^{\nu}(f)$ and therefore, $\pi_T^{\mu}(f) \xrightarrow{L^p(\nu)} \pi_T^{\nu}(f)$ for all $p \ge 1$.

Remark 3: The problem of stochastic stability is a special case when the filtration $\mathcal{Z}$ is independent of X (e.g., $\mathcal{Z}$ is trivial). The counterpart of the Wonham filter is the Kolmogorov's forward equation

$$\mathrm{d}\pi_t(f) = \pi_t(Af) \tag{4}$$

With an initialization $\pi_0 = \mu \in \mathcal{P}(\mathbb{S})$, the solution $\pi^{\mu} := \{\pi_t^{\mu} \in \mathcal{P}(\mathbb{S}) : t \ge 0\}$ is now a deterministic process (which serves to simplify the problem considerably). Adapting (3) to this case, the basic problem of stochastic stability is to show $\pi_T^{\mu}(f) \to \bar{\mu}(f)$ for all $f \in C_b(\mathbb{S})$.

III. Main assumption: Poincaré inequality (PI)

A. Standard form of PI

The carré du champ operator is defined as

$$\Gamma(f)(x) := \sum_{j \in \mathbb{S}} A(x,j)(f(x) - f(j))^2, \ x \in \mathbb{S}$$

For a deterministic function $f \in L^2(\mathbb{S})$, the energy and variance are defined as

$$\mathrm{enr}_0^{\bar{\mu}}(f) := \mathsf{E}^{\bar{\mu}}(\Gamma(f)(X_T)) = \sum_x \bar{\mu}(x)\Gamma(f)(x)$$
$$\mathrm{var}_0^{\bar{\mu}}(f) := \mathsf{E}^{\bar{\mu}}(|f(X_T) - \bar{\mu}(f)|^2) = \sum_x \bar{\mu}(x)|f(x) - \bar{\mu}(f)|^2$$

The standard form of the PI relates the two as follows:

$$\mathrm{PI}(\bar{\mu}, A): \quad \mathrm{enr}_0^{\bar{\mu}}(f) \ge c_0\ \mathrm{var}_0^{\bar{\mu}}(f) \quad \forall\ f \in L^2(\mathbb{S})$$

where $c_0 > 0$. The PI is a standard assumption in the theory of Markov processes to show stochastic stability [20].

B. Conditional form of PI

For a random function $F \in L^2_{\mathcal{Z}_T}(\mathbb{S})$, the energy and variance are defined as follows:

$$\begin{aligned}&\text{(energy)} \quad \text{enr}_T^{\bar\mu}(F) := \mathsf{E}^{\bar\mu}(\Gamma(F)(X_T))\\ &\text{(variance)} \quad \text{var}_T^{\bar\mu}(F) := \mathsf{E}^{\bar\mu}(|F(X_T) - \pi_T^{\bar\mu}(F)|^2)\end{aligned}$$

Definition 2: A Markov process is said to satisfy the *conditional Poincaré inequality* (PI) with a constant c if

$$\text{PI}(\bar\mu, A; \mathcal{Z}): \quad \text{enr}_T^{\bar\mu}(F) \geq c \ \text{var}_T^{\bar\mu}(F) \quad \forall\, F \in L^2_{\mathcal{Z}_T}(\mathbb{S}),\ \forall\, T \geq 0$$

C. Examples

Example 1: Suppose $A = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$ is irreducible. Then $\lambda_1 > 0$ and $\lambda_2 > 0$. Observe that

$$\begin{aligned}&\lambda_1 \pi_T^{\bar\mu}(1)(F(1) - F(2))^2 + \lambda_2 \pi_T^{\bar\mu}(2)(F(1) - F(2))^2\\ &\geq (\lambda_1 + \lambda_2)\big(\pi_T^{\bar\mu}(1)(F(1) - \pi_T^{\bar\mu}(F))^2 + \pi_T^{\bar\mu}(2)(F(2) - \pi_T^{\bar\mu}(F))^2\big)\end{aligned}$$

and therefore upon taking expectations on both sides,

$$\text{enr}_T^{\bar\mu}(F) \geq (\lambda_1 + \lambda_2)\, \text{var}_T^{\bar\mu}(F)$$

Hence, the conditional PI holds for every irreducible 2-state Markov chain with a constant $c = (\lambda_1 + \lambda_2)$. Note that the best constant for standard PI is $c_0 = 2(\lambda_1 + \lambda_2)$.

This simple example admits the following generalizations described in the proposition with proofs in the Appendix A.

Proposition 1: The conditional PI holds with the following constants (provided these are positive)

$$c = \sum_j \min_{i \in \mathbb{S}:\ i \neq j} A(i,j), \quad c = \min_{i \neq j} \sqrt{A(i,j)}\sqrt{A(j,i)}$$

The first of the two formulae in Prop. 1 is related to the Doeblin type strong mixing condition [5, Assumption 4.3.24]. The second formula is the same as [4, Theorem 6], [9, Theorem 4.3] and [13, Corollary 2.3.2].

In this paper, the constant c of the conditional PI is shown to play the same role for the filter as the constant c of the standard PI does for the Markov process. (Both give the rate of convergence.) Moreover, stronger results are possible with weaker notions of the conditional PI (see Prop. 2 and Example 2).

Remark 4: One may conjecture that the filter "inherits" the PI from the underlying Markov process. Note that the definition 2 is stated for a general class of filtrations (not necessarily defined according to the model (1)). In the general settings, the conditional PI holds for deterministic functions $f \in L^2(\mathbb{S})$. This is because

$$\text{enr}_T^{\bar\mu}(f) \geq c_0\, \mathsf{E}^{\bar\mu}(|f(X_T) - \bar\mu(f)|^2) \geq c_0\ \text{var}_T^{\bar\mu}(f)$$

This shows that the PI and also the constant c is inherited on the subspace $L^2(\mathbb{S}) \subset L^2_{\mathcal{Z}_T}(\mathbb{S})$ of deterministic functions. However, with general types of filtrations, it may not hold for random functions. A counterexample appears in the Appendix B.

In the following, a proof of the filter stability is presented based on the conditional PI. The proof of stochastic stability arises as a special case, and is included in Appendix C. The two sections are self-contained and may be read independently of the other. However, a reader may benefit from reading the two sections simultaneously.

IV. Filter Stability

For a random function $F \in L^2_{\mathcal{Z}_T}(\mathbb{S})$, the conditional energy and the conditional variance are defined as follows:

$$\begin{aligned}&\text{(cond. energy)} \quad \mathcal{E}_T^{\bar\mu}(F) := \mathsf{E}^{\bar\mu}(\Gamma(F)(X_T)|\mathcal{Z}_T)\\ &\text{(cond. var.)} \quad \mathcal{V}_T^{\bar\mu}(F) := \mathsf{E}^{\bar\mu}(|F(X_T) - \pi_T^{\bar\mu}(F)|^2|\mathcal{Z}_T)\end{aligned}$$

Upon taking expectations

$$\text{enr}_T^{\bar\mu}(F) = \mathsf{E}^{\bar\mu}(\mathcal{E}_T^{\bar\mu}(F)), \quad \text{var}_T^{\bar\mu}(F) = \mathsf{E}^{\bar\mu}(\mathcal{V}_T^{\bar\mu}(F))$$

A. Duality

In our prior work [21], the following backward stochastic differential equation (BSDE) constrained optimal control problem is introduced. The significance of the problem is that it is a dual of the nonlinear filtering problem.

Dual optimal control problem:

$$\underset{U \in \mathcal{U}}{\text{Min}}\ \ \mathsf{J}_T^{\bar\mu}(U) = |Y_0(X_0) - \bar\mu(Y_0)|^2 + \mathsf{E}^{\bar\mu}\Big(\int_0^T \ell(Y_t, V_t, U_t; X_t)\,\mathrm{d}t\Big) \tag{5a}$$

$$\begin{aligned}\text{Subj.} \ -\mathrm{d}Y_t(x) &= \big((AY_t)(x) + h(x)(U_t + V_t(x))\big)\,\mathrm{d}t - V_t^\top(x)\,\mathrm{d}Z_t\\ Y_T(x) &= F(x)\ \ \forall\, x \in \mathbb{S} \quad \text{(given)}\end{aligned} \tag{5b}$$

where $\ell(y,v,u;x) := \Gamma(y)(x) + |u + v(x)|_R^2$, $\mathcal{U} := L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$, and $F \in L^2_{\mathcal{Z}_T}(\mathbb{S})$.

The existence and uniqueness of the optimal control follows from the standard results in the BSDE constrained optimal control theory [21]. The solution, including the formula for optimal control, is described in [19, Theorem 1]. Let $U^{\text{opt}} := \{U_t^{\text{opt}} \in \mathbb{R}^m : 0 \leq t \leq T\}$ be the optimal control input and $(Y,V) := \{(Y_t, V_t) \in C(\mathbb{S}) \times C(\mathbb{S})^m : 0 \leq t \leq T\}$ be the associated $\mathcal{Z}$-adapted (optimal) trajectory (the solution of the BSDE $(Y,V) \in L^2_{\mathcal{Z}}([0,T]; C(\mathbb{S}) \times C(\mathbb{S})^m)$). Then

1. [19, Theorem 2]: The conditional mean

$$\pi_T^{\bar\mu}(Y_T) = \bar\mu(Y_0) - \int_0^T U_t^{\text{opt}}\,\mathrm{d}Z_t$$

2. [19, Theorem 5]: Define a $\mathcal{Z}$-adapted process $M := \{M_t : 0 \leq t \leq T\}$ as follows:

$$M_t := \mathcal{V}_t^{\bar\mu}(Y_t) - \int_0^t \Big(\mathcal{E}_s^{\bar\mu}(Y_s) + \sum_x \pi_s^{\bar\mu}(x)|U_s^{\text{opt}} + V_s(x)|_R^2\Big)\,\mathrm{d}s$$

Then M is a $\mathsf{P}^{\bar\mu}$-martingale.

3. Therefore, $\mathsf{E}^{\bar\mu}(M_T) = \mathsf{E}^{\bar\mu}(M_0)$, which is expressed as

$$\text{var}_0^{\bar\mu}(Y_0) + \int_0^T \text{enr}_t^{\bar\mu}(Y_t) + \mathsf{E}^{\bar\mu}(|U_t^{\text{opt}} + V_t(X_t)|_R^2)\,\mathrm{d}t = \text{var}_T^{\bar\mu}(Y_T)$$

$$\therefore, \quad \text{var}_0^{\bar\mu}(Y_0) + \int_0^T \text{enr}_t^{\bar\mu}(Y_t)\,\mathrm{d}t \leq \text{var}_T^{\bar\mu}(Y_T)$$

and using the conditional PI

$$\mathrm{var}_0^{\bar\mu}(Y_0) \le e^{-cT}\,\mathrm{var}_T^{\bar\mu}(Y_T) \tag{6}$$

Equation (6) is the backward inequality for the variance of the dual process (this is the only place where the conditional PI is used). The counterpart for a Markov process is the inequality (11) for the dual process in Appendix C.

A more general result, described in the following proposition, is obtained by considering the martingale M directly. Its proof appears in Appendix D.

Proposition 2: Suppose $\beta = \{\beta_t : t \ge 0\}$ is any non-negative $\mathcal{Z}$-adapted process such that

$$\mathcal{E}_t^{\bar\mu}(f) \ge \beta_t\,\mathcal{V}_t^{\bar\mu}(f) \quad \mathsf{P}^{\bar\mu}-\text{a.s.} \quad \forall\, f \in L^2(\mathbb{S}),\ t \ge 0 \tag{7}$$

Then the backward inequality is of the form

$$\mathrm{var}_0^{\bar\mu}(Y_0) \le \mathsf{E}^{\bar\mu}\left(e^{-\int_0^T \beta_t\,\mathrm{d}t}\,\mathcal{V}_T^{\bar\mu}(Y_T)\right) \tag{8}$$

Consequently, if $\frac{1}{T}\int_0^T \beta_t\,\mathrm{d}t \to c$ (a deterministic constant), then the backward inequality (6) for the variance is obtained asymptotically. The following example shows how to choose β to obtain an asymptotic formula for the convergence rate.

Example 2: It is a straightforward calculation to verify

$$\mathcal{E}_t^{\bar\mu}(f) \ge \left(\sum_i \pi_t^{\bar\mu}(i) \min_{j\in\mathbb{S}:\ i\ne j} A(i,j)\right)\mathcal{V}_t^{\bar\mu}(f), \quad \forall\, t \ge 0$$

Set $\beta_t = \sum_i \pi_t^{\bar\mu}(i)\min_{j:i\ne j} A(i,j)$. Using [9, Eq. (5.15)], it is known that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \beta_t\,\mathrm{d}t = \sum_i \bar\mu(i) \min_{j\in\mathbb{S}:\,i\ne j} A(i,j) \quad \text{a.s.}$$

The righthand-side then gives the asymptotic constant c for the variance inequality (6). This formula for the asymptotic convergence rate of the Wonham filter can be found in [9, Theorem 4.2].

Remark 5: Inequality (7) is the pathwise version of the conditional PI. Note the inequality needs to be specified only for deterministic functions. Because of its pathwise nature, the inequality then also holds for random functions $F \in L^2_{\mathcal{Z}_t}(\mathbb{S})$. A formal definition of pathwise PI is stated next.

Definition 3: A Markov process is said to satisfy the *pathwise Poincaré inequality* with a constant c if (7) holds and

$$\beta_t \;\ge\; c \;>\; 0 \quad \mathsf{P}^{\bar\mu}-\text{a.s.},\ t \ge 0$$

The following proposition shows that conditional PI and pathwise PI are equivalent. Its proof appears in Appendix E.

Proposition 3: The conditional PI holds with a constant c if and only if the pathwise PI holds with a constant c.

Remark 6: For the conditional PI to hold, $\pi_t^{\bar\mu}$ must therefore satisfy the standard form of the PI with a uniform constant c:

$$\sum_x \pi_t^{\bar\mu}(x)\Gamma(f)(x) \;\ge\; c\,\sum_x \pi_t^{\bar\mu}(x)|f(x) - \pi_t^{\bar\mu}(f)|^2 \quad \mathsf{P}^{\bar\mu}-\text{a.s.}$$

for all $f \in L^2(\mathbb{S})$ and for all $t \ge 0$. This is a very stringent requirement that greatly limits the application of this paper: It is easy to come up with examples where the standard form of the PI holds for $\bar\mu$ but not for all $\pi_t^{\bar\mu}$. The weaker form presented in Prop. 2, specifically (8), is more general. However, it is not clear how to obtain useful asymptotic bounds for $\frac{1}{T}\int_0^T \beta_t\,\mathrm{d}t$, beyond the result in Example 2.

B. Y_T as likelihood ratio

In the remainder of this section π^μ and $\pi^{\bar\mu}$ are the solutions of the Wonham filter (4), with initialization $\pi_0 = \mu$ and $\pi_0 = \bar\mu$, respectively. The likelihood ratio is the random function

$$\gamma_T(x) := \frac{\pi_T^\mu(x)}{\pi_T^{\bar\mu}(x)} \quad \text{for } x \in \mathbb{S}$$

The function is well-defined because $\pi_T^{\bar\mu}(x) > 0$ for all $x \in \mathbb{S}$ [9, Remark 3]. Its conditional mean and variance are as follows:

$$\text{mean}: \qquad \pi_T^{\bar\mu}(\gamma_T) = \sum_x \pi_T^{\bar\mu}(x)\gamma_T(x) = 1$$

$$\text{variance}: \quad \mathrm{var}_T^{\bar\mu}(\gamma_T) = \mathsf{E}^{\bar\mu}(|\gamma_T(X_T) - 1|^2) = \mathsf{E}^{\bar\mu}(\pi_T^\mu(\gamma_T) - 1)$$

To derive the formula for the variance note

$$\begin{aligned}\mathrm{var}_T^{\bar\mu}(\gamma_T) + 1 &= \mathsf{E}^{\bar\mu}(|\gamma_T(X_T)|^2) = \mathsf{E}^{\bar\mu}(\mathsf{E}^{\bar\mu}(|\gamma_T(X_T)|^2|\mathcal{Z}_T)) \\ &= \mathsf{E}^{\bar\mu}(\pi_T^{\bar\mu}(\gamma_T^2)) = \mathsf{E}^{\bar\mu}(\pi_T^\mu(\gamma_T))\end{aligned}$$

Useful formulae for $\pi_T^\mu(\gamma_T)$ are obtained by setting $Y_T = \gamma_T$ in the dual optimal control problem. The results are presented in the following proposition whose proof appears in the Appendix F. To state the proposition, we need the exponential $\mathsf{P}^{\bar\mu}$-martingale

$$A_t := \exp\Big(\int_0^t D_s(\,\mathrm{d}Z_s - \pi_s^{\bar\mu}(h)\,\mathrm{d}s) - \tfrac{1}{2}\int_0^t D_s^2\,\mathrm{d}s\Big), \quad t \ge 0$$

where the difference $D_t := \pi_t^\mu(h) - \pi_t^{\bar\mu}(h)$. ($\{A_t : t \ge 0\}$ is in fact the change of measures between the conditional laws [22, Theorem 3.1], but we do not make use of this interpretation here.)

Proposition 4: Consider the dual optimal control problem with $Y_T := \gamma_T$. Then

1) The optimal control $U^{\text{opt}} = 0$ a.s., and the optimal trajectory (Y, V) is the solution of the BSDE

$$\begin{aligned}-\mathrm{d}Y_t(x) &= ((AY_t)(x) + h(x)V_t(x))\,\mathrm{d}t - V_t(x)\,\mathrm{d}Z_t, \\ Y_T(x) &= \gamma_T(x)\ \ \forall\, x \in \mathbb{S}\end{aligned} \tag{9}$$

2) The process $\pi_t^{\bar\mu}(Y_t) \equiv 1$ for all $0 \le t \le T$. Therefore

$$\pi_T^{\bar\mu}(\gamma_T) = \bar\mu(Y_0) = 1$$

3) The stochastic process $\{\pi_t^\mu(Y_t) : 0 \le t \le T\}$ is a P^μ-mg. Consequently

$$\mathsf{E}^\mu(\pi_T^\mu(\gamma_T)) = \mu(Y_0)$$

4) The stochastic process $\{A_t\pi_t^\mu(Y_t) : 0 \le t \le T\}$ is a $\mathsf{P}^{\bar\mu}$-mg. Consequently

$$\mathsf{E}^{\bar\mu}(A_T\pi_T^\mu(\gamma_T)) = \mu(Y_0)$$

Now, using the inequality (6) together with the result from either part (3) or part (4), it is a straightforward calculation to derive the following forward inequality for the variance:

$$R_T \ \mathrm{var}_T^{\bar{\mu}}(\gamma_T) \le e^{-cT} \ \mathrm{var}_0^{\bar{\mu}}(\gamma_0) \tag{10}$$

where R_T is a factor obtained from either part (3) or part (4) of the Proposition. The formulae for the two cases are:

- R_T using part (3):

$$R_T = \left(\frac{\mathsf{E}^{\mu}(\pi_T^{\mu}(\gamma_T) - 1)}{\mathsf{E}^{\bar{\mu}}(\pi_T^{\mu}(\gamma_T) - 1)} \right)^2$$

- R_T using part (4):

$$R_T = \left(\frac{\mathsf{E}^{\bar{\mu}}(A_T(\pi_T^{\mu}(\gamma_T) - 1))}{\mathsf{E}^{\bar{\mu}}(\pi_T^{\mu}(\gamma_T) - 1)} \right)^2$$

The calculation for (10) appears in the Appendix F after the proof of the Prop. 4.

C. Filter stability

We are interested in the difference

$$\begin{aligned} \pi_T^{\mu}(f) - \pi_T^{\bar{\mu}}(f) &= \pi_T^{\bar{\mu}}((\gamma_T - 1)(f - \bar{\mu}(f))) \\ &= \mathsf{E}^{\bar{\mu}}((\gamma_T(X_T) - 1)(f(X_T) - \bar{\mu}(f))|\mathcal{Z}_T) \end{aligned}$$

Taking expectations of the absolute value of both sides and using Cauchy-Schwarz

$$(\mathsf{E}^{\bar{\mu}}(|\pi_T^{\mu}(f) - \pi_T^{\bar{\mu}}(f)|))^2 \le \mathrm{var}_T^{\bar{\mu}}(\gamma_T)\,\mathrm{var}_0^{\bar{\mu}}(f)$$

Combined with the forward inequality (10) for the variance, we have proved the following main result for filter stability.

Theorem 1: Suppose the filter satisfies the conditional PI with constant c. Then

$$R_T(\mathsf{E}^{\bar{\mu}}(|\pi_T^{\mu}(f) - \pi_T^{\bar{\mu}}(f)|))^2 \le e^{-cT} \ \mathrm{var}_0^{\bar{\mu}}(\gamma_0)\,\mathrm{var}_0^{\bar{\mu}}(f)$$

Consequently if $R_T \ge a^2 > 0$ then

$$(\mathsf{E}^{\bar{\mu}}(|\pi_T^{\mu}(f) - \pi_T^{\bar{\mu}}(f)|))^2 \le \frac{1}{a^2} e^{-cT} \ \mathrm{var}_0^{\bar{\mu}}(\gamma_0)\,\mathrm{var}_0^{\bar{\mu}}(f)$$

For equivalent measures, a conservative lower bound for R_T is given in the following proposition, whose proof appears in the Appendix G.

Proposition 5: Suppose $a = \min_x \frac{\mu(x)}{\bar{\mu}(x)} > 0$. Then $R_T \ge a^2$.

D. Forward variance inequality and filter stability

Equation (10) is the key variance inequality to obtain the filter stability result. Its counterpart for stochastic stability is (12). For the ease of reader, these are stated again:

$$\begin{aligned} \text{Markov process:} &\quad \mathrm{var}_0^{\bar{\mu}}(\gamma_T) \le e^{-cT} \ \mathrm{var}_0^{\bar{\mu}}(\gamma_0) \\ \text{Wonham filter:} &\quad R_T \ \mathrm{var}_T^{\bar{\mu}}(\gamma_T) \le e^{-cT} \ \mathrm{var}_0^{\bar{\mu}}(\gamma_0) \end{aligned}$$

The first of these inequalities is readily verified from a direct calculation, which in fact is a standard proof to prove stochastic stability using the PI (see [1, Theorem 4.2.5]):

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{var}_0^{\bar{\mu}}(\gamma_t) = -\mathrm{enr}_0^{\bar{\mu}}(\gamma_t)$$

and using PI, the variance inequality for the Markov process follows. In contrast, a duality based proof of the variance inequality, described in Appendix C, is more involved.

For the filter, the variance inequality (10) is new. One may ask whether (10) can also be derived more directly (without the use of duality)? A direct calculation shows that

$$\mathrm{d}\mathcal{V}_t^{\bar{\mu}}(\gamma_t) = -\mathcal{E}_t^{\bar{\mu}}(\gamma_t)\,\mathrm{d}t + C_t\,\mathrm{d}t + (\mathsf{P}^{\mu}\text{-mg. increment})$$

where the coefficient

$$C_t := \pi_t^{\bar{\mu}}\big(\gamma_t^2(h - \pi_t^{\mu}(h))\big)(\pi_t^{\bar{\mu}}(h) - \pi_t^{\mu}(h))$$

It is not clear how the equation can be simplified to obtain (10). It may be possible to derive (10) through a clever choice of an integrating factor. However, we have not yet been successful in this endeavor. Because the BSDE (5b) and the Zakai equation are dual (see [23, Prop. 1]), it will be helpful to relate this work to the study of the Lyapunov exponents of the Zakai equation [4], [6], [7].

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APPENDIX

A. Proof of Prop. 1

Because the formulae are of independent interest, their proofs are included as part of two examples which are described below. Together with Examples 1 and 2, they comprise the four examples in the paper where the conditional PI holds.

Example 3: A Markov process is *Doeblin* if there exist a state $j^* \in \mathbb{S}$ such that $A(x,j^*) > c$ for all $x \in \mathbb{S} \setminus \{j^*\}$. Then

$$\begin{aligned}\mathrm{enr}_T^{\bar\mu}(F) &= \mathsf{E}^{\bar\mu}\Big(\sum_{j\in\mathbb{S}} A(X_T,j)(F(X_T)-F(j))^2\Big)\\ &\geq c\,\mathsf{E}^{\bar\mu}\big((F(X_T)-F(j^*))^2\big) \geq c\,\mathrm{var}_T^{\bar\mu}(F)\end{aligned}$$

Therefore, the conditional PI holds for Doeblin chains. This admits a straightforward generalization that gives the first of the two formulae in Prop. 1:

$$\begin{aligned}\mathrm{enr}_T^{\bar\mu}(F) &\geq \mathsf{E}^{\bar\mu}\Big(\sum_{j\in\mathbb{S}} \min_{i:\, i\neq j} A(i,j)\,(F(X_T)-F(j))^2\Big)\\ &\geq \Big(\sum_{j\in\mathbb{S}} \min_{i:\, i\neq j} A(i,j)\Big)\,\mathrm{var}_T^{\bar\mu}(F)\end{aligned}$$

Example 4: Using the tower property of conditional expectation

$$\mathrm{enr}_T^{\bar\mu}(F) = \mathsf{E}^{\bar\mu}\Big(\sum_{i,j\in\mathbb{S}} \pi_T^{\bar\mu}(i)A(i,j)(F(i)-F(j))^2\Big)$$

$$\mathrm{var}_T^{\bar\mu}(F) = \mathsf{E}^{\bar\mu}\Big(\sum_{i,j\in\mathbb{S}} \pi_T^{\bar\mu}(i)\pi_T^{\bar\mu}(j)(F(i)-F(j))^2\Big)$$

Because algebraic mean dominates the geometric mean

$$\begin{aligned}&\sum_{i,j\in\mathbb{S}} \pi_T^{\bar\mu}(i)A(i,j)(F(i)-F(j))^2\\ &\quad= \sum_{i,j\in\mathbb{S}} \frac{1}{2}\big(\pi_T^{\bar\mu}(i)A(i,j)+\pi_T^{\bar\mu}(j)A(j,i)\big)(F(i)-F(j))^2\\ &\quad\geq \sum_{i,j\in\mathbb{S}} \sqrt{\pi_T^{\bar\mu}(i)\pi_T^{\bar\mu}(j)}\sqrt{A(i,j)A(j,i)}(F(i)-F(j))^2\\ &\quad\geq c\sum_{i,j\in\mathbb{S}} \pi_T^{\bar\mu}(i)\pi_T^{\bar\mu}(j)(F(i)-F(j))^2\end{aligned}$$

where we used the fact that $\sqrt{x} \geq x$ for $0 \leq x \leq 1$. Taking expectations of both sides gives the second of the two fromulae in Prop. 1.

B. A counterexample for conditional PI

Example 5: This is the famous counterexample of the filtering theory [18, pp. 9-10]. The state-space $\mathbb{S} = \{1,2,3,4\}$ and the rate matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\\ 1 & 0 & 0 & -1\end{pmatrix}$$

whose unique invariant measure $\bar\mu = [\frac14,\frac14,\frac14,\frac14]$. The standard PI holds with a constant $c = 2$.

Consider a sigma-algebra $\mathcal{G} = \sigma([X_T \in \{1,3\}])$ along with a $\mathcal{G}$-measurable function:

$$F(\cdot) = \begin{cases}\begin{pmatrix}1 & 1 & -1 & -1\end{pmatrix} & \text{if } X_T \in \{1,3\}\\ \begin{pmatrix}-1 & 1 & 1 & -1\end{pmatrix} & \text{if } X_T \in \{2,4\}\end{cases}$$

Then the conditional distribution

$$\pi_T^{\bar\mu}(\cdot) = \mathsf{E}^{\bar\mu}(1_{[X_T=\cdot]} \mid \mathcal{G}) = \begin{cases}\begin{pmatrix}\frac12 & 0 & \frac12 & 0\end{pmatrix} & \text{if } X_T \in \{1,3\}\\ \begin{pmatrix}0 & \frac12 & 0 & \frac12\end{pmatrix} & \text{if } X_T \in \{2,4\}\end{cases}$$

The conditional mean $\pi_T^{\bar\mu}(F) = 0$, the conditional variance $\pi_T^{\bar\mu}(F^2) = 1$, and therefore the variance $\mathrm{var}_T^{\bar\mu}(F) = 1$. On the other hand, the energy $\mathrm{enr}_T^{\bar\mu}(F) = 0$. Therefore, the conditional PI does not hold for this example.

C. Stochastic stability

In this section, we specialize the results of Sec. IV to the problem of stochastic stability of Markov processes.

Dual process: Introduce a deterministic dual process $y = \{y_t \in L^2(\mathbb{S}) : 0 \leq t \leq T\}$ as a solution of the backward ode

$$-\frac{\mathrm{d}}{\mathrm{d}t}y_t = Ay_t, \quad y_T \text{ fixed (given)}$$

Then a standard application of Itô product formula gives

$$y_T(X_T) = y_0(X_0) + \int_0^T \sum_{x\in\mathbb{S}} y_t(x)\,\mathrm{d}N_t(x)$$

where $\{N_t : t \geq 0\}$ is a martingale. Taking an expectation

$$\mathsf{E}^{\bar\mu}(y_T(X_T)) = \bar\mu(y_0)$$

and the equation for the variance is

$$\mathrm{var}_0^{\bar\mu}(y_0)+\int_0^T \mathrm{enr}_0^{\bar\mu}(y_t)\,\mathrm{d}t=\mathrm{var}_0^{\bar\mu}(y_T)$$

Using the PI it follows

$$\mathrm{var}_0^{\bar\mu}(y_0)\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(y_T) \tag{11}$$

This is counterpart of (6). This inequality is the only place in the stability proof where the PI is used.

y_T as likelihood ratio: In the remainder of this section π^μ and $\pi^{\bar\mu}$ are the solutions of the Kolmogorov's equation (4), with initialization $\pi_0=\mu$ and $\pi_0=\bar\mu$, respectively.

Consider the likelihood ratio $\gamma_T(x):=\frac{\pi_T^\mu(x)}{\bar\mu(x)}$ for $x\in\mathbb{S}$. The ratio is well-defined because $\bar\mu(x)>0$ for all $x\in\mathbb{S}$. The mean and variance of γ_T are as follows:

$$\begin{aligned}&\text{mean}: && \bar\mu(\gamma_T)=\sum_x\bar\mu(x)\gamma_T(x)=1\\ &\text{variance}: && \mathrm{var}_0^{\bar\mu}(\gamma_T)=\bar\mu(\gamma_T^2)-1=\pi_T^\mu(\gamma_T)-1\end{aligned}$$

The dual process is used to express $\pi_T^\mu(\gamma_T)$ in terms of the initial measure μ. The counterpart of Prop. 4 is as follows:

Proposition 6: Consider the dual process with terminal condition $y_T:=\gamma_T$. Then

$$\bar\mu(\gamma_T)=\bar\mu(y_0),\quad \pi_T^\mu(\gamma_T)=\mu(y_0)$$

Proof: Let $\{\pi_t:0\le t\le T\}$ be the solution of the Kolmogorov's forward equation then

$$\pi_T(y_T)=\pi_0(y_0)$$

The two formulae follow by using $\pi_0=\bar\mu$ and $\pi_0=\mu$. ■

Using the inequality (11) together with the result of the proposition, the following counterpart of (10) is obtained:

$$\mathrm{var}_0^{\bar\mu}(\gamma_T)\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(\gamma_0) \tag{12}$$

The calculation is included at the end of this section.

Stochastic stability: Using Cauchy-Schwarz, the difference squared

$$|\pi_T^\mu(f)-\bar\mu(f)|^2=|\bar\mu((\gamma_T-1)(f-\bar\mu(f)))|^2\le\mathrm{var}_0^{\bar\mu}(\gamma_T)\,\mathrm{var}_0^{\bar\mu}(f)$$

Combining this with (12), we have the following result on stochastic stability:

Theorem 2: Suppose X is a Markov process with an everywhere positive invariant measure $\bar\mu$ whose generator satisfies the PI with constant c. Then

$$|\pi_T^\mu(f)-\bar\mu(f)|^2\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(\gamma_0)\,\mathrm{var}_0^{\bar\mu}(f)$$

Calculation for (12): Using the second formula from Prop. 6

$$\mathrm{var}_0^{\bar\mu}(\gamma_T)=\pi_T^\mu(\gamma_T)-1=\mu(y_0)-1=\bar\mu\Big(\big(\frac{\mu}{\bar\mu}-1\big)(y_0-1)\Big)$$

and using Cauchy-Schwarz

$$(\mathrm{var}_0^{\bar\mu}(\gamma_T))^2\le\bar\mu\Big(\big(\frac{\mu}{\bar\mu}-1\big)^2\Big)\,\bar\mu\big((y_0-1)^2\big)=\mathrm{var}_0^{\bar\mu}(\gamma_0)\ \mathrm{var}_0^{\bar\mu}(y_0)$$

Finally, use the inequality (11) for the dual process:

$$\begin{aligned}&\mathrm{var}_0^{\bar\mu}(y_0)\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(\gamma_T)\\ \therefore\ &(\mathrm{var}_0^{\bar\mu}(\gamma_T))^2\le\mathrm{var}_0^{\bar\mu}(\gamma_0)\ \mathrm{var}_0^{\bar\mu}(y_0)\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(\gamma_0)\,\mathrm{var}_0^{\bar\mu}(\gamma_T)\\ \therefore\ &\mathrm{var}_0^{\bar\mu}(\gamma_T)\le e^{-cT}\,\mathrm{var}_0^{\bar\mu}(\gamma_0)\end{aligned}$$

The proves the inequality in (12).

D. *Proof of Prop. 2*

Using the definitions for M and β

$$\begin{aligned}M_t&=\mathcal{V}_t^{\bar\mu}(Y_t)-\int_0^t\Big(\mathcal{E}_s^{\bar\mu}(Y_s)+\sum_x\pi_s^{\bar\mu}(x)|U_s^{\mathrm{opt}}+V_s(x)|_R^2\Big)\,\mathrm{d}s\\ &\le\mathcal{V}_t^{\bar\mu}(Y_t)-\int_0^t\mathcal{E}_s^{\bar\mu}(Y_s)\,\mathrm{d}s\ \le\ \mathcal{V}_t^{\bar\mu}(Y_t)-\int_0^t\beta_s\mathcal{V}_s^{\bar\mu}(Y_s)\,\mathrm{d}s\end{aligned}$$

That is

$$M_t\le\mathcal{V}_t^{\bar\mu}(Y_t)-\int_0^t\beta_s\mathcal{V}_s^{\bar\mu}(Y_s)\,\mathrm{d}s\quad \mathsf{P}^{\bar\mu}-\text{a.s.}\quad\text{for } 0\le t\le T \tag{13}$$

Set $\Phi_t=e^{-\int_0^t\beta_s\,\mathrm{d}s}$ and multiply both sides by the integrating factor $\beta_t\Phi_t$ to obtain

$$\beta_t\Phi_tM_t\le\beta_t\Phi_t\Big(\mathcal{V}_t^{\bar\mu}(Y_t)-\int_0^t\beta_s\mathcal{V}_s^{\bar\mu}(Y_s)\,\mathrm{d}s\Big)$$

and upon integrating from 0 to T

$$\int_0^T\beta_t\Phi_tM_t\,\mathrm{d}t\le\Phi_T\int_0^T\beta_t\mathcal{V}_t^{\bar\mu}(Y_t)\,\mathrm{d}t$$

Because $\mathrm{d}\Phi_t=-\beta_t\Phi_t\,\mathrm{d}t$

$$-\Phi_TM_T+M_0+\int_0^T\Phi_t\,\mathrm{d}M_t\le\Phi_T\int_0^T\beta_t\mathcal{V}_t^{\bar\mu}(Y_t)\,\mathrm{d}t$$

and because $M_0=\mathcal{V}_0^{\bar\mu}(Y_0)=\mathrm{var}_0^{\bar\mu}(Y_0)$

$$\begin{aligned}\mathrm{var}_0^{\bar\mu}(Y_0)+\int_0^T\Phi_t\,\mathrm{d}M_t&\le\Phi_T\Big(\int_0^T\beta_s\mathcal{V}_s^{\bar\mu}(Y_s)\,\mathrm{d}s+M_T\Big)\\ &\le\Phi_T\mathcal{V}_T^{\bar\mu}(Y_T)\end{aligned}$$

where (13) is used (with $t=T$) to obtain the final inequality. Take expectations of both sides to obtain (8).

E. *Proof of Prop. 3*

Taking an expectation of both sides of (7), using $\beta_t>c$, one obtains the conditional PI. Therefore, pathwise PI with a constant c implies conditional PI with the constant c. Conversely, consider a random function $F=1_Bf$ where the set $B\in\mathcal{Z}_T$ and $f\in L^2(\mathbb{S})$. Using the property of the conditional expectation $\mathcal{E}_T^{\bar\mu}(1_Bf)=1_B\mathcal{E}_T^{\bar\mu}(f)$ and $\mathcal{V}_T^{\bar\mu}(1_Bf)=1_B\mathcal{V}_T^{\bar\mu}(f)$, and therefore conditional PI implies

$$\mathsf{E}^{\bar\mu}\big(1_B\mathcal{E}_T^{\bar\mu}(f)\big)\ \ge\ c\,\mathsf{E}^{\bar\mu}\big(1_B\mathcal{V}_T^{\bar\mu}(f)\big)$$

Since B is arbitrary, pathwise PI follows.

F. Proof of Prop. 4 and (10)

1) With $Y_T = \gamma_T$, the conditional expectation

$$\pi_T^{\bar{\mu}}(Y_T) = \sum_{x\in\mathbb{S}} \pi_T^{\bar{\mu}}(x)\gamma_T(x) = \sum_{x\in\mathbb{S}} \pi_T^{\mu}(x) = 1$$

Therefore U_t^{opt} satisfies

$$1 = \bar{\mu}(Y_0) - \int_0^T U_t^{\text{opt}}\,\mathrm{d}Z_t \quad \mathsf{P}^{\bar{\mu}} - \text{a.s.}$$

If Z was a w.p. then it follows from the representation theorem [24, Theorem 5.18] that $U_t^{\text{opt}} \equiv 0$ a.s. for $0 \le t \le T$ and $\bar{\mu}(Y_0) = 1$. Moreover, the representation is unique in $L^2_{\mathcal{Z}}([0,T])$. We can not apply the representation theorem directly because Z is not a $\mathsf{P}^{\bar{\mu}}$-w.p. However, it is w.p. under the Girsanov change of measure [25, p. 85]. Since the two measures are equivalent, the representation also holds for $\mathsf{P}^{\bar{\mu}}$.

2) In [19, Theorem 2], it is proved that along the optimal trajectory

$$\pi_t^{\bar{\mu}}(Y_t) = \bar{\mu}(Y_0) - \int_0^t U_s^{\text{opt}}\,\mathrm{d}Z_s, \quad \text{for } 0 \le t \le T$$

Therefore, $\pi_t^{\bar{\mu}}(Y_t) = \bar{\mu}(Y_0) = 1$.

3) Using the equation of the Wonham filter (2) for π^μ and the BSDE (9) for Y, a direct calculation (included in Appendix H)) shows

$$\mathrm{d}(\pi_t^\mu(Y_t)) = C_t\big(\mathrm{d}Z_t - \pi_t^\mu(h)\,\mathrm{d}t\big), \quad 0 \le t \le T \tag{14}$$

where the coefficient

$$C_t = \pi_t^\mu(hY_t) - \pi_t^\mu(Y_t)\pi_t^\mu(h) + \pi_t^\mu(V_t), \quad 0 \le t \le T$$

4) This part also follows from a direct calculation (included in Appendix H)) to show that

$$\mathrm{d}\big(A_t\pi_t^\mu(Y_t)\big) = A_t(D_t\pi_t^\mu(Y_t) + C_t)(\mathrm{d}Z_t - \pi_t^{\bar{\mu}}(h)\,\mathrm{d}t) \tag{15}$$

where C_t is as defined in the proof of part 3. This completes the proof of the four parts of the Proposition.

We next derive the inequality in (10). Two derivations are provided starting from results in part 3 and part 4:

Derivation of (10) **from part 3 of Prop. 4:** Since $\{\pi_t^\mu(Y_t) : 0 \le t \le T\}$ is P^μ-mg

$$\mathsf{E}^\mu(\pi_T^\mu(\gamma_T) - 1) = \mu((Y_0 - 1)) = \bar{\mu}((\frac{\mu}{\bar{\mu}} - 1)(Y_0 - 1))$$

and therefore using Cauchy-Schwarz

$$|\mathsf{E}^\mu(\pi_T^\mu(\gamma_T) - 1)|^2 \le \text{var}_0^{\bar{\mu}}(\gamma_0)\ \text{var}_0^{\bar{\mu}}(Y_0)$$

Now we use the inequality (6) for the dual process

$$\text{var}_0^{\bar{\mu}}(Y_0) \le e^{-cT}\,\text{var}_T^{\bar{\mu}}(\gamma_T)$$
$$\therefore, \quad |\mathsf{E}^\mu(\pi_T^\mu(\gamma_T) - 1)|^2 \le e^{-cT}\,\text{var}_0^{\bar{\mu}}(\gamma_0)\,\text{var}_T^{\bar{\mu}}(\gamma_T)$$

which is expressed as

$$R_T\ \text{var}_T^{\bar{\mu}}(\gamma_T) \le e^{-cT}\,\text{var}_0^{\bar{\mu}}(\gamma_0)$$

with

$$R_T := \left(\frac{\mathsf{E}^\mu(\pi_T^\mu(\gamma_T) - 1)}{\mathsf{E}^{\bar{\mu}}(\pi_T^\mu(\gamma_T) - 1)}\right)^2$$

Derivation of (10) **from part 4 of Prop. 4:** Since $\{A_t\pi_t^\mu(Y_t) : 0 \le t \le T\}$ is $\mathsf{P}^{\bar{\mu}}$-mg

$$\mathsf{E}^{\bar{\mu}}(A_T(\pi_T^\mu(\gamma_T) - 1)) = \mu((Y_0 - 1)) = \mu((\frac{\mu}{\bar{\mu}} - 1)(Y_0 - 1))$$

and therefore using Cauchy-Schwarz

$$|\mathsf{E}^{\bar{\mu}}(A_T(\pi_T^\mu(\gamma_T) - 1))|^2 \le \text{var}_0^{\bar{\mu}}(\gamma_0)\ \text{var}_0^{\bar{\mu}}(Y_0)$$

Now we use the inequality (6) for the dual process

$$\text{var}_0^{\bar{\mu}}(Y_0) \le e^{-cT}\,\text{var}_T^{\bar{\mu}}(\gamma_T)$$
$$\therefore, \quad |\mathsf{E}^{\bar{\mu}}(A_T(\pi_T^\mu(\gamma_T) - 1))|^2 \le e^{-cT}\,\text{var}_0^{\bar{\mu}}(\gamma_0)\,\text{var}_T^{\bar{\mu}}(\gamma_T)$$

which is expressed as

$$R_T\ \text{var}_T^{\bar{\mu}}(\gamma_T) \le e^{-cT}\,\text{var}_0^{\bar{\mu}}(\gamma_0)$$

with

$$R_T := \left(\frac{\mathsf{E}^{\bar{\mu}}(A_T(\pi_T^\mu(\gamma_T) - 1))}{\mathsf{E}^{\bar{\mu}}(\pi_T^\mu(\gamma_T) - 1)}\right)^2$$

G. Proof of Prop. 5

Consider the random variable $S_T := \pi_T^\mu(\gamma_T) - 1$. The ratio $R_T = \left(\frac{\mathsf{E}^\mu(S_T)}{\mathsf{E}^{\bar{\mu}}(S_T)}\right)^2$. Using the law of total probability

$$\mathsf{E}^\mu(S_T) = \sum_x \mathsf{E}^{\delta_x}(S_T)\mu(x), \quad \mathsf{E}^{\bar{\mu}}(S_T) = \sum_x \mathsf{E}^{\delta_x}(S_T)\bar{\mu}(x)$$

Therefore,

$$\mathsf{E}^\mu(S_T) = \sum_x \mathsf{E}^{\delta_x}(S_T)\frac{\mu(x)}{\bar{\mu}(x)}\bar{\mu}(x) \ge a\mathsf{E}^{\bar{\mu}}(S_T)$$

The bound follows.

H. Additional calculations

Calculation for (14)**:** For this and the next calculation, π^μ is a solution of the Wonham filter (2) and Y is a solution of the BSDE (9). Using the Itô-Wentzell formula for measures [26, Theorem 1.1]:

$$\begin{aligned}
\mathrm{d}\pi_t^\mu(Y_t) &= \pi_t^\mu\big(hY_t + V_t\big)\,\mathrm{d}Z_t - \pi_t^\mu(Y_t)\pi_t^\mu(h)\,\mathrm{d}Z_t \\
&\quad + \pi_t^\mu(Y_t)\pi_t^\mu(h)\pi_t^\mu(h)\,\mathrm{d}t - \pi_t^\mu\big(hY_t + V_t\big)\pi_t^\mu(h)\,\mathrm{d}t \\
&= \big(\pi_t^\mu(hY_t) - \pi_t^\mu(Y_t)\pi_t^\mu(h) + \pi_t^\mu(V_t)\big)\big(\mathrm{d}Z_t - \pi_t^\mu(h)\,\mathrm{d}t\big) \\
&= C_t\big(\mathrm{d}Z_t - \pi_t^\mu(h)\,\mathrm{d}t\big)
\end{aligned}$$

Calculation for (15)**:** The exponential martingale A_t is the Doléans exponential of $\int_0^t D_s(\mathrm{d}Z_s - \pi_s^{\bar{\mu}}(h)\,\mathrm{d}s)$, and therefore its differential form is given by

$$\mathrm{d}A_t = A_tD_t(\mathrm{d}Z_t - \pi_t^{\bar{\mu}}(h)\,\mathrm{d}t)$$

Applying Itô's lemma together with (14) yields

$$\begin{aligned}
\mathrm{d}\big(A_t\pi_t^\mu(Y_t)\big) &= A_tD_t\pi_t^\mu(Y_t)(\mathrm{d}Z_t - \pi_t^{\bar{\mu}}(h)\,\mathrm{d}t) \\
&\quad + A_tC_t\big(\mathrm{d}Z_t - \pi_t^\mu(h)\,\mathrm{d}t\big) + A_tC_tD_t\,\mathrm{d}t \\
&= A_t\big(D_t\pi_t^\mu(Y_t) + C_t\big)(\mathrm{d}Z_t - \pi_t^{\bar{\mu}}(h)\,\mathrm{d}t)
\end{aligned}$$