Convergence Analysis of Nonconvex Distributed Stochastic Zeroth-order Coordinate Method

Shengjun Zhang, Yunlong Dong, Dong Xie, Lisha Yao, Colleen P. Bailey, Shengli Fu

Abstract—This paper investigates the stochastic distributed nonconvex optimization problem of minimizing a global cost function formed by the summation of n local cost functions. We solve such a problem by involving zeroth-order (ZO) information exchange. In this paper, we propose a ZO distributed primal—dual coordinate method (ZODIAC) to solve the stochastic optimization problem. Agents approximate their own local stochastic ZO oracle along with coordinates with an adaptive smoothing parameter. We show that the proposed algorithm achieves the convergence rate of $\mathcal{O}(\sqrt{p}/\sqrt{T})$ for general nonconvex cost functions. We demonstrate the efficiency of proposed algorithms through a numerical example in comparison with the existing state-of-the-art centralized and distributed ZO algorithms.

I. INTRODUCTION

In this paper, we investigate stochastic distributed nonconvex optimization problems with only zeroth-order (ZO) information available. Such problems can be mathematically summarized in the form:

$$\min_{x \in \mathbb{R}^p} f(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\xi_i} [F_i(x, \xi_i)], \tag{1}$$

where n is the total number of agents, x is the decision variable, ξ_i is a random variable with dimension p, and $F_i(\cdot,\xi_i):\mathbb{R}^p\to\mathbb{R}$ is the stochastic function. Agent i is only able to access its own stochastic ZO information $F_i(x,\xi_i)$. For each agent i, the local cost function $f_i(x)$ is the expectation of the ZO information $\mathbb{E}_{\xi_i}[F_i(x,\xi_i)]$. Agents communicate with their neighbors via an undirected communication network graph \mathcal{G} .

Many algorithms based on first-order gradient information have been proposed in the literature and applied to various applications. Unfortunately, in many scenarios, the deceptively simple gradient information is not available or too expensive [1]–[3]. For instance, in

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simulation based optimization problems [4], the gradient information of objective functions is not available. In the machine learning community, universal attacking of deep neural networks is considered a black-box optimization problem [5]–[7], where it is too difficult to derive the explicit form of the gradient. Moreover, in the era of big data, people are dealing with complex data generating processes problems, however, the cost functions of such problems cannot be expressed explicitly [8]. In addition, decentralized optimization methods in general perform better than centralized ones in terms of robustness, data privacy and computation reduction [9]–[11].

Starting from early 1960s, derivative-free optimization (DFO) has been applied in several numerical and statistical problems [12]–[14]. With the rise of machine learning in the past decades, DFO has gained more attention and been investigated deeply. Recently, the most popular DFO method is utilizing the ZO information, which is treated as the counterpart of the firstorder gradient. In recent years, distributed optimization problems obtained more and more attention as they can be applied into massive networked systems including power systems, sensor networks, smart buildings, and smart manufacturing [11]. More specifically, [15]–[18] focus specifically on distributed ZO gradient descent methods. Yuan et al proposed distributed ZO with the push-sum technique [19], Yu et al extended mirror descent algorithm to distributed settings [20], and Tang et al provided distributed ZO gradient tracking algorithms [21]. Both Hajinezhad et al and Yi et al utilized primaldual techniques combined with ZO information [22], [23] and Beznosikov et al considered distributed ZO sliding algorithms [24].

Most of the aforementioned algorithms can handle the deterministic form of (1), e.g. $\min_{x \in \mathbb{R}^p} f(x) = \frac{1}{n} \sum_{i=1}^n F_i(x)$, where $F_i(x)$ is a deterministic function. For stochastic distributed settings in the exact form of (1), Hajjinezhad et al are able to solve, however, it requires a very high sampling size of $\mathcal{O}(T)$ to achieve the convergence rate of $\mathcal{O}(p^2n/T)$, which is not practically suitable for high dimensional decision variables [22].

In this paper, we propose a <u>ZO distributed primal-dual</u> coordinate method (ZODIAC) to solve the stochastic optimization problem (1). To our best knowledge, com-

pared to other existing ZO distributed algorithms, ZO-DIAC is the only one estimating ZO oracle along with coordinates, which improves the gradient estimation error [25]. Compared to [22], ZODIAC has lower sample requirements and is favorable for large-scale distributed optimization problems in practice. We show that our algorithm finds a stationary point with a convergence rate of $\mathcal{O}(\sqrt{p}/\sqrt{T})$ for general nonconvex cost functions using a fixed stepsize, which is faster than the centralized ZO algorithms in [7], [26]–[31] and the distributed ZO primal algorithm in [21].

The rest of this paper is organized as follows. Section II introduces some preliminary concepts. Sections III introduces ZODIAC and analyzes its convergence properties. Simulations are presented in Section IV. Finally, concluding remarks are offered in Section V.

Notations: \mathbb{N}_0 and \mathbb{N}_+ denote the set of nonnegative and positive integers, respectively. [n] denotes the set $\{1,\ldots,n\}$ for any $n\in\mathbb{N}_+$. $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices. \mathbb{B}^p and \mathbb{S}^p are the unit ball and sphere centered around the origin in \mathbb{R}^p under Euclidean norm, respectively. Given a differentiable function f, ∇f denotes the gradient of f. $\mathbf{1}_{n}$ ($\mathbf{0}_{n}$) denotes the column one (zero) vector of dimension n. col (z_1, \ldots, z_k) is the concatenated column vector of vectors $z_i \in \mathbb{R}^{p_i}, i \in [k]$. I_n is the n-dimensional identity matrix. Given a vector $[x_1,\ldots,x_n]^{\top} \in \mathbb{R}^n$, $\operatorname{diag}([x_1,\ldots,x_n])$ is a diagonal matrix with the *i*-th diagonal element being x_i . The notation $A \otimes B$ denotes the Kronecker product of matrices A and B. Moreover, we denote $\boldsymbol{x} = \operatorname{col}(x_1, \dots, x_n), \, \bar{\boldsymbol{x}} = \frac{1}{n} (\boldsymbol{1}_n^\top \otimes \boldsymbol{I}_p) \boldsymbol{x}, \, \bar{\boldsymbol{x}} =$ $\mathbf{1}_n \otimes \bar{x}$. $\rho(\cdot)$ stands for the spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues.

II. PRELIMINARIES

The following section discusses some background on graph theory, smooth functions, the gradient estimator, and additional assumptions used in this paper.

A. Graph Theory

Agents communicate with their neighbors through an underlying network, which is modeled by an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\ldots,n\}$ is the agent set, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the edge set, and $(i,j)\in\mathcal{E}$ if agents i and j can communicate with each other. For an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, let $\mathcal{A}=(a_{ij})$ be the associated weighted adjacency matrix with $a_{ij}>0$ if $(i,j)\in\mathcal{E}$ if $a_{ij}>0$ and zero otherwise. It is assumed that $a_{ii}=0$ for all $i\in[n]$. Let $\deg_i=\sum\limits_{j=1}^n a_{ij}$ denotes the weighted degree of vertex i. The degree matrix of graph \mathcal{G} is $\mathrm{Deg}=\mathrm{diag}([\deg_1,\cdots,\deg_n])$. The Laplacian matrix is $L=(L_{ij})=\mathrm{Deg}-\mathcal{A}$. Additionally, we

denote $K_n = I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^{\top}$, $L = L \otimes I_p$, $K = K_n \otimes I_p$, $H = \frac{1}{n} (\mathbf{1}_n \mathbf{1}_n^{\top} \otimes I_p)$. Moreover, from Lemmas 1 and 2 in [32], we know there exists an orthogonal matrix $[r \ R] \in \mathbb{R}^{n \times n}$ with $r = \frac{1}{\sqrt{n}} \mathbf{1}_n$ and $R \in \mathbb{R}^{n \times (n-1)}$ such that $R\Lambda_1^{-1} R^{\top} L = LR\Lambda_1^{-1} R^{\top} = K_n$, and $\frac{1}{\rho(L)} K_n \leq R\Lambda_1^{-1} R^{\top} \leq \frac{1}{\rho_2(L)} K_n$, where $\Lambda_1 = \mathrm{diag}([\lambda_2, \dots, \lambda_n])$ with $0 < \lambda_2 \leq \dots \leq \lambda_n$ being the eigenvalues of the Laplacian matrix L.

B. Smooth Function

Definition 1. A function $f(x): \mathbb{R}^p \to \mathbb{R}$ is smooth with constant $L_f > 0$ if it is differentiable and

$$\|\nabla f(x) - \nabla f(y)\| \le L_f \|x - y\|, \ \forall x, y \in \mathbb{R}^p.$$
 (2)

C. Gradient Approximation

Denote a random subset of the coordinates $S \subseteq \{1, 2, ..., p\}$ where the cardinality of S is $|S| = n_c$. We provide two options of gradient approximation, denoted g_i^e and defined by (3) and (4).

$$g_i^e = \frac{p}{n_c} \sum_{i \in \mathcal{S}} \frac{(F(x + \delta_i e_i, \xi) - F(x, \xi))}{\delta_i} e_i$$
 (3)

$$g_i^e = \frac{p}{n_c} \sum_{i \in \mathcal{S}} \frac{(F(x + \delta_i e_i, \xi) - F(x - \delta_i e_i, \xi))}{2\delta_i} e_i \quad (4)$$

The coordinates are sampled uniformly, i.e. $\Pr(i \in \mathcal{S}) = n_c/p$, which guarantees that both (3) and (4) are *unbiased* estimators of the *full* coordinate gradient estimator $\sum_{i=1}^d \frac{(F(x+\delta_i e_i,\xi)-F(x-\delta_i e_i,\xi))}{2\delta_i}e_i$ [33].

D. Assumptions

Assumption 1. The undirected graph G is connected.

Assumption 2. The optimal set \mathbb{X}^* is nonempty and the optimal value $f^* > -\infty$.

Assumption 3. For almost all ξ_i , the stochastic ZO oracle $F_i(\cdot, \xi_i)$ is smooth with constant $L_f > 0$.

Assumption 4. The stochastic gradient $\nabla_x F_i(x, \xi_i)$ has bounded variance for any jth coordinate of x, i.e., there exists $\zeta \in \mathbb{R}$ such that $\mathbb{E}_{\xi_i}[(\nabla_x F_i(x, \xi_i) - \nabla f_i(x))_j^2] \leq \zeta^2$, $\forall i \in [n], \ \forall j \in [p], \ \forall x \in \mathbb{R}^p$. It also implies that $\mathbb{E}_{\xi_i}[\|\nabla_x F_i(x, \xi_i) - \nabla f_i(x)\|^2] \leq \sigma_1^2 \triangleq p\zeta^2$, $\forall i \in [n], \ \forall x \in \mathbb{R}^p$.

Assumption 5. Local cost functions are similar, i.e., there exists $\sigma_2 \in \mathbb{R}$ such that $\|\nabla f_i(x) - \nabla f(x)\|^2 \le \sigma_2^2$, $\forall i \in [n], \ \forall x \in \mathbb{R}^p$.

Remark 1. There is no assumption made on convexity. Assumption 1 and 2 are basic and common in optimization literature. Assumptions 3 and 4 are standard for solving ZO stochastic optimization problems. Assumption 5 is slightly weaker than stating each ∇f_i is

bounded, which is commonly used in finite-sum type ZO optimization literature.

III. PROPOSED ZODIAC ALGORITHM

A. Algorithm Description

In order to handle stochastic optimization problems, we propose the ZODIAC algorithm, where we consider the novel distributed primal-dual scheme [34] with the stochastic coordinate estimators (3) and (4), summarized in Algorithm 1.

$$x_{i,k+1} = x_{i,k} - \eta \left(\alpha \sum_{j=1}^{n} L_{ij} x_{j,k} + \beta v_{i,k} + g_{i,k}^{e} \right),$$
(5a)

$$v_{i,k+1} = v_{i,k} + \eta \beta \sum_{j=1}^{n} L_{ij} x_{j,k},$$

$$\forall x_{i,0} \in \mathbb{R}^p, \sum_{j=1}^{n} v_{j,0} = \mathbf{0}_p, \ \forall i \in [n]. \quad (5b)$$

Algorithm 1 ZODIAC

- 1: **Input**: positive number α , β , η , and positive sequences $\{\delta_{i,k}\}$.
- 2: **Initialize**: $x_{i,0} \in \mathbb{R}^p$ and $v_{i,0} = \mathbf{0}_p$, $\forall i \in [n]$.
- 3: **for** $k = 0, 1, \dots$ **do**
- 4: **for** i = 1, ..., n in parallel **do**
- 5: Broadcast $x_{i,k}$ to \mathcal{N}_i and receive $x_{j,k}$ from $j \in \mathcal{N}_i$:
- 6: Select coordinates independently and uniformly;
- 7: Select $\xi_{i,k}$ independently;
- 8: **Option 1:** sample $F_i(x_{i,k} + \delta_{i,k}e_{i,k}, \xi_{i,k})$, and $F_i(x_{i,k}, \xi_{i,k})$;
- 9: Update $x_{i,k+1}$ by (5a) with (3);
- 10: **Option 2:** sample $F_i(x_{i,k} + \delta_{i,k}e_{i,k}, \xi_{i,k})$ and $F_i(x_{i,k} \delta_{i,k}e_{i,k}, \xi_{i,k})$;
- 11: Update $x_{i,k+1}$ by (5a) with (4);
- 12: Update $v_{i,k+1}$ by (5b).
- 13: end for
- 14: end for
- 15: Output: $\{\boldsymbol{x}_k\}$.

Algorithm (1) can be written in compact form as

$$egin{aligned} & oldsymbol{x}_{k+1} = oldsymbol{x}_k - \eta(\alpha oldsymbol{L} oldsymbol{x}_k + eta oldsymbol{v}_k + oldsymbol{g}_k^e), & ext{(6a)} \ & oldsymbol{v}_{k+1} = oldsymbol{v}_k + \eta eta oldsymbol{L} oldsymbol{x}_k, & orall oldsymbol{x}_0 \in \mathbb{R}^{np}, & \sum_{i=1}^n v_{i,0} = oldsymbol{0}_p. \end{aligned}$$

B. Convergence Analysis

Theorem 1. Suppose Assumptions 1–5 hold. For any given $T > n^3/p$, let $\{x_k, k = 0, ..., T\}$ be the output generated by Algorithm 1 with

$$\alpha = \kappa_1 \beta, \ \beta = \frac{\kappa_2 \sqrt{pT}}{\sqrt{n}}, \ \eta = \frac{\kappa_2}{\beta},$$

$$\delta_{i,k} \le \frac{\kappa_\delta}{p^{\frac{1}{4}} n^{\frac{1}{4}} (k+1)^{\frac{1}{4}}}, \ \forall k \le T,$$
(7)

where $\kappa_1 > \frac{1}{\rho_2(L)} + 1$, $\kappa_2 \in \left(0, \min\left\{\frac{(\kappa_1 - 1)\rho_2(L) - 1}{\rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1}, \frac{1}{5}\right\}\right)$, and $\kappa_{\delta} > 0$, then we have.

$$\frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = \mathcal{O}(\frac{\sqrt{p}}{\sqrt{T}}) + \mathcal{O}(\frac{n}{T}), \quad (8a)$$

$$\mathbb{E}[f(\bar{x}_T)] - f^* = \mathcal{O}(1),\tag{8b}$$

$$\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2 \right] = \mathcal{O}(\frac{n}{T}). \tag{8c}$$

Before proving Theorem 1, we introduce the following lemmas.

Lemma 1. Consider $f(x) = \mathbb{E}_{\xi}[F(x,\xi)]$, we have the following relationship,

$$\mathbb{E}\left[\|g_{i}^{e}\|^{2}\right]$$

$$\leq 2(p-1)\|\nabla f(x)\|^{2} + 2p\sigma_{1}^{2} + \frac{3p^{2}}{n_{c}}\left(\zeta^{2} + \frac{L_{f}^{2}\delta_{k}^{2}}{2}\right)$$

$$+ \frac{p^{2}L_{f}^{2}\delta_{k}^{2}}{2} \tag{9}$$

where $\delta_k = \max\{\delta_i\}, i \in [p].$

Proof. Apply the proposition III.2 in [33] and consider the coordinates are picked uniformly, then we have

$$\mathbb{E}\left[\|g_{i}^{e} - \nabla f(x)\|^{2}\right]$$

$$\leq \sum_{i=1}^{p} p\left[2\left(\nabla f(x)\right)_{i}^{2} + \frac{3}{n_{c}}\left(\zeta^{2} + \frac{L^{2}\delta_{i}^{2}}{2}\right) + \frac{L^{2}\delta_{i}^{2}}{2}\right]$$

$$-2\left\|\nabla f(x)\right\|^{2}$$
(10)

We can easily get Eq. (9) by simplifying the above inequality. \Box

Lemma 2. Suppose Assumptions 3–5 hold. Let $\{x_k\}$ be the sequence generated by Algorithm 1, $\boldsymbol{g}_k^e = \operatorname{col}(g_{1,k}^e,\ldots,g_{n,k}^e)$, $\boldsymbol{g}_k^0 = n\nabla f(\bar{\boldsymbol{x}}_k)$, $\bar{\boldsymbol{g}}_k^0 = \boldsymbol{H}\boldsymbol{g}_k^0 = \boldsymbol{1}_n \otimes \nabla f(\bar{\boldsymbol{x}}_k)$, then

$$\mathbb{E}\Big[\|\boldsymbol{g}_{k}^{e}\|^{2}\Big] \leq 6(p-1)\|\bar{\boldsymbol{g}}_{k}^{0}\|^{2} + 6(p-1)L_{f}^{2}\|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2} + 6n(p-1)\sigma_{2}^{2} + \frac{3np^{2}}{n_{c}}\left(\zeta^{2} + \frac{L_{f}^{2}\delta_{k}^{2}}{2}\right)$$

$$+2np\sigma_1^2 + \frac{np^2L_f^2\delta_k^2}{2}$$
 (11a)

$$\|\boldsymbol{g}_{k+1}^{0}\|^{2} \le 3(\eta^{2}L_{f}^{2}\|\boldsymbol{g}_{k}^{e}\|^{2} + n\sigma_{2}^{2} + \|\bar{\boldsymbol{g}}_{k}^{0}\|^{2}).$$
 (11b)

Proof. (i) Eq. (11a) is due to Lemma 1, Cauchy-Schwarz inequality and Assumption 5.

(ii) Eq. (11b) is established by Cauchy-Schwarz inequality, Assumption 3 and 5.

Lemma 3. Suppose Assumptions 1–5 hold, and we have fixed parameters $\alpha = \kappa_1 \beta$, β , and $\eta = \frac{\kappa_2}{\beta}$, where β is large enough, $\kappa_1 > \frac{1}{\rho_2(L)} + 1$ and $\kappa_2 \in \left(0, \min\{\frac{(\kappa_1 - 1)\rho_2(L) - 1}{\rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1}, \frac{1}{5}\}\right)$ are constants. Let $\{x_k\}$ be the sequence generated by Algorithm 1, then

$$\mathbb{E}[W_{k+1}] \leq W_k - \kappa_4 \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 - \frac{1}{2}(\kappa_2 - 5\kappa_2^2) \|\boldsymbol{v}_k + \frac{1}{\beta}\boldsymbol{g}_k^0\|_{\boldsymbol{K}}^2 - \frac{1}{8}\eta \|\bar{\boldsymbol{g}}_k^0\|^2 + \mathcal{O}(np)\eta^2 + \mathcal{O}(np^2)\eta\delta_k^2,$$
(12a)

$$\mathbb{E}[W_{4,k+1}] \le W_{4,k} + 2\eta L_f^2 \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 - \frac{1}{8}\eta \|\bar{\boldsymbol{g}}_k^0\|^2 + \mathcal{O}(p)\eta^2 + \mathcal{O}(np)\eta\delta_k^2.$$
(12b)

Proof. We provide the proof of Lemma 3 in the appendix.

We are now ready to prove Theorem 1.

Proof. Denote

$$\hat{V}_k = \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + \left\|\boldsymbol{v}_k + \frac{1}{\beta_k}\boldsymbol{g}_k^0\right\|_{\boldsymbol{K}}^2 + n(f(\bar{x}_k) - f^*).$$

We have

$$= \frac{1}{2} \|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2} + \frac{1}{2} \|\boldsymbol{v}_{k} + \frac{1}{\beta_{k}} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{Q} + \kappa_{1} \boldsymbol{K}}^{2} + \boldsymbol{x}_{k}^{\top} \boldsymbol{K} \left(\boldsymbol{v}_{k} + \frac{1}{\beta_{k}} \boldsymbol{g}_{k}^{0} \right) + n(f(\bar{x}_{k}) - f^{*})$$

$$\geq \frac{1}{2} \|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2} + \frac{1}{2} \left(\frac{1}{\rho(L)} + \kappa_{1} \right) \|\boldsymbol{v}_{k} + \frac{1}{\beta_{k}} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2}$$

$$- \frac{1}{2\kappa_{1}} \|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2} - \frac{1}{2} \kappa_{1} \|\boldsymbol{v}_{k} + \frac{1}{\beta_{k}} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} + n(f(\bar{x}_{k}) - f^{*})$$

$$\geq \min \left\{ \frac{1}{2\rho(L)}, \frac{\kappa_{1} - 1}{2\kappa_{1}} \right\} \hat{V}_{k} \geq 0, \tag{13}$$

Additionally, we can get $W_k \leq (\frac{\kappa_1+1}{2} + \frac{1}{2\rho_2(L)})\hat{V}_k$. Consider that $\beta = \kappa_2 \sqrt{pT}/\sqrt{n}$ and $T > n^3/p$, we

know that Lemma 3 are satisfied. So (12a) and (12b) hold. Summing (12a) over $k \in [0, T]$ and applying (13), we have

$$\frac{1}{T+1} \sum_{k=0}^{T} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \|x_{i,k} - \bar{x}_k\|^2\right]$$

$$\leq \frac{1}{\kappa_4} \left(\frac{W_0}{n(T+1)} + \frac{\mathcal{O}(n)\eta \delta_k^2}{T} + \frac{\mathcal{O}(n/p)\eta^2 \kappa_\delta}{\sqrt{T(T+1)}} \right) \\
= \mathcal{O}(\frac{n}{T}), \tag{14}$$

where $W_0 = \mathcal{O}(n)$, $\frac{W_0}{n(T+1)} = \mathcal{O}(\frac{1}{T})$, $\frac{n\mathcal{O}(p^2)\eta\delta_k^2}{T}$ $\mathcal{O}(\frac{n}{T})$, and $\frac{\mathcal{O}(n/p)\eta^2\kappa_\delta}{\sqrt{T(T+1)}} = \mathcal{O}(\frac{n}{pT})$, which gives (8c). From (12b), (7), and (13), summing (12b) over $k \in$

[0,T] similar to the way to get (8c), we have

$$\frac{1}{T+1} \sum_{k=0}^{T} \mathbb{E}[\|\nabla f(\bar{x}_{k})\|^{2}] = \frac{1}{n(T+1)} \sum_{k=0}^{T} \mathbb{E}[\|\bar{\boldsymbol{g}}_{k}^{0}\|^{2}]$$

$$\leq 8 \left(\frac{W_{4,0}}{n(T+1)\eta} + \frac{2L_{f}^{2}}{n(T+1)} \sum_{k=0}^{T} \mathbb{E}[\|\boldsymbol{x}_{k}\|_{K}^{2}] + \frac{\mathcal{O}(p)}{n} + \frac{\mathcal{O}(\sqrt{np})}{\sqrt{n(T+1)}} \right). \tag{15}$$

Noting that $\eta = \kappa_2/\beta_k = \sqrt{n}/\sqrt{pT}$, and $n/T < \sqrt{p}/\sqrt{nT}$ due to $T > n^3/p$, from (15) and (14), we

$$\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2] = \mathcal{O}(\frac{\sqrt{p}}{\sqrt{T}}) + \mathcal{O}(\frac{n}{T}),$$

which gives (8a).

Summing (12b) over $k \in [0, T]$, and using (7) yield

$$n(\mathbb{E}[f(\bar{x}_{T+1})] - f^*) = \mathbb{E}[W_{4,T+1}]$$

$$\leq W_{4,0} + \frac{2\sqrt{n}}{\sqrt{pT}} L_f^2 \sum_{k=0}^T \|\boldsymbol{x}_k\|_{\mathbf{K}}^2 + n\mathcal{O}(p)\eta^2 \frac{T+1}{T} + \mathcal{O}(np)\eta \delta_k^2 \sqrt{\frac{T+1}{T}}.$$
(16)

Noting that $W_{4,0} = \mathcal{O}(n)$ and $\sqrt{n}n/\sqrt{pT} < 1$ due to $T > n^3/p$, from (14) and (16), we have $\mathbb{E}[f(\bar{x}_{T+1})]$ – $f^* = \mathcal{O}(1)$, which gives (8b).

IV. NUMERICAL EXAMPLES

We consider a benchmark non-linear least square problem from the literature [6], [30]. The local cost function is given as $f_i(\mathbf{x}) = (y_i - \phi(\mathbf{x}; \mathbf{a}_i))^2$ for $i \in [n]$, $-\frac{1}{2\kappa_1} \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 - \frac{1}{2}\kappa_1 \|\boldsymbol{v}_k + \frac{1}{\beta_k} \boldsymbol{g}_k^0\|_{\boldsymbol{K}}^2 + n(f(\bar{x}_k) - f^*) \text{ where } \phi(\mathbf{x}; \mathbf{a}_i) = \frac{1}{1 + e^{-\mathbf{a}_i^T \mathbf{x}}}, \xi_i \text{ follows a standard normal standard norm$ distribution $\mathcal{N}_i(0, 0.01)$. To prepare the synthetic dataset, we randomly draw samples \mathbf{a}_i from $\mathcal{N}(\mathbf{0}, \mathbf{I})$ and set an optimal vector $\mathbf{x_{opt}} = \mathbf{1}$. The label is $y_i = 1$ if $\phi(\mathbf{x}_{opt}; \mathbf{a}_i) \geq 0.5$ and 0 otherwise. The training set has 2000 samples and the test set has 200 samples. We set the dimension d of a_i to 100, the batch size is 1, and the total iteration number is 50000. As suggested in the work [30], the smooth parameter $\delta = \frac{10}{\sqrt{Td}}$ The communication topology of 10 agents is generated randomly following the Erdős - Rényi model with the connection probability of 0.4.

We compare the proposed ZODIAC algorithm with the two estimator options, (3) and (4), against the current state-of-the-art centralized and distributed ZO algorithms: ZO-SGD [26], ZO-SCD [27], distributed ZO gradient tracking algorithm (ZO-GDA) [21] and ZONE-M [22]. The hyper-parameters used in the experiments are well-tuned based on performance and provided in Table I. The test accuracy of each algorithm is summarized in Table II. From Fig. 1, we can see that ZODIAC outperforms the existing algorithms and achieve better loss results. Additionally, both ZODIAC implementations have higher accuracy. Moreover, we provide the error of the gradient estimation in ZODICA in Fig. 2.

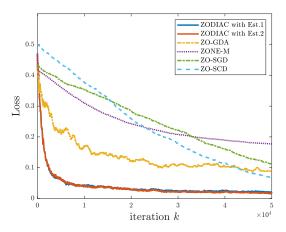


Fig. 1: Training loss evaluations.

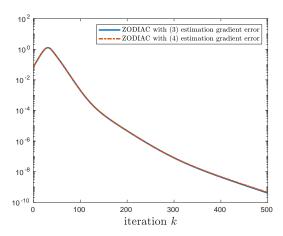


Fig. 2: The error of the gradient estimation.

¹ZODIAC is tested under the same parameters for both estimators (3) and (4).

TABLE I: Parameters for Binary Classification

Algorithm	Parameters
ZODIAC ¹	$\eta = 0.08, \alpha = 4, \beta = 3$
ZO-SGD [26]	$\mu = 0.01$
ZO-SCD [27]	$\mu = 0.01$
ZO-GDA [21]	$\eta = 0.08/k^{10^{-5}}$
ZONE-M [22]	$\rho = 0.1\sqrt{k}$

TABLE II: Accuracy

Algorithm	Accuracy(%)
ZODIAC with (3)	99.0
ZODIAC with (4)	98.5
ZO-SGD [26]	85.5
ZO-SCD [27]	91.0
ZO-GDA [21]	91.0
ZONE-M [22]	89.5

V. CONCLUSIONS

In this paper, we investigated the stochastic distributed nonconvex optimization problem and proposed a stochastic coordinate method within a primal–dual scheme, ZODIAC. We demonstrated that the proposed algorithm achieves the convergence rate of $\mathcal{O}(\sqrt{p}/\sqrt{T})$ for general nonconvex cost functions. Additionally, we illustrated the efficacy and accuracy of ZODIAC through a benchmark example in comparison with the existing state-of-the-art centralized and distributed ZO algorithms.

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APPENDIX

PROOF OF LEMMA 3

Proof. Consider the following Lyapunov candidate function

$$W_{k} = \underbrace{\frac{1}{2} \|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2}}_{W_{1,k}} + \underbrace{\frac{1}{2} \|\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0}\|_{\boldsymbol{Q} + \kappa_{1} \boldsymbol{K}}^{2}}_{W_{2,k}} + \underbrace{\boldsymbol{x}_{k}^{\top} \boldsymbol{K} \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0}\right)}_{W_{3,k}} + \underbrace{n(f(\bar{x}_{k}) - f^{*})}_{W_{4,k}}$$
(17)

where $Q = R\Lambda_1^{-1}R^{\top} \otimes I_p$. Additionally, we denote $g_{i,k}^s = \nabla \mathbb{E}[f_i(x + \delta_{i,k}e_i)], \ \boldsymbol{g}_k^s = \operatorname{col}(g_{1,k}^s, \dots, g_{n,k}^s), \ \bar{\boldsymbol{g}}_k^s = \boldsymbol{H}\boldsymbol{g}_k^s, \ \bar{\boldsymbol{g}}_k^e = \frac{1}{n}(\boldsymbol{1}_n^{\top} \otimes I_p)\boldsymbol{g}_k^e, \ \text{and} \ \bar{\boldsymbol{g}}_k^e = \boldsymbol{1}_n \otimes \bar{\boldsymbol{g}}_k^e = \boldsymbol{H}\boldsymbol{g}_k^e$

(i) We have

$$\mathbb{E}[W_{1,k+1}] = \mathbb{E}\left[\frac{1}{2}\|\boldsymbol{x}_{k+1}\|_{\boldsymbol{K}}^{2}\right]$$

$$\stackrel{\text{Eq. } 6a}{=} \mathbb{E}\left[\frac{1}{2}\|\boldsymbol{x}_{k} - \eta(\alpha \boldsymbol{L}\boldsymbol{x}_{k} + \beta \boldsymbol{v}_{k} + \boldsymbol{g}_{k}^{e})\|_{\boldsymbol{K}}^{2}\right]$$

$$\stackrel{\text{(a)}}{=} \mathbb{E}\left[\frac{1}{2}\|\boldsymbol{x}_{k}\|_{\boldsymbol{K}}^{2} - \eta\alpha\|\boldsymbol{x}_{k}\|_{\boldsymbol{L}}^{2} + \frac{1}{2}\eta^{2}\alpha^{2}\|\boldsymbol{x}_{k}\|_{\boldsymbol{L}^{2}}^{2}\right]$$

$$- \eta\beta\boldsymbol{x}_{k}^{\top}(\boldsymbol{I}_{np} - \eta\alpha\boldsymbol{L})\boldsymbol{K}\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{e}\right)$$

$$+ \frac{1}{2}\eta^{2}\beta^{2}\|\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{e}\|_{\boldsymbol{K}}^{2}\right]$$

$$\stackrel{\text{(b)}}{=} W_{1,k} - \|\boldsymbol{x}_{k}\|_{\eta\alpha\boldsymbol{L} - \frac{1}{2}\eta^{2}\alpha^{2}\boldsymbol{L}^{2}}^{2}$$

$$- \eta\beta\boldsymbol{x}_{k}^{\top}(\boldsymbol{I}_{np} - \eta\alpha\boldsymbol{L})\boldsymbol{K}\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{s}\right)$$

$$+ \frac{1}{2}\eta^{2}\beta^{2}\mathbb{E}\left[\|\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0} + \frac{1}{\beta}\boldsymbol{g}_{k}^{e} - \frac{1}{\beta}\boldsymbol{g}_{k}^{0}\|_{\boldsymbol{K}}^{2}\right]$$

$$\stackrel{\text{(c)}}{\leq} W_{1,k} - \|\boldsymbol{x}_{k}\|_{\eta\alpha\boldsymbol{L} - \frac{1}{2}\eta^{2}\alpha^{2}\boldsymbol{L}^{2}}^{2}$$

$$-\eta \beta \boldsymbol{x}_{k}^{\top} \boldsymbol{K} \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right)$$

$$+ \frac{1}{2} \eta \| \boldsymbol{x}_{k} \|_{\boldsymbol{K}}^{2} + \frac{1}{2} \eta \| \boldsymbol{g}_{k}^{s} - \boldsymbol{g}_{k}^{0} \|^{2}$$

$$+ \frac{1}{2} \eta^{2} \alpha^{2} \| \boldsymbol{x}_{k} \|_{\boldsymbol{L}^{2}}^{2} + \frac{1}{2} \eta^{2} \beta^{2} \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2}$$

$$+ \frac{1}{2} \eta^{2} \alpha^{2} \| \boldsymbol{x}_{k} \|_{\boldsymbol{L}^{2}}^{2} + \frac{1}{2} \eta^{2} \| \boldsymbol{g}_{k}^{s} - \boldsymbol{g}_{k}^{0} \|^{2}$$

$$+ \eta^{2} \beta^{2} \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} + \eta^{2} \mathbb{E} [\| \boldsymbol{g}_{k}^{e} - \boldsymbol{g}_{k}^{0} \|^{2}]$$

$$\stackrel{\text{(d)}}{\leq} W_{1,k} - \| \boldsymbol{x}_{k} \|_{\eta \alpha \boldsymbol{L} - \frac{1}{2} \eta \boldsymbol{K} - \frac{3}{2} \eta^{2} \alpha^{2} \boldsymbol{L}^{2} - \eta (1 + 5 \eta) L_{f}^{2} \boldsymbol{K}$$

$$- \eta \beta \boldsymbol{x}_{k}^{\top} \boldsymbol{K} \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right) + \left\| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right\|_{\frac{3}{2} \eta^{2} \beta^{2} \boldsymbol{K}}^{2}$$

$$+ n L_{f}^{2} \eta \left[\frac{p}{4} + (\frac{p}{4} + 4) \eta \right] \delta_{k}^{2} + 2 \eta^{2} \mathbb{E} [\| \boldsymbol{g}_{k}^{e} \|^{2}], \quad (18)$$

where (a) holds due to Lemma 1 and 2 in [32]; (b) holds due to $\mathbb{E}[\boldsymbol{g}_k^e] = \boldsymbol{g}_k^s$ and that $x_{i,k}$ and $v_{i,k}$ are independent of $u_{i,k}$ and $\xi_{i,k}$; (c) holds due to the Cauchy–Schwarz inequality and $\rho(\mathbf{K}) = 1$; (d) holds $\begin{array}{l} \text{due to } \|\boldsymbol{g}_k^s - \boldsymbol{g}_k^0\|^2 \leq 2L_f^2 \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + \frac{np}{2}L_f^2\delta_k^2 \text{ and } \\ \mathbb{E}[\|\boldsymbol{g}_k^0 - \boldsymbol{g}_k^e\|^2] \leq 4L_f^2 \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + 4nL_f^2\delta_k^2 + 2\mathbb{E}[\|\boldsymbol{g}_k^e\|^2]. \end{array}$

$$W_{2,k+1} = \frac{1}{2} \left\| \boldsymbol{v}_{k+1} + \frac{1}{\beta} \boldsymbol{g}_{k+1}^{0} \right\|_{\boldsymbol{Q} + \kappa_{1} \boldsymbol{K}}^{2}$$
 (19)

$$\begin{array}{l} \overset{\text{Eq.}}{=} \frac{6b}{2} \frac{1}{2} \left\| v_k + \frac{1}{\beta} g_k^0 + \eta \beta L x_k + \frac{1}{\beta} (g_{k+1}^0 - g_k^0) \right\|_{Q + \kappa_1 K}^2 \\ \overset{\text{(e)}}{=} W_{2,k} + \eta \beta x_k^\top (K + \kappa_1 L) \left(v_k + \frac{1}{\beta} g_k^0 \right) \\ + \left\| x_k \right\|_{\frac{2}{2} \eta^2 \beta^2 (L + \kappa_1 L^2)}^2 + \frac{1}{2\beta^2} \left\| g_{k+1}^0 - g_k^0 \right\|_{Q + \kappa_1 K}^2 \\ + \frac{1}{\beta} \left(v_k + \frac{1}{\beta} g_k^0 + \eta \beta L x_k \right)^\top (Q + \kappa_1 K) (g_{k+1}^0 - g_k^0) \\ & + \left\| x_k \right\|_{\frac{2}{2} \eta^2 \beta^2 (L + \kappa_1 L^2)}^2 + \frac{1}{2\beta^2} \left\| g_{k+1}^0 - g_k^0 \right\|_{Q + \kappa_1 K}^2 \\ + \frac{1}{\beta} \left(v_k + \frac{1}{\beta} g_k^0 + \eta \beta L x_k \right)^\top (Q + \kappa_1 K) (g_{k+1}^0 - g_k^0) \end{aligned} \qquad (iii) \text{ We have}$$

$$\begin{array}{l} \overset{\text{(f)}}{\leq} W_{2,k} + \eta \beta x_k^\top (K + \kappa_1 L) \left(v_k + \frac{1}{\beta} g_k^0 \right) \\ + \left\| x_k \right\|_{\frac{2}{2} \eta^2 \beta^2 (L + \kappa_1 L^2)}^2 + \frac{1}{2\beta^2} \left\| g_{k+1}^0 - g_k^0 \right\|_{Q + \kappa_1 K}^2 \end{aligned} \qquad \begin{array}{l} \overset{\text{(iii)}}{\leq} W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ + \left\| x_k \right\|_{\frac{2}{2} \eta^2 \beta^2 (L + \kappa_1 L^2)}^2 + \frac{1}{2\beta^2} \left\| g_{k+1}^0 - g_k^0 \right\|_{Q + \kappa_1 K}^2 \end{aligned} \qquad \begin{array}{l} \overset{\text{(iii)}}{\leq} W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right) \\ & & & & & \\ W_{3,k+1} = x_{k+1}^\top K \left(v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \right)$$

$$+ \frac{1}{\beta^{2}} \left(1 + \frac{1}{2\eta}\right) \left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right) \|\boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0}\|^{2}$$

$$\stackrel{(i)}{\leq} W_{2,k} + \eta \beta \boldsymbol{x}_{k}^{\top} (\boldsymbol{K} + \kappa_{1} \boldsymbol{L}) \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0}\right)$$

$$+ \|\boldsymbol{x}_{k}\|_{\eta^{2}\beta^{2}(\boldsymbol{L} + \kappa_{1} \boldsymbol{L}^{2})}^{2} + \left\|\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0}\right\|_{\frac{1}{2}\eta(\boldsymbol{Q} + \kappa_{1} \boldsymbol{K})}^{2}$$

$$+ \frac{\eta}{\beta^{2}} \left(\eta + \frac{1}{2}\right) \left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right) L_{f}^{2} \|\bar{\boldsymbol{g}}_{k}^{e}\|^{2}, \tag{20}$$

where (e) holds due to (a) holds due to Lemma 1 and 2 in [32]; (f) holds due to the Cauchy-Schwarz inequality; (g) holds due to Lemma 1 and 2 in [32]; (h) holds due to $\rho(Q + \kappa_1 K) \leq \rho(Q) + \kappa_1 \rho(K)$, and $\rho(\boldsymbol{K}) = 1$; (i) holds due to $\|\boldsymbol{g}_{k+1}^0 - \boldsymbol{g}_k^0\|^2 \le$ $\eta^2 L_f^2 \|\bar{\boldsymbol{g}}_k^e\|^2 \le \eta^2 L_f^2 \|\boldsymbol{g}_k^e\|^2.$

Moreover, we have the following two inequalities

$$\|\boldsymbol{g}_{k+1}^{0}\|_{\boldsymbol{Q}+\kappa_{1}\boldsymbol{K}}^{2} \leq \left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right)\|\boldsymbol{g}_{k+1}^{0}\|^{2}.$$
 (21)

$$\left\|\boldsymbol{v}_{k} + \frac{1}{\beta_{k}}\boldsymbol{g}_{k}^{0}\right\|_{\boldsymbol{Q}+\kappa_{1}\boldsymbol{K}}^{2} \leq \left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right) \left\|\boldsymbol{v}_{k} + \frac{1}{\beta_{k}}\boldsymbol{g}_{k}^{0}\right\|_{\boldsymbol{K}}^{2}.$$
(22)

Then, from (19)–(22), we have

$$W_{2,k+1}$$

$$\leq W_{2,k} + \eta \beta \boldsymbol{x}_{k}^{\top} (\boldsymbol{K} + \kappa_{1} \boldsymbol{L}) \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right)$$

$$+ \frac{1}{2} \eta \left(\frac{1}{\rho_{2}(L)} + \kappa_{1} \right) \left\| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right\|_{\boldsymbol{K}}^{2}$$

$$+ \left\| \boldsymbol{x}_{k} \right\|_{\eta^{2} \beta^{2}(\boldsymbol{L} + \kappa_{1} \boldsymbol{L}^{2})}^{2}$$

$$+ \frac{\eta}{\beta^{2}} \left(\eta + \frac{1}{2} \right) \left(\frac{1}{\rho_{2}(L)} + \kappa_{1} \right) L_{f}^{2} \| \bar{\boldsymbol{g}}_{k}^{e} \|^{2}. \tag{23}$$

(iii) We have

$$W_{3,k+1} = \boldsymbol{x}_{k+1}^{\top} \boldsymbol{K} \left(\boldsymbol{v}_{k+1} + \frac{1}{\beta} \boldsymbol{g}_{k+1}^{0} \right).$$
 (24)

$$\mathbb{E}\left[W_{3,k+1}\right]$$

$$\stackrel{\text{Eq. } 6}{=} \mathbb{E}\left[\left(\boldsymbol{x}_{k} - \eta(\alpha \boldsymbol{L}\boldsymbol{x}_{k} + \beta \boldsymbol{v}_{k} + \boldsymbol{g}_{k}^{0} + \boldsymbol{g}_{k}^{e} - \boldsymbol{g}_{k}^{0})\right)^{\top}$$

$$\boldsymbol{K}\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0} + \eta\beta \boldsymbol{L}\boldsymbol{x}_{k} + \frac{1}{\beta}(\boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0})\right)\right]$$

$$\stackrel{\text{(j)}}{=} \boldsymbol{x}_{k}^{\top}(\boldsymbol{K} - \eta(\alpha + \eta\beta^{2})\boldsymbol{L})\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0}\right) + \|\boldsymbol{x}_{k}\|_{\eta\beta(\boldsymbol{L} - \eta\alpha\boldsymbol{L}^{2})}^{2}$$

$$+ \frac{1}{\beta}\boldsymbol{x}_{k}^{\top}(\boldsymbol{K} - \eta\alpha\boldsymbol{L})\mathbb{E}[\boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0}] - \eta\beta\|\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0}\|_{\boldsymbol{K}}^{2}$$

$$- \eta\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0}\right)^{\top}\boldsymbol{K}\mathbb{E}[\boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0}]$$

$$- \eta(\boldsymbol{g}_{k}^{s} - \boldsymbol{g}_{k}^{0})^{\top}\boldsymbol{K}\left(\boldsymbol{v}_{k} + \frac{1}{\beta}\boldsymbol{g}_{k}^{0} + \eta\beta\boldsymbol{L}\boldsymbol{x}_{k}\right)$$

$$- \frac{1}{\beta}\mathbb{E}[\eta(\boldsymbol{g}_{k}^{e} - \boldsymbol{g}_{k}^{0})^{\top}\boldsymbol{K}(\boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0})]$$

$$\stackrel{\text{(k)}}{\leq} \boldsymbol{x}_{k}^{\top} (\boldsymbol{K} - \eta \alpha \boldsymbol{L}) \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right) + \frac{1}{2} \eta^{2} \beta^{2} \| \boldsymbol{L} \boldsymbol{x}_{k} \|^{2} \\
+ \frac{1}{2} \eta^{2} \beta^{2} \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} + \| \boldsymbol{x}_{k} \|_{\eta \beta (\boldsymbol{L} - \eta \alpha \boldsymbol{L}^{2})}^{2} \\
+ \frac{1}{2} \eta \| \boldsymbol{x}_{k} \|_{\boldsymbol{K}}^{2} + \frac{1}{2 \eta \beta^{2}} \mathbb{E} [\| \boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0} \|^{2} \\
+ \frac{1}{2} \eta^{2} \alpha^{2} \| \boldsymbol{L} \boldsymbol{x}_{k} \|^{2} + \frac{1}{2 \beta^{2}} \mathbb{E} [\| \boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0} \|^{2}] \\
- \eta \beta \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} \\
+ \frac{1}{2} \eta^{2} \beta^{2} \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} + \frac{1}{2 \beta^{2}} \mathbb{E} [\| \boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0} \|^{2}] \\
+ \frac{1}{2} \eta \| \boldsymbol{g}_{k}^{s} - \boldsymbol{g}_{k}^{0} \|^{2} + \frac{1}{2} \eta \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\boldsymbol{K}}^{2} \\
+ \frac{1}{2} \eta^{2} \| \boldsymbol{g}_{k}^{s} - \boldsymbol{g}_{k}^{0} \|^{2} + \frac{1}{2} \eta^{2} \beta^{2} \| \boldsymbol{L} \boldsymbol{x}_{k} \|^{2} \\
+ \frac{1}{2} \eta^{2} \mathbb{E} [\| \boldsymbol{g}_{k}^{e} - \boldsymbol{g}_{k}^{0} \|^{2}] + \frac{1}{2 \beta^{2}} \mathbb{E} [\| \boldsymbol{g}_{k+1}^{0} - \boldsymbol{g}_{k}^{0} \|^{2}] \\
\stackrel{\text{(i)}}{\leq} W_{3,k} - \eta \alpha \boldsymbol{x}_{k}^{\top} \boldsymbol{L} \left(\boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \right) \\
+ \| \boldsymbol{x}_{k} \|_{\eta(\beta \boldsymbol{L} + \frac{1}{2} \boldsymbol{K}) + \eta^{2} (\frac{1}{2} \alpha^{2} - \alpha \beta + \beta^{2}) \boldsymbol{L}^{2} + \eta (1 + 3 \eta) L_{f}^{2} \boldsymbol{K} \\
+ \eta^{2} \left[1 + (\frac{1}{2 \eta \beta^{2}} + \frac{3}{2 \beta^{2}}) L_{f}^{2} \right] \mathbb{E} [\| \boldsymbol{g}_{k}^{e} \|^{2}] \\
+ \eta L_{f}^{2} \eta \left[\frac{p}{4} + (\frac{p}{4} + 2) \eta \right] \delta_{k}^{2} \\
- \| \boldsymbol{v}_{k} + \frac{1}{\beta} \boldsymbol{g}_{k}^{0} \|_{\eta(\beta - \frac{1}{2} - \eta \beta^{2}) \boldsymbol{K}}^{2}. \tag{25}$$

where (j) holds since $K_nL=LK_n=L$, $\mathbb{E}[\boldsymbol{g}_k^e]=\boldsymbol{g}_k^s$, and that $x_{i,k}$ and $v_{i,k}$ are independent; (k) holds due to the Cauchy–Schwarz inequality, the Jensen's inequality, and $\rho(\boldsymbol{K})=1$; (l) holds due to $\|\boldsymbol{g}_k^s-\boldsymbol{g}_k^0\|^2 \leq 2L_f^2\|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + \frac{np}{2}L_f^2\delta_k^2$ and $\mathbb{E}[\|\boldsymbol{g}_k^0-\boldsymbol{g}_k^e\|^2] \leq 4L_f^2\|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + 4nL_f^2\delta_k^2 + 2\mathbb{E}[\|\boldsymbol{g}_k^e\|^2]$..

(iv) We have

$$\begin{split} &\mathbb{E}[W_{4,k+1}] = \mathbb{E}[n(f(\bar{x}_{k+1}) - f^*)] = \mathbb{E}[\tilde{f}(\bar{x}_{k+1}) - nf^*] \\ &= \mathbb{E}[\tilde{f}(\bar{x}_k) - nf^* + \tilde{f}(\bar{x}_{k+1}) - \tilde{f}(\bar{x}_k)] \\ &\stackrel{\text{(m)}}{\leq} \mathbb{E}[\tilde{f}(\bar{x}_k) - nf^* - \eta(\bar{g}_k^e)^{\top} g_k^0 + \frac{1}{2} \eta^2 L_f \|\bar{g}_k^e\|^2] \\ &\stackrel{\text{(n)}}{=} W_{4,k} - \eta(\bar{g}_k^s)^{\top} \bar{g}_k^0 + \frac{1}{2} \eta^2 L_f \mathbb{E}[\|\bar{g}_k^e\|^2] \\ &\stackrel{\text{(o)}}{=} W_{4,k} - \frac{1}{2} \eta(\bar{g}_k^s)^{\top} (\bar{g}_k^s + \bar{g}_k^0 - \bar{g}_k^s) \\ &- \frac{1}{2} \eta(\bar{g}_k^s - \bar{g}_k^0 + \bar{g}_k^0)^{\top} \bar{g}_k^0 + \frac{1}{2} \eta^2 L_f \mathbb{E}[\|\bar{g}_k^e\|^2] \\ &\stackrel{\text{(p)}}{\leq} W_{4,k} - \frac{1}{4} \eta(\|\bar{g}_k^s\|^2 - \|\bar{g}_k^0 - \bar{g}_k^s\|^2 + \|\bar{g}_k^0\|^2 \\ &- \|\bar{g}_k^0 - \bar{g}_k^s\|^2) + \frac{1}{2} \eta^2 L_f \mathbb{E}[\|\bar{g}_k^e\|^2] \\ &\stackrel{\text{(q)}}{\leq} W_{4,k} - \frac{1}{4} \eta \|\bar{g}_k^s\|^2 + \|x_k\|_{\eta L_f^2 K}^2 \\ &+ \frac{np}{4} \eta L_f^2 \delta_k^2 - \frac{1}{4} \eta \|\bar{g}_k^0\|^2 + \frac{1}{2} \eta^2 L_f \mathbb{E}[\|\bar{g}_k^e\|^2], \end{cases} \tag{26} \end{split}$$

where (m) holds since that \tilde{f} is smooth; (n) holds due to $\mathbb{E}[\boldsymbol{g}_k^e] = \boldsymbol{g}_k^s, \, x_{i,k}$ and $v_{i,k}$ are independent; (o) holds due to $(\bar{\boldsymbol{g}}_k^s)^{\top} \boldsymbol{g}_k^0 = (\boldsymbol{g}_k^s)^{\top} \boldsymbol{H} \boldsymbol{g}_k^0 = (\boldsymbol{g}_k^s)^{\top} \boldsymbol{H} \boldsymbol{H} \boldsymbol{g}_k^0 = (\bar{\boldsymbol{g}}_k^s)^{\top} \bar{\boldsymbol{g}}_k^0$; (p) holds due to the Cauchy–Schwarz inequality; and (q) holds due to $\|\boldsymbol{g}_k^s - \boldsymbol{g}_k^0\|^2 \leq 2L_f^2 \|\boldsymbol{x}_k\|_{\boldsymbol{K}}^2 + \frac{np}{2}L_f^2 \delta_k^2$.

(v) Define $W_{k+1} = \sum_{i=1}^{4} W_{i,k+1}$ and then we have the following inequality holds.

$$\mathbb{E}[W_{k+1}] \leq W_{k} - \|x_{k}\|_{\eta\alpha L - \frac{1}{2}\eta K - \frac{3}{2}\eta^{2}\alpha^{2}L^{2} - \eta(1+5\eta)L_{f}^{2}K} \\
+ \|v_{k} + \frac{1}{\beta}g_{k}^{0}\|_{\frac{3}{2}\eta^{2}\beta^{2}K}^{2} + nL_{f}^{2}\eta \left[\frac{p}{4} + (\frac{p}{4} + 4)\eta\right]\delta_{k}^{2} \\
+ 2\eta^{2}\mathbb{E}[\|g_{k}^{e}\|^{2}] + \frac{1}{2}\eta\left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right)\|v_{k} + \frac{1}{\beta}g_{k}^{0}\|_{K}^{2} \\
+ \|x_{k}\|_{\eta^{2}\beta^{2}(L+\kappa_{1}L^{2})}^{2} \\
+ \frac{\eta}{\beta^{2}}\left(\eta + \frac{1}{2}\right)\left(\frac{1}{\rho_{2}(L)} + \kappa_{1}\right)L_{f}^{2}\mathbb{E}[\|\bar{g}_{k}^{e}\|^{2}] \\
+ \|x_{k}\|_{\eta(\beta L + \frac{1}{2}K) + \eta^{2}(\frac{1}{2}\alpha^{2} - \alpha\beta + \beta^{2})L^{2} + \eta(1+3\eta)L_{f}^{2}K} \\
+ \eta^{2}\left[1 + \left(\frac{1}{2\eta\beta^{2}} + \frac{3}{2\beta^{2}}\right)L_{f}^{2}\right]\mathbb{E}[\|g_{k}^{e}\|^{2}] \\
+ nL_{f}^{2}\eta\left[\frac{p}{4} + \left(\frac{p}{4} + 2\right)\eta\right]\delta_{k}^{2} \\
- \|v_{k} + \frac{1}{\beta}g_{k}^{0}\|_{\eta(\beta - \frac{1}{2} - \eta\beta^{2})K}^{2} - \frac{1}{4}\eta\|\bar{g}_{k}^{s}\|^{2} \\
+ \|x_{k}\|_{\eta L_{f}^{2}K}^{2}K \\
+ \frac{np}{4}\eta L_{f}^{2}\delta_{k}^{2} - \frac{1}{4}\eta\|\bar{g}_{k}^{0}\|^{2} + \frac{1}{2}\eta^{2}L_{f}\mathbb{E}[\|\bar{g}_{k}^{e}\|^{2}] \\
\stackrel{\text{(c)}}{\leq} W_{k} - \|x_{k}\|_{\eta M_{1} - \eta^{2}M_{2} - b_{1}K}^{2} - \|v_{k} + \frac{1}{\beta}g_{k}^{0}\|_{b_{2}^{0}K}^{2} \\
- \eta\left(\frac{1}{4} - 6c_{1}(p - 1)\right)\|\bar{g}_{k}^{0}\|^{2} \\
+ c_{1}\left[6(p - 1)\sigma_{2}^{2} + \left(\frac{3}{n_{c}} + 2\right)p\sigma_{1}^{2}\right] + \underbrace{c_{3}\eta\delta_{k}^{2}L_{f}^{2}}_{\mathcal{O}(np^{2})\eta\delta_{k}^{2}}^{2}$$

$$(27)$$

where (r) holds due to (11a), (11b), $\alpha = \kappa_1 \beta$, $\eta = \frac{\kappa_2}{\beta}$, and

$$\begin{split} \boldsymbol{M}_{1} &= (\alpha - \beta) \boldsymbol{L} - \left(1 + 3L_{f}^{2} + \frac{6L_{f}^{4}\kappa_{1}}{\beta^{2}}(p - 1)\right) \boldsymbol{K}, \\ \boldsymbol{M}_{2} &= \beta^{2} \boldsymbol{L} + (2\alpha^{2} + \beta^{2}) \boldsymbol{L}^{2} + 8L_{f}^{2} \boldsymbol{K} \\ &+ 6(p - 1)\left(3 + \frac{1}{2}L_{f} + \frac{2L_{f}^{2}}{\beta^{2}}\kappa_{1} + \frac{L_{f}^{2}}{2\beta^{2}}\right) \boldsymbol{K}, \\ \kappa_{3} &= \frac{1}{\rho_{2}(L)} + \kappa_{1} + 1, \\ b_{2}^{0} &= \frac{1}{2}\eta(2\beta - \kappa_{3}) - 2.5\kappa_{2}^{2}, \\ b_{1} &= 6p\kappa_{3}L_{f}^{4} \frac{\eta}{\beta^{2}} + 12p(\kappa_{3} + 1)L_{f}^{4} \frac{\eta^{2}}{\beta^{2}}, \end{split}$$

$$c_{1} = \left(3 + \frac{1}{2}L_{f} + \frac{2L_{f}^{2}}{\beta^{2}}\kappa_{1} + \frac{L_{f}^{2}}{2\beta^{2}}\right)n\eta^{2} + \frac{L_{f}^{2}\kappa_{1}}{\beta^{2}}n\eta,$$

$$c_{2} = \frac{3}{4}pn + \eta n(\frac{p}{2} + 6) + \frac{p^{2}L_{f}^{2}}{2\beta^{2}}\kappa_{1},$$

$$c_{3} = c_{2} + \left(c_{1} - \frac{L_{f}^{2}\kappa_{1}}{\beta^{2}}n\eta\right)p^{2}\eta.$$

Consider $p \ge 1$, $\alpha = \kappa_1 \beta$, $\kappa_1 > 1$, β is large enough, and $\eta = \frac{\kappa_2}{\beta}$, we have

$$\eta \mathbf{M}_1 \ge [(\kappa_1 - 1)\rho_2(L) - 1]\kappa_2 \mathbf{K}. \tag{28}$$

$$\eta^2 \mathbf{M}_2 \le [\rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1]\kappa_2^2 \mathbf{K}.$$
(29)

$$b_2^0 \ge \frac{1}{2}(\kappa_2 - 5\kappa_2^2). \tag{30}$$

From (27)–(30), let $\kappa_4 = [(\kappa_1-1)\rho_2(L)-1]\kappa_2 - [\rho(L)+(2\kappa_1^2+1)\rho(L^2)+1]\kappa_2^2$ we know that (12a) holds. Similar to the way to get (12a), we have (12b).