

A relaxed-inertial forward-backward-forward algorithm for Stochastic Generalized Nash equilibrium seeking

Shisheng Cui¹, Barbara Franci², Sergio Grammatico², Uday V. Shanbhag¹ and Mathias Staudigl³

Abstract—In this paper we propose a new operator splitting algorithm for distributed Nash equilibrium seeking under stochastic uncertainty, featuring relaxation and inertial effects. Our work is inspired by recent deterministic operator splitting methods, designed for solving structured monotone inclusion problems. The algorithm is derived from a forward-backward-forward scheme for solving structured monotone inclusion problems featuring a Lipschitz continuous and monotone game operator. To the best of our knowledge, this is the first distributed (generalized) Nash equilibrium seeking algorithm featuring acceleration techniques in stochastic Nash games without assuming cocoercivity. Numerical examples illustrate the effect of inertia and relaxation on the performance of our proposed algorithm.

I. INTRODUCTION

A stochastic generalized Nash equilibrium problem (SGNEP) describes a subclass of competitive multi-agent optimization problems in which local unilateral minimization of an agent-specific expectation-valued cost function subject to system-wide shared coupling constraints. Due to the presence of the uncertainty and the shared constraints, computing a SGNE is generally rather challenging. However, these problems have recently received the attention of the system and control community, especially because of their applicability [1], [2], [3], [4], [5] to relevant problems in the engineering sciences. An important class of models in this context is that of networked Cournot games with market capacity constraints and uncertainty in demand and capacity [6]. Instances of these models arise in transportation systems, where the drivers' perception of travel-time is a possible source of uncertainty [7], electricity markets where companies dispatch electricity without an a priori knowledge of actual demand [8], and natural gas markets where the companies participate in a bounded capacity market [9].

Typically, two key concerns arise in any attempt to deal with uncertainty. First, often the distribution of the random noise is not known to the agent, thus making the computation of the cost function impossible. Second, even if

the distribution of the stochastic uncertainty is known or predictable from, say, historical data, a key complication arises when trying to compute the expected value (and its gradients). Costly simulation-based integration techniques required employment each time an agent is asked to compute its decision variable, imposing significant computational burden on each agent. A versatile alternative to such approaches is provided by stochastic approximation theory (SA), under which agents' draw fresh samples at each iteration.

With the aim of boosting the performance of distributed Nash equilibrium seeking algorithms, Yi and Pavel [10] introduced a preconditioned forward-backward splitting with inertial effects in a completely deterministic environment where agents receive perfect feedback information. The possibility of including inertia in the basic forward-backward scheme has received some attention in the field already before (see e.g. [11], [12], [13], [14]). The common motivation of all these contributions is to exploit momentum effects to accelerate the numerical schemes, inspired by Nesterov's accelerated gradient method [15] for convex optimization. However, in the context of distributed computation of Nash equilibria, the role of inertial and acceleration effects is not well understood. This applies in particular to situations where the game data are subject to stochastic uncertainty so that agents have only noisy information available in their decision-making process. Even in the most general problem where one's aim is to solve a stochastic monotone inclusion [16], [17], [18], [19], standard acceleration techniques have not received much attention. Our aim is to shed some light on this highly understudied question and prove some interesting properties about accelerated game dynamics. This paper departs from recent progress made in the field of splitting algorithms for stochastic variational problems, summarized in [20], [21] and [19], which contain new asymptotic and non-asymptotic results on stochastic sampling-based algorithms under weaker hypothesis than usually assumed in the computational game theory literature. In these seminal contributions, stochastic versions of Tseng's modified extragradient (hitherto forward-backward-forward) algorithm [22], [23] have been introduced. The importance of this alternative splitting technique in the context of distributed Nash equilibrium seeking has been emphasized in [24]. This work extends all these seminal contributions via an explicit study of the effects of acceleration parameters. The numerical scheme presented in this paper is provably convergent (in an almost sure sense) without assuming cocoercivity of the game operator, and can be implemented via a disciplined mini-batch stochastic approximation tech-

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nology, distributed over a network of competing agents. The main result of this work gives a precise set of parameter sequences ensuring convergence of the game play to the set of *variational equilibria*, an important subset of generalized Nash equilibria with a clear economic interpretation [25]. Our work extends the recent results reported in [26], reliant on forward-backward splitting ideas, and thus require co-coercivity, as well as the stochastic extragradient method introduced in [27], where no joint coupling constraints are considered.

A. Basic Notation

\mathbb{R} denotes the set of real numbers and $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the standard inner product and $\|\cdot\|$ represents the associated Euclidean norm. We indicate a (symmetric and) positive definite matrix A , i.e., $x^\top A x > 0$, with $A \succ 0$. Given a matrix $\Phi \succ 0$, we define the Φ -induced inner product as $\langle x, y \rangle_\Phi = \langle \Phi x, y \rangle$ and the norm as $\|x\|_\Phi = \sqrt{\langle \Phi x, x \rangle}$. $A \otimes B$ indicates the Kronecker product between matrices A and B . $\mathbf{0}_m$ ($\mathbf{1}_m$) indicates the vector with m entries all equal to 0 (1). Given $x_1, \dots, x_N \in \mathbb{R}^n$, $\mathbf{x} := \text{col}(x_1, \dots, x_N) = [x_1^\top, \dots, x_N^\top]^\top$.

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued operator. The domain of T are defined by $\text{dom} T = \{x \in \mathbb{R}^n \mid T(x) \neq \emptyset\}$. The set of zeros of T is $\text{Zer}(T) = \{x \in \mathbb{R}^n \mid 0 \in T(x)\}$. The resolvent of the operator T is $J_T = (\text{Id} + T)^{-1}$, where Id indicates the identity operator. An operator T is monotone if $\langle T(x) - T(y), x - y \rangle \geq 0$ and it is Lipschitz continuous if, for some $\beta > 0$, $\|T(x) - T(y)\| \leq \beta \|x - y\|$ for all $x, y \in \text{dom} T$. A monotone operator is maximally monotone if its graph is not properly contained in the graph of another monotone operator.

Given a proper, lower semi-continuous, and convex function g , the subdifferential is the operator $\partial g(x) := \{u \in \Omega \mid (\forall y \in \Omega) : \langle y - x, u \rangle + g(x) \leq g(y)\}$. The proximal operator is defined as $\text{prox}_g(v) := \text{argmin}_{u \in \Omega} \{g(u) + \frac{1}{2} \|u - v\|^2\} = J_{\partial g}(v)$. ι_C is the indicator function of the set C , i.e., $\iota_C(x) = 1$ if $x \in C$ and $\iota_C(x) = 0$ otherwise. The set-valued mapping $N_C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator for the the set C , i.e., $N_C(x) = \emptyset$ if $x \notin C$, $\{v \in \mathbb{R}^n \mid \sup_{z \in C} v^\top (z - x) \leq 0\}$ otherwise.

All randomness is modeled on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k \geq 0}$.

II. MATHEMATICAL SETUP

A. Generalized Nash equilibrium problems

We consider a game where each agent $i \in \mathcal{I} = \{1, \dots, N\}$ chooses an action $u_i \in \mathbb{R}^{d_i}$. Let $\mathbf{u} = \text{col}(u_1, \dots, u_N)$ and $d \equiv \sum_{i=1}^N d_i$. Each agent i has a local cost function $\mathbb{J}_i : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$ of the form

$$\mathbb{J}_i(u_i, \mathbf{u}_{-i}) = f_i(u_i, \mathbf{u}_{-i}) + g_i(u_i). \quad (1)$$

where $\mathbf{u}_{-i} = \text{col}(\{u_j\}_{j \neq i})$ is the vector of all decision variables except for u_i , and $g_i : \mathbb{R}^{d_i} \rightarrow \bar{\mathbb{R}}$ is a local idiosyncratic cost function. The function \mathbb{J}_i in (1) has the typical splitting into smooth and non-smooth parts.

Standing Assumption 1: For each $i \in \mathcal{I}$, the function g_i in (1) is proper, convex and lower semi-continuous and $\text{dom}(g_i) = U_i \subseteq \mathbb{R}^{d_i}$ is (nonempty) compact and convex. \square

Examples for the nonsmooth part are indicator functions to enforce local constraints, or penalty functions that promote sparsity, or other desirable structure.

We assume that the function $f_i(u_i, \mathbf{u}_{-i})$ depends on the own action u_i and a subset of the others actions $\{u_j\}_{j \in \mathcal{N}_i^A}$, where the set $\mathcal{N}_i^A \subset \mathcal{I}$ is the *interaction neighborhood* of agent i . Furthermore, we assume convexity and differentiability, as usual in the generalized Nash equilibrium problem (GNEP) literature [28], [29], [1].

Standing Assumption 2: For each $i \in \mathcal{I}$ and for all \mathbf{u}_{-i} , the function $f_i(\cdot, \mathbf{u}_{-i})$ in (1) is convex and continuously differentiable. \square

We assume that the game displays joint convexity with affine coupling constraints defining the collective feasible set

$$\mathcal{C} = \{\mathbf{u} \in \mathbf{U} \mid D\mathbf{u} - b \leq \mathbf{0}_m\}, \quad (2)$$

where $\mathbf{U} = U_1 \times \dots \times U_N$, $D = [D_1 \mid \dots \mid D_N] \in \mathbb{R}^{m \times d}$ and $b = \sum_{i=1}^N b_i \in \mathbb{R}^m$. Each matrix $D_i \in \mathbb{R}^{m \times d_i}$ defines how agent i is involved in the coupling constraints. Given the strategies of all other agents \mathbf{u}_{-i} , the set of feasible actions of player i is defined as the following set-valued map.

$$\mathcal{C}_i(\mathbf{u}_{-i}) = \{u_i \in U_i \mid D_i u_i - b_i \leq \sum_{j \neq i}^N (b_j - D_j u_j)\}. \quad (3)$$

Standing Assumption 3: The global feasible set \mathcal{C} in (2) satisfies Slater's constraint qualification. \square

For $i \in \mathcal{I}$, the i th agent solves the following parametrized optimization problem.

$$\forall i \in \mathcal{I} : \begin{cases} \min_{u_i \in \mathbb{R}^{d_i}} & \mathbb{J}_i(u_i, \mathbf{u}_{-i}) \\ \text{s.t.} & u_i \in \mathcal{C}_i(\mathbf{u}_{-i}). \end{cases} \quad (4)$$

The usual solution concept for the game with coupling constraints in (4) is that of *generalized Nash equilibrium* (GNE) [29], [30], i.e., an N -tuple $\mathbf{u}^* = \text{col}(u_1^*, \dots, u_N^*) \in \mathbf{U}$ such that for all $i \in \mathcal{I}$,

$$\mathbb{J}_i(u_i^*, \mathbf{u}_{-i}^*) \leq \inf \{\mathbb{J}_i(u_i, \mathbf{u}_{-i}^*) \mid u_i \in \mathcal{C}_i(\mathbf{u}_{-i}^*)\}.$$

Our computational approach for solving the GNEP in (4) makes use of the Karush-Kuhn-Tucker (KKT) conditions characterizing the unilateral optimization of the agents. To achieve a numerically tractable framework, we impose some conditions on the model concerning the monotonicity and Lipschitz continuity of the mapping that collects the local pseudogradients of the agents.

Standing Assumption 4: The pseudogradient mapping

$$F(\mathbf{u}) = \text{col}(\nabla_{u_1} f_1(\mathbf{u}), \dots, \nabla_{u_N} f_N(\mathbf{u})) \quad (5)$$

is monotone and ℓ -Lipschitz continuous. \square

The KKT conditions corresponding to the game in (4) are necessary and sufficient for characterizing a tuple of strategies to be a GNE. Among all possible GNEs of the game, we focus on the computation of *variational equilibria* (v-GNE), i.e. a GNE in which all agents share consensus on the dual variables [29, Theorem 3.1], [31, Theorem 3.1] which is, in turn, a solution of the variational system

$$\forall i \in \mathcal{I} : \begin{cases} \mathbf{0}_{d_i} \in \nabla_{u_i} f_i(u_i^*, \mathbf{u}_{-i}^*) + \partial g_i(u_i^*) + D_i^\top \lambda^* \\ \mathbf{0}_m \in \mathbb{N}_{\mathbb{R}_{\geq 0}^m}(\lambda^*) - (D\mathbf{u}^* - b). \end{cases} \quad (6)$$

for some $\lambda^* \in \mathbb{R}_{\geq 0}^m$.

B. Distributed GNE via operator splitting

A key challenge one faces in any computational approach in Nash equilibrium problems is to resolve the question how players access the decision variables of the other agents. An attractive approach for resolving this issue is the distributed operator splitting approach pioneered in [10].

We allow each agent to have information on his own local problem data only, i.e., \mathbb{J}_i , U_i , D_i and b_i . Moreover, each agent i controls its local decision u_i and a local copy $\lambda_i \in \mathbb{R}_{\geq 0}^m$ of dual variables, as well as a local auxiliary variable $\mu_i \in \mathbb{R}^m$ used to enforce consensus of the dual variables. To reach such consensus, we let the agents exchange information via an undirected weighted communication graph represented by its weighted adjacency matrix $\mathbf{W} = [w_{i,j}] \in \mathbb{R}^{N \times N}$. We assume $w_{ij} > 0$ iff (i, j) is an edge in the communication graph. The set of neighbors of agent i in the communication graph is $\mathcal{N}_i^\lambda = \{j \mid w_{i,j} > 0\}$.

Standing Assumption 5: The adjacency matrix \mathbf{W} of the communication graph is symmetric and irreducible. \square

Let us define the weighted Laplacian as $\mathbf{L} = \text{diag}\{(\mathbf{W}\mathbf{1}_N)_1, \dots, (\mathbf{W}\mathbf{1}_N)_N\} - \mathbf{W}$. It holds that $\mathbf{L}^\top = \mathbf{L}$, $\text{null}(\mathbf{L}) = \{a\mathbf{1}_N, a \in \mathbb{R}\}$ and that, given Standing Assumption 5, \mathbf{L} is positive semi-definite with real and distinct eigenvalues $0 = s_1 < s_2 \leq \dots \leq s_N$. Moreover, given the maximum (weighted) degree of the graph, $\Delta := \max_{i \in \mathcal{I}} (\mathbf{W}\mathbf{1}_N)_i$, it holds that $\Delta \leq s_N \leq 2\Delta$. Denoting by $\kappa = |\mathbf{L}|$, it holds that $\kappa \leq 2\Delta$ [32]. We define the tensorized Laplacian as the matrix $\bar{\mathbf{L}} = \mathbf{L} \otimes \mathbf{I}_m$. We set $\bar{b} = (b_1, \dots, b_N)^\top$, $\mathbf{u} = \text{col}(u_1, \dots, u_N)$ and similarly $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$. As the state variable, we consider the triple $\mathbf{x} = (\mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^{mN} \times \mathbb{R}^{mN}$ and endow \mathcal{X} with the product topology. Let $\mathbf{D} = \text{diag}\{D_1, \dots, D_N\}$. Then, we define the maximally monotone operators

$$V(\mathbf{x}) = \begin{bmatrix} F(\mathbf{u}) + \mathbf{D}^\top \boldsymbol{\lambda} \\ \bar{\mathbf{L}} \boldsymbol{\lambda} \\ \bar{b} + \bar{\mathbf{L}}(\boldsymbol{\lambda} - \boldsymbol{\mu}) - \bar{\mathbf{D}} \mathbf{u} \end{bmatrix}, \quad (7)$$

$$T(\mathbf{x}) = G(\mathbf{u}) \times \{\mathbf{0}_{Nm}\} \times \mathbb{N}_{\mathbb{R}_{\geq 0}^{mN}}(\boldsymbol{\lambda}), \quad (8)$$

where $G(\mathbf{u}) = \partial g_1(u_1) \times \dots \times \partial g_N(u_N)$. Let us summarize the properties of the operators above.

Lemma 1: The following statements hold:

- (i) $V : \mathcal{X} \rightarrow \mathcal{X}$ is maximally monotone and $\ell_V = (\ell + 2\kappa + |\mathbf{D}|)$ -Lipschitz continuous.
- (ii) $T : \mathcal{X} \rightrightarrows \mathcal{X}$ is maximally monotone.

Proof: (i) We can split the operator V into the parts $V_1(\mathbf{x}) = \text{col}(F(\mathbf{u}), \mathbf{0}_{mN}, \bar{b} + \bar{\mathbf{L}}\boldsymbol{\lambda})$ and $V_2(\mathbf{x}) = \text{col}(\mathbf{D}^\top \boldsymbol{\lambda}, \bar{\mathbf{L}}\boldsymbol{\lambda}, -\bar{\mathbf{D}}\mathbf{u} - \bar{\mathbf{L}}\boldsymbol{\mu})$, which are maximally monotone by [23, Prop. 20.23], [23, Cor. 20.28]. Furthermore, similarly to [24, Lemma 1] V_1 is $\ell_1 = (\ell + \kappa)$ -Lipschitz continuous and V_2 is $\ell_2 = (|\mathbf{D}| + \kappa)$ -Lipschitz continuous. Hence, V is $\ell_1 + \ell_2 = \ell_V$ -Lipschitz continuous.

- (ii) It follows from [10, Lemma 5], [24, Lemma 1]. \blacksquare

The splitting $V+T$ encodes a distributed version of the KKT conditions for v-GNE (4). In particular, it can be shown that the zeros of the maximally monotone inclusion $V+T$ are in correspondence with variational equilibria of the Nash game.

Proposition 1: The set $\text{zer}(V+T)$ coincides with the set of v-GNE of the game satisfying the KKT conditions in (6).

Proof: This follows from [10, Thm. 2] or [24, Lemma 3]. \blacksquare

C. Stochastic GNEPs

Stochastic uncertainty affecting the decision problem of agent i is modeled by a random variable $\xi_i : \Omega \rightarrow \Xi_i$, where $\Xi_i \subset \mathbb{R}^{q_i}$ is a given measurable set. We assume that the uncertainty enters the model in the smooth part of the agents' optimization problem, i.e., for each $i \in \mathcal{I}$

$$\mathbb{J}_i(u_i, \mathbf{u}_{-i}) = \mathbb{E}[\hat{f}_i(\mathbf{u}, \xi_i)] + g_i(u_i). \quad (9)$$

It follows that the local optimization problems in (4) describes a stochastic programming problem, parameterized by the decisions of the opponents \mathbf{u}_{-i} .

Let $k \in \mathbb{N}$ denote the iteration count of our computational procedure. We assume that at each round k agent i is able to generate a random sample $\xi_{i,k} = (\xi_{i,k}^{(t)})_{t=1}^{S_k}$, consisting of i.i.d copies of the random element ξ_i . This sample is used to construct an agent-specific gradient estimator of the form

$$\hat{F}_{i,k}(\mathbf{u}, \xi_{i,k}) = \frac{1}{S_k} \sum_{t=1}^{S_k} \nabla_{u_i} \hat{f}_i(\mathbf{u}, \xi_{i,k}^{(t)}), \quad (10)$$

where $S_k \geq 1$ is the size of the data sample. (10) is an example of a mini-batch estimator, which interpolates between cheap sampling and precision. The degree of precision is regulated via the batch size sequence $\{S_k\}_k$. Dynamically adjusting the size of the batch simulates an online variance reduction mechanism, which plays a key role in our convergence analysis of the distributed operator splitting algorithm to come. Mini-batch samples are prominent in simulation-based optimization, where taking repeated samples of stochastic gradients is computationally cheap [33], [27], [21], [19], [34].

Standing Assumption 6: The batch size $(S_k)_{k \geq 1}$ is increasing and such that $\sum_{k \in \mathbb{N}} \frac{1}{S_k} < \infty$. \square

Under the prevailing i.i.d. assumption, it holds true that $\mathbb{E}[\hat{F}_{i,k}(\mathbf{u}, \xi_{i,k}) | \mathbf{u}] = F_i(\mathbf{u})$ for all $i \in \mathcal{I}$ and all $\mathbf{u} \in \mathbb{R}^d$. Hence, the random variable (10) is an *unbiased* estimator of the individual payoff gradient at each action profile \mathbf{u} . Upon defining the random operator

$$\hat{V}_k(\mathbf{x}, \xi_k) = \begin{bmatrix} \hat{F}_k(\mathbf{u}, \xi_k) + \mathbf{D}^\top \boldsymbol{\lambda} \\ \bar{\mathbf{L}} \boldsymbol{\lambda} \\ \bar{b} + \bar{\mathbf{L}}(\boldsymbol{\lambda} - \boldsymbol{\mu}) - \bar{\mathbf{D}} \mathbf{u} \end{bmatrix}, \quad (11)$$

with $\xi_k = \text{col}(\xi_{i,k})_{i \in \mathcal{I}}$, we see that $\mathbb{E}[\hat{V}_k(\mathbf{x}, \xi_k) | \mathbf{x}] = V(\mathbf{x})$ for all $\mathbf{x} = (\mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{X}$.

Fundamental to the analysis of stochastic approximation algorithms is the control of the stochastic error, defined for all $k \in \mathbb{N}$ as

$$\varepsilon_k(\mathbf{x}, \xi_k) = \hat{V}_k(\mathbf{x}, \xi_k) - V(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{X}. \quad (12)$$

Algorithm 1 Distributed Relaxed Inertial Stochastic Forward Backward Forward (RISFBF)

Initialization: $u_{i,0} \in \mathbb{R}^{d_i}$, $\lambda_{i,0} \in \mathbb{R}_{\geq 0}^m$, and $\mu_{i,0} \in \mathbb{R}^m$.

Iteration k : Agent i

(1) Perform inertia step:

$$\begin{aligned} u_{i,k}^{in} &= u_{i,k} + \alpha(u_{i,k} - u_{i,k-1}) \\ \mu_{i,k}^{in} &= \mu_{i,k} + \alpha(\mu_{i,k} - \mu_{i,k-1}) \\ \lambda_{i,k}^{in} &= \lambda_{i,k} + \alpha(\lambda_{i,k} - \lambda_{i,k-1}). \end{aligned}$$

(2) Receives $u_{j,k}^{in}$ for $j \in \mathcal{N}_i^A$, $\lambda_{j,k}^{in}$ and $\mu_{j,k}^{in}$ for $j \in \mathcal{N}_i^\lambda$ and update

$$\begin{aligned} u_{i,k}^{md} &= \text{prox}_{\gamma_i g_i} [u_{i,k}^{in} - \gamma_i (\hat{F}_{i,k}(\mathbf{u}_k^{in}, \xi_k) + D_i^\top \lambda_{i,k})] \\ \mu_{i,k}^{md} &= \mu_{i,k}^{in} + \sigma_i \sum_j w_{i,j} (\lambda_{j,k}^{in} - \lambda_{i,k}^{md}) \\ \lambda_{i,k}^{md} &= \Pi_{\mathbb{R}_{\geq 0}^m} \{ \lambda_{i,k}^{in} + \tau_i (D_i u_{i,k}^{in} - b_i) \\ &\quad + \tau \sum_j w_{i,j} [(\mu_{i,k}^{in} - \mu_{j,k}^{in}) - (\lambda_{i,k}^{in} - \lambda_{j,k}^{in})] \} \end{aligned}$$

(3) Receives $u_{j,k}^{md}$ for $j \in \mathcal{N}_i^A$, $\lambda_{j,k}^{md}$ and $\mu_{j,k}^{md}$ for $j \in \mathcal{N}_i^\lambda$ and performs a relaxation step:

$$\begin{aligned} u_{i,k+1} &= (1 - \rho_k) u_{i,k}^{in} + \rho_k [u_{i,k}^{md} + \gamma_i (\hat{F}_{i,k}(\mathbf{u}_k^{in}, \xi_{i,k}) + \\ &\quad - \hat{F}_{i,k}(\mathbf{u}_k^{md}, \eta_{i,k})) + \gamma_i D_i^\top (\lambda_{i,k}^{in} - \lambda_{i,k}^{md})] \\ \mu_{i,k+1} &= (1 - \rho_k) \mu_{i,k}^{in} + \rho_k \{ \mu_{i,k}^{md} + \\ &\quad \sigma_i \sum_j w_{i,j} [(\lambda_{i,k}^{in} - \lambda_{j,k}^{in}) - (\lambda_{i,k}^{md} - \lambda_{j,k}^{md})] \} \\ \lambda_{i,k+1} &= (1 - \rho_k) \lambda_{i,k}^{in} + \rho_k \{ [\lambda_{i,k}^{md} + \tau_i D_i (u_{i,k}^{in} - u_{i,k}^{md}) \\ &\quad - \tau_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} [(\mu_{i,k}^{md} - \mu_{j,k}^{md}) - (\mu_{i,k}^{in} - \mu_{j,k}^{in})] \\ &\quad + \tau_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} [(\lambda_{i,k}^{in} - \lambda_{j,k}^{in}) - (\lambda_{i,k}^{md} - \lambda_{j,k}^{md})] \} \end{aligned}$$

Standing Assumption 7: There exists $\sigma > 0$ such that for all $k \in \mathbb{N}$, the stochastic error is such that the following hold \mathbb{P} -a.s. .

$$\mathbb{E}_{\mathbb{P}}[\varepsilon_k(\mathbf{x}, \xi_k) \mid \mathbf{x}] = 0, \text{ and} \quad (13)$$

$$\mathbb{E}_{\mathbb{P}}[\|\varepsilon_k(\mathbf{x}, \xi_k)\|^2 \mid \mathbf{x}] \leq \frac{\sigma^2}{S_k}. \quad (14)$$

□

Remark 1: Assumption 7 is rather mild and standard in stochastic optimization [35], [26] while Condition (13) means that the random operator $\hat{V}_k(\mathbf{x}, \xi_k)$ is a conditionally unbiased estimator of $V(\mathbf{x})$. Note that (14) can be satisfied if the sequence of martingale difference errors $\hat{F}_k(\mathbf{u}, \xi) - F(\mathbf{u})$ satisfies a uniform variance bound [35], [26].

III. A DISTRIBUTED ALGORITHM

With the intent of boosting the convergence of distributed Nash seeking algorithms, we propose a relaxed inertial forward-backward-forward algorithm (RISFBF), presented in Algorithm 1. Using operator-theoretic notation, the numeri-

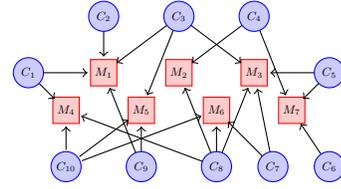


Fig. 1: Networked Cournot game: an edge from C_i to M_j means that company i sells energy in market j .

cal scheme can be restated compactly as

$$\begin{cases} Z_k = X_k + \alpha_k (X_k - X_{k-1}), \\ Y_k = \mathbf{J}_{\Psi^{-1}T} (Z_k - \Psi^{-1} \hat{V}_k(Z_k, \xi_k)), \\ X_{k+1} = (1 - \rho_k) Z_k + \\ \quad \rho_k [Y_k - \Psi^{-1} (\hat{V}_k(Y_k, \eta_k) - \hat{V}_k(Z_k, \xi_k))], \end{cases} \quad (15)$$

where $Z_k = (\mathbf{u}_k^{in}, \boldsymbol{\mu}_k^{in}, \boldsymbol{\lambda}_k^{in})$, $X_k = (\mathbf{u}_k, \boldsymbol{\mu}_k, \boldsymbol{\lambda}_k)$ and $Y_k = (\mathbf{u}_k^{md}, \boldsymbol{\mu}_k^{md}, \boldsymbol{\lambda}_k^{md})$. The random sequence $\eta_k = (\eta_{i,k})_{i \in \mathcal{I}}$ is another i.i.d. random sample, generated independently by each agent after the first updating step in Algorithm 1 is completed. The preconditioning matrix

$$\Psi = \text{diag}(\boldsymbol{\gamma}^{-1}, \boldsymbol{\sigma}^{-1}, \boldsymbol{\tau}^{-1}) \quad (16)$$

collects all agent-specific step sizes, so that $\boldsymbol{\gamma} = \text{diag}\{\gamma_1 \mathbf{I}_{d_1}, \dots, \gamma_N \mathbf{I}_{d_N}\}$ is a block-diagonal matrix with $\gamma_i > 0$ (analogously, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$).

The iterations involve first an inertial step in the primal-dual space. Then, there is a proximal step corresponding to a gradient-based update given the stochastic estimate of the pseudogradient and the local estimate of the dual variable, followed by a consensus-enforcing estimate merging the values of the dual variables of neighboring agents, and a dual update in the spirit of Lagrangian methods. The last step is a weighted average between the inertial iterate Z_k and the forward update Y_k . The algorithm is distributed and involves communication only in terms of dual variables. This fact makes the scheme very attractive for decentralized implementations in large networked game-theoretic settings.

Standing Assumption 4 and Lemma 1 imply that V is monotone and $\ell_{V,\Psi} = \ell_V / \lambda_{\min}(\Psi)$ -Lipschitz continuous in the Ψ -induced norm [24].

Theorem 1: Suppose ν is a positive scalar where $0 < \nu < 1$. Let $\lambda_{\min}(\Psi) \in (0, \frac{1-\nu}{2\ell_V})$, $0 < \alpha_k \leq \bar{\alpha} < 1$ and $\rho_k = \frac{(3-\nu)(1-\bar{\alpha})^2}{2(2\alpha_k^2 - \alpha_k + 1)(1 + \ell_{V,\Psi})}$. Then, the sequence $(\mathbf{u}_k)_{k \geq 1}$ generated by Algorithm 1 converges almost surely to a ν -GNE of the game in (4).

Proof: See Section V-B. ■

Remark 2: The classic SFBF algorithm [21] is obtained as a special case by taking $\alpha = 0$ and $\rho_k = 1$. Under noise-free feedback, this scheme would coincide with the operator-splitting approach of [24].

IV. NUMERICAL RESULTS

In this section, we report the results of some numerical simulations to illustrate the improved performance of the

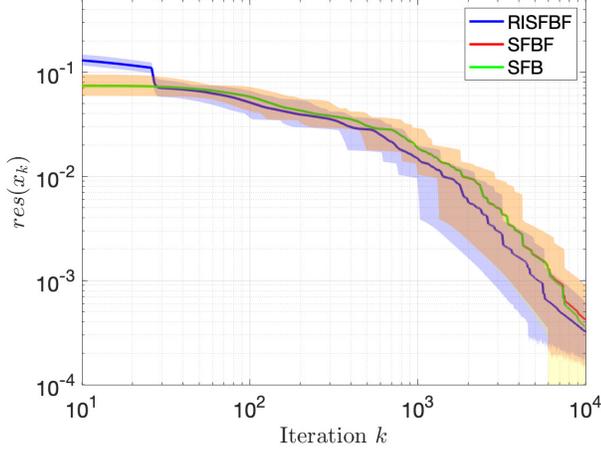


Fig. 2: Residual distance of the primal variable form the solution.

RISFBF algorithm (Algorithm 1) compared to the classic SFBF [35], [24] and with the preconditioned SFB [26], [10].

Let us consider a networked Cournot problem with market capacity constraints as, for instance, the electricity market or the gas market, inspired by [36]. We suppose that there are $N = 10$ firms selling energy in $m = 7$ markets. Not every company sells quantities on each market. Instead, we let $\mathcal{M}_i \subseteq \{1, \dots, m\}$ denote the subset of markets firm i is active on. Each company has a cost of production $c_i(u_i) = c_i^\top u_i$ where $c_i \in \mathbb{R}^{d_i}$ is chosen according to a truncated normal distribution, i.e., $[c_i]_j = \max(N(2, 1), 0.6)$. Moreover, each market j has an inverse demand function $P_j(\mathbf{u}, \xi) = q_j + p_j(\xi)[S_j(\mathbf{u})]^\sigma$ where $q_j = 400$ and $p_j(\xi)$ depends on the unknown random variable, e.g, the overall demand. The values of $p_j(\xi)$ are randomly generated with a normal distribution with mean 0.02 and bounded variance. The variable $S_j(\mathbf{u}) = \sum_{i \in \mathcal{I}} [u_i]_j$ couples the actions of the companies and it represents the total energy sold in market j . Hence, the cost function of each company is $\mathbb{J}_i(u_i, \mathbf{u}_{-i}) = c_i(u_i) - \sum_{j \in \mathcal{M}_i} \mathbb{E}[P_j(\mathbf{u}, \xi)[u_i]_j]$. The corresponding pseudogradient mapping is monotone, according to [36, Section 4], for $1 < \sigma \leq 3$. Therefore, we fix $\sigma = 1.2$. Moreover, we suppose that the companies have a limited production, i.e., $0 \leq [u_i]_j \leq \theta_{i,j}$ with $\theta_{i,j} = \max(N(250, 50), 0)$ for $j \in \mathcal{M}_i$. This can be incorporated by setting $g_i(u_i) = \sum_{j \in \mathcal{N}_i^A} \iota_{[0, \theta_{i,j}]}([u_i]_j)$. Similarly, the markets have a bounded capacity $b_j \in [5, 10]$, $j = 1, \dots, m$, and the coupling between the companies can be retrieved from Figure 1.

The plot in Figure 2 shows the performance, in terms of the residual, of our proposed algorithm in comparison with the SFBF and SFB algorithms. The residual mapping is defined as $\text{res}(x^k) = \|x^k - \text{proj}_{\mathcal{C}}(x^k - F(x^k))\|$ and it measures the distance of the primal variable from being a Nash equilibrium. The thick line indicates the average performance and the transparent area is the variability over 10 simulations. The acceleration parameter is updated according to $\alpha_k = \bar{\alpha}(1 - \frac{1}{k+1})$ with $\bar{\alpha} = 0.1$ and the relaxation

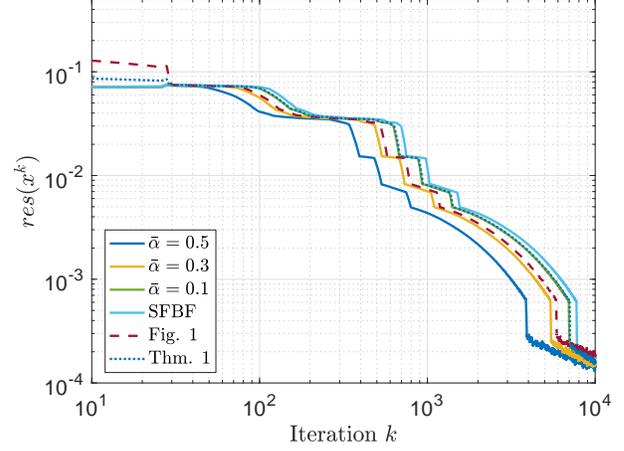


Fig. 3: Residual distance of the primal variable form the solution for the RISFBF algorithm varying the inertial parameter.

parameter is $\rho_k = \frac{(3-\nu)(1-\bar{\alpha})^2}{2(2\bar{\alpha}_k^2 - \alpha_k + 1)(1+\ell_{V,\Psi})}$, with $\nu = 0.01$. Figure 3 shows how the performance changes varying the inertial parameter $\bar{\alpha}$ while $\rho_k = 1$ is fixed. For the sake of comparison, we also include the performance with the same parameters as in Figure 2, the updating rule for ρ_k as in Theorem 1 and the SFBF ($\alpha_k = 0$, $\rho_k = 0$).

V. ANALYSIS

A. Preparatory facts

To simplify the analysis, let us define the random processes $\mathcal{A}_k := \hat{V}_k(Z_k, \xi_k)$ and $\mathcal{B}_k := \hat{V}_k(Y_k, \eta_k)$. Define the sub-sigma algebra $\mathcal{F}_k := \sigma(\mathbf{x}_0, \xi_0, \dots, \xi_{k-1}, \eta_0, \dots, \eta_{k-1})$, and $\mathcal{G}_k := \sigma(\mathcal{F}_k \cup \sigma(\xi_k))$. We introduce the centered error processes $U_k := \mathcal{A}_k - \mathbb{E}[\mathcal{A}_k | \mathcal{F}_k]$ and $W_k := \mathcal{B}_k - \mathbb{E}[\mathcal{B}_k | \mathcal{G}_k]$. Note that Standing Assumption 7 implies that $\mathbb{E}[U_k | \mathcal{F}_k] = \mathbb{E}[W_k | \mathcal{F}_k] = 0$ and that $(\mathbb{E}[\|U_k\|_{\Psi^{-1}}^2 | \mathcal{F}_k])_{k \geq 1}$ and $(\mathbb{E}[\|W_k\|_{\Psi^{-1}}^2 | \mathcal{F}_k])_{k \geq 1}$ are summable sequences.

Define the residual function for the monotone inclusion as $r_\Psi(x) = \|x - \text{J}_{\Psi^{-1}T}(x - \Psi^{-1}V(x))\|$. For every $\Psi > 0$, $x \in \text{zer}(V + T) \Leftrightarrow r_\Psi(x) = 0$.

Lemma 2: For $x, y \in \mathbb{X}$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, it holds that $\|\alpha x + \beta y\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 - \alpha\beta \|x - y\|^2$.

Lemma 3 (Robbins-Siegmund): [37, Lemma 11, page 50]. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_k)_{k \geq 0}, \mathbb{P})$ be a discrete stochastic basis. Let $(\alpha_k)_{k \in \mathbb{N}}$, $(\theta_k)_{k \in \mathbb{N}}$, $(\eta_k)_{k \in \mathbb{N}}$ and $(\chi_k)_{k \in \mathbb{N}}$ be non-negative processes such that $\sum_k \eta_k < \infty$, $\sum_k \chi_k < \infty$ and let

$$\forall k \in \mathbb{N}, \quad \mathbb{E}[\alpha_{k+1} | \mathcal{F}_k] + \theta_k \leq (1 + \chi_k)\alpha_k + \eta_k \quad a.s.$$

Then $\sum_k \theta_k < \infty$ and $(\alpha_k)_{k \in \mathbb{N}}$ converges a.s. to a non-negative random variable.

Lemma 4: For all $k \geq 1$ we have

$$-\|Z_k - Y_k\|_{\Psi}^2 \leq \|U_k\|_{\Psi^{-1}}^2 - \frac{1}{2}r_{\Psi}^2(Z_k). \quad (17)$$

Proof: By definition

$$\begin{aligned} \frac{1}{2}r_{\Psi}^2(Z_k) &= \frac{1}{2}\|Z_k - \text{J}_{\Psi^{-1}T}(Z_k - \Psi^{-1}V(Z_k))\|^2 \\ &\leq \|Z_k - Y_k\|_{\Psi}^2 \end{aligned}$$

$$\begin{aligned}
& + \|\mathbb{J}_{\Psi^{-1}T}(Z_k - \Psi^{-1}\mathcal{A}_k) - \mathbb{J}_{\Psi^{-1}T}(Z_k - \Psi^{-1}V(Z_k))\|_{\Psi}^2 \\
& \leq \|Z_k - Y_k\|_{\Psi}^2 + \|U_k\|_{\Psi^{-1}}^2
\end{aligned}$$

where the last inequality uses the non-expansiveness of the resolvent operator $\mathbb{J}_{\Psi^{-1}T}$ under the norm $\|\cdot\|_{\Psi}$. ■

B. Convergence analysis of RISFBF algorithm

Define the stochastic processes

$$\Delta M_k := \frac{(3-\nu)\rho_k}{1+\ell_{V,\Psi}} \|\mathbf{e}_k\|^2 + \nu\rho_k \|U_k\|_{\Psi^{-1}}^2, \quad (18)$$

$$\Delta N_k(p) := 2\rho_k \langle W_k, p - Y_k \rangle, \quad (19)$$

with $\mathbf{e}_k := W_k - U_k$. We start proving the following fundamental inequality.

Lemma 5 (Fundamental Recursion): Fix $p \in \text{zer}(V + T)$ arbitrary, and set $R_k = Y_k + \Psi^{-1}(\mathcal{A}_k - \mathcal{B}_k)$. For all $k \geq 0$, it holds true that

$$\begin{aligned}
& \|X_{k+1} - p\|_{\Psi}^2 \leq (1 + \alpha_k) \|X_k - p\|_{\Psi}^2 - \alpha_k \|X_{k-1} - p\|_{\Psi}^2 \\
& + \Delta M_k + \Delta N_k(p) - \frac{\nu\rho_k}{2} r_{\Psi}^2(Z_k) \\
& + \alpha_k \|X_k - X_{k-1}\|_{\Psi}^2 \left(2\alpha_k + \frac{3-\nu(1-\alpha_k)}{2\rho_k(1+\ell_{V,\Psi})} \right) \\
& - (1 - \alpha_k) \left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1 \right) \|X_{k+1} - X_k\|_{\Psi}^2.
\end{aligned}$$

Proof: Start by observing

$$\begin{aligned}
\|Z_k - p\|_{\Psi}^2 & = \|Z_k - Y_k + Y_k - R_k + R_k - p\|_{\Psi}^2 \\
& = \|Z_k - Y_k\|_{\Psi}^2 - \|Y_k - R_k\|_{\Psi}^2 + \|R_k - p\|_{\Psi}^2 \\
& + 2\langle Z_k - R_k, Y_k - p \rangle_{\Psi}.
\end{aligned}$$

Since

$$\begin{aligned}
\|Y_k - R_k\|_{\Psi}^2 & = \|\Psi^{-1}(\mathcal{A}_k - \mathcal{B}_k)\|_{\Psi}^2 \\
& = \|V(Y_k) - V(Z_k) + W_k - U_k\|_{\Psi^{-1}}^2 \\
& \leq 2\ell_{V,\Psi}^2 \|Y_k - Z_k\|_{\Psi}^2 + 2\|W_k - U_k\|_{\Psi^{-1}}^2 \\
& = 2\ell_{V,\Psi}^2 \|Y_k - Z_k\|_{\Psi}^2 + 2\|\mathbf{e}_k\|_{\Psi^{-1}}^2. \quad (20)
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Z_k - p\|_{\Psi}^2 & \stackrel{(20)}{\geq} (1 - 2\ell_{V,\Psi}^2) \|Z_k - Y_k\|_{\Psi}^2 - 2\|\mathbf{e}_k\|_{\Psi^{-1}}^2 \\
& + \|R_k - p\|_{\Psi}^2 + 2\langle Z_k - R_k, Y_k - p \rangle_{\Psi}.
\end{aligned}$$

Using the definition $Y_k \triangleq \mathbb{J}_{\Psi^{-1}T}(Z_k - \Psi^{-1}\mathcal{A}_k)$, we get

$$\begin{aligned}
Y_k + \Psi^{-1}T(Y_k) & \ni (Z_k - \Psi^{-1}\mathcal{A}_k) \\
\text{or } T(Y_k) & \ni \Psi(Z_k - Y_k - \Psi^{-1}\mathcal{A}_k)
\end{aligned}$$

Since $p \in \text{zer}(T + V)$, $(p, \mathbf{0}) \in \text{gr}(T + V)$, implying that $\mathbf{0} - V(p) \in T(p)$. Consequently, by monotonicity of T , we have that

$$\begin{aligned}
& \langle \Psi(Z_k - Y_k - \Psi^{-1}\mathcal{A}_k) + V(p), Y_k - p \rangle \geq 0 \\
& \text{or } \langle Z_k - Y_k - \Psi^{-1}(\mathcal{A}_k - \mathcal{B}_k), Y_k - p \rangle_{\Psi} \\
& \geq \langle V(Y_k) - V(p), Y_k - p \rangle + \langle \mathcal{B}_k - V(Y_k), Y_k - p \rangle.
\end{aligned}$$

By definition, $R_k = Y_k + \Psi^{-1}(\mathcal{A}_k - \mathcal{B}_k)$, $W_k = \mathcal{B}_k - V(Y_k)$, and $V(Y_k) = \mathbb{E}[\mathcal{B}_k | \mathcal{G}_k]$, we have that

$$\begin{aligned}
\langle Z_k - R_k, Y_k - p \rangle_{\Psi} & \geq \langle V(Y_k) - V(p), Y_k - p \rangle \\
& + \langle W_k, Y_k - p \rangle
\end{aligned}$$

Since V is a monotone operator, this implies $\langle Z_k - R_k, Y_k - p \rangle_{\Psi} \geq \langle W_k, Y_k - p \rangle$. Whence,

$$\begin{aligned}
\|Z_k - p\|_{\Psi}^2 & \geq (1 - 2\ell_{V,\Psi}^2) \|Y_k - Z_k\|_{\Psi}^2 + \|R_k - p\|_{\Psi}^2 \\
& - 2\|\mathbf{e}_k\|_{\Psi^{-1}}^2 + 2\langle W_k, Y_k - p \rangle.
\end{aligned}$$

Rearranging, we arrive at

$$\begin{aligned}
\|R_k - p\|_{\Psi}^2 & \leq \|Z_k - p\|_{\Psi}^2 + 2\|\mathbf{e}_k\|_{\Psi^{-1}}^2 \\
& - (1 - 2\ell_{V,\Psi}^2) \|Y_k - Z_k\|_{\Psi}^2 + 2\langle W_k, p - Y_k \rangle. \quad (21)
\end{aligned}$$

Next, we use Lemma 2 to arrive at

$$\begin{aligned}
\|X_{k+1} - p\|_{\Psi}^2 & = \|(1 - \rho_k)Z_k + \rho_k R_k - p\|_{\Psi}^2 \\
& = (1 - \rho_k) \|Z_k - p\|_{\Psi}^2 + \rho_k \|R_k - p\|_{\Psi}^2 \\
& - \rho_k(1 - \rho_k) \|R_k - Z_k\|_{\Psi}^2 \\
& = (1 - \rho_k) \|Z_k - p\|_{\Psi}^2 + \rho_k \|R_k - p\|_{\Psi}^2 - \frac{1-\rho_k}{\rho_k} \|X_{k+1} - Z_k\|_{\Psi}^2 \\
& \leq \|Z_k - p\|_{\Psi}^2 - \frac{1-\rho_k}{\rho_k} \|X_{k+1} - Z_k\|_{\Psi}^2 + 2\lambda^2 \rho_k \|\mathbf{e}_k\|_{\Psi^{-1}}^2 \\
& - \rho_k(1 - 2\ell_{V,\Psi}^2) \|Z_k - Y_k\|_{\Psi}^2 - 2\rho_k \langle W_k, Y_k - p \rangle \\
& = \|Z_k - p\|_{\Psi}^2 - \frac{1-\rho_k}{\rho_k} \|X_{k+1} - Z_k\|_{\Psi}^2 - 2\rho_k \langle W_k, Y_k - p \rangle \\
& - \rho_k((1 - \nu) - 2\ell_{V,\Psi}^2) \|Z_k - Y_k\|_{\Psi}^2 - \nu\rho_k \|Y_k - Z_k\|_{\Psi}^2 \\
& + 2\rho_k \|\mathbf{e}_k\|_{\Psi^{-1}}^2.
\end{aligned}$$

Using (17), this implies

$$\begin{aligned}
\|X_{k+1} - p\|_{\Psi}^2 & \leq \|Z_k - p\|_{\Psi}^2 - \frac{1-\rho_k}{\rho_k} \|X_{k+1} - Z_k\|_{\Psi}^2 \\
& - \rho_k((1 - \nu) - 2\ell_{V,\Psi}^2) \|Z_k - Y_k\|_{\Psi}^2 - \frac{\nu\rho_k}{2} r_{\Psi}^2(Z_k) \\
& - 2\rho_k \langle W_k, Y_k - p \rangle + 2\rho_k \|\mathbf{e}_k\|_{\Psi^{-1}}^2 + \nu\rho_k \|U_k\|_{\Psi^{-1}}^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{1}{\rho_k} \|X_{k+1} - Z_k\|_{\Psi} & = \|R_k - Z_k\|_{\Psi} \\
& \leq \|\mathcal{B}_k - \mathcal{A}_k\|_{\Psi^{-1}} + \|Y_k - Z_k\|_{\Psi} \\
& \leq (1 + \ell_{V,\Psi}) \|Y_k - Z_k\|_{\Psi} + \|\mathbf{e}_k\|_{\Psi^{-1}},
\end{aligned}$$

which implies

$$\frac{1}{2\rho_k^2} \|X_{k+1} - Z_k\|_{\Psi}^2 \leq (1 + \ell_{V,\Psi})^2 \|Y_k - Z_k\|_{\Psi}^2 + \|\mathbf{e}_k\|_{\Psi^{-1}}^2.$$

Multiplying both sides by $\frac{\rho_k(1-\nu-2\ell_{V,\Psi})}{1+\ell_{V,\Psi}}$, we obtain

$$\begin{aligned}
& \frac{1-\nu-2\ell_{V,\Psi}}{2\rho_k} (1 + \ell_{V,\Psi}) \|X_{k+1} - Z_k\|_{\Psi}^2 \\
& \leq \rho_k(1 - \nu - 2\ell_{V,\Psi})(1 + \ell_{V,\Psi}) \|Y_k - Z_k\|_{\Psi}^2 \\
& + \frac{\rho_k(1-\nu-2\ell_{V,\Psi})}{1+\ell_{V,\Psi}} \|\mathbf{e}_k\|_{\Psi^{-1}}^2.
\end{aligned}$$

Rearranging terms, and noting that $(1 - \nu - 2\ell_{V,\Psi})(1 + \ell_{V,\Psi}) \leq 1 - \nu - 2\ell_{V,\Psi}^2$, the above estimate becomes

$$\begin{aligned}
& -\rho_k(1 - \nu - 2\ell_{V,\Psi}^2) \|Y_k - Z_k\|_{\Psi}^2 \\
& \leq -\frac{1-\nu-2\ell_{V,\Psi}}{2\rho_k} (1 + \ell_{V,\Psi}) \|X_{k+1} - Z_k\|_{\Psi}^2 + \frac{\rho_k(1-\nu-2\ell_{V,\Psi})}{1+\ell_{V,\Psi}} \|\mathbf{e}_k\|_{\Psi^{-1}}^2.
\end{aligned}$$

Substituting this bound into the first majorization of the anchor process $\|X_{k+1} - p\|_{\Psi}^2$, we see

$$\begin{aligned}
\|X_{k+1} - p\|_{\Psi}^2 & \leq \|Z_k - p\|_{\Psi}^2 + \nu\rho_k \|U_k\|_{\Psi^{-1}}^2 - \frac{\nu\rho_k}{2} r_{\Psi}^2(Z_k) \\
& - \left(\frac{1-\rho_k}{\rho_k} + \frac{1-\nu-2\ell_{V,\Psi}}{2\rho_k(1+\ell_{V,\Psi})} \right) \|X_{k+1} - Z_k\|_{\Psi}^2 \\
& + \rho_k \|\mathbf{e}_k\|_{\Psi^{-1}}^2 \left(2 + \frac{1-\nu-2\ell_{V,\Psi}}{1+\ell_{V,\Psi}} \right) - 2\rho_k \langle W_k, Y_k - p \rangle.
\end{aligned}$$

Observe that

$$\begin{aligned} \|X_{k+1} - Z_k\|_{\Psi}^2 &\geq (1 - \alpha_k)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &\quad + (\alpha_k^2 - \alpha_k)\|X_k - X_{k-1}\|_{\Psi}^2, \end{aligned} \quad (22)$$

$$\begin{aligned} \|Z_k - p\|_{\Psi}^2 &= (1 + \alpha_k)\|X_k - p\|_{\Psi}^2 - \alpha_k\|X_{k-1} - p\|_{\Psi}^2 \\ &\quad + \alpha_k(1 + \alpha_k)\|X_k - X_{k-1}\|_{\Psi}^2. \end{aligned} \quad (23)$$

Choose parameters α_k and ρ_k such that $\frac{3-\nu-2\rho_k(1+\ell_{V,\Psi})}{2\rho_k(1+\ell_{V,\Psi})} > 0$. Then, using both of these relations in the last estimate for $\|X_{k+1} - p\|_{\Psi}^2$, we arrive at

$$\begin{aligned} \|X_{k+1} - p\|_{\Psi}^2 &\leq (1 + \alpha_k)\|X_k - p\|_{\Psi}^2 - \alpha_k\|X_{k-1} - p\|_{\Psi}^2 \\ &\quad + \alpha_k(1 + \alpha_k)\|X_k - X_{k-1}\|_{\Psi}^2 - 2\rho_k\langle W_{k+1}, Y_k - p \rangle \\ &\quad - \frac{\nu\rho_k}{2}r_{\Psi}^2(Z_k) + \frac{(3-\nu)\rho_k}{1+\ell_{V,\Psi}}\|e_k\|_{\Psi}^2 + \nu\rho_k\|U_k\|_{\Psi}^2 \\ &\quad - \left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\left[(1 - \alpha_k)\|X_{k+1} - X_k\|_{\Psi}^2\right. \\ &\quad \left. + (\alpha_k^2 - \alpha_k)\|X_k - X_{k-1}\|_{\Psi}^2\right]. \end{aligned}$$

Using the respective definitions of the stochastic increments $\Delta M_{k+1}, \Delta N_k(p)$ in (18) and (19), we arrive at

$$\begin{aligned} \|X_{k+1} - p\|_{\Psi}^2 &\leq (1 + \alpha_k)\|X_k - p\|_{\Psi}^2 - \alpha_k\|X_{k-1} - p\|_{\Psi}^2 \\ &\quad + \Delta M_k + \Delta N_k(p) - \frac{\nu\rho_k}{2}r_{\Psi}^2(Z_k) \\ &\quad + \alpha_k\|X_k - X_{k-1}\|_{\Psi}^2 \left(2\alpha_k + \frac{(3-\nu)(1-\alpha_k)}{2\rho_k(1+\ell_{V,\Psi})}\right) \\ &\quad - (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \end{aligned}$$

Rearranging the fundamental recursion, we see

$$\begin{aligned} \|X_{k+1} - p\|_{\Psi}^2 - \alpha_k\|X_k - p\|_{\Psi}^2 &+ (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &\leq \|X_k - p\|_{\Psi}^2 - \alpha_k\|X_{k-1} - p\|_{\Psi}^2 + \Delta M_k + \Delta N_k(p) \\ &\quad + (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_k - X_{k-1}\|_{\Psi}^2 - \frac{\nu\rho_k}{2}r_{\Psi}^2(Z_k) \\ &\quad + \left(2\alpha_k^2 + (1 - \alpha_k)\left(1 - \frac{(3-\nu)(1-\alpha_k)}{2\rho_k(1+\ell_{V,\Psi})}\right)\right)\|X_k - X_{k-1}\|_{\Psi}^2. \end{aligned}$$

Suppose $(\alpha_k)_k$ is a non-decreasing sequence satisfying $0 < \alpha_k \leq \bar{\alpha} < 1$ and $\rho_k = \frac{(3-\nu)(1-\bar{\alpha})^2}{2(2\alpha_k^2 - \alpha_k + 1)(1+\ell_{V,\Psi})}$. Since $\rho_k \leq \frac{(3-\nu)(1-\alpha_k)^2}{2(2\alpha_k^2 - \alpha_k + 1)(1+\ell_{V,\Psi})}$, we claim that

$$\begin{aligned} \|X_{k+1} - p\|_{\Psi}^2 - \alpha_k\|X_k - p\|_{\Psi}^2 &+ (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &\geq \|X_{k+1} - p\|_{\Psi}^2 - \alpha_k\|X_k - p\|_{\Psi}^2 \\ &\quad + (1 - \alpha_k)\left(\frac{2\alpha_k^2 - \alpha_k + 1}{(1-\alpha_k)^2} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \geq 0. \end{aligned}$$

To see this, observe that for any $\alpha > 0$,

$$\begin{aligned} \|X_{k+1} - p\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 &+ (1 - \alpha)\left(\frac{2\alpha^2 - \alpha + 1}{(1-\alpha)^2} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &> \|X_{k+1} - p\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 \\ &\quad + (1 - \alpha)\left(\frac{\alpha^2 - \alpha + 1}{(1-\alpha)^2} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &= \|X_{k+1} - p\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 \end{aligned}$$

$$\begin{aligned} &- \left(\frac{\alpha^2 - \alpha + 1}{1-\alpha} - \frac{1-2\alpha+\alpha^2}{1-\alpha}\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &= (\alpha + (1 - \alpha))\|X_{k+1} - p\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 \\ &\quad + \left(\alpha + \frac{\alpha^2}{1-\alpha}\right)\|X_{k+1} - X_k\|_{\Psi}^2 \\ &\geq \alpha\|X_{k+1} - p\|_{\Psi}^2 + \alpha\|X_{k+1} - X_k\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 \\ &\quad + 2\alpha\|X_{k+1} - p\|_{\Psi} \cdot \|X_{k+1} - X_k\|_{\Psi} \\ &= \alpha(\|X_{k+1} - p\|_{\Psi} + \|X_{k+1} - X_k\|_{\Psi})^2 - \alpha\|X_k - p\|_{\Psi}^2 \\ &\geq \alpha\|X_{k+1} - p + X_k - X_{k+1}\|_{\Psi}^2 - \alpha\|X_k - p\|_{\Psi}^2 = 0 \end{aligned}$$

where the third inequality follows from Young's inequality. Under this specific coupling of the inertial and relaxation parameters, it holds that $2\alpha_k^2 + (1 - \alpha_k)\left(1 - \frac{(3-\nu)(1-\alpha_k)}{2\rho_k(1+\ell_{V,\Psi})}\right) \leq 0$. Now, let $H_k(p) = \|X_k - p\|_{\Psi}^2 - \alpha_k\|X_{k-1} - p\|_{\Psi}^2 + (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_k - X_{k-1}\|_{\Psi}^2$, and $\delta_k = \frac{\nu\rho_k}{2}r_{\Psi}^2(Z_k) - \left(2\alpha_k^2 + (1 - \alpha_k)\left(1 - \frac{(3-\nu)(1-\alpha_k)}{2\rho_k(1+\ell_{V,\Psi})}\right)\right)\|X_k - X_{k-1}\|_{\Psi}^2$. Then, $(1 - \alpha_{k+1})\left(\frac{3-\nu}{2\rho_{k+1}(1+\ell_{V,\Psi})} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2 \leq (1 - \alpha_k)\left(\frac{3-\nu}{2\rho_k(1+\ell_{V,\Psi})} - 1\right)\|X_{k+1} - X_k\|_{\Psi}^2$. Therefore, for all $k \geq 0$, we conclude

$$\mathbb{E}[H_{k+1}(p) | \mathcal{F}_k] \leq H_k(p) - \delta_k(p) + \mathbb{E}[\Delta M_k | \mathcal{F}_k].$$

Using Standing Assumption 7, we deduce that $(\mathbb{E}[\Delta M_k | \mathcal{F}_k])_{k \in \mathbb{N}}$ is summable, and thus we can apply Lemma 3 to the above recursion. Hence, we readily deduce the existence of an a.s. finite limiting random variable $H_{\infty}(p)$ such that $\mathbb{P}(\lim_{k \rightarrow \infty} H_k(p) = H_{\infty}(p)) = 1$ and $\sum_{k \in \mathbb{N}} \delta_k(p)_{k \in \mathbb{N}} < \infty$. Therefore, there exists a measurable set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ it holds true that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|X_k(\omega) - X_{k-1}(\omega)\|_{\Psi} &= 0 \text{ and} \\ \lim_{k \rightarrow \infty} \frac{\nu\rho_k}{2}r_{\Psi}^2(Z_k(\omega)) &= 0. \end{aligned}$$

Assuming that $\liminf_{k \rightarrow \infty} \rho_k > 0$, we conclude that $\lim_{k \rightarrow \infty} r_{\Psi}^2(Z_k) = 0$ \mathbb{P} -a.s. Therefore, we conclude that $(X_k)_{k \geq 1}$ converges a.s. to a limiting random variable with values in $\text{zer}(V + T)$.

VI. CONCLUSION

In the context of shared constraint variants of stochastic generalized Nash equilibrium problems, the convergence of the forward-backward-forward algorithm can be boosted via an integrated acceleration-relaxation procedure. In the presence of stochastic uncertainty, convergence can be proved assuming only monotonicity and Lipschitz continuity of the expected-valued operator. Specifically, our main result is the claim of almost sure global convergence of the trajectory of actions to the set of variational equilibria. In future research, we aim to investigate the question how one may relax monotonicity and Lipschitz continuity assumptions even further, derive rate statements, and examine how partial information may be introduced into the algorithm.

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