

Additive Networks of Chen-Fliess Series: Local Convergence and Relative Degree

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Abstract—Given an additive network of input-output systems where each node of the network is modeled by a locally convergent Chen-Fliess series, two basic properties of the network are established. First, it is shown that every input-output map between a given pair of nodes has a locally convergent Chen-Fliess series representation. Second, sufficient conditions are given under which the input-output map between a pair of nodes has a well defined relative degree as defined by its generating series. This analysis leads to the conclusion that this relative degree property is generic in a certain sense.

I. INTRODUCTION

Networks of nonlinear dynamical systems appear in many fields, especially in the natural sciences where the nonlinearity is often a key feature in generating the observed behavior [5], [13]. The vast majority of analysis of such networks is done in a finite dimensional state space setting using coupled systems of ordinary differential equations. In [8], however, the authors describe an alternative approach which uses only input-output models at each node of the network in the form of a locally convergent Chen-Fliess series [3], [4]. These weighted infinite sums of iterated integrals provide a convenient algebraic framework for describing the network's behavior without relying on any particular choice of coordinates as in the state space setting. Series coefficients for each node can be estimated via system identification techniques [10]. Computational tools were developed in [8] to determine, for example, how an input injected at one node affects the output observed at another node. Nevertheless, there are still a number of open questions regarding the basic properties of such networks. The focus here will be on so called *additive* networks, where the outputs of the nodes are simply added together and injected into other nodes, including self-loops. Other classes of aggregation functions such the multiplication of node outputs will not be addressed here.

This paper has two goals. The first goal to address the open problem stated in [8] regarding whether an additive network of locally convergent Chen-Fliess series always yields mappings between nodes which have locally convergent Chen-Fliess series representations. This hypothesis will be proved to be true and is independent of the network's topology. The approach taken is to identify for a given network an

associated *maximal network* whose growth bounds on the coefficients of the generating series between nodes upper bound all the growth bounds of the original network and are much easier to determine using conventional methods as presented in [18]. The particular growth bound derived turns out to be exactly equivalent to one discovered for a class of unity feedback systems described in [19]. The second goal is to provide sufficient conditions under which the input-output map between a pair of nodes has well defined relative degree as defined by its generating series [6], [9]. A simple counterexample will be given first to show that this property can fail to hold in certain situations. The proofs of the sufficient conditions rely on identifying certain properties first described in [9] in relation to a subgraph connecting a given input node and output node. It is also shown, however, that this relative degree property is *generic* in a certain sense. Namely, if the generating series for every node has relative degree and the connection strengths between the nodes are random, then every node pair has a generating series with well defined relative degree with probability one. An obvious application for this result is in the context of feedback linearization for networks [15], however, that application will not be pursued here.

The paper is organized as follows. To keep the presentation as self-contained as possible, the required preliminaries are briefly summarized in Section II. The question regarding the convergence of Chen-Fliess series for mappings between nodes is addressed in Section III. The subsequent section treats the property of relative degree. The paper's conclusions are summarized in the final section.

II. PRELIMINARIES

An *alphabet* $X = \{x_0, x_1, \dots, x_m\}$ is any nonempty and finite set of noncommuting symbols referred to as *letters*. A *word* $\eta = x_{i_1} \cdots x_{i_k}$ is a finite sequence of letters from X . The number of letters in a word η , written as $|\eta|$, is called its *length*. The empty word, \emptyset , is taken to have length zero. The collection of all words having length k is denoted by X^k . Define $X^* = \bigcup_{k \geq 0} X^k$, which is a monoid under the concatenation (Cauchy) product. Any mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*. Often c is written as the formal sum $c = \sum_{\eta \in X^*} \langle c, \eta \rangle \eta$, where the *coefficient* $\langle c, \eta \rangle \in \mathbb{R}^\ell$ is the image of $\eta \in X^*$ under c . The *support* of c , $\text{supp}(c)$, is the set of all words having nonzero coefficients. A series c is *proper* when $\emptyset \notin \text{supp}(c)$. The set of all noncommutative formal power series over the alphabet X is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{R}^\ell \langle X \rangle$. For any $c, d \in$

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$\mathbb{R}\langle\langle X \rangle\rangle$, the scalar product is $\langle c, d \rangle := \sum_{\eta \in X^*} \langle c, \eta \rangle \langle d, \eta \rangle$, provided the sum is finite. The set $\mathbb{R}^\ell\langle\langle X \rangle\rangle$ is an associative \mathbb{R} -algebra under the concatenation product and an associative and commutative \mathbb{R} -algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words $x_i\eta, x_j\xi \in X^*$:

$$(x_i\eta) \sqcup (x_j\xi) = x_i(\eta \sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup \xi),$$

where $x_i, x_j \in X$ and with $\eta \sqcup \emptyset = \emptyset \sqcup \eta = \eta$ [3].

A. Chen-Fliess series

Given any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ one can associate a causal m -input, ℓ -output operator, F_c , in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$ -norm for a measurable real-valued function, u_i , defined on $[t_0, t_1]$. Let $L_{\mathfrak{p}}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each word $\eta = x_i\bar{\eta} \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \rightarrow C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0) d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The *Chen–Fliess series* corresponding to $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} \langle c, \eta \rangle E_\eta[u](t, t_0)$$

[3]. If there exist real numbers $K, M > 0$ such that

$$|\langle c, \eta \rangle| \leq KM^{|\eta|} |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$ and some $S > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [11]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent series* is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$, and F_c is referred to as a *Fliess operator*.

B. System interconnections

Given Fliess operators F_c and F_d , where $c, d \in \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup d}$, respectively [3]. When Fliess operators F_c and F_d with $c \in \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}_{LC}^m\langle\langle X \rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} \langle c, \eta \rangle \psi_d(\eta) (\mathbf{1})$$

[2]. Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X \rangle\rangle$ to the vector space endomorphisms on $\mathbb{R}\langle\langle X \rangle\rangle$, $\text{End}(\mathbb{R}\langle\langle X \rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ$

$\psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i \sqcup e)$, $i = 0, 1, \dots, m$ for any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where d_i is the i -th component series of d ($d_0 := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$.

C. Relative degree of a generating series

Let $X = \{x_0, x_1\}$. Following [6], a series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r if and only if it has the decomposition

$$c = c_N + Kx_0^{r-1}x_1 + x_0^{r-1}e$$

for some $K \neq 0$ and proper $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \text{supp}(e)$. This definition of relative degree is consistent with the classical definition whenever $y = F_c[u]$ is realizable [6], [7]. The following results will be of central importance in the work that follows.

Theorem 1: [9] If $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have distinct relative degrees r_c and r_d , respectively, then $c+d$ has relative degree $\min(r_c, r_d)$. On the other hand, if $r_c = r_d =: r$, then $c+d$ has relative degree r if and only if $\langle c, x_0^{r-1}x_1 \rangle + \langle d, x_0^{r-1}x_1 \rangle \neq 0$.

Corollary 1: If c_1, c_2, \dots, c_m have relative degree r_1, r_2, \dots, r_m , respectively, with $r_i \neq r_j$ when $i \neq j$, then the relative degree of $c_1 + c_2 + \dots + c_m$ is $\min_i(r_i)$.

Corollary 2: Suppose c_1, c_2, \dots, c_m have relative degree r_1, r_2, \dots, r_m , respectively. Let s_j denote the multiplicity of relative degree r_j . If for each $s_j > 1$ the series $c_{k_1}, c_{k_2}, \dots, c_{k_{s_j}}$ having relative degree r_j satisfy

$$\langle c_{k_1}, x_0^{r_j-1}x_1 \rangle + \langle c_{k_2}, x_0^{r_j-1}x_1 \rangle + \dots + \langle c_{k_{s_j}}, x_0^{r_j-1}x_1 \rangle \neq 0,$$

then the relative degree of $c_1 + c_2 + \dots + c_m$ is $\min_i(r_i)$.

Theorem 2: [9] If $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ have relative degrees r_c and r_d , respectively, then $r_{c \circ d}$ has relative degree $r_c + r_d$.

D. Formal realizations and representations

It is shown in [14] that a given Chen-Fliess series $y = F_c[u]$ can be written in terms of a state z evolving on a formal Lie group $\mathcal{G}(X)$ with Lie algebra $\widehat{\mathcal{L}}(X)$ and output map $y = \langle c, z \rangle$. This notion of a *universal control system* was generalized in [8] as follows to describe networks of Chen-Fliess series.

Definition 1: Let V_i be a vector field on $\mathcal{G}^n(X) := \mathcal{G}(X) \times \mathcal{G}(X) \times \dots \times \mathcal{G}(X)$, $i = 0, 1, \dots, m$ with

$$V_i : \mathcal{G}^n(X) \rightarrow T_z \mathcal{G}^n(X)$$

$$z = (z_1, \dots, z_n) \mapsto V_i(z) = (V_{i1}(z)z_1, \dots, V_{in}(z)z_n),$$

where $V_{ij}(z(t)) \in \widehat{\mathcal{L}}(X)$. The j -th component of the corresponding state equation on $\mathcal{G}^n(X)$ is

$$\dot{z}_j = \sum_{i=0}^m V_{ij}(z) z_j u_{ij}, \quad z_j(0) = z_{j0}.$$

Given $\hat{c}_k \in \mathbb{R}_{LC}^{\otimes n}\langle\langle X \rangle\rangle$, $k = 1, 2, \dots, \ell$, the k -th output equation is defined to be

$$y_k = \hat{c}_k(z).$$

Collectively, (V, z_0, \hat{c}) is a *formal realization* on $\mathcal{G}^n(X)$ of the formal input-output map $u \mapsto y$.

Analogous to the standard finite dimensional theory [12], [16], a series $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$ is said to have a *formal representation* when there exists a formal realization with the property that every coefficient of c can be written in terms of iterated Lie derivatives of the vectors fields acting on the output map and evaluated at z_0 , i.e., $\langle c, x_{i_1} \cdots x_{i_k} \rangle = L_{V_{i_k}} \cdots L_{V_{i_1}} \hat{c}(z_0)$.

III. ADDITIVE NETWORKS OF CHEN-FLIESS SERIES: LOCAL CONVERGENCE

In this section it is shown that every network of additively interconnected locally convergent Fliess operators has the property that the input-output maps between any two nodes can be represented by a locally convergent Fliess operator. The first definition describes the specific class of networks under consideration.

Definition 2: A set of m single-input, single-output Chen-Fliess series mapping u_i to y_i with generating series $c_i \in \mathbb{R}_{LC} \langle \langle X_i \rangle \rangle$, where $X_i = \{x_0, x_i\}$ is said to be an *additively interconnected network* \mathcal{N}_m with weighting matrix $W \in \mathbb{R}^{m \times m}$ if $u_i = v_i + \sum_{j=1}^m W_{ij} y_j$, $i = 1, 2, \dots, m$.

A network \mathcal{N}_m can therefore be viewed as a directed graph connecting m nodes, where the i -th node corresponds to a Chen-Fliess series with generating series c_i , $i = 1, 2, \dots, m$. Henceforth, it will be assumed that the connection weights are normalized so that $W_{ij} \in [0, 1]$, $i, j = 1, 2, \dots, m$. The following theorem follows directly from Theorem 5.1 in [8].

Theorem 3: The input-output map $v_i \mapsto y_j$ in any additively interconnected network \mathcal{N}_m has generating series $d_{ji} \in \mathbb{R} \langle \langle X_i \rangle \rangle$ which can be computed from a formal representation in terms of the vector fields

$$V_0(z) = \begin{bmatrix} x_0 z_1 \\ x_0 z_2 \\ \vdots \\ x_0 z_m \end{bmatrix} + \text{diag}(x_1 z_1, \dots, x_m z_m) W \begin{bmatrix} \langle c_1, z_1 \rangle \\ \langle c_2, z_2 \rangle \\ \vdots \\ \langle c_m, z_m \rangle \end{bmatrix}$$

$$V_i(z) = x_i z_i e_i$$

acting on $\hat{c}_j = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes c_j \otimes \mathbf{1} \cdots \otimes \mathbf{1} \in \mathbb{R}_{LC}^{\otimes m} \langle \langle X \rangle \rangle$ (c_j appears in the j -th position) and evaluated at $z_{j0} = \mathbf{1}$, $i, j = 1, 2, \dots, m$.

The next theorem states the main convergence result concerning additive networks.

Theorem 4: If \mathcal{N}_m is an additively interconnected network where the generating series for each node $c_i \in \mathbb{R}_{LC} \langle \langle X_i \rangle \rangle$, then the generating series for every input-output map $d_{ji} \in \mathbb{R}_{LC} \langle \langle X_i \rangle \rangle$. More specifically, if K_i, M_i denote the growth constants for c_i , then for all $i, j = 1, 2, \dots, m$

$$|\langle d_{ji}, \eta \rangle| < KM^{|\eta|} |\eta|!, \quad \forall \eta \in X^*$$

for some $K > 0$ and any $M > M_{\text{inf}}$, where

$$M_{\text{inf}} = \frac{\bar{M}}{1 - m\bar{K} \ln \left(1 + \frac{1}{m\bar{K}}\right)} \quad (1)$$

with $\bar{K} = \max_i K_i$ and $\bar{M} = \max_i M_i$.

Proof: It is first shown that each generating series d_{ji} is locally convergent. Consider the case where every

node series $c_i \in \mathbb{R}_{LC} \langle \langle X_i \rangle \rangle$ is a *maximal series* $\bar{c}_i := \sum_{\eta \in X^*} K_i M_i^{|\eta|} |\eta|! \eta$. That is, every coefficient of \bar{c}_i is growing at its maximal rate. While $y_i = F_{c_i}[u_i]$ may not have a finite dimensional state space realization, it is easily shown that a maximal series has the realization

$$\dot{z}_i = \frac{M_i}{K_i} z_i^2 (1 + u_i), \quad z_i(0) = K_i, \quad y_i = z_i$$

[19, Lemma 3]. Therefore, the corresponding network can be realized by

$$\dot{z}_i = \frac{M_i}{K_i} z_i^2 \left(1 + \sum_{j=1}^m W_{ij} z_j + v_i\right), \quad z_i(0) = K_i, \quad y_i = z_i, \quad (2)$$

$i = 1, 2, \dots, m$. As this realization of the input-output map $v \mapsto y$ is polynomial, it is clearly real analytic. Therefore, every generating series for $v_i \mapsto y_j$, say \bar{d}_{ji} , must be locally convergent [18, Lemma 4.2]. The claim now is that d_{ji} must also be locally convergent since $|\langle d_{ji}, \eta \rangle| \leq \langle \bar{d}_{ji}, \eta \rangle$ for all $\eta \in X^*$. This inequality is most easily deduced from the formal realization of $v_i \mapsto y_j$ given in Theorem 3, where the Lie derivatives used to compute the coefficients of d_{ji} will all be upper bounded in magnitude by the Lie derivatives computed using maximal series.

Next, a suitable geometric growth constant for the network \mathcal{N}_m is determined. First observe that the growth constants \bar{K} and \bar{M} constitute a worst case maximum growth rate for every node in the network. In light of the formal representation of any d_{ji} in Theorem 3, the growth rate of d_{ji} is upper bounded by the growth rate of the natural response $\langle \bar{d}_{ji}, x_0^k \rangle = L_{V_0}^k \hat{c}_j(\mathbf{1})$, $k \geq 0$, where $W_{ij} = 1$ for all i, j , and every non-trivial component of \hat{c}_j is the maximal series $\bar{c} = \sum_{\eta \in X^*} \bar{K} \bar{M}^{|\eta|} |\eta|! \eta$. (See [19, Lemma 7] for an alternative approach when $m = 1, 2$.) From the symmetry of such a *maximal network*, $z_i = z_j$ for all i, j . Applying these conditions to (2), the natural response at each node is given by the solution of the Abel differential equation

$$\dot{z} = \frac{\bar{M}}{\bar{K}} (z^2 + m z^3), \quad z(0) = \bar{K}. \quad (3)$$

It can be directly verified that this equation has the solution

$$z(t) = \frac{-\frac{1}{m}}{1 + \mathcal{W} \left[-\left(1 + \frac{1}{m\bar{K}}\right) \exp \left(\frac{\bar{M}}{m\bar{K}} t - \left(1 + \frac{1}{m\bar{K}}\right) \right) \right]},$$

where \mathcal{W} denotes the Lambert W -function, that is, the inverse of the function $f(x) = x \exp(x)$ corresponding to the principal branch of this multi-valued function [1]. As \mathcal{W} is known to be holomorphic on the complex plane, $z(t)$ will therefore be analytic at $t = 0$. The corresponding Taylor series has a radius of convergence determined by the singularity nearest to the origin, in this case

$$t^* = \frac{1}{\bar{M}} \left(1 - m\bar{K} \ln \left(1 + \frac{1}{m\bar{K}}\right)\right).$$

Applying a well known theorem from complex analysis (see [20, Theorem 2.4.3]) gives the infimum of all geometric growth constants for the maximal network, namely $M_{\text{inf}} =$

TABLE I
INTEGER SEQUENCES GENERATED BY MAXIMAL ADDITIVE NETWORK
WITH UNITY GROWTH CONSTANTS

m	a_n	M_{inf}	\hat{M}_n
1	1, 2, 10, 82, 938, 13778, 247210, ...	3.2589	3.22634
2	1, 3, 24, 318, 5892, 140304, ...	5.2891	5.23618
3	1, 4, 44, 804, 20556, 675588, ...	7.3017	7.22873
4	1, 5, 70, 1630, 53120, 2225480, ...	9.3088	9.21567
5	1, 6, 102, 2886, 114294, 5819190, ...	11.3132	11.2001
6	1, 7, 140, 4662, 217308, 13022688, ...	13.3163	13.1831

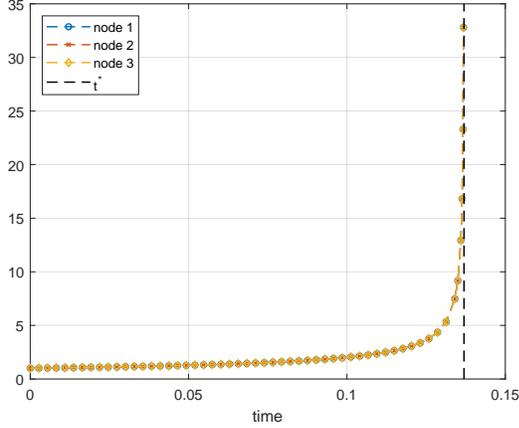


Fig. 1. Natural response of three node maximal network in Example 1.

$1/t^*$. (Note that the function $\lambda(x) = 1 - x \ln(1 + 1/x)$ is a decreasing function, which further justifies using the maximum K_i in the network as the worst case.) Since for any $M > M_{\text{inf}}$ there is a $K > 0$ to upper bound the fastest coefficient growth in the maximal network, the generating series for every node in the original network must also be upper bounded by this growth rate. ■

It is worth noting that (1) is in fact identical to the growth constant identified for unity feedback systems with m inputs as described in [19, Corollary 2]. While the network topologies are clearly distinct, this point of tangency is derived from the fact that unity feedback systems and additive maximal networks both have natural responses satisfying (3).

Example 1: Consider a maximal additive network where $K_i = M_i = 1$, $i = 1, 2, \dots, m$. The Taylor series of the natural response has integer coefficients a_n , $n \geq 1$ as shown in Table I. The coefficients when $m = 1$ correspond to the OEIS integer sequence A112487 [17]. The table also shows the growth rate M_{inf} computed from (1) and an estimate of the growth constant M computed from $\hat{M}_n = na_n/a_{n-1}$ when $n = 50$. The corresponding three node network was simulated in MatLab for the zero input case. The node responses, which are identical, are shown in Figure 1. Since the coefficients of every generating series are positive, it is known that the natural response of every node will have a finite escape time at $t = t^*$ (see [19, Theorem 11]). In this case, $t^* = 1/M_{\text{inf}} = 0.1379$, which is what was observed in the simulation. □

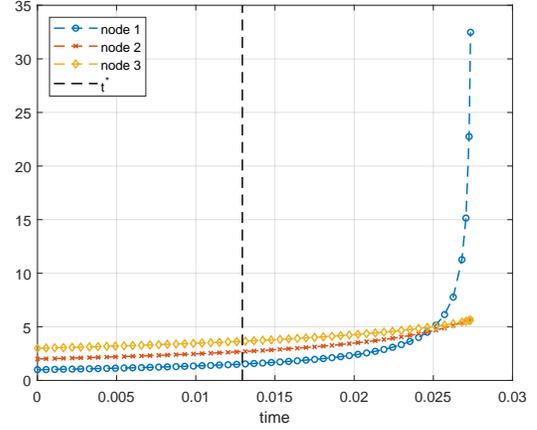


Fig. 2. Natural response of three node network in Example 2.

Example 2: Consider a three node additive network involving maximal series with $K_i = i$, $M_i = 5 - i$ and

$$W = \begin{bmatrix} 1 & 0.5 & 1 \\ 1 & 1 & 0 \\ 0.25 & 1 & 1 \end{bmatrix}.$$

Thus, $\bar{K} = 3$, $\bar{M} = 4$, and $M_{\text{inf}} = 77.2867$. The node natural responses are shown in Figure 2. As this network is not maximal, $t^* = 1/M_{\text{inf}} = 0.01294$ provides only a lower bound on the escape times of each node. □

IV. ADDITIVE NETWORKS OF CHEN-FLIESS SERIES: RELATIVE DEGREE

In this section the following question is addressed: When does the generating series of the mapping $v_i \mapsto y_j$ in an additively interconnected network \mathcal{N}_m have a well defined relative degree? The treatment starts with the easiest case first as described next. It is assumed throughout that \mathcal{N}_m is comprised of systems with generating series c_i which have relative degree r_i for $i = 1, 2, \dots, m$.

Definition 3: The i -th node in a network \mathcal{N}_m is said to be *fully connected* if $W_{ij} \neq 0$ for all $j \neq i$. A network \mathcal{N}_m is said to be *fully connected* if every node is fully connected.

Note that self-loops, i.e., when $W_{ii} \neq 0$, are not important in the present context as proportional output feedback is easily shown to preserve relative degree [9].

Theorem 5: If the i -th node in \mathcal{N}_m is fully connected, then the generating series d_{ji} for mapping $v_i \mapsto y_j$ has relative degree $r_{ji} = r_j + r_i$.

Proof: Observe that the full output at node j is

$$\begin{aligned} y_j &= F_{c_j} \left[v_j + \sum_{k=1}^m W_{jk} y_k \right] \\ &= F_{c_j} \left[v_j + \sum_{k,l=1}^m W_{jk} F_{d_{kl}} [v_l] \right]. \end{aligned}$$

For any $i \neq j$, that part of y_j in response to v_i acting alone (i.e., $v_l = 0$ for $l \neq i$) is given by

$$y_j = F_{c_j} \left[W_{ji} F_{c_i} [v_i] + \sum_{\substack{k=1 \\ k \neq i}}^m W_{jk} F_{d_{ki}} [v_i] + \sum_{\substack{k,l=1 \\ l \neq i}}^m W_{jk} F_{d_{kl}} [0] \right].$$

Note that for all $k \neq i$, $\text{supp}(d_{ki}) \subseteq x_0^r X^*$, where $r \geq r_i + 1$, since v_i passes through F_{c_i} in every path leading to the j -th node. In which case, the argument of F_{c_j} above has a generating series with relative degree r_i . The conclusion then follows immediately from Theorem 2. ■

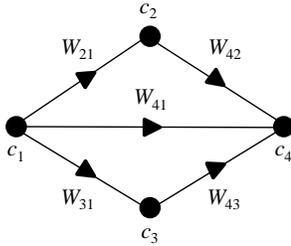


Fig. 3. Four node network in Example 3.

Example 3: Consider the network shown in Figure 3. The corresponding weighting matrix is

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ W_{21} & 0 & 0 & 0 \\ W_{31} & 0 & 0 & 0 \\ W_{41} & W_{42} & W_{43} & 0 \end{bmatrix}.$$

The network is clearly *not* fully connected, but node 4 is fully connected assuming $W_{4j} \neq 0$, $j = 1, 2, 3$. Therefore, applying the theorem above gives, for example, that $r_{41} = r_4 + r_1$.

Suppose now that $W_{41} = 0$ so that the theorem no longer applies. Further assume that $r_2 = r_3 = r$. Observe that

$$u_4 = W_{42} F_{c_2} [W_{21} F_{c_1} [u_1]] + W_{43} F_{c_3} [W_{31} F_{c_1} [v_1]],$$

and thus,

$$d_{41} = c_4 \circ [W_{42}(c_2 \circ (W_{21}c_1)) + W_{43}(c_3 \circ (W_{31}c_1))].$$

Both $c_2 \circ (W_{21}c_1)$ and $c_3 \circ (W_{31}c_1)$ have relative degree $r + r_1$, but d_{41} can fail to have relative degree. As a simple example, suppose $c_1 = c_2 = c_4 = x_1$ and $c_3 = -x_1$ so that $d_{41} = (W_{42}W_{21} - W_{43}W_{31})x_0^2x_1$. If W is such that $W_{42}W_{21} = W_{43}W_{31}$, then $d_{41} = 0$ does not have relative degree. On the other hand, if the symmetry condition $r_2 = r_3$ is broken, then it follows that d_{41} has relative degree $r_{41} = r_4 + \min(r_2, r_3) + r_1$. □

The final case in the example above suggests a sufficient condition for the general case. Namely, in the absence of these degenerate situations where a node is presented with an input whose underlying generating series does not have relative degree, the relative degree for d_{ji} will be well defined

and determined by a path from node i to node j whose *accumulated* relative degrees is minimal. To make this claim more precise, the following language adapted from signal flow graph theory will be useful.

Let \mathcal{N}_m be a given additive network. An *edge* is a directed line segment connecting two nodes. A *path* is a continuous set of edges connecting two nodes in \mathcal{N}_m and traversed in the direction indicated. A *forward path* is a path in which no node is encountered more than once. A *loop* is a path that originates and ends on the same node in which no node is encountered more than once. Finally, the *subgraph* G_{ji} from node i to node j is the simple graph (i.e., all loops are omitted) consisting of all forward paths connecting node i and node j .

The following theorems provide a sufficient condition under which the relative degree is well defined for a given input-output map $v_i \mapsto y_j$ in an additive network. Given a subgraph G_{ji} , the *accumulated relative degree* of node i is $r_i^+ = r_i$. If node $k \neq i$ in G_{ji} has N incoming edges from nodes i_1, i_2, \dots, i_N with accumulated relative degrees $r_{i_1}^+, r_{i_2}^+, \dots, r_{i_N}^+$, respectively, then the *accumulated relative degree* at node k is

$$r_k^+ = r_k + \min\{r_{i_1}^+, r_{i_2}^+, \dots, r_{i_N}^+\}.$$

Note this definition does not imply that any mappings defined by the network have relative degree, it simply computes the *potential* relative degree of such a mapping should it be well defined.

Theorem 6: Let i and j be fixed nodes in \mathcal{N}_m . If at every node $l \notin \{i, j\}$ the accumulated relative degrees of the nodes from every incoming edge are distinct, then the generating series d_{ji} for $v_i \mapsto y_j$ in \mathcal{N}_m has well defined relative degree equivalent to $r_{ji} = r_j^+$.

Proof: As feedback loops do not affect the relative degree of any forward path, it is sufficient to consider only the subgraph G_{ji} . The claim then follows directly from Corollary 1, Theorem 2, and the definition of accumulated relative degree. ■

The distinctness condition in the above theorem can be relaxed by utilizing instead the condition in Corollary 2.

Theorem 7: Let i and j be fixed nodes in \mathcal{N}_m . If at every node $l \notin \{i, j\}$ the accumulated relative degrees of the nodes from every incoming edge satisfy the condition in Corollary 2, then the generating series d_{ji} for $v_i \mapsto y_j$ in \mathcal{N}_m has well defined relative degree equivalent to $r_{ji} = r_j^+$.

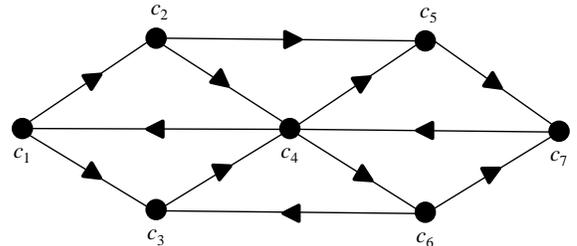


Fig. 4. Network in Example 4.

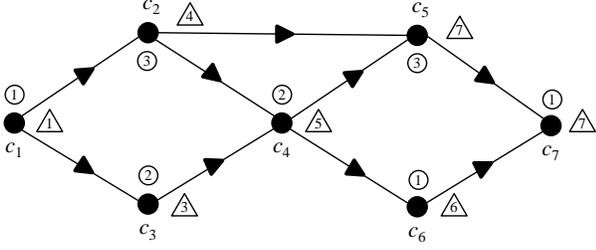


Fig. 5. Subgraph of forward paths for $v_1 \mapsto y_7$ in Example 4. The relative degree of each generating series c_i is the circled number. The accumulated relative degree at each node is the number in the triangle.

Example 4: Consider the network shown in Figure 4, where each weight $W_{ij} \in \{0, 1\}$ (i.e., 0 \sim not connected, 1 \sim connected), and the generating series for the nodes are:

$$\begin{aligned} c_1 &= K_1 x_1 + 2x_0 x_1 \\ c_2 &= x_0 + K_2 x_0^2 x_2 \\ c_3 &= K_3 x_0 x_3 + 3x_0^2 x_3^2 \\ c_4 &= 1 + K_4 x_0 x_4 - x_0^2 x_4 x_0 \\ c_5 &= 4x_0 + K_5 x_0^2 x_5 - 2x_0^4 x_5 \\ c_6 &= K_6 x_6 - x_6^2 \\ c_7 &= x_0 + 2 + K_7 x_7 + 4x_0 x_7 \end{aligned}$$

with $K_i \neq 0$ in every case. The subgraph of forward paths is shown in Figure 5. The relative degree of the generating series at each node is the circled number shown next to each node. The accumulated relative degree at each node is the number in the triangle. The goal is to determine the relative degree of the mapping $v_1 \mapsto y_7$, provided it is well defined. Observe that only nodes 4, 5 and 7, have more than one incoming edge. In each case, the accumulated relative degrees are distinct, namely, 3, 4; 4, 5; and 6, 7, respectively. Therefore, Theorem 6 applies, and $r_{71} = 7$. To independently verify this claim, the generating series d_{71} was computed using the full network via Theorem 3 with the aid of Mathematica and found to be

$$d_{71} = d_{71,N} + K_1 K_3 K_4 K_6 K_7 x_0^6 x_1 + x_0^6 e,$$

where

$$\begin{aligned} d_{71,N} &= x_0 + (4K_7 + K_6 K_7) x_0^2 + (16 + 4K_6 - K_7) x_0^3 + \\ &\quad (-4 + K_5 K_7 + 2K_4 K_6 K_7) x_0^4 + (4K_5 + 8K_4 K_6 - \\ &\quad 8K_4 K_7 + K_5 K_7 + 2K_4 K_6 K_7) x_0^5 + \dots \\ e &= (4K_1 K_3 K_4 K_6 - 6K_1 K_3 K_4 K_7 + K_1 K_2 K_5 K_7 + \\ &\quad K_1 K_2 K_4 K_6 K_7 + 2K_3 K_4 K_6 K_7) x_0 x_1 + \dots \end{aligned}$$

The relative degree of d_{71} is 7 as expected. \square

An additive network \mathcal{N}_m is said to have *complete relative degree* if every mapping $v_i \mapsto y_j$, $i, j = 1, 2, \dots, m$ has relative degree. From Theorem 5 it is immediate that fully connected networks have this property. Another class of networks sharing this property is given in the following theorem. It states that in some sense the property of a network having complete relative degree is *generic*.

Theorem 8: Consider an additive network \mathcal{N}_m where the weighting matrix has entries $W_{ij} \in \{0, 1\}$. If the unity weights are replaced with continuous random variables, then every sample network has complete relative degree.

Proof: At any given node, the incoming nodes may or may not have distinct accumulated relative degree. In the case where they do, then Theorem 6 applies, otherwise, Theorem 7 applies provided the condition for multiplicities greater than one can be met. Specifically, at node k with incoming edges from nodes j_1, j_2, \dots, j_N with accumulated relative degrees $r_{j_1}^+, r_{j_2}^+, \dots, r_{j_N}^+$, it is required that if r_j^+ is repeated $s_j > 1$ times then

$$\begin{aligned} W_{ij(1)} \langle d_{j(1)i}, x_0^{r_j^+ - 1} x_i \rangle + W_{ij(2)} \langle d_{j(2)i}, x_0^{r_j^+ - 1} x_i \rangle + \dots \\ + W_{ij(s_j)} \langle d_{j(s_j)i}, x_0^{r_j^+ - 1} x_i \rangle \neq 0, \end{aligned}$$

where $j(l) \in \{j_1, j_2, \dots, j_N\}$, and the W_{ijl} are random variables with any continuous distribution(s). But this condition is always true with probability one, and hence, the theorem is proved. \blacksquare

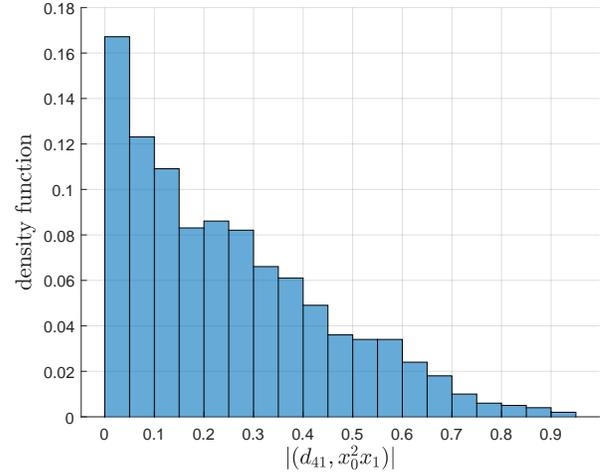


Fig. 6. Estimate of density function for $|(d_{41}, x_0^2 x_1)|$ in Example 5.

Example 5: Reconsider Example 3, where $W_{41} = 0$ and now W_{21}, W_{31}, W_{42} and W_{43} are i.i.d. random variables with a uniform distribution on $[0, 1]$. An estimate of the density function for the random variable $|(d_{41}, x_0^2 x_1)|$ is shown in Figure 6. In every case of the 1000 random networks generated, d_{41} had relative degree $r = 3$ as expected. \square

V. CONCLUSIONS

Two basic properties were established for an additive network of input-output systems where each node of the network is modeled by a convergent Chen-Fliess series. First it was shown that every input-output map between a pair of nodes has a locally convergence Chen-Fliess series representation. An explicit and in some cases achievable growth bound on the coefficients was computed using the notion of a maximal network. Second, sufficient conditions were given under which the input-output map between a pair

of nodes has a well defined relative degree as defined by its generating series. This analysis led to the conclusion that this relative degree property is generic when the connection strengths between nodes are randomized.

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