A Dual Characterization of the Stability of the Wonham Filter

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Abstract— This paper revisits the classical question of the stability of the nonlinear Wonham filter. The novel contributions of this paper are two-fold: (i) definition of the stabilizability for the (control-theoretic) dual to the nonlinear filter; and (ii) the use of this definition to obtain conclusions on the stabilizability of the Wonham filter. Specifically, it is shown that the stabilizability of the dual system is necessary for filter stability and conversely stabilizability implies that the filter asymptotically detects the correct ergodic class. The formulation and the proofs are based upon a recently discovered duality result whereby the nonlinear filtering problem is cast as a stochastic optimal control problem for a backward stochastic differential equation (BSDE). The control-theoretic proof techniques and results may be viewed as a generalization of the classical work on the stability of the Kalman filter.

I. INTRODUCTION

Viewed from a certain lens, the story of stochastic filter stability is a story of two parts: (i) stability of the Kalman filter where control-theoretic definitions and methods are paramount; and (ii) stability of the nonlinear filter where there is no hint of such methods (with one notable exception [1]). Arguably, the control techniques are useful for the analysis of Kalman filter, because of the classical dual relationship between observability and controllability of deterministic linear systems. This dual relationship extends to the stochastic linear Gaussian settings: In Kalman's celebrated paper with Bucy, it is shown that the Kalman filter is dual to a deterministic linear quadratic (LQ) optimal control problem. The relationship is useful in two ways: (i) Asymptotic stability of the filter is related to the asymptotic stability of the dual optimal control system; and (ii) Necessary and sufficient conditions for the same are stabilizability for the optimal control problem, and (because of the dual relationship) detectability for the filter. Notably, duality explains why, with the time arrow reversed, the covariance update equation of the Kalman filter is the same as the dynamic Riccati equation (DRE) of optimal control. In practical terms, asymptotic stability of the Kalman filter is deduced by establishing an asymptotic limit for the value function of the dual LQ problem [2, Ch. 9].

Our goal in this paper is to extend these classical control theoretic techniques for the stability analysis of the nonlinear Wonham filter. Specifically, we are interested in obtaining necessary and sufficient conditions for filter stability. Our focus is on the so called non-ergodic signal case and we are interested in a minimal condition that a model should satisfy for the filter stability to hold. In the filter stability literature, this condition is referred to as detectability [3].

This problem has a rich and storied history; cf., [4], [5] and references therein. For the non-ergodic signal case, a notable early contribution is [6] where formulae for the relative entropy are derived and it is shown that the relative entropy is a Lyapunov function for the filter. Our paper is closely inspired by [7] who are the first to formulate certain "identifying conditions" that are shown to be sufficient for the stability of Wonham filter. These conditions are formulated in terms of the model parameters (transition matrix and the observation function). The definition of observability and detectability appears in [3], [8]. For the Wonham filter, the subspace of observable functions is completely characterized in terms of model parameters. Detectability is shown to be both necessary and sufficient for filter stability. Extensions to these definitions have recently appeared in [9], [10].

The paper has a single contribution given as Thm. 1: We define the stabilizability property of the dual system and relate this property to the asymptotic stability of the filter. Stabilizability is shown to be the dual to the detectability definition of [3]. The overall development - introduction of the dual system, stabilizability definition, and its use the in the filter stability analysis - has close parallels to the Kalman filter stability theory. This connection is explained using several remarks in the paper. While the narrow focus of this paper is on the non-ergodic signal case (where stabilizability is non-trivial), a companion paper presents filter stability results for the ergodic signal case [11]. The analysis in both these papers is based upon a recently discovered duality result whereby the nonlinear filtering problem is cast as a stochastic optimal control problem for a backward stochastic differential equation (BSDE) [12].

Both the optimal control formulation and its use in obtaining the filter stability proofs are new. While [1] also employed a dual optimal control problem, it is completely different from the dual formulation used here. As explained in our earlier papers [12], [13], [14], our formulation is a generalization of the Kalman-Bucy duality while [1] is a generalization of the minimum energy or maximum likelihood duality (see also [15]). Notably, the classical proofs of the stability of the Kalman filter are based on the original Kalman-Bucy duality (see e.g. [2]). Our proofs can thus be viewed as a generalization of the linear stability theory.

The outline of the remainder of this paper is as follows: The problem formulation appears in Sec. II. The background on duality and stabilizability is in Sec. III. The main result and its proof are in Sec. IV and Sec. V, respectively.

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II. PROBLEM FORMULATION

Notation: The state-space $S := \{1, 2, \dots, d\}$ is finite. The set of probability vectors on S is denoted by $\mathcal{P}(S)$: $\mu \in \mathcal{P}(S)$ if $\mu(x) \ge 0$ and $\sum_{x \in S} \mu(x) = 1$. The space of deterministic functions on S is identified with \mathbb{R}^d : Any function $f : S \to \mathbb{R}$ is determined by its value f(x) at $x \in S$. For a measure $\mu \in \mathcal{P}(S)$ and a function $f \in \mathbb{R}^d$, $\mu(f) := \sum_x \mu(x)f(x)$. For two vectors $f, h \in \mathbb{R}^d$, fh denotes the element-wise (Hadamard) product: (fh)(x) := f(x)h(x) and similarly $f^2 = ff$. The vector of all ones is denoted as 1 and $f|_{1^\perp} := f - \frac{1}{d}(1^\top f)$. For a subset $D \subset S$, 1_D denotes the indicator function with support on D.

A. Filtering model

Consider a pair of continuous-time stochastic processes (X,Z) defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$:

(1) The state $X = \{X_t \in \mathbb{S} : t \ge 0\}$ is a Markov process with initial condition $X_0 \sim \mu \in \mathcal{P}(\mathbb{S})$ (prior) and the generator (row stochastic rate matrix) $A \in \mathbb{R}^{d \times d}$. The finite state-space is partitioned into *m* ergodic classes $\mathbb{S} = \bigcup_{k=1}^m \mathbb{S}_k$ such that $\mathbb{P}([X_t \in \mathbb{S}_l] | [X_0 \in \mathbb{S}_k]) = 0$ for all $t \ge 0$ and $l \ne k$. For a function $f \in \mathbb{R}^d$, the carré du champ operator $\Gamma : \mathbb{R}^d \to \mathbb{R}^d$ is defined according to

$$\Gamma(f)(x) \coloneqq \sum_{j \in \mathbb{S}} A(x, j) (f(x) - f(j))^2 \quad \text{for } x \in \mathbb{S}$$

(2) The observation process $Z = \{Z_t \in \mathbb{R} : t \ge 0\}$ is defined according to the following model:

$$Z_t := \int_0^t h(X_s) \,\mathrm{d}s + W_t$$

where $h: \mathbb{S} \to \mathbb{R}$ is the observation function and $W = \{W_t \in \mathbb{R} : t \ge 0\}$ is a Wiener process (w.p.) that is assumed to be independent of *X*. The scalar-valued observation model is considered for notational ease. The covariance of *W* is denoted as *R* which is assumed to be positive. The filtration generated by *Z* is denoted as $\mathcal{Z} := \{Z_t : 0 \le t \le T\}$ where $Z_t = \sigma(\{Z_s : 0 \le s \le t\}).$

Function spaces: To stress the prior μ , the probability measure is denoted P^{μ} and the expectation is $\mathsf{E}^{\mu}(\cdot)$. The space of square-integrable \mathcal{Z}_T -measurable random functions on \mathbb{S} is denoted $L^2_{\mathcal{Z}_T}(\mathbb{S})$: $F \in L^2_{\mathcal{Z}_T}(\mathbb{S})$ if F is \mathcal{Z}_T -measurable and $\mathsf{E}^{\mu}(|F(X_T)|^2) < \infty$. The space of \mathcal{Z} -adapted square-integrable S-valued stochastic processes is denoted $L^2_{\mathcal{Z}}([0,T];S)$. Typical examples of S are (i) \mathbb{R} for real-valued, and (ii) \mathbb{R}^d for vector-valued stochastic processes.

The filtering problem is to compute the conditional distribution (posterior), denoted $\pi_t^{\mu} \in \mathcal{P}(\mathbb{S})$, of the state X_t given \mathcal{Z}_t . For $f \in \mathbb{R}^d$, $\pi_t^{\mu}(f) \coloneqq \mathsf{E}^{\mu}(f(X_t)|\mathcal{Z}_t)$ is the object of interest.

B. Definition of filter stability

In the finite state-space settings of this paper, the Wonham filter is a stochastic differential equation (SDE):

$$\mathrm{d}\pi_t = A^\top \pi_t \,\mathrm{d}t + \left(\mathrm{diag}(h) - \pi_t h^\top\right) \pi_t R^{-1} \left(\mathrm{d}Z_t - \pi_t(h) \,\mathrm{d}t\right) \quad (1)$$

With an initialization $\pi_0 = v \in \mathcal{P}(\mathbb{S})$, the solution of the Wonham filter is denoted as $\pi^v := \{\pi_t^v \in \mathcal{P}(\mathbb{S}) : t \ge 0\}$. The (true) posterior $\pi^{\mu} := \{\pi_t^{\mu} \in \mathcal{P}(\mathbb{S}) : t \ge 0\}$ results from the choice of the initial condition $\pi_0 = \mu$.

Definition 1: The Wonham filter is *stable*, in the sense of weak convergence in L^2 , if for each $f \in \mathbb{R}^d$

$$\Xi^{\mu}(|\pi_{t}^{\mu}(f) - \pi_{t}^{\nu}(f)|^{2}) \longrightarrow 0 \quad \text{as } t \to \infty$$
(2)

whenever $\mu \ll v$.

III. DUALITY AND STABILIZABILITY

A. Background on duality for nonlinear filtering

In our recent work [12], a dual optimal control problem is introduced. It is based upon the following backward stochastic differential equation (BSDE):

$$-dY_t(x) = ((AY_t)(x) + h(x)U_t + h(x)V_t(x)) dt - V_t(x) dZ_t,$$

$$Y_T(x) = f(x) \quad \forall x \in \mathbb{S}$$
(3)

where the control input $U = \{U_t \in \mathbb{R} : 0 \le t \le T\}$ is in $L^2_{\mathcal{Z}}([0,T];\mathbb{R}) =: \mathcal{U}$. Such controls are referred to as *admissible*. The solution $(Y,V) := \{(Y_t,V_t) \in \mathbb{R}^d \times \mathbb{R}^d : 0 \le t \le T\}$ is in $L^2_{\mathcal{Z}}([0,T];\mathbb{R}^d \times \mathbb{R}^d)$. That is, the solution is forward-adapted to the filtration \mathcal{Z} .

Consider next the following estimator (for the random variable $f(X_T)$):

$$S_T = \pi_0(Y_0) - \int_0^T U_t \, \mathrm{d}Z_t \tag{4}$$

The estimator is parameterized with an admissible control $U \in \mathcal{U}$ and a given $\pi_0 \in \mathcal{P}(\mathbb{S})$. Y_0 is obtained by solving the BSDE (3) with the control input U and $Y_T = f$.

By an application of the Itô-Wentzell theorem (see [12, Appendix A])

$$f(X_T) - S_T = Y_0(X_0) - \pi_0(Y_0) + \int_0^T (U_t + V_t(X_t)) dW_t + Y_t^{\mathsf{T}} dN_t$$

where N is a P^{μ} -martingale. Upon squaring and taking expectation, we obtain the duality relationship

$$\mathsf{E}^{\mu}(|f(X_T) - S_T|^2) = \mathsf{J}_T^{\mu}(U; f) + |\pi_0(Y_0) - \mu(Y_0)|^2 \quad (5)$$

where

$$\mathsf{J}_{T}^{\mu}(U;f) := \mathsf{E}^{\mu}\Big(|Y_{0}(X_{0}) - \mu(Y_{0})|^{2} + \int_{0}^{T} \ell(Y_{t}, V_{t}, U_{t}; X_{t}) \,\mathrm{d}t\Big)$$

where $\ell(y, v, u; x) = \Gamma(y)(x) + |u + v(x)|_R^2$; cf., [12, Eq. (6)]. In [12], (5) is referred to as the *duality principle*.

The dual optimal control problem is to choose a control $U \in \mathcal{U}$ such that $J_T^{\mu}(U; f)$ is minimized subject to the BSDE constraint (3). The minimum value is denoted as $J_T^{\mu}(f)$ or more simply as J_T^{μ} if there is no chance of ambiguity.

The existence and uniqueness of the optimal control follows from the standard results in the BSDE constrained optimal control theory [16]. The solution, including the formula for optimal control, is described in [12, Theorem 1]. Some technical background is also included in the Appendix A of this paper. Let $U^{\mu} := \{U_t^{\mu} : 0 \le t \le T\}$ be the optimal control and $(Y^{\mu}, V^{\mu}) := \{(Y^{\mu}_t, V^{\mu}_t) : 0 \le t \le T\}$ is the associated (optimal) trajectory. Then:

(1) [12, Theorem 2]: The optimal control gives the conditional mean

$$\pi_T^{\mu}(f) = \mu(Y_0^{\mu}) - \int_0^T U_t^{\mu} \, \mathrm{d}Z_t, \quad \mathsf{P}^{\mu} - \mathrm{a.s.} \tag{6}$$

(2) [12, Theorem 5]: The optimal value function gives the variance

$$\mathsf{J}_{T}^{\mu}(f) = \mathsf{E}^{\mu}(|f(X_{T}) - \pi_{T}^{\mu}(f)|^{2})$$

Using this formula, the optimal value is uniformly bounded by $\frac{1}{4}|\operatorname{osc}(f)|^2$ where $\operatorname{osc}(f) \coloneqq \max_{i,j\in\mathbb{S}} |f(i) - f(j)|$.

B. Stabilizability of the BSDE

Definition 2: For the BSDE (3), the *controllable subspace* is defined as:

$$\mathcal{C}_T := \{ y_0 \in \mathbb{R}^d \mid \exists c \in \mathbb{R}, U \in \mathcal{U} \text{ s.t. } Y_0 = y_0 \text{ and } Y_T = c1 \}$$

(Note c, y_0 are deterministic and U is an admissible stochastic process.) If $C_T = \mathbb{R}^d$ then the BSDE is said to be *controllable*.

Because $1 \in C_T$, it is a non-trivial subspace of \mathbb{R}^d (even with h = 0). An explicit characterization of the controllable subspace is given in the following:

Proposition 1: [14, Theorem 2] For any positive value of terminal time T, C_T is the smallest such subspace $C \subseteq \mathbb{R}^d$ that satisfies the following two properties:

(i) The constant function $1 \in C$; and (ii) If $g \in C$ then $Ag \in C$ and $gh \in C$. Explicitly,

$$C \coloneqq \operatorname{span} \{ 1, h, Ah, A^2h, A^3h, \dots, \\ h^2, A(h^2), h(Ah), A^2(h^2), \dots, \\ h^3, (Ah)(h^2), hA(h^2), \dots \}$$

Remark 1: The subspace C is identical to the space of "observable functions" in [3, Lemma 9]. An explanation for this correspondence is provided in [14] where it is shown that the BSDE is the dual of the Zakai equation of filtering.

Because *A* is a stochastic matrix, its eigenvalues are either in the open left half-plane or at zero. To define stabilizability, consider first the zero subspace:

$$S_0 \coloneqq \{ y \in \mathbb{R}^d \mid Ay = 0 \}$$

Definition 3: The BSDE (3) is stabilizable if $S_0 \subset C$.

Proposition 2: Consider the BSDE (3). Then

(i) If S has a single ergodic class then BSDE is stabilizable.
(ii) If S = ∪^m_{k=1}S_k is partitioned into *m* ergodic classes then the BSDE is stabilizable if and only if the indicator functions 1_{S_k} ∈ C for k = 1,2,...,m.

IV. MAIN RESULT

Theorem 1: Suppose the Wonham filter is stable. Then the BSDE (3) is stabilizable. Conversely, if the BSDE is stabilizable then

$$\pi_T^{\nu}(1_{\mathbb{S}_k}) \xrightarrow{(T \to \infty)} 1_{\mathbb{S}_k}(X_0) \quad \mathsf{P}^{\mu}\text{-a.s.}$$

whenever $\mu \ll \nu$. (That is, the filter asymptotically detects the correct ergodic class.)

It is shown in Appendix C that for any given $v \in \mathcal{P}(\mathbb{S})$, one can pick $\{v_k \in \mathcal{P}(\mathbb{S}) : 1 \le k \le m\}$ such that v_k has support on \mathbb{S}_k and

$$\pi_T^{\nu}(f) = \sum_{k=1}^m \pi_T^{\nu}(1_{\mathbb{S}_k}) \pi_T^{\nu_k}(f)$$
(7)

Using Thm. 1, the problem of filter stability reduces to the problem of filter stability for each ergodic class. This is also the justification for treating the ergodic and non-ergodic signal cases separately. It is known that, for the type of observations considered here, the filter "inherits" the stability property of the Markov process [7, Theorem 4.2]. The proof of this is far from straightforward and spurred much research during the first decade [4], [5], [17]. Assuming it to be true, from Thm. 1 it follows that stabilizability is both a necessary and sufficient condition for filter stability.

The dual optimal control approach of this paper is also useful to the study of filter stability of ergodic signals (e.g., to obtain results on asymptotic convergence of $\pi_T^{\nu_k}(f)$ above). This is the subject of a companion paper published in the proceedings of the conference [11].

Remark 2: The sufficient condition stated in [7, Theorem 4.4] correctly stress the importance of the "identifying" property of the filter to identify the correct ergodic class [7, Lemma 6.3]. Subsequently, the definition of detectability was first introduced in [3], [4]. For the Wonham filter, the detectability property was shown to be equivalent to filter stability [3, Theorem 2].

Because of Remark 1, the stabilizability definition taken together with the result in Theorem 1 represent dual counterparts of these earlier definitions and results. Additional details on duality between controllability of the BSDE and observability of the filter can be found in our prior paper [14, Prop. 2].

It is also worthwhile to note that the appropriate observability and detectability definitions were discovered only after a decade of intense research; c.f., [4], [5]. In contrast, using duality these definitions are obtained quite naturally.

Remark 3: For the stability of the Kalman filter, the importance of detectability is well known. Most proofs of filter stability rely on the analysis of the dual LQ optimal control problem [2, Ch. 9], [18, Sec. 2]. Stabilizability of the dual system is then *actually* the condition that is used to obtain the proof of Kalman filter stability. Our results can thus be viewed as generalization of the Kalman filter stability theory.

V. PROOF OF THE MAIN RESULT

A. Probability spaces

Recall P^{μ} is the probability measure indicative of the fact that $X_0 \sim \mu$. For $\mu \ll \nu$, consider a probability measure P^{ν} on the common measurable space (Ω, \mathcal{F}) as in Sec. II-A. It is noted that

$$\frac{\mathrm{d}\mathsf{P}^{\mu}}{\mathrm{d}\mathsf{P}^{\nu}}(\omega) = \sum_{x\in\mathbb{S}} \frac{\mu(x)}{\nu(x)} \mathbf{1}_{[X_0=x]}(\omega)$$

Then (X,Z) have the same transition law and if $X_0 \sim v$ under P^v then $X_0 \sim \mu$ under P^{μ} . The expectation under P^v is denoted $\mathsf{E}^v(\cdot)$. The solution of the Wonham filter $\pi_t^{\mu} = \mathsf{E}^{\mu}(X_t | \mathcal{Z}_t)$ and $\pi_t^v = \mathsf{E}^v(X_t | \mathcal{Z}_t)$.

In the settings of this paper, a filter is obtained by solving the dual optimal control problem. A user who (incorrectly) believes the prior to be v solves the optimal control problem under the (incorrect) measure P^{v} :

$$\mathsf{J}_{T}^{\nu}(U;f) = \mathsf{E}^{\nu}\Big(|Y_{0}(X_{0}) - \nu(Y_{0})|^{2} + \int_{0}^{T} \ell(Y_{t}, V_{t}, U_{t}; X_{t}) \,\mathrm{d}t\Big)$$
(8)

subject to the BSDE constraint (3). Note the two changes: the expectation is now with respect to P^{ν} and $\nu(Y_0)$ appears in the terminal cost (first of the two terms). The optimal control for this problem is denoted U^{ν} and the associated optimal trajectory is (Y^{ν}, V^{ν}) . The counterpart of (6) is

$$\pi_T^{\nu}(f) = \nu(Y_0^{\nu}) - \int_0^T U_t^{\nu} dZ_t \quad \mathsf{P}^{\nu} - \text{a.s.}$$
(9)

and $J_T^{\nu}(f) = \mathsf{E}^{\nu}(|f(X_T) - \pi_T^{\nu}(f)|^2) \le \frac{1}{4}|\operatorname{osc}(f)|^2$ for all $T \ge 0$.

Before the estimator is assessed with respect to the (correct) measure P^{μ} , there are three technical concerns:

- 1) Existence and uniqueness of the optimal control U^{ν} and the solution (Y^{ν}, V^{ν}) for all $T \ge 0$. This ensures in particular that the righthand-side (9) is well-defined.
- 2) Admissibility of the optimal control U^{ν} with respect to P^{μ} . This ensures that the optimal control can be assessed using P^{μ} whenever $\mu \ll \nu$.
- 3) Apriori bounds and continuity properties for the value $J_T^{\mu}(U^{\nu})$ as $\mu \to \nu$.

Appendix A contains the requisite technical background that also serves to address these concerns.

B. Relationship of duality to filter stability

For any $S_T \in L^2_{Z_T}$, the projection theorem gives

$$\mathsf{E}^{\mu}(|f(X_T) - S_T|^2) = \mathsf{J}_T^{\mu} + \mathsf{E}^{\mu}(|\pi_T^{\mu}(f) - S_T|^2)$$

and using the duality formula (5) in the left-hand side

$$\mathsf{E}^{\mu}(|\pi_{T}^{\mu}(f) - S_{T}|^{2}) = (\mathsf{J}_{T}^{\mu}(U) - \mathsf{J}_{T}^{\mu}) + |\pi_{0}(Y_{0}) - \mu(Y_{0})|^{2}$$

With $\pi_0 = v$ and $U = U^v$, the estimate $S_T = \pi_T^v(f)$ and therefore

$$\mathsf{E}^{\mu}(|\pi_{T}^{\mu}(f) - \pi_{T}^{\nu}(f)|^{2}) = (\mathsf{J}_{T}^{\mu}(U^{\nu}) - \mathsf{J}_{T}^{\mu}) + |\nu(Y_{0}^{\nu}) - \mu(Y_{0}^{\nu})|^{2}$$

Both the terms on the righthand-side are non-negative (the first term so because J_T^{μ} is the minimum value and U^{ν} is P^{μ} -admissible). Therefore, the limit of the lefthand-side (as $T \to \infty$) is 0 if and only if each of the two terms on the righthand-side individually approach 0. We state the result as a proposition.

Proposition 3: The filter is stable if and only if

$$\mu(Y_0^{\nu}) - \nu(Y_0^{\nu}) \stackrel{(T \to \infty)}{\longrightarrow} 0$$
 (10a)

$$\mathsf{J}_{T}^{\mu}(U^{\nu}) - \mathsf{J}_{T}^{\mu} \stackrel{(T \to \infty)}{\longrightarrow} 0 \tag{10b}$$

whenever $\mu \ll v$.

Remark 4: These conditions are the nonlinear counterparts of the sufficient conditions for the stability of the Kalman filter in [18, Theorem 2.3].

- 1) Equation (10a) means that the optimal control system is asymptotically stable. That is, $Y_0 \rightarrow$ (const.)1 as $T \rightarrow \infty$. This is also the reason why the stabilizability condition is important to the problem of filter stability. The condition plays the same role in linear and nonlinear settings.
- 2) Equation (10b) means that the value converges to the optimal value. Since the optimal value $J_T^{\mu}(f)$ has the interpretation of the minimum variance, its convergence is analogous to the convergence of the solution of the DRE in the Kalman filter. In linear settings, the latter is deduced by establishing an asymptotic limit for the value function of the dual optimal control problem [2, Sec. 9.4].

C. Proof of necessity in Thm. 1

In this subsection, we prove the necessity part of Thm. 1: That is, if the filter is stable then is the BSDE (3) is stabilizable. The argument rests on the result described in the following proposition whose proof appears in the Appendix D.

Proposition 4: Suppose the BSDE (3) is not stabilizable. Then there exists an $f \in C^{\perp}$ such that for any T and any $U \in \mathcal{U}$, the solution to the BSDE (3) is given by

$$Y_0 = y_0 + f$$

where $y_0 \in C$ (and can depend upon *T* and *U*). Since $1 \in C$, this implies $|Y_0 - \frac{1}{d}(1^{\mathsf{T}}Y_0)1| \ge |f|$, i.e., Y_0 is uniformly bounded away from the subspace of constant vectors.

The proof of necessity follows from Prop. 4: Pick any $v \in \mathcal{P}(\mathbb{S})$ such that 0 < v(x) < 1 for all $x \in \mathbb{S}$ and set $\mu = v + \varepsilon f$ where ε is chosen sufficiently small such that $\mu \in \mathcal{P}(\mathbb{S})$. Then $|\mu(Y_0) - v(Y_0)| = \varepsilon |f|^2$. Applying Prop. 3, the filter is not stable for this choice of μ and v.

D. Completing the proof of Thm. 1

We first state a technical lemma that is used in the proof. The proof of the Lemma appears in Appendix E. *Lemma 1:* Suppose $\bar{\mu}$ is an invariant measure of *A* (i.e., $A^{\top}\bar{\mu} = 0$). Then for each fixed $f \in \mathbb{R}^d$

- (i) The sequence $\{J_T^{\mu}(f): T \ge 0\}$ is bounded, non-negative, and non-increasing in *T*. Therefore, $J_T^{\bar{\mu}}(f)$ converges as $T \to \infty$. Denote the limit as $J_{\infty}^{\bar{\mu}}(f)$.
- (ii) For a given $\mu \in \mathcal{P}(\mathbb{S})$, denote $\mu_T := e^{A^\top T} \mu$. Suppose $\mu_T \to \overline{\mu}$ as $T \to \infty$. Then

$$\limsup_{T \to \infty} \mathsf{J}^{\mu}_{T}(f) \le \mathsf{J}^{\bar{\mu}}_{\infty}(f)$$

We now complete the proof of Thm. 1: That is, we show that if $1_{\mathbb{S}_k} \in \mathcal{C}$ then

$$\pi_T^{\nu}(1_{\mathbb{S}_k}) \xrightarrow{(T \to \infty)} 1_{\mathbb{S}_k}(X_0) \quad \mathsf{P}^{\mu}\text{-a.s.}$$

The proof is in the following three steps:

1. In step 1, we show that $\pi_T^{\nu}(1_{\mathbb{S}_k})$ converges P^{ν} -a.s.

1. In step 2, we show that if $1_{\mathbb{S}_k} \in \mathcal{C}$ then $J_T^{\mu}(1_{\mathbb{S}_k}) \to 0$ as $T \to \infty$ where $\overline{\mu}$ is any invariant measure of *A*. We use part (i) of Lemma 1 to prove this result.

3. In step 3, we combine the conclusions of steps 1 and 2 to prove the result. We use part (ii) of Lemma 1 to prove this result.

Step 1: Consider the Wonham filter (1) with $\pi_0 = v$. Since $A1_{\mathbb{S}_k} = 0$, $\{\pi_T^v(1_{\mathbb{S}_k}) : T \ge 0\}$ is a bounded P^v -martingale and therefore converges P^v -a.s. (Therefore, the a.s. convergence does not require stabilizability of the model.)

Step 2: Suppose $\bar{\mu}$ is any invariant measure. Then J_T^{μ} is monotone (part (i) of Lemma 1). In the following, we construct a sequence of admissible control input $\{U^{(T)}: T = 1, 2, \cdots\}$ such that $J_T^{\bar{\mu}}(U^{(T)}; \mathbb{1}_{\mathbb{S}_k}) \to 0$ as $T \to \infty$. Since $J_T^{\bar{\mu}}(\mathbb{1}_{\mathbb{S}_k})$ is the minimum value this implies $J_T^{\bar{\mu}}(\mathbb{1}_{\mathbb{S}_k}) \to 0$ as $T \to \infty$ (for this particular sub-sequence). Since $J_T^{\bar{\mu}}$ is monotone, the limit exists and equals this sub-sequential limit.

Suppose $1_{\mathbb{S}_k} \in C$. Then we claim that there exists an admissible control $U^{(1)} := \{U_t^{(1)} : 0 \le t \le 1\}$ and a constant $c \in \mathbb{R}$ such that $Y_T^{(1)} = 1_{\mathbb{S}_k}$ and $Y_0^{(1)} = c1$. The claim follows from the definition of controllable space. A proof of the claim appears at the end of the proof. Assuming the claim to be true for now, denote the associated solution of the BSDE (3) as $(Y^{(1)}, V^{(1)}) := \{(Y_t^{(1)}, V_t^{(1)}) : 0 \le t \le 1\}$.

Since *Z* is a w.p. under the Girsanov change of measure, there exists a functional $\phi(\cdot, \cdot) : [0,1] \times C([0,1];\mathbb{R}) \to \mathbb{R}$ such that

 $U_t^{(1)} = \phi(t, \{Z_s : 0 \le s \le t\}) \quad \mathsf{P}^{\bar{\mu}} - \mathrm{a.s.}$

Now consider the following control over the time-horizon [0,n]: For $l = 0, 1, 2, \dots, n-1$

$$U_t^{(n)} := \frac{1}{n} \phi(t - l, \{Z_s : l \le s \le t\}), \quad t \in (l, (l+1)]$$

Such a control input is clearly admissible. With $Y_n = 1_{\mathbb{S}_k}$, one obtains the following solution $(Y^{(n)}, V^{(n)}) := \{(Y_t^{(n)}, V_t^{(n)}) : 0 \le t \le n\}$ of the BSDE (3):

$$V_t^{(n)} \stackrel{\text{(d)}}{=} \frac{1}{n} V_{t-l}^{(1)}, \quad t \in (l, (l+1)]$$

for $l = 0, 1, 2, \dots, n-1$, $V_0^{(n)} = V_0^{(1)}$, and

$$Y_{t}^{(n)} \stackrel{\text{(d)}}{=} \begin{cases} \frac{1}{n}Y_{t}^{(1)} + \frac{n-1}{n}\mathbf{1}_{\mathbb{S}_{k}} & \text{if } t \in ((n-1),n] \\ \frac{1}{n}Y_{t}^{(1)} + \frac{n-2}{n}\mathbf{1}_{\mathbb{S}_{k}} + \frac{c}{n}\mathbf{1} & \text{if } t \in ((n-2), (n-1)] \\ \vdots & \vdots \\ \frac{1}{n}Y_{t}^{(1)} + \frac{1}{n}\mathbf{1}_{\mathbb{S}_{k}} + \frac{(n-2)c}{n}\mathbf{1} & \text{if } t \in (1,2] \\ \frac{1}{n}Y_{t}^{(1)} + \frac{(n-1)c}{n}\mathbf{1} & \text{if } t \in (0,1] \end{cases}$$

and $Y_0^{(n)} = c1$.

Since $Y_0^{(n)} = c1$, the terminal cost $|Y_0^{(n)}(X_0) - \bar{\mu}(Y_0^{(n)})|^2 = 0$. And since $X_t \sim \bar{\mu}$, for $l = 0, 1, 2, \dots, n-1$:

$$\mathsf{E}^{\bar{\mu}} \Big(\int_{l}^{(l+1)} \Gamma(Y_{t}^{(n)})(X_{t}) + |U_{t}^{(n)} + V_{t}^{(n)}(X_{t})|_{R}^{2} \,\mathrm{d}t \Big)$$

= $\frac{1}{n^{2}} \mathsf{E}^{\bar{\mu}} \Big(\int_{0}^{1} \Gamma(Y_{t}^{(1)})(X_{t}) + |U_{t}^{(1)} + V_{t}^{(1)}(X_{t})|_{R}^{2} \,\mathrm{d}t \Big)$

Therefore,

$$\mathsf{J}_{n}^{\bar{\mu}}(U^{(n)}) = \frac{1}{n} \mathsf{J}_{1}^{\bar{\mu}}(U^{(1)})$$

and thus the optimal value

$$\mathsf{J}_{n}^{\bar{\mu}} \leq \mathsf{J}_{n}^{\bar{\mu}}(U^{(n)}) = \frac{1}{n} \mathsf{J}_{1}^{\bar{\mu}}(U^{(1)}) \longrightarrow 0 \quad \text{as} \quad n \to \infty \tag{11}$$

Step 3: Suppose $v \in \mathcal{P}(\mathbb{S})$ and $1_{\mathbb{S}_k} \in \mathcal{C}$. In this final step, we show that $J_T^v \to 0$ and

$$\pi_T^{\nu}(1_{\mathbb{S}_k}) \xrightarrow{(T \to \infty)} 1_{\mathbb{S}_k}(X_0) \quad \mathsf{P}^{\nu}\text{-a.s.}$$

Let $\bar{\mu}_l \in \mathcal{P}(\mathbb{S})$ be the invariant measure for the l^{th} -ergodic class and $a_l \coloneqq v(1_{\mathbb{S}_l})$ for $l = 1, 2, \dots, m$. Choose the invariant measure as follows:

$$\bar{\mu} = a_1\bar{\mu}_1 + a_2\bar{\mu}_2 + \dots + a_m\bar{\mu}_m$$

From step 2, we know that $J_{\infty}^{\bar{\mu}} = 0$. Also, $v_T := e^{A^{\top}T} v \to \bar{\mu}$ as $T \to \infty$. Therefore, using part (ii) of Lemma 1,

$$\limsup_{T \to \infty} \mathsf{J}_T^{\mathsf{V}} \le \mathsf{J}_\infty^{\bar{\mu}} = 0$$

which shows that $J_T^{\nu} \to 0$ as $T \to \infty$.

Since S_k is an ergodic class,

$$\mathsf{J}_{T}^{\nu} = \mathsf{E}^{\nu} (|1_{\mathbb{S}_{k}}(X_{T}) - \pi_{T}^{\nu}(1_{\mathbb{S}_{k}})|^{2}) = \mathsf{E}^{\nu} (|1_{\mathbb{S}_{k}}(X_{0}) - \pi_{T}^{\nu}(1_{\mathbb{S}_{k}})|^{2})$$

By Fatou's lemma,

$$\mathsf{E}^{\nu}\left(\liminf_{T\to\infty}|\pi_{T}^{\nu}(1_{\mathbb{S}_{k}})-1_{\mathbb{S}_{k}}(X_{0})|^{2}\right)\leq\lim_{T\to\infty}\mathsf{J}_{T}^{\nu}=0$$

In step 1, we showed that $\pi_T^{\nu}(1_{\mathbb{S}_k})$ has an a.s. limit. So, limit is replaced as

$$\lim_{T\to\infty} |\pi_T^{\nu}(1_{\mathbb{S}_k}) - 1_{[X_0\in\mathbb{S}_k]}|^2 = 0 \quad \mathsf{P}^{\nu}\text{-a.s.}$$

and therefore also P^{μ} -a.s. whenever $\mu \ll v$.

Proof of the claim in step 2: Suppose $f \in C_T$. Since C_T is A-invariant (see Prop. 1), $e^{AT} f \in C$. Therefore, from definition of C, there is a deterministic constant $c \in \mathbb{R}$ and an admissible control $U \in \mathcal{U}$ such that the solution of the BSDE (3) is obtained with $Y_T = c1$ and $Y_0 = e^{AT} f$. Now

consider a second solution of the BSDE (3) with $Y_T = c1 + f$ and zero control input. Since A1 = 0, this second solution is $(Y_t, V_t) = (c1 + e^{A(T-t)}f, 0)$ for $t \in [0, T]$. By linearity, we subtract the two solutions to show that with $Y_T = f$ and control -U, one obtains $Y_0 = c1$.

VI. CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

Nonlinear filtering is an old subject. It is also notoriously difficult, which is why it remains an exciting research domain with many open questions remaining. This paper presents a new attack on filter stability, which we hope will open new avenues for research on filter performance and design. There are several avenues for future work:

Although we prove a filter stability result when the dual model is stabilizable, a more nuanced understanding is possible through the consideration of the controllable subspace C of the BSDE. Specifically, C is the space of functions for which the filter forgets the initial condition. A duality-based proof of this remains open.

Another important question is to relate our work to the deterministic definitions. Given the importance of the IOSS definitions of observability for deterministic models, it is of interest to investigate the dual optimal control problem in its small noise limit (in particular as $R \downarrow 0$).

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APPENDIX

A. Technical background

In this section, we describe some technical background on well-posedness of the optimal control problem (8) with respect to P^{v} . We assume $\mu, \mathsf{v} \in \mathsf{P}(\mathbb{S})$ and $\mu \ll \mathsf{v}$.

Define the innovation process $I^{v} := \{I_{t}^{v} \in \mathbb{R} : t \in [0, T]\}$ and the covariance process $\Sigma^{v} := \{\Sigma_{t}^{v} \in \mathbb{R}^{d \times d} : 0 \le t \le T\}$ as follows:

(innovation):
$$I_t^{\mathcal{V}} \coloneqq Z_t - \int_0^t \pi_s^{\mathcal{V}}(h) \, \mathrm{d}s$$

(covariance): $\Sigma_t^{\mathcal{V}} \coloneqq \mathrm{diag}(\pi_t^{\mathcal{V}}) - \pi_t^{\mathcal{V}}(\pi_t^{\mathcal{V}})^{\top}$

It is known that I^{ν} is P^{ν} -w.p. and moreover the filtration generated by Z and I are identical [19].

The following identity is proved in [12, Theorem 5]: For any admissible control input U

$$J_T^{\nu}(U) = \mathsf{E}^{\nu} \Big(Y_t^{\top} \Sigma_t^{\nu} Y_t + \int_t^T \ell(Y_s, V_s, U_s; X_s) \, \mathrm{d}s \Big) \\ + \mathsf{E}^{\nu} \Big(\int_0^t \left| U_s + (h^{\top} \Sigma_s^{\nu} Y_s + \pi_s^{\nu}(V_s)) \right|^2 \, \mathrm{d}s \Big)$$

From this identity, the following conclusions are deduced:

- (i) The minimum value $J_T^{\nu} = \mathsf{E}^{\nu}(f^{\top}\Sigma_T^{\nu}f)$.
- (ii) The optimal control is of the feedback form

$$U_t^{\mathcal{V}} = -h^{\mathsf{T}} \Sigma_t^{\mathcal{V}} Y_t - \pi_t^{\mathcal{V}} (V_t), \quad \le t \le T$$
(12)

Using the optimal control law (12) in the BSDE (3) results in the following linear feedback control system:

$$-dY_t = \left(\left(A - hh^{\mathsf{T}} \Sigma_t^{\mathsf{V}} \right) Y_t + h \cdot V_t - h\pi_t^{\mathsf{V}} \left(V_t \right) - V_t \pi_t^{\mathsf{V}} \left(h \right) \right) dt - V_t dI_t^{\mathsf{V}}$$

$$Y_T = f \tag{13}$$

The following proposition provides the answer to the three concerns in Sec. V-A:

Proposition 5: Consider the linear BSDE (13) with $f \in \mathbb{R}^d$ and T > 0. Then

(i) There exists a unique solution $(Y^{\nu}, V^{\nu}) \in L^{2}_{\mathcal{Z}}([0, T]; \mathbb{R}^{d} \times \mathbb{R}^{d})$. The optimal value $J^{\nu}_{T}(U^{\nu}; f) \leq \frac{1}{4} |\operatorname{osc}(f)|^{2}$.

(ii) Suppose $\mu \ll v$. The optimal control U^{v} is in $L^{2}_{\mathcal{Z}}([0,T];\mathbb{R})$ also with respect to the P^{μ} -measure (so it is admissible). The value

$$J_T^{\mu}(U^{\nu};f) + |\mu(Y_0^{\nu}) - \nu(Y_0^{\nu})|^2 \le \max_{x \in \mathbb{S}} \frac{\mu(x)}{\nu(x)} \frac{1}{4} |\operatorname{osc}(f)|^2$$

(iii) (Continuity property) Consider a family $\{v_n \in P(\mathbb{S}) : n = 1, 2, \cdots\}$ such that $v_n \ll v$ and $v_n \rightarrow v$. Then

$$\left|\mathsf{J}_{T}^{\nu_{n}}(U^{\nu};f)-\mathsf{J}_{T}^{\nu}(U^{\nu};f)\right|\longrightarrow0\tag{CP}$$

where the convergence is uniform in T.

Proof:

Part (i): The optimal control system (13) is a linear BSDE with random but bounded coefficients. The coefficients are bounded because each element of π_t^{ν} and Σ_t^{ν} is in [0,1]. Part (i) follows from existence uniqueness theory of linear BSDE [20, Theorem 7.2.2]. Because $J_T^{\nu} = E^{\nu}(f^{\top}\Sigma_T^{\nu}f)$, the uniform bound readily follows.

Part (ii): Since $\mu \ll v$, $\mathsf{P}^{\mu} \ll \mathsf{P}^{\nu}$ with $\frac{\mathrm{d}\mathsf{P}^{\mu}}{\mathrm{d}\mathsf{P}^{\nu}} = \frac{\mu(X_0)}{\nu(X_0)}$. So,

$$\mathsf{E}^{\mu} \Big(\int_{0}^{T} U_{t}^{2} \, \mathrm{d}t \Big) = \mathsf{E}^{\nu} \Big(\frac{\mu(X_{0})}{\nu(X_{0})} \int_{0}^{T} U_{t}^{2} \, \mathrm{d}t \Big)$$
$$\leq \max_{x \in \mathbb{S}} \frac{\mu(x)}{\nu(x)} \mathsf{E}^{\nu} \Big(\int_{0}^{T} U_{t}^{2} \, \mathrm{d}t \Big)$$

Therefore if U is P^{v} -admissible then it is also P^{μ} -admissible. From duality (5),

$$\mathsf{J}_{T}^{\mu}(U^{\nu};f) + |\mu(Y_{0}^{\nu}) - \nu(Y_{0}^{\nu})|^{2} = \mathsf{E}^{\mu}(|f(X_{T}) - \pi_{T}^{\nu}(f)|^{2})$$

and using the change of measure the righthand-side

$$\mathsf{E}^{\mu}\big(|f(X_T) - \pi_T^{\nu}(f)|^2\big) \le \max_{x \in \mathbb{S}} \frac{\mu(x)}{\nu(x)} \mathsf{E}^{\nu}\big(|f(X_T) - \pi_T^{\nu}(f)|^2\big)$$

The result follows from using the bound from part (i).

Part (iii): We have

$$J_T^{\nu_n}(U^{\nu}; f) = \mathsf{E}^{\nu_n} \Big(|Y_0^{\nu}(X_0) - \nu_n(Y_0^{\nu})|^2 + \int_0^T \ell(Y_t^{\nu}, V_t^{\nu}, U_t^{\nu}; X_t) \, \mathrm{d}t \Big)$$

We show $J_T^{\nu_n}(U) \longrightarrow J_T^{\nu}(U)$ if $\nu_n \rightarrow \nu$. We consider each of the two terms:

1) The first term is written as $Y_0^{\nu \top} \Sigma_0^{\nu_n} Y_0^{\nu}$. We have

$$\begin{split} \|\Sigma_{0}^{\nu_{n}} - \Sigma_{0}^{\nu}\|_{2} &= \|\operatorname{diag}(\nu_{n} - \nu) - (\nu_{n}\nu_{n}^{\top} - \nu\nu)\|_{2} \\ &\leq \|\operatorname{diag}(\nu_{n} - \nu)\|_{2} + \|\nu_{n}\nu_{n}^{\top} - \nu\nu^{\top}\|_{2} \\ &\leq 3\|\nu_{n} - \nu\|_{\infty} \xrightarrow{(n \to \infty)} 0 \end{split}$$

By part (i) $Y_0^{\nu} \Sigma_0^{\nu} Y_0^{\nu}$ is uniformly bounded so the limit is well-defined.

2) For the integral term, let $\xi_T := \int_0^T \ell(Y_t^v, V_t^v, U_t^v; X_t) dt$. Then

$$\mathsf{E}^{\mathsf{v}}(\xi_{T}) = \sum_{i=1}^{d} v_{n}(i) \mathsf{E}^{\delta_{i}}(\xi_{T}), \ \mathsf{E}^{v_{n}}(\xi_{T}) = \sum_{i=1}^{d} v_{n}(i) \mathsf{E}^{\delta_{i}}(\xi_{T})$$

Since $\mathsf{E}^{v}(\xi_T) \leq \mathsf{J}_T^{v}$,

$$|\mathsf{E}^{\mathbf{v}_n}(\xi_T) - \mathsf{E}^{\mathbf{v}}(\xi_T)| = \sum_{i=1}^d |\mathbf{v}_n(i) - \mathbf{v}(i)| \mathsf{E}^{\delta_i}(\xi_T) \longrightarrow 0$$

and the convergence is uniform in T.

B. Proof of Proposition 2

- (i) For the ergodic case, 0 is a simple eigenvalue of the matrix A and S₀ = span{1}.
- (ii) Suppose $\mathbb{S} = \bigcup_{k=1}^{m} \mathbb{S}_k$ is a ergodic partition. By choosing an appropriate coordinate, the rate matrix

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}$$

where A_k has a simple eigenvalue at 0 with the associated eigenvector $1_{\mathbb{S}_k}$. (This is so because \mathbb{S}_k is an ergodic class.) Therefore,

$$S_0 = \operatorname{span}\{1_{\mathbb{S}_1}, 1_{\mathbb{S}_2}, \cdots, 1_{\mathbb{S}_m}\}$$

C. Proof of the splitting (7)

Suppose $\mathbb{S} = \bigcup_{k=1}^{m} \mathbb{S}_k$ is an ergodic partition. For each such ergodic class with $\mathsf{P}^{\mathsf{v}}([X_0 \in \mathbb{S}_k]) > 0$ define

$$v_k(x) \coloneqq \begin{cases} \frac{v(i)}{\mathsf{P}^v([X_0 \in \mathbb{S}_k])} & \text{if } x \in \mathbb{S}_k \\ 0 & \text{if } x \notin \mathbb{S}_k \end{cases}$$

Clearly $v_k \ll v$ and

$$\frac{\mathrm{d}\mathsf{P}^{v_k}}{\mathrm{d}\mathsf{P}^{v}}(\omega) = \sum_{x \in \mathbb{S}} \frac{v_k(x)}{v(x)} \mathbb{1}_{[X_0 = x]}(\omega) = \frac{\mathbb{1}_{[X_0(\omega) \in \mathbb{S}_k]}(\omega)}{\mathsf{P}^{v}([X_0 \in \mathbb{S}_k])}$$

An application of the Bayes' formula gives

$$\mathsf{E}^{\mathsf{v}_{k}}(f(X_{T})|\mathcal{Z}_{T}) = \frac{\mathsf{E}^{\mathsf{v}}(f(X_{T})\mathbf{1}_{[X_{0}\in\mathbb{S}_{k}]}|\mathcal{Z}_{T})}{\mathsf{E}^{\mathsf{v}}(\mathbf{1}_{[X_{0}\in\mathbb{S}_{k}]}|\mathcal{Z}_{T})}$$

and therefore

$$\mathsf{E}^{\mathsf{v}}\big(f(X_T)\mathbf{1}_{[X_0\in\mathbb{S}_k]}|\mathcal{Z}_T\big) = \pi_T^{\mathsf{v}}(\mathbf{1}_{\mathbb{S}_k})\pi_T^{\mathsf{v}_k}(f) \tag{14}$$

where we have used the fact that

$$\mathbb{1}_{[X_0(\omega)\in\mathbb{S}_k]}(\omega) = \mathbb{1}_{[X_T(\omega)\in\mathbb{S}_k]}(\omega) \quad \mathsf{P}^{\nu}-a.s.$$

Note that the identity (14) is true for all $k = 1, 2, \dots, m$. (If $\mathsf{P}^{v}([X_0 \in \mathbb{S}_k]) = 0$ then both sides are zero.) Upon summing the identity over the index *k*, one arrives at

$$\pi_T^{\mathcal{V}}(f) = \sum_{k=1}^m \pi_T^{\mathcal{V}}(1_{\mathbb{S}_k}) \pi_T^{\mathcal{V}_k}(f)$$

If $\mathsf{P}^{v}([X_0 \in \mathbb{S}_k]) = 0$ then take v_k to be any probability measure with support on \mathbb{S}_k (e.g., the invariant measure for the restriction of the Markov process on \mathbb{S}_k).

D. Proof of Prop. 4

Suppose C is the controllable subspace with dimension strictly less than d. Consider the splitting $\mathbb{R}^d = C \oplus C^{\perp}$ and an associated orthogonal transformation $\mathcal{T} : C \oplus C^{\perp} \to \mathbb{R}^d$ such that

$$Y_t = \mathcal{T}\bar{Y}_t, \ V_t = \mathcal{T}\bar{V}_t \quad \text{where} \quad \bar{Y}_t = \begin{bmatrix} \bar{Y}_t^c \\ \bar{Y}_t^{uc} \end{bmatrix}, \ \bar{V}_t = \begin{bmatrix} \bar{V}_t^c \\ \bar{Y}_t^{uc} \end{bmatrix}$$

With respect to the new coordinates, the BSDE (3) becomes

$$-\mathrm{d}\bar{Y}_t = \left(\bar{A}\bar{Y}_t + \bar{h}U_t + \bar{K}\bar{V}_t\right)\mathrm{d}t - \bar{V}_t\,\mathrm{d}Z_t, \quad \bar{Y}_T = \mathcal{T}^{-1}f = \bar{f} = \begin{bmatrix} \bar{f}^c\\\bar{f}^{uc} \end{bmatrix}$$

where (because C is A- and diag(h)-invariant) the matrices

$$\bar{A} = \mathcal{T}^{-1}A\mathcal{T} = \begin{bmatrix} \bar{A}_c & * \\ 0 & \bar{A}_{uc} \end{bmatrix}, \quad \bar{K} = \mathcal{T}^{-1}\operatorname{diag}(h)\mathcal{T} = \begin{bmatrix} \bar{K}_c & * \\ 0 & \bar{K}_{uc} \end{bmatrix}$$

and $\bar{h} = \mathcal{T}^{-1}h = \begin{bmatrix} \bar{h}_c \\ 0 \end{bmatrix}$. On the \mathcal{C}^{\perp} subspace

$$-\mathrm{d}\bar{Y}_t^{uc} = \left(\bar{A}_{uc}\bar{Y}_t^{uc} + K_{uc}\bar{V}_t^{uc}\right)\mathrm{d}t - \bar{V}_t^{uc}\mathrm{d}Z_t, \quad \bar{Y}_T^{uc} = \bar{f}^{uc}$$

whose unique solution is given by

$$\bar{Y}_{t}^{uc} = e^{\bar{A}_{uc}(T-t)} \bar{f}^{uc}, \quad \bar{V}_{t}^{uc} \equiv 0, \quad t \in [0,T]$$
(15)

This is so because f is a deterministic function.

If the BSDE is not stabilizable, then there exists a non-zero vector $\bar{\eta}$ such that $\bar{A}_{uc}\bar{\eta} = 0$. Set

$$\bar{f} = \begin{bmatrix} 0\\ \bar{\eta} \end{bmatrix} \implies \bar{Y}_0 = \begin{bmatrix} *\\ \bar{\eta} \end{bmatrix}$$

Since $1 \in C$, the length of the vector $Y_0|_{1^{\perp}}$ is at least $|\bar{\eta}| = |\bar{f}| = |f|$.

E. Proof of Lemma 1

The proof of the lemma requires a technical construction. Consider the time horizon $[0, T_1 + T_2]$. If $X_0 \sim \mu$ then $X_{T_1} \sim e^{A^{\top}T_1}\mu =: \mu_{T_1}$. This is useful to relate the properties of $J_{T_1+T_2}^{\mu}(\cdot)$ and $J_{T_2}^{\mu_{T_1}}(\cdot)$. For this purpose, consider first the time horizon $[T_1, T_1 + T_2]$. Over this time horizon, introduce the filtration

$$\tilde{\mathcal{Z}}_{t-T_1} := \{ Z_t - Z_{T_1} : T_1 \le t \le T_1 + T_2 \}$$

For a control $\tilde{U} \in L^2_{\tilde{Z}}([0,T_2])$, let $\{(\tilde{Y}_t, \tilde{V}_t) : t \in [0,T_2]\}$ denote the solution of the BSDE (3) with $\tilde{Y}_{T_2} = f$. The control \tilde{U} is extended to the time-horizon $[0,T_1+T_2]$ as follows:

$$U_{t} = \begin{cases} 0 & 0 \le t < T_{1} \\ \tilde{U}_{t-T_{1}} & T_{1} \le t \le T_{1} + T_{2} \end{cases}$$
(16)

The control $U \in \mathcal{U}$ and yields the following solution of the BSDE (3):

$$(Y_t, V_t) = \begin{cases} (e^{A(T_1 - t)} \tilde{Y}_0, 0) & 0 \le t < T_1 \\ (\tilde{Y}_{t - T_1}, \tilde{V}_{t - T_1}) & T_1 \le t \le T_1 + T_2 \end{cases}$$

Under this definition, we claim that

$$\mathsf{J}_{T_1+T_2}^{\mu}(U) = \mathsf{J}_{T_2}^{\mu_{T_1}}(\tilde{U}) \tag{17}$$

and then the two results in the lemma are direct consequences of the claim:

Part (i): Take $\mu = \overline{\mu}$ and $\widetilde{U} = U^{\overline{\mu}}$. Then

$$\mathsf{J}_{T_{1}+T_{2}}^{\bar{\mu}} \leq \mathsf{J}_{T_{1}+T_{2}}^{\bar{\mu}}(U) \stackrel{\mathrm{Eq.}\,(17)}{=} \mathsf{J}_{T_{2}}^{\bar{\mu}}(U^{\bar{\mu}}) = \mathsf{J}_{T_{2}}^{\bar{\mu}}$$

where we used the facts that (i) $\mu_{T_1} = e^{A^{\top}T_1}\bar{\mu} = \bar{\mu}$ because $\bar{\mu}$ is the invariant measure; and (ii) $U^{\bar{\mu}}$ is the optimal control for the $J_{T_2}^{\bar{\mu}}(\cdot)$ problem. Therefore, $J_T^{\bar{\mu}}$ is monotone in *T* and converges as $T \to \infty$. Denote the limit as $J_{\infty}^{\bar{\mu}}$.

Part (ii): Let $T = T_1 + T_2$. For $\mu \in \mathcal{P}(\mathbb{S})$, with $\tilde{U} = U^{\bar{\mu}}$

$$J_T^{\mu} \le J_{T_1+T_2}^{\mu}(U) \stackrel{\text{Eq.}(17)}{=} J_{T_2}^{\mu_{T_1}}(U^{\bar{\mu}})$$
(18)

We have

$$|\mathsf{J}_{T_{2}}^{\mu_{T_{1}}}(U^{\bar{\mu}}) - \mathsf{J}_{\infty}^{\bar{\mu}}| \leq |\mathsf{J}_{T_{2}}^{\mu_{T_{1}}}(U^{\bar{\mu}}) - \mathsf{J}_{T_{2}}^{\bar{\mu}}(U^{\bar{\mu}})| + |\mathsf{J}_{T_{2}}^{\bar{\mu}}(U^{\bar{\mu}}) - \mathsf{J}_{\infty}^{\bar{\mu}}|$$

The second term on the righthand-side does not depend upon T_1 . Because $U^{\bar{\mu}}$ is the optimal control input, this term goes to zero as $T_2 \rightarrow \infty$: That is, given $\varepsilon > 0$, there exist an n_2 such that

$$|\mathsf{J}_{T_2}^{\mu}(U^{\mu}) - \mathsf{J}_{\infty}^{\mu}| \le \varepsilon \quad \forall T_2 \ge n_2$$

Now fix $T_2 = n_2$ and apply continuity property (CP) to the first term on the righthand-side: There exists $n_1 = n_1(n_2)$ such that

$$\left|\mathsf{J}_{n_{2}}^{\mu_{T_{1}}}(U^{\bar{\mu}})-\mathsf{J}_{n_{2}}^{\bar{\mu}}(U^{\bar{\mu}})\right|\leq\varepsilon\quad\forall T_{1}\geq n_{1}$$

Combine these inequalities concludes for all $T_1 \ge n_1$,

$$\mathsf{J}_{n_2}^{\mu_{T_1}}(U^{\bar{\mu}}) \leq \mathsf{J}_{\infty}^{\bar{\mu}} + 2\varepsilon$$

From (18), $J_T^{\mu} \leq J_{n_2}^{\mu_{T-n_2}} (U^{\bar{\mu}})$. Therefore, for all $T \geq n_1 + n_2$,

$$\mathsf{J}_{T}^{\mu} \leq \mathsf{J}_{n_{2}}^{\mu_{T-n_{2}}} \left(U^{\bar{\mu}} \right) \leq \mathsf{J}_{\infty}^{\bar{\mu}} + 2\varepsilon$$

Since ε is arbitrary, the result follows.

It remains to prove the claim (17). We have

$$J_T^{\mu}(U) = \mathsf{E}^{\mu} \Big(|Y_0(X_0) - \mu(Y_0)|^2 + \int_0^{T_1} \Gamma(Y_t)(X_t) \, \mathrm{d}t \Big) \\ + \quad \mathsf{E}^{\mu} \Big(\int_{T_1}^{T_1 + T_2} \Gamma(Y_t)(X_t) + |U_t + V_t(X_t)|_R^2 \, \mathrm{d}t \Big)$$

Each of the two terms is simplified separately.

Consider the control input *U* defined according to (16). Since $Y_{T_1} = \tilde{Y}_0$ is a deterministic function and the control is set to be zero, $V_t = 0$ on $0 \le t < T_1$ and the BSDE becomes ODE:

$$-\frac{\mathrm{d}}{\mathrm{d}t}Y_t = AY_t, \quad Y_{T_1} = \tilde{Y}_0$$

A straightforward calculation shows that

$$E^{\mu} \Big(|Y_0(X_0) - \mu(Y_0)|^2 + \int_0^{T_1} \Gamma(Y_t)(X_t) dt \Big)$$

= $E^{\mu} \Big(|Y_{T_1}(X_{T_1}) - \mu_{T_1}(Y_{T_1})|^2 \Big) = E^{\mu_{T_1}} \Big(|\tilde{Y}_0(X_0) - \mu_{T_1}(\tilde{Y}_0)|^2 \Big)$

The second term

$$\mathsf{E}^{\mu} \Big(\int_{T_1}^{T_1+T_2} \Gamma(Y_t)(X_t) + |U_t + V_t(X_t)|_R^2 \, \mathrm{d}t \Big)$$

= $\mathsf{E}^{\mu} \Big(\int_{T_1}^{T_1+T_2} \Gamma(\tilde{Y}_{t-T_1})(X_t) + |\tilde{U}_{t-T_1} + \tilde{V}_{t-T_1}(X_t)|_R^2 \, \mathrm{d}t \Big)$
= $\mathsf{E}^{\mu_{T_1}} \Big(\int_0^{T_2} \Gamma(\tilde{Y}_t)(X_t) + |\tilde{U}_t + \tilde{V}_t(X_t)|_R^2 \, \mathrm{d}t \Big)$

Combining the results of the two calculations yields:

$$\begin{aligned} \mathsf{J}_{T}^{\mu}(U) &= \mathsf{E}^{\mu_{T_{1}}}\Big(|\tilde{Y}_{0}(X_{0}) - \mu_{T_{1}}(\tilde{Y}_{0})|^{2}\Big) \\ &+ \mathsf{E}^{\mu_{T_{1}}}\Big(\int_{0}^{T_{2}} \Gamma(\tilde{Y}_{t})(\tilde{X}_{t}) + |\tilde{U}_{t} + \tilde{V}_{t}(\tilde{X}_{t})|_{R}^{2} \,\mathrm{d}t\Big) \\ &= \mathsf{J}_{T_{2}}^{\mu_{T_{1}}}(\tilde{U}) \end{aligned}$$