Joint Nonanticipative Rate Distortion Function for a Tuple of Random Processes with Individual Fidelity Criteria

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Abstract— The joint nonanticipative rate distortion function (NRDF) for a tuple of random processes with individual fidelity criteria is considered. Structural properties of optimal test channel distributions are derived. Further, for the application example of the joint NRDF of a tuple of jointly multivariate Gaussian Markov processes with individual square-error fidelity criteria, a realization of the reproduction processes which induces the optimal test channel distribution is derived, and the corresponding joint NRDF is characterized. The analysis of the simplest example, of a tuple of scalar correlated Markov processes, illustrates many of the challenging aspects of such problems.

I. INTRODUCTION

Presently of much interest in information theory and in the theory of information transmission for control systems applications, is the Gorbunov and Pinsker [1] nonanticipatory epsilon entropy and message generation rates of a discretetime random process $X^n \stackrel{\triangle}{=} \{X_1, X_2, \ldots, X_n\}, X_t(\omega), t =$ $1, 2, \ldots, n, \omega \in \Omega$ taking values in X, with joint probability distribution \mathbf{P}_{X^n} , subject to a fidelity criterion,

$$\frac{1}{n} \mathbf{E} \Big\{ d_n(X^n, Y^n) \Big\} \le \epsilon, \quad d_n(x^n, y^n) \in [0, \infty)$$
(I.1)

of reconstructing X^n by another random process $Y^n \triangleq \{Y_1, Y_2, \ldots, Y_n\}, Y_t(\omega), t = 1, 2, \ldots, n, \omega \in \Omega$ taking values in $\mathbb{Y} \subseteq \mathbb{X}$. Nonanticipatory entropy is often described under the designated name nonanticipative or sequential rate distortion function (RDF) [2]–[6]. However, the name for this quantity does alter the situation, that it corresponds to a variant of Shannon's [7] "rate of creating information with respect to a fidelity", often designated by the name rate distortion function (RDF) [8]. Shannon's RDF is the information theoretic definition of the operational definition, "the optimal performance theoretically attainable" (OPTA) (i.e., the infimum of rates of creating information) by noncausal codes subject to a fidelity.

Gorbunov's and Pinsker's [1] nonanticipatory epsilon en-

tropy of a process X^n with distribution \mathbf{P}_{X^n} , is defined by

$$R_{X^{n}}(\epsilon) \stackrel{\Delta}{=} \inf_{\mathbf{P}_{X^{n},Y^{n}}} \int \log\left(\frac{\mathbf{P}_{Y^{n},X^{n}}}{\mathbf{P}_{Y^{n}} \times \mathbf{P}_{Y^{n}}}\right) \mathbf{P}_{X^{n},Y^{n}} \quad (I.2)$$

subject to the average disrortion (I.1) and (I.3)

causality,
$$\mathbf{P}_{Y^t|X^n} = \mathbf{P}_{Y^t|X^t}, t = 1, \dots n$$
 (I.4)

where the infimum is taken over all joint distributions \mathbf{P}_{X^n,Y^n} such that the X marginal distribution is the fixed distribution \mathbf{P}_{X^n} , and the fidelity and causality are satisfied. Shannon's RDF corresponds to $R_{X^n}(\epsilon)$ without the causality restriction (I.4). $\mathbf{P}_{Y^n|X^n}$ and $\mathbf{P}_{X^n|Y^n}$ are known as, the forward test channel and the backward test channel, respectively, of reconstructing X^n by Y^n subject to fidelity (I.1). Over the years, $R_{X^n}(\epsilon)$ is applied in the following areas.

- 1) Quantification of the rate loss of the OPTA by causal codes [9] and zero-delay codes [10], [11], with respect to noncausal codes, for Gaussian Markov processes X^n with square-error fidelity [3]. The construction of causal and zero-delay codes [12], based on subtractive dither with uniform scalar quantization (SDUSQ) [13].
- Necessary and sufficient conditions to stabilize unstable linear Gaussian control systems over finite rate, noiseless or noisy, communications channels, and to design controllers, encoders and decoders subject to finite rate constraints [2], [14].
- 3) Synthesize recursive, causal filters of Gaussian Markov processes subject to square-error fidelity [4], [6].

However, the complete characterization of the multivariate Gaussian Markov process X^n with square-error fidelity, i.e., the specification of the realization of Y^n and its structural properties, which induces the optimal test channel, was only recently completed in [15], although the problem was posed and solved for the scalar Gaussian Markov source by Gorbunov and Pinsker in [16].

In this paper we formulate and analyze the nonanticipatory epsilon entropy, designated henceforth by the name joint nonanticipative rate distortion function (NRDF), of a tuple of processes (X_1^n, X_2^n) , when each process is assigned an individual fidelity criterion, as shown in Fig. I.1.

Our interest in this problem is motivated by the classical joint compression problem of a tuple of jointly independent and identically distributed processes (X_1^n, X_2^n) with individual fidelity criteria, introduced in [17]. As pointed out in [17], contrary to the classical joint RDF of a tuple process, viewed as a single process, $X^n = (X_1^n, X_2^n)$, with a single fidelity criterion assigned to X^n , the classical joint RDF with

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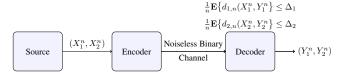


Fig. I.1: Lossy Compression of correlated sources with individual distortion criteria.

individual fidelity criteria aims at the design of encoders with cooperation, and hence fundamentally different from the former. Indeed, inherent in the optimal test channel distribution is the encoder cooperation, which is absent in the classical joint RDF with a single fidelity criterion. The additional level of complexity of the encoder cooperation is demonstrated in [17], through the calculation of the classical joint RDF of a tuple of scalar, jointly independent and identically distributed Gaussian RVs with individual squareerror distortion criteria. The recent treatment in [18], of the classical joint RDF for a tuple of multivariate jointly independent and identically distributed Gaussian processes with individual square-error distortion criteria, demonstrated additional challenges, which are due to the consideration of the multivariate analog of [17], and which are absent in the classical water-filling solution of the analogous RDF with a single fidelity criterion. Another application of the classical joint RDF of a tuple of processes with individual fidelity criteria is the Gray and Wyner [19] source coding for a simple network, where this joint RDF is needed to characterize the rate region [20].

A. Problem Statement and Main Results

We consider a tuple of random processes $X_i^n \stackrel{\triangle}{=} \{X_{i,1}, \ldots, X_{i,n}\}, i = 1, 2,$

$$X_{i,t}: \Omega \to \mathbb{X}_i, \quad t = 1, \dots, n, \quad i = 1, 2$$
(I.5)

where \mathbb{X}_i , i = 1, 2 are metric spaces, with corresponding tuple of reproduction processes $Y_i^n \stackrel{\triangle}{=} \{Y_{i,1}, \ldots, Y_{i,n}\}, i = 1, 2,$

$$Y_{i,t}: \Omega \to \mathbb{Y}_i, \quad t = 1, \dots, n, \quad i = 1, 2$$
(I.6)

where $\mathbb{Y}_i \subseteq \mathbb{X}_i$, i = 1, 2. The reproduction tuple satisfies two individual fidelity criteria of reconstructing X_i^n by Y_i^n , i = 1, 2, defined by the measurable functions $d_{i,n} : \mathbb{X}_i^n \times \mathbb{Y}_i^n \to [0, \infty), i = 1, 2$,

$$\frac{1}{n} \mathbf{E} \Big\{ d_{i,n}(X_i^n, Y_i^n) \Big\} \le \Delta_i, \quad i = 1, 2, \quad (I.7)$$

$$d_{i,n}(x_i^n, x_i^n) = \sum_{t=1}^{n} \rho_t(x_t, y_t), \quad i = 1, 2.$$
 (I.8)

Given the fixed joint distribution $\mathbf{P}_{X_1^n,X_2^n}$ of the tuple (X_1^n,X_2^n) , we define the joint distribution of $(X_1^n,X_2^n,Y_1^n,Y_2^n)$, using the forward $\mathbf{P}_{Y_1^n,Y_2^n|X_1^n,X_2^n}$ and backward $\mathbf{P}_{X_1^n,X_2^n|Y_1^n,Y_2^n}$ test channel distributions by

$$\mathbf{P}_{X_{1}^{n},X_{2}^{n},Y_{1}^{n},Y_{2}^{n}} = \mathbf{P}_{Y_{1}^{n},Y_{2}^{n}|X_{1}^{n},X_{2}^{n}} \otimes \mathbf{P}_{X_{1}^{n},X_{2}^{n}}$$
(I.9)

$$= \mathbf{P}_{X_1^n, X_2^n | Y_1^n, Y_2^n} \otimes \mathbf{P}_{Y_1^n, Y_2^n} \qquad (I.10)$$

where \otimes denotes the compound probability operator.

Let $I(X_1^n X_2^n; Y_1^n, Y_2^n)$ denote the mutual information between the tuple (X_1^n, X_2^n) and its reproduction tuple (Y_1^n, Y_2^n) , defined by [21]

$$I(X_{1}^{n}, X_{2}^{n}; Y_{1}^{n}, Y_{2}^{n}) \stackrel{\triangle}{=} \int \log \left(\frac{\mathbf{P}_{X_{1}^{n}, X_{2}^{n}, Y_{1}^{n}, Y_{2}^{n}}}{\mathbf{P}_{X_{1}^{n}, X_{2}^{n}} \times \mathbf{P}_{Y_{1}^{n}, Y_{2}^{n}}} \right)$$
$$\mathbf{P}_{X_{1}^{n}, X_{2}^{n}, Y_{1}^{n}, Y_{2}^{n}}.$$
(I.11)

Let $Q_{X_1^n,X_2^n}^S(\Delta_1,\Delta_2)$ denote the fidelity constraint set of the two individual distortions, defined by

$$\mathcal{Q}^{S}_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) \stackrel{\triangle}{=} \left\{ \mathbf{P}_{Y_{1}^{n},Y_{2}^{n}|X_{1}^{n},X_{2}^{n}} \middle| \text{ the } \mathbb{X}_{1}^{n} \times \mathbb{X}_{2}^{n} - \right.$$

marginal of the joint dist. (I.9) is $\mathbf{P}_{X_1^n,X_2^n}$, and

$$\frac{1}{n} \mathbf{E} \Big\{ d_{i,n}(X_i^n, Y_i^n) \Big\} \le \Delta_i, \ i = 1, 2 \Big\}.$$
(I.12)

The joint NRDF for the tuple (X_1^n, X_2^n) with individual fidelity criteria is defined by

$$R_{X_1^n, X_2^n}(\Delta_1, \Delta_2) \stackrel{\triangle}{=} \inf_{\substack{\mathcal{Q}_{X_1^n, X_2^n}^s(\Delta_1, \Delta_2): (\mathbb{C}) \text{ holds}} \left\{ I(X_1^n, X_2^n; Y_1^n, Y_2^n) \right\}$$
(I.13)

where the infimum is taken over all joint distributions $\mathbf{P}_{X_1^n,X_2^n,Y_1^n,Y_2^n} \in \mathcal{Q}_{X_1^n,X_2^n}^S(\Delta_1,\Delta_2)$ such that following condition holds:

(C) for each $t \in \{1, ..., n\}$, the process (Y_1^t, Y_2^t) is conditionally independent of $(X_{1,t+1}^n, X_{2,t+1}^2)$ conditioned on (X_1^t, X_2^t) , that is,

$$\mathbf{P}_{Y_1^t, Y_2^t | X_1^n, X_2^n} = \mathbf{P}_{Y_1^t, Y_2^t | X_1^t, X_2^t}, \quad t = 1, \dots, n-1 \quad (I.14)$$

equivalently expressed as a Markov chain (\leftrightarrow)

$$(X_{1,t+1}^n, X_{2,t+1}^n) \leftrightarrow (X_1^t, X_2^t) \leftrightarrow (Y_1^t, Y_2^t), \ t = 1, \dots, n-1.$$

Conditional independence (I.14) is a "causality condition" of the reproduction distribution.

The main result of the first part of the paper, is

(R1) the structural properties of optimal test channel distributions, and realizations of the reproduction processes.

In the second part of the paper, we analyze the joint NRDF $R_{X_1^n,X_2^n}(\Delta_1, \Delta_2)$, for a tuple of jointly multivariate Gaussian Markov processes (X_1^n, X_2^n) , and two square-error distortion functions, defined by

$$X_{i,t}: \Omega \to \mathbb{X}_i \stackrel{\bigtriangleup}{=} \mathbb{R}^{p_i}, \quad t = 1, \dots, n,$$
(I.15)

$$\mathbf{P}_{X_{1,t},X_{2,t}|X_{1}^{t-1},X_{2}^{t-1}} = \mathbf{P}_{X_{1,t},X_{2,t}|X_{1,t-1},X_{2,t-1}}, \quad (I.16)$$

$$(X_{1}, X_{2}, X_{2}, Y_{2}, Y_{2},$$

$$(X_{1,t}, X_{2,t}) \in G(0, Q_{(X_{1,t}, X_{2,t})}), \tag{1.17}$$

$$Q_{(X_{1,t},X_{2,t})} \stackrel{\triangle}{=} \mathbf{E} \left\{ \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix}^{\mathsf{T}} \right\}$$
(I.18)

$$Y_{i,t}: \Omega \to \mathbb{Y}_i \stackrel{\Delta}{=} \mathbb{R}^{p_i}, \quad i = 1, 2, \tag{I.19}$$

$$d_{i,n}(x_i^n, y_i^n) = \frac{1}{n} \sum_{t=1}^n ||x_{i,t} - y_{i,t}||_{\mathbb{R}^{p_i}}^2, \quad i = 1, 2.$$
 (I.20)

Here $X \in G(0, Q_X)$ means X is a Gaussian RV, with zero mean and covariance matrix $Q_X \succeq 0$. Our main contributions include,

(R2) realizations of optimal reproduction process (Y_2^n, Y_2^n) , and its structural properties, and

(R3) characterization of joint NRDF $R_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$.

II. JOINT NONANTICIPATIVE RDF WITH INDIVIDUAL FIDELITY CRITERIA

A. Notation

 $\mathbb{R} \triangleq (-\infty, \infty), \ \mathbb{Z} \triangleq \{\dots, -1, 0, 1, \dots\}, \ \mathbb{Z}_0 \triangleq \{0, 1, 2, \dots\}, \ \mathbb{N} \triangleq \{1, 2, \dots\}, \ \mathbb{N}^n \triangleq \{1, \dots, n\}, \ n \in \mathbb{N}.$ For any matrix $A \in \mathbb{R}^{p \times m}, (p, m) \in \mathbb{N} \times \mathbb{N}$, we denote its transpose by A^{T} , its pseudoinverse by $A^{\dagger} \in \mathbb{R}^{m \times p}$, and for m = p, we denote its trace by $\operatorname{tr}(A)$. The *n* by *n* identity (resp. zero) matrix is represented by I_n (resp. 0_n). $\mathcal{S}^{p \times p}_+$ denotes the set of symmetric positive semidefinite matrices. The statement $A \succeq B$ (resp. $A \succ B$) means that A - B is symmetric positive semidefinite (resp. positive definite).

Denote an arbitrary set or space by \mathbb{U} and the product space formed by $n \in \mathbb{N}$ copies of it by $\mathbb{U}^n \stackrel{\triangle}{=} \times_{t=1}^n \mathbb{U}$. $u^n \in \mathbb{U}^n$ denotes the set of *n*-tuples $u^n \stackrel{\triangle}{=} (u_1, u_2, \dots, u_n)$, where $u_k \in \mathbb{U}, k = 1, \dots, n$ are its coordinates.

Denote a probability space by $(\Omega, \mathcal{F}, \mathbb{P})$. For a sub-sigmafield $\mathcal{G} \subseteq \mathcal{F}, A \in \mathcal{F}$, we denote by $\mathbb{P}(A|\mathcal{G}) = \mathbb{P}(A|\mathcal{G})(\omega), \omega \in \Omega$ the conditional probability of A given \mathcal{G} . For a tuple of real-valued RVs (RV) $X : \Omega \to \mathbb{X}, Y : \Omega \to \mathbb{Y}$, where $(\mathbb{X}, \mathcal{B}(\mathbb{X})), (\mathbb{Y}, \mathcal{B}(\mathcal{Y}))$ are measurable spaces, we denote the measure (resp. joint distribution, if \mathbb{X}, \mathbb{Y} are Euclidean spaces) induced by RVs (X, Y) on $\mathbb{X} \times \mathbb{Y}$ by $\mathbf{P}(dx, dy)$ (resp. $\mathbf{P}_{X,Y}$), and their marginals on \mathbb{X} and \mathbb{Y} by $\mathbf{P}(dx)$ and $\mathbf{P}(dy)$ (resp. \mathbf{P}_X and \mathbf{P}_Y), respectively. We denote the conditional distribution of RV X conditioned on Y by $\mathbf{P}_{X|Y}$ or $\mathbf{P}(dx|y)$, if Y = y is fixed.

For a triple of real-valued RVs $X : \Omega \to \mathbb{X}, Y : \Omega \to \mathbb{Y}, Z : \Omega \to \mathbb{Z}$, we say that RVs (Y, Z) are conditional independent given RV X if $\mathbf{P}_{Y,Z|X} = \mathbf{P}_{Y|X}\mathbf{P}_{Z|X}$ -a.s (almost surely) or equivalently $\mathbf{P}_{Z|X,Y} = \mathbf{P}_{Z|X}$ -a.s; the specification a.s is often omitted. We often denote the above conditional independence by the Markov chain (MC) $Y \leftrightarrow X \leftrightarrow Z$.

The conditional covariance of the two-component vector RV $X = (X_1^{\mathsf{T}}, X_2^{\mathsf{T}})^{\mathsf{T}}, X_i : \Omega \to \mathbb{R}^{p_i}, i = 1, 2$ conditioned on the two-component vector $Y = (Y_1^{\mathsf{T}}, Y_2^{\mathsf{T}})^{\mathsf{T}}, Y_i : \Omega \to \mathbb{R}^{p_i}, i = 1, 2$ is denoted by $Q_{(X_1, X_2)|Y} \stackrel{\triangle}{=} \operatorname{cov} (X, X | Y) \succeq 0$, where

$$Q_{(X_1,X_2)|Y} = \begin{pmatrix} Q_{X_1|Y} & Q_{X_1,X_2|Y} \\ Q_{X_1,X_2|Y}^{\mathsf{T}} & Q_{X_2|Y} \end{pmatrix} \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)},$$

$$Q_{X_1,X_2|Y} \stackrel{\triangle}{=} \operatorname{cov}\left(X_1,X_2\middle|Y\right).$$

$$\stackrel{(1)}{=} \mathbf{E}\left\{\left(X_1 - \mathbf{E}\left\{X_1\middle|Y\right\}\right)\left(X_2 - \mathbf{E}\left\{X_2\middle|Y\right\}\right)^{\mathsf{T}}\right\}$$

$$= \mathbf{E}\left\{E_1E_2^{\mathsf{T}}\right\}, \quad E_i \stackrel{\triangle}{=} X_i - \mathbf{E}\left\{X_i\middle|Y\right\}, \quad i = 1, 2$$

$$\equiv \Sigma_{E_1,E_2} \qquad (\text{II.1})$$

and where (1) holds if (X_1, X_2, Y_1, Y_2) is jointly Gaussian. Similarly for $Q_{X_i|Y}$, i = 1, 2. Consequently, for jointly Gaussian RVs (X_1, X_2, Y_1, Y_2) , and the two-component vector RV $E \stackrel{\triangle}{=} (E_1^{\mathsf{T}}, E_2^{\mathsf{T}})^{\mathsf{T}}$, we have $Q_{(X_1, X_2)|Y} = \Sigma_{(E_1, E_2)}$.

B. Equivalent Sequential Formula of Joint NRDF

First, we give the sequential equivalent of the joint NRDF $R_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$. We make use of the following lemma.

Lemma 1: [4], [6] Conditional independence conditions The following statements are equivalent $\forall n \in \mathbb{N}$. MC1: $\mathbf{P}_{V^n | V^n | X^n} = \bigotimes_{t=1}^{n} \mathbf{P}_{V = V | V^{t-1} | V^{t-1} | V^{t-1} | V^{t}}$.

 $\begin{array}{lll} \text{MC1:} \ \mathbf{P}_{Y_1^n,Y_2^n|X_1^n,X_2^n} = \otimes_{t=1}^n \mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1},X_1^t,X_2^t}.\\ \text{MC2:} \ (Y_{1,t},Y_{2,t}) &\leftrightarrow & (X_1^t,X_2^t,Y_1^{t-1},Y_2^{t-1}) &\leftrightarrow \\ (X_{1,t+1}^n,X_{2,t+1}^n) \text{ forms a MC, for each } t = 1,\ldots,n-1.\\ \text{MC3:} \ (Y_1^t,Y_2^t) &\leftrightarrow (X_1^t,X_2^t) &\leftrightarrow (X_{1,t+1},X_{2,t+1}) \text{ forms a } \\ \text{MC, for each } t = 1,\ldots,n-1. \end{array}$

 $\begin{array}{ll} \mathsf{MC4:} & (X_{1,t+1}^n,X_{2,t+1}^n) \leftrightarrow (X_1^t,X_2^t) \leftrightarrow (Y_1^t,Y_2^t) \text{ forms a} \\ \mathsf{MC, for each } t=1,\ldots,n-1. \end{array}$

By Lemma 1, Condition (C) is equivalent to MC1, and the joint distribution of $(X_1^n, X_2^n, Y_1^n, Y_2^n)$, is expressed as

$$\begin{aligned} \mathbf{P}_{X_{1}^{n},X_{2}^{n},Y_{1}^{n},Y_{2}^{n}} &= \mathbf{P}_{Y_{1,n},Y_{2,n}|Y_{1}^{n-1},Y_{2}^{n-1},X_{1}^{n},X_{2}^{n}} \\ &\otimes \mathbf{P}_{X_{1,n},X_{2,n}|Y_{1}^{n-1},Y_{2}^{n-1},X_{1}^{n-1},X_{2}^{n-1}} \cdots \\ &\otimes \mathbf{P}_{Y_{1,2},Y_{2,2}|Y_{1,1},Y_{2,1},X_{1}^{2},X_{2}^{2}} \otimes \mathbf{P}_{X_{1,2},X_{2,2}|X_{1,1},X_{2,1}} \\ &\otimes \mathbf{P}_{Y_{1,1},Y_{2,1}|X_{1,1},X_{2,1}} \otimes \mathbf{P}_{X_{1,1},X_{2,1}} \end{aligned}$$
(II.2)

The information measure $I(X_1^n, X_2^n; Y_1^n, Y_2^n)$ in (I.13) is expressed sequentially as,

$$I(X_1^n, X_2^n; Y_1^n, Y_2^n) = \mathbf{E} \left\{ \sum_{t=1}^n \log \left(\frac{\mathbf{P}_{Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1}, X_1^t, X_2^t}}{\mathbf{P}_{Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1}}} \right) \right\}$$
$$= \sum_{t=1}^n I(X_1^t, X_2^t; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1}).$$

The joint NRDF $R_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$ of (I.13) subject to condition (C) is expressed sequentially as follows.

$$R_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$$
(II.3)
=
$$\inf_{\mathcal{Q}_{X_1^n, X_2^n}(\Delta_1, \Delta_2)} \left\{ \sum_{t=1}^n I(X_1^t, X_2^t; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1}) \right\}$$

where

$$\mathcal{Q}_{X_1^n, X_2^n}(\Delta_1, \Delta_2) \triangleq \left\{ \mathbf{P}_{X_1^t, X_2^t, Y_1^t, Y_2^t}, t = 1, \dots, n \right|$$
(II.2) holds, the $\mathbb{X}_1^n \times \mathbb{X}_2^n$ - marginal is $\mathbf{P}_{X_1^n, X_2^n}$,
 $\frac{1}{n} \mathbf{E} \left\{ d_{i,n}(X_i^n, Y_i^n) \right\} \leq \Delta_i, i = 1, 2 \right\}.$
(II.4)

It can be shown that

$$\begin{aligned} \mathcal{Q}_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) &= \Big\{ \mathbf{P}_{Y_{1,t}Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1},X_{1}^{t},X_{2}^{t}}, t = 1,\dots,n \\ \text{ the } \mathbb{X}_{1}^{n} \times \mathbb{X}_{2}^{n} - \text{ marginal of (II.2) is } \mathbf{P}_{X_{1}^{n},X_{2}^{n}}, \\ \frac{1}{n} \mathbf{E} \Big\{ d_{i,n}(X_{i}^{n},Y_{i}^{n}) \Big\} \leq \Delta_{i}, i = 1,2 \Big\}. \end{aligned}$$
(II.5)

C. Information Structures of Sequential Joint RDF for a Tuple of Markov Processes

The main result of this section is Theorem 2, which identifies structural properties of the realizations (Y_1^n, Y_2^n) , of the test channels that minimize $\sum_{t=1}^n I(X_1^t, X_2^t; Y_{1,t}, Y_{2,t}|Y_1^{t-1}, Y_2^{t-1})$, when the joint process (X_1^n, X_2^n) is Markov and the fidelity is defined with respect to the square-error.

First, we recall a preliminary result, of a structural property of test channel distributions from the set $Q_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$.

Theorem 1: [6], [22]

Consider the joint NRDF of (II.3), and assume the joint process (X_1^n, X_2^n) is Markov, that is, $\forall t \in \mathbb{N}^n$

$$\mathbf{P}_{X_{1,t},X_{2,t}|X_1^{t-1},X_2^{t-1}} = \mathbf{P}_{X_{1,t},X_{2,t}|X_{1,t-1},X_{2,t-1}}.$$
 (II.6)

Then the joint NRDF is given by

$$R_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) = \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2})} \left\{$$
(II.7)

$$\mathbf{E}\left\{\sum_{t=1}^{n}\log\left(\frac{\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1},X_{1,t},X_{2,t}}}{\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1}}}\right)\right\}\right\}$$
(II.8)

$$= \inf_{\mathcal{M}_{X_{1}^{n}, X_{2}^{n}}(\Delta_{1}, \Delta_{2})} \left\{ \sum_{t=1}^{n} I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | Y_{1}^{t-1}, Y_{2}^{t-1}) \right\}$$

where

$$\mathcal{M}_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) \triangleq \left\{ \mathbf{P}_{Y_{1,t}Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1},X_{1,t},X_{2,t}}, \\ t = 1,\ldots,n \middle| \text{ the } \mathbb{X}_{1}^{n} \times \mathbb{X}_{2}^{n} - \text{marginal corresp. to (II.6)} \\ \frac{1}{n} \mathbf{E} \left\{ d_{i,n}(X_{i}^{n},Y_{i}^{n}) \right\} \leq \Delta_{i}, i = 1,2 \right\},$$
(II.9)
$$\mathbf{P}_{X_{1}^{t},X_{2}^{t},Y_{1}^{t},Y_{2}^{t}} = \mathbf{P}_{X_{1,1},X_{2,1}} \otimes \mathbf{P}_{Y_{1,1},Y_{2,1}|X_{1,1},X_{2,1}} \\ \otimes_{i=1}^{t} \left(\mathbf{P}_{X_{1,t},X_{2,t}|X_{1}^{i-1},X_{2}^{i-1}} \otimes \mathbf{P}_{Y_{1,i},Y_{2,i}|Y_{1}^{i-1},Y_{2}^{i-1},X_{1}^{t},X_{2}^{t}} \right), \\ \mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1}} = \int_{\mathbb{R}_{t} \cup \mathbb{R}} \mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1},X_{1,t}X_{2,t}} \right\}$$

$$\otimes \mathbf{P}_{X_{1,t},X_{2,t}|Y_1^{t-1},Y_2^{t-1}}$$
 (II.10)

Next, we identify an important structural property of the optimal reproduction process (Y_1^n, Y_2^n) .

Theorem 2: Structural property of reproduction process Consider the statement of Theorem 1 and the joint NRDF $R_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$ of (II.7) for the Markov (X_1^n, X_2^n) . (a) Define

$$\begin{split} \widehat{X}_{i,t} &= g_{i,t}(Y_1^t, Y_2^t), \; \forall t \in \mathbb{N}^n, \; i = 1, 2, \\ g_{i,t} : \mathbb{Y}_1^t \times \mathbb{Y}_2^t \to \mathbb{Y}_i, \; g_{i,t}(\cdot) \; \text{are meas. functions,} \; i = 1, 2. \end{split}$$
(II.11)

Then, the following inequality holds for t = 1, ..., n:

$$I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1}) \geq I(X_{1,t}, X_{2,t}; \hat{X}_{1,t}, \hat{X}_{2,t} | Y_1^{t-1}, Y_2^{t-1}).$$
(II.12)

Moreover, if there exist $(\widehat{X}_{1,t}, \widehat{X}_{2,t})$ such that the functions $g_{i,t}(\cdot, \cdot)$ satisfy for i = 1, 2

$$g_{i,t}(Y_1^t, Y_2^t) = \mathbf{E} \Big\{ X_{i,t} \Big| Y_1^t, Y_2^t \Big\} = Y_{i,t}, \ \forall t \in \mathbb{N}^n, \ \text{(II.13)}$$

then the inequality in (II.12) holds with equality.

(b) Let $\mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{Y}_1 \times \mathbb{Y}_2 = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, $(p_1, p_2) \in \mathbb{Z}_+$. For all measurable functions $h_{i,t}(Y_1^t, Y_2^t)$, i = 1, 2 then

$$\mathbf{E}\left\{\left|\left|X_{i,t}-h_{i,t}(Y_{1}^{t},Y_{2}^{t})\right|\right|_{\mathbb{R}^{p_{i}}}^{2}\right\}$$

$$\geq \mathbf{E}\left\{\left|\left|X_{i,t}-\mathbf{E}\left\{X_{i,t}\middle|Y_{1}^{t},Y_{2}^{t}\right\}\right|\right|_{\mathbb{R}^{p_{i}}}^{2}\right\}, \quad \forall t \in \mathbb{N}^{n}, \ i=1,2.$$

(c) Suppose $\mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{Y}_1 \times \mathbb{Y}_2 = \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, $(p_1, p_2) \in \mathbb{Z}_+$, and (II.13) holds. Then the joint RDF given by (II.8) is characterized by

$$R_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) = \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{cm}(\Delta_{1},\Delta_{2})} \left\{ \sum_{t=1}^{n} I(X_{1,t},X_{2,t};Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1}) \right\}$$
(II.14)

where $\mathcal{M}_{X_1^n,X_2^n}^{cm}(\Delta_1,\Delta_2) \subseteq \mathcal{M}_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$, with the additional restriction $g_{i,t}(Y_1^t,Y_2^t) = \mathbf{E}\left\{X_{i,t}\middle|Y_1^t,Y_2^t\right\} = Y_{i,t}, -a.s., \forall t \in \mathbb{N}^n \text{ for } i = 1, 2.$

Proof: (a) By properties of mutual information, we have

$$I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1})$$
(II.15)

$$\stackrel{(1)}{=} I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t}, \hat{X}_{1,t}, \hat{X}_{2,t} | Y_1^{t-1}, Y_2^{t-1}) \quad (II.16)$$

$$\stackrel{(2)}{=} I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | \hat{X}_{1,t}, \hat{X}_{2,t}, Y_1^{t-1}, Y_2^{t-1})$$

$$+ I(X_{1,t}, X_{2,t}; \hat{X}_{1,t}, \hat{X}_{2,t} | Y_1^{t-1}, Y_2^{t-1})$$

$$\stackrel{(3)}{\geq} I(X_{1,t}, X_{2,t}; \hat{X}_{1,t}, \hat{X}_{2,t} | Y_1^{t-1}, Y_2^{t-1}), \quad (II.17)$$

where (1) is due to \hat{X}_i , i = 1, 2, are functions of (Y_1^t, Y_2^t) , (2) is due to the chain rule of mutual information, and (3) is due to $I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | \hat{X}_{1,t}, \hat{X}_{2,t}, Y_1^{t-1}, Y_2^{t-1}) \geq$ 0. Thus, (II.12) is obtained. Furthermore, if $\hat{X}_{i,t} =$ $g_{i,t}(Y_1^t, Y_2^t) = Y_{i,t} - a.s, i = 1, 2$ hold, then $I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | \hat{X}_{1,t}, \hat{X}_{2,t}, Y_1^{t-1}, Y_2^{t-1}) = 0$, and hence the inequality (II.17) become equality. (b) The inequality is well-known, due to the orthogonal projection theorem. (c) This is due to (a), (b), and the fact that the fidelity constraints hold with equality.

Remark 1: For a tuple of Gaussian Markov processes, (X_1^n, X_2^n) , Theorem 1 and Theorem 2, are used in the remaining paper to characterize joint NRDF.

III. JOINT NRDF OF MULTIVARIATE GAUSSIAN MARKOV PROCESSES WITH INDIVIDUAL MSE DISTORTION CRITERIA

For the rest of the paper we consider the tuple of multivariate Gaussian Markov process of Definition 1.

Definition 1: A tuple of multivariate Gaussian Markov process, $X_t = (X_{1,t}^{\mathsf{T}}, X_{2,t}^{\mathsf{T}})^{\mathsf{T}}$, $X_{i,t} : \Omega \to \mathbb{R}^{p_i}, t = 0, \dots, n$, i = 1, 2, is defined for $t = 1, \dots, n - 1$ by the recursion

$$\begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \end{pmatrix} = A_t \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} + B_t \begin{pmatrix} W_{1,t+1} \\ W_{2,t+1} \end{pmatrix}, \quad (\text{III.1})$$

where (i) $A_t \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}, B_t \in \mathbb{R}^{(p_1+p_2)\times(q_1+q_2)}$ are non-random matrices; (ii) $\{W_{i,t} : t = 2, ..., n-1\}$ is an \mathbb{R}^{q_i} -valued independent Gaussian process, for $i = 1, 2, W_t = (W_{1,t}^{\mathsf{T}}, W_{2,t}^{\mathsf{T}})^{\mathsf{T}} \in G(0, Q_{(W_{1,t}, W_{2,t})}), Q_{(W_{1,t}, W_{2,t})} \succeq 0,$ independent of X_1 ; (iii) $X_1 \in \mathbb{R}^{p_1+p_2}$ is Gaussian $X_1 \in G(0, Q_{(X_{1,1}, X_{2,1})}), Q_{(X_{1,1}, X_{2,1})} \succeq 0.$

Definition 2: Define

$$\begin{aligned} X_t &\stackrel{\triangle}{=} \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix}, \ Y_t \stackrel{\triangle}{=} \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}, \quad \forall t \in \mathbb{N}^n, \\ E_t \stackrel{\triangle}{=} \begin{pmatrix} E_{1,t} \\ E_{2,t} \end{pmatrix}, \ E_t^- \stackrel{\triangle}{=} \begin{pmatrix} E_{1,t} \\ E_{2,t}^- \end{pmatrix}, \\ E_{i,t} \stackrel{\triangle}{=} X_{i,t} - \widehat{X}_{i,t|t}, \ E_{i,t}^- \stackrel{\triangle}{=} X_{i,t} - \widehat{X}_{i,t|t-1}, i = 1, 2, \\ \widehat{X}_{t|s} \stackrel{\triangle}{=} \mathbf{E} \Big\{ \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} \Big| Y^s \Big\} = \begin{pmatrix} \widehat{X}_{1,t|s} \\ \widehat{X}_{2,t|s} \end{pmatrix}, \ \forall (t,s) \in \mathbb{N}^n \times \mathbb{N}^n \end{aligned}$$

and the mean-square errors

$$\Sigma_{(E_{1,t},E_{2,t})} \stackrel{\triangle}{=} \mathbf{E} \Big\{ E_t E_t^{\mathsf{T}} \Big\}, \quad \forall t \in \mathbb{N}^n,$$
(III.2)

$$\Sigma_{(E_{1,t}^-, E_{2,t}^-)} \stackrel{\triangle}{=} \mathbf{E} \left\{ E_t^- \left(E_t^- \right)^{\mathsf{T}} \right\} \quad \forall t \in \mathbb{N}^n$$
(III.3)

where for t = 1, $\Sigma_{(E_{1,1}^-, E_{2,1}^-)} \stackrel{\triangle}{=} Q_{(X_{1,1}, X_{2,1})}$.

Next, we present another structural property. the tuple of Gaussian Markov process subject to two square-error distortion criteria.

Theorem 3: Consider the joint NRDF $R_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$ of (II.3) for the tuple of multivariate Gaussian Markov process of Definition 1, with individual distortion criteria, $d_{i,n}(x_1^n,y_1^n) \stackrel{\triangle}{=} \frac{1}{n} \sum_{t=1}^n ||x_{1,t} - y_{1,t}||_{\mathbb{R}^{p_i}}^2, i = 1, 2.$ The following hold.

(a) The minimizing element of the set $Q_{X_1^n, X_2^n}(\Delta_1, \Delta_2)$ is jointly Gaussian $\mathbf{P}_{X_1^t, X_2^t, Y_1^t, Y_2^t} = \mathbf{P}_{X_1^t, X_2^t, Y_1^t, Y_2^t}^G$, $t = 1, \ldots, n$, and it is induced by the parametric realization

$$\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = H_t \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} + \begin{pmatrix} g_{1,t}(Y_1^{t-1}, Y_2^{t-1}) \\ g_{2,t}(Y_1^{t-1}, Y_2^{t-1}) \end{pmatrix} + \begin{pmatrix} V_{1,t} \\ V_{2,t} \end{pmatrix},$$
(III.4)
$$= H_t X_t + \left(I_{p_1+p_2} - H_t \right) \widehat{X}_{t|t-1} + V_t$$
(III.5)

where

$$H_{t} = \begin{pmatrix} H_{11,t} & H_{12,t} \\ H_{21,t} & H_{22,t} \end{pmatrix} \in \mathbb{R}^{(p_{1}+p_{2})\times(p_{1}+p_{2})} \text{ are nonrandom,}$$
(III.6)

$$g_{i,t}(Y_1^{t-1}, Y_2^{t-1}) = \hat{X}_{i,t|t-1} - \begin{pmatrix} H_{i1,t} & H_{i2,t} \end{pmatrix} \hat{X}_{t|t-1},$$
(III.7)
$$\hat{X}_{t|t-1} = \mathbf{E} \Big\{ X_t \Big| Y_1^{t-1}, Y_2^{t-1} \Big\} = A_{t-1} \begin{pmatrix} \hat{X}_{1,t-1|t-1} \\ \hat{X}_{2,t-1|t-1} \end{pmatrix}$$

$$V_t = (V_{1,t}^{\mathsf{T}}, V_{2,t}^{\mathsf{T}})^{\mathsf{T}} \in G(0, Q_{(V_{1,t}, V_{2,t})}), Q_{(V_{1,t}, V_{2,t})} \succeq 0,$$

$$V_t \text{ is indep. of } X_1 \text{ and } W_s = (W_{1,s}^{\mathsf{T}}, W_{2,s}^{\mathsf{T}})^{\mathsf{T}}, s = 1, \dots, t.$$
(III.8)

Moreover,

(i) the test channel, denoted by $\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1},X_{1,t},X_{2,t}}^G$, is parametrized by $(H_t, Q_{(V_{1,t},V_{2,t})}), t = 1, \ldots, n$, and satisfies

$$\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1},X_{1,t},X_{2,t}}^{G} (\text{III.9}) \\
= Q_{t}(dy_{1,t},dy_{2,t}|\widehat{x}_{1,t-1|t-1},\widehat{x}_{2,t-1|t-1},x_{1,t},x_{2,t})$$

(ii) for each $t = 1, \ldots, n$ the pay-off satisfies

$$I(X_1^t, X_2^t; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1})$$
(III.10)

$$= I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1})$$
(III.11)
= $I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t} | Y_1^{t-1}, Y_2^{t-1})$

$$\widehat{X}_{1,t-1|t-1}, \widehat{X}_{2,t-1|t-1}), \qquad \text{(III.12)}$$

$$= I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t}, \hat{X}_{1,t|t}, \hat{X}_{2,t|t}|Y_1^{t-1}, Y_2^{t-1}, \\ \hat{X}_{1,t|t-1}, \hat{X}_{2,t|t-1}),$$
(III.13)

$$= I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t}|Y_1^{t-1}, Y_2^{t-1}, \\ \widehat{X}_{1,t-1|t-1}, \widehat{X}_{2,t-1|t-1}, \widehat{X}_{1,t|t}, \widehat{X}_{2,t|t}) \\ + I(X_{1,t}, X_{2,t}; \widehat{X}_{1,t|t}, \widehat{X}_{2,t|t}|Y_1^{t-1}, Y_2^{t-1}, \\ \widehat{X}_{1,t-1|t-1}, \widehat{X}_{2,t-1|t-1})$$
(III.14)

$$\geq I(X_{1,t}, X_{2,t}; \widehat{X}_{1,t|t}, \widehat{X}_{2,t|t}| Y_1^{t-1}, Y_2^{t-1}, \widehat{X}_{1,t|t-1}, \widehat{X}_{2,t|t-1}),$$
(III.15)

and equality holds in (III.15) if

$$\widehat{X}_{i,t|t} = \mathbf{E} \Big\{ X_{i,t} \Big| Y_1^t, Y_2^t \Big\} = Y_{i,t} - a.s., \quad i = 1, 2.$$
(III.16)

(b) Consider the realization of part (a). A sufficient condition for (III.16) to hold is,

$$\mathbf{E}\left\{X_{t}\middle|Y_{1}^{t}, Y_{2}^{t}\right\} = \mathbf{E}\left\{X_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right\} \\
+ \operatorname{cov}\left(X_{t}, Y_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right)\operatorname{cov}\left(Y_{t}, Y_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right)^{\dagger} \\
\left(Y_{t} - \mathbf{E}\left\{Y_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right\}\right) = Y_{t} - a.s. \quad \text{(III.17)}$$

for $t = 1, ..., provided such a \left(H_t, Q_{(V_{1,t}, V_{2,t})}\right)$ exists. Moreover, if the pseudoinverse $\operatorname{cov}\left(Y_t, Y_t | Y_1^{t-1}, Y_2^{t-1}\right)^{\dagger} = \operatorname{cov}\left(Y_t, Y_t | Y_1^{t-1}, Y_2^{t-1}\right)^{-1}$ i.e., the inverse exists, then the following, Conditions 1 and 2, are sufficient for (III.17) to hold for t = 1, ..., n.

(1)
$$\operatorname{cov}\left(X_{t}, Y_{t}|Y_{1}^{t-1}, Y_{2}^{t-1}\right) = \operatorname{cov}\left(Y_{t}, Y_{t}|Y_{1}^{t-1}, Y_{2}^{t-1}\right)$$

(III.18)
(2) $\mathbf{E}\left\{X_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right\} = \mathbf{E}\left\{Y_{t}\middle|Y_{1}^{t-1}, Y_{2}^{t-1}\right\}.$ (III.19)

Proof: (a) The fact that a jointly Gaussian distribution is optimal, is shown similar to the classical RDF of Gaussian random processes with square error fidelity, and follows from [6]. Hence, the test channel distribution is conditionally Gaussian, i.e., $\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1},X_{1,t},X_{2,t}} = \mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1},X_{1,t},X_{2,t}}^G$, with linear conditional mean and nonrandom conditional covariance. Such a distribution is induced by the parametric realization (III.4) with linear $g_{i,t}(\cdot), i = 1, 2$. (III.7) follows from the joint NRDF given by (II.7), because for each t, the payoff $\mathbf{E}\left\{\log\left(\frac{\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1},X_{1,t},X_{2,t}}{\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_1^{t-1},Y_2^{t-1}}}\right)\right\}$ does not depend on $g_{i,t}(\cdot)$, and the average distortions $\mathbf{E}\left\{\sum_{t=1}^{n} ||X_{i,t} - Y_{i,t}||_{\mathbb{R}^{p_i}}^2\right\}, i = 1, 2$ is minimized by $g_{i,t}(\cdot)$ given by (III.7). (i) The test channel distribution (III.9) follows from the realization and properties of conditional mutual information, and the equality (III.14) follows from the chain rule of conditional mutual information [23]. Inequality (III.15) is due to the nonnegative property of conditional mutual information

$$I(X_{1,t}, X_{2,t}; Y_{1,t}, Y_{2,t}|Y_1^{t-1}, Y_2^{t-1}, \\ \hat{X}_{1,t-1|t-1}, \hat{X}_{2,t-1|t-1}, \hat{X}_{1,t|t}, \hat{X}_{2,t|t}) \ge 0.$$
(III.20)

Moreover, if (III.16) holds, then the value of the left hand side of (III.20) is zero, and the inequality (III.15) holds with equality. (b) Since $(X_1^n, X_2^n, Y_1^n, Y_2^n)$ is jointly Gaussian, by mean-square estimation theory follows that if (III.17) holds then (III.16) holds. If the stated inverse exists then (III.18), (III.19) imply (III.17).

Next, we establish existence of the tuple $(H, Q_{(V_{1,t}, V_{2,t})})$ such that equality holds in (III.16), which is essential to characterize the joint NRDF.

Theorem 4: Consider the joint NRDF $R_{X_1^n,X_2^n}(\Delta_1, \Delta_2)$ of (II.3) for the tuple of multivariate Gaussian Markov process of Definition 1, with individual distortion criteria, $d_{i,n}(x_1^n, y_1^n) \triangleq \frac{1}{n} \sum_{t=1}^n ||x_{1,t} - y_{1,t}||_{\mathbb{R}^{p_i}}^2$, i = 1, 2. The following hold.

(a) The optimal test channel distribution of the joint NRDF is conditionally Gaussian, $\mathbf{P}_{Y_{1,t},Y_{2,t}|Y_{1,t-1},Y_{2,t-1},X_{1,t},X_{2,t}}^{G}$, induced by (X_{1}^{n}, X_{2}^{n}) and the realization, for $t = 1, \ldots, n$,

$$\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = H_t \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} + \left(I_{p_1+p_2} - H_t \right) A_{t-1} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} V_{1,t} \\ V_{2,t} \end{pmatrix}$$

$$= H_t X_t + \left(I_{p_1+p_2} - H_t \right) A_{t-1} Y_{t-1} + V_t \quad (\text{III.22})$$

where the matrices, $(H_t, Q_{(V_{1,t}, V_{2,t})})$ satisfy,

$$H_{t}\Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} = \Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} - \Sigma_{(E_{1,t}, E_{2,t})}$$
(III.23)
$$= \Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} H_{t}^{\mathsf{T}} \succeq 0,$$
$$Q_{(V_{1,t}, V_{2,t})} = H_{t}\Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} - H_{t}\Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} H_{t}^{\mathsf{T}} \succeq 0,$$
$$\Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} = A_{t-1}\Sigma_{(E_{1,t-1}, E_{2,t-1})} A_{t-1}^{\mathsf{T}}$$
$$+ B_{t-1}Q_{(W_{1,t}, W_{2,t})} B_{t-1}^{\mathsf{T}}, \quad t = 2, \dots, n \quad \text{(III.24)}$$
$$\Sigma_{(E_{1,t}^{-}, E_{2,1}^{-})} = Q_{(X_{1,1}, X_{2,1})}. \quad \text{(III.25)}$$

If $Q_{(X_{1,1},X_{2,1})}$ and $B_{t-1}Q_{(W_{1,t},W_{2,t})}B_{t-1}^{T}, t = 2, ..., n$ are full rank matrices then

$$Q_{(V_{1,t},V_{2,t})} = \Sigma_{(E_{1,t}E_{2,t})} - \Sigma_{(E_{1,t}E_{2,t})} \left(\Sigma_{(E_{1,t}^{-},E_{2,t}^{-})} \right)^{-1} \Sigma_{(E_{1,t}E_{2,t})} \succeq 0.$$
(III.26)

(b) The characterization of the NRDF is given by

$$R_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) = \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{G}(\Delta_{1},\Delta_{2})} \left\{ \sum_{t=1}^{n} I(X_{1,t},X_{2,t};Y_{1,t},Y_{2,t}|Y_{1,t-1},Y_{2,t-1}) \right\}$$
(III.27)
$$= \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{G}(\Delta_{1},\Delta_{2})} \left\{ \frac{1}{2} \sum_{t=1}^{n} \log \left(\frac{|\Sigma_{(E_{1,t}^{-},E_{2,t}^{-})}|}{|\Sigma_{(E_{1,t},E_{2,t})}|} \right) \right\}$$
(III.28)

where the constraint set is

$$\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{G}(\Delta_{1},\Delta_{2}) \stackrel{\Delta}{=} \left\{ \Sigma_{(E_{1,t},E_{2,t})} \in \mathcal{S}_{+}^{p \times p}, \ t = 1,\ldots,n \right|$$

$$\Sigma_{(E_{1,t},E_{2,t})} \preceq \Sigma_{(E_{1,t}^{-},E_{2,t}^{-})}, \ t = 1,\ldots,n, \ \text{(III.24)}, \ \text{(III.25)}$$

$$\frac{1}{n} \sum_{t=1}^{n} \operatorname{tr}(\Sigma_{E_{1,t}}) \leq \Delta_{1}, \ \frac{1}{n} \sum_{t=1}^{n} \operatorname{tr}(\Sigma_{E_{2,t}}) \leq \Delta_{2} \right\}. \ \text{(III.29)}$$

Proof: First, note that by Theorem 3.(a), the joint NRDF is also expressed as

$$R_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) = \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{G}(\Delta_{1},\Delta_{2})} \left\{ \text{(III.30)} \right.$$
$$\left. \sum_{t=1}^{n} I(X_{1,t},X_{2,t};Y_{1,t},Y_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1}) \right\}$$
$$= \inf_{\mathcal{M}_{X_{1}^{n},X_{2}^{n}}^{G}(\Delta_{1},\Delta_{2})} \mathbf{E} \left\{ \sum_{t=1}^{n} \log \left(\frac{\mathbf{P}_{X_{1,t},X_{2,t}|Y_{1}^{t-1},Y_{2}^{t-1}}}{\mathbf{P}_{X_{1,t},X_{2,t}|Y_{1}^{t},Y_{2}^{t}}} \right) \right\}$$
(III.31)

where $\mathcal{M}_{X_1^n,X_2^n}^G(\Delta_1,\Delta_2)$ is the subset of $\mathcal{M}_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$ defined by (II.9), generated by jointly Gaussian distributions $\mathbf{P}_{X_1^t,X_2^t,Y_1^t,Y_2^t}^G$, $t = 1, \ldots, n$ and $\mathbf{P}_{X_{1,t},X_{2,t}|Y_1^{t-1},Y_2^{t-1}}^G$, $\mathbf{P}_{X_{1,t},X_{2,t}|Y_1^t,Y_2^t}^G$ denote conditionally Gaussian distributions, obtained from the realization (III.4)-(III.8). By properties of jointly Gaussian random processes, then

$$\operatorname{cov}\left(X_{t}, X_{t} \middle| Y^{t}\right) = \mathbf{E}\left\{E_{t}\left(E_{t}\right)^{\mathsf{T}}\right\} = \Sigma_{\left(E_{1,t}E_{2,t}\right)}$$
$$\operatorname{cov}\left(X_{t}, X_{t} \middle| Y^{t-1}\right) = \mathbf{E}\left\{E_{t}^{-}\left(E_{t}^{-}\right)^{\mathsf{T}}\right\} = \Sigma_{\left(E_{1,t}^{-}, E_{2,t}^{-}\right)}$$

Clearly, $\Sigma_{(E_{1,t}^-, E_{2,t}^-)}$ is given by (III.24).

(a) Realization (III.21) and specifically, (III.23)-(III.25) are

obtained by applying Theorem 3.(b) so that (III.16) holds. The conditions of Theorem 3.(b) give rise to the equations of $(H_t, Q_{(V_{1,t}, V_{2,t})})$ as specified. (b) This follows directly by using the realization of part (a) to calculate (III.31).

The next theorem gives the Kuhn-Tucker conditions of the optimization problem of Theorem 4.(b).

 $R_{X_1,X_2}(\Delta_1,\Delta_2)$ of Theorem 5: Consider \triangleq Theorem 4.(b), defined by (III.28) and assume \overline{Q}_t $B_t Q_{(W_{1,t+1},W_{2,t+1})} B_t^{\mathsf{T}} \succ 0$, and $R_{X_1^n,X_2^n}(\Delta_1,\Delta_2) < +\infty$. The Lagrange functional is,

$$\mathcal{L} \stackrel{\Delta}{=} \sum_{t=1}^{n} \left\{ \frac{1}{2} \log \left(\frac{\left| \Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} \right|}{\left| \Sigma_{(E_{1,t}, E_{2,t})} \right|} \right) + \operatorname{tr} \left(\Theta_t \left(\Sigma_{(E_{1,t}, E_{2,t})} - \Sigma_{(E_{1,t}^{-}, E_{2,t}^{-})} \right) \right) - \operatorname{tr} \left(V_t \Sigma_{(E_{1,t}, E_{2,t})} \right) \right\} \\ + \lambda_1 \left(\sum_{t=1}^{n} \operatorname{tr} \left(\Sigma_{E_{1,t}} \right) - n \Delta_1 \right) + \lambda_2 \left(\sum_{t=1}^{n} \operatorname{tr} \left(\Sigma_{E_{2,t}} \right) - n \Delta_2 \right) \right)$$

where $\Theta_t \succeq 0$, $V_t \succeq 0$, $\lambda_i \in [0, \infty)$, i = 1, 2. The optimal $\{\Sigma_{(E_{1,t},E_{2,t})} : t = 1,...,n\}$ $\mathcal{M}^G_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$ for $R_{X_1^n,X_2^n}(\Delta_1,\Delta_2)$ is found as follows. (i) Stationarity:

$$-\frac{1}{2}\Sigma_{(E_{1,t},E_{2,t})}^{-1} + \begin{bmatrix} \lambda_1 I_{p_1} & 0\\ 0 & \lambda_2 I_{p_2} \end{bmatrix} + \Theta_t + V_t = 0.$$
(III.32)

(ii) Complementary Slackness:

$$\lambda_1 \left(\sum_{t=1}^n \operatorname{tr} \left(\Sigma_{E_{1,t}} \right) - n \Delta_1 \right) = 0, \qquad (\text{III.33})$$

$$\lambda_2 \left(\sum_{t=1}^n \operatorname{tr} \left(\Sigma_{E_{2,t}} \right) - n \Delta_2 \right) = 0, \qquad \text{(III.34)}$$

$$\operatorname{tr}\left(V_{t}\Sigma_{(E_{1,t},E_{2,t})}\right) = 0, \quad t = 1,\dots,n, \quad (\text{III.35})$$

$$\operatorname{tr}\left(\Theta_t\left(\Sigma_{(E_{1,t},E_{2,t})} - \Sigma_{(E_{1,t}^-,E_{2,t}^-)}\right)\right) = 0. \quad (\text{III.36})$$

(iii) Primal Feasibility: Defined by $\mathcal{M}_{X_1^n,X_2^n}^G(\Delta_1,\Delta_2.$ (iv) Dual Feasibility: $\lambda_1 \geq 0, \ \lambda_2 \geq 0, \ \Theta_t \succeq 0, \ V_t \succeq$ $0, t = 1, \ldots, n.$

Moreover, the following hold.

(a) $V_t = 0$ for t = 1, ..., n and

For
$$t = n$$
:

$$\Sigma_{(E_{1,n},E_{2,n})} = \frac{1}{2} \left(\begin{bmatrix} \lambda_1 I_{p_1} & 0\\ 0 & \lambda_2 I_{p_2} \end{bmatrix} + \Theta_n \right)^{-1} \succ 0.$$
(III.37)

For
$$t = n - 1, ..., 1$$
:

$$\Sigma_{(E_{1,t}, E_{2,t})} + \Sigma_{(E_{1,t}, E_{2,t})} \overline{Q}_t \Sigma_{(E_{1,t}, E_{2,t})}$$

$$- \frac{1}{2} \left(\begin{bmatrix} \lambda_1 I_{p_1} & 0 \\ 0 & \lambda_2 I_{p_2} \end{bmatrix} + \Theta_t - A_t^{\mathsf{T}} \Theta_{t+1} A_t \right)^{-1} = 0$$
(III.38)

(b) If $\Sigma_{(E_{1,t}^-, E_{2,t}^-)} - \Sigma_{(E_{1,t}, E_{2,t})} \succ 0$ for all $t = 1, \dots, n$ then $\Theta_t = 0$ for all $t = 1, \ldots, n$, and

Proof: The derivation is utilizes [6, Theorem 5.3].

Example 1: To illustrate fundamental challenges, we consider $X_{i,t}: \Omega \to \mathbb{R}, i = 1, 2$ and $Y_{i,t}: \Omega \to \mathbb{R}, i = 1, 2$ for t = 1, ..., n. For simplicity, assume $\overline{Q}_t = \text{diag}(q_1, q_2) \succ$ 0, i.e., $q_1 > 0$, $q_2 > 0$, and $A_t = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} t = 0, \dots, n$, and $X_{1,1}, X_{2,1}$ are independent with variances σ_1^2, σ_2^2 , respectively. By Theorem 5 item (b), we have: For t = n:

$$\Sigma_{(E_{1,n},E_{2,n})} = \begin{bmatrix} \frac{1}{2\lambda_1} & 0\\ 0 & \frac{1}{2\lambda_2} \end{bmatrix}, \Sigma_{E_{1,n},E_{2,n}} = 0.$$
(III.39)
For $t = 1, \dots, n-1$:

 $\Sigma_{(E_{1,t},E_{2,t})} + \Sigma_{(E_{1,t},E_{2,t})}\overline{Q}_{t}\Sigma_{(E_{1,t},E_{2,t})} - \begin{bmatrix} \frac{1}{2\lambda_{1}} & 0\\ 0 & \frac{1}{2\lambda_{2}} \end{bmatrix} = 0$

The set of equations for $t = 1, \ldots, n - 1$, are

$$\Sigma_{E_1,t} + \Sigma_{E_1,t}^2 q_1 + \Sigma_{E_{1,t},E_2,t}^2 q_2 - \frac{1}{2\lambda_1} = 0 \qquad \text{(III.40)}$$

$$\sum_{E_{1,t},E_2,t} (1 + \sum_{E_1,t} q_1 + \sum_{E_2,t} q_2) = 0$$
(III.41)

$$\Sigma_{E_2,t} + \Sigma_{E_2,t}^2 q_2 + \Sigma_{E_{1,t},E_2,t}^2 q_1 - \frac{1}{2\lambda_2} = 0 \qquad \text{(III.42)}$$

By (III.41), $\Sigma_{E_1,E_2,t} = 0$ or $\Sigma_{E_1,t}q_1 + \Sigma_{E_2,t}q_2 = -1$. The later cannot hold because the left hand side is always positive, hence $\Sigma_{E_{1,t},E_2,t} = 0$ and the optimal matrices $\Sigma_{(E_{1,t},E_{2,t})}$ are diagonal for all $t = 1, \ldots, n-1$. Then by (III.40) and (III.42) the positive solutions are

$$\Sigma_{E_1,t} = \frac{-1 + \sqrt{1 + \frac{2q_1}{\lambda_1}}}{2q_1}, \ \Sigma_{E_2,t} = \frac{-1 + \sqrt{1 + \frac{2q_2}{\lambda_2}}}{2q_2}$$

For t = 0, ..., n using the above values we can determine $\lambda_i \geq 0, i = 1, 2$, using the average distortions as follows.

$$\frac{1}{2\lambda_i} + (n-1)\left(\frac{-1+\sqrt{1+\frac{2q_i}{\lambda_i}}}{2q_i}\right) = n\Delta_i, \quad i = 1, 2$$
(III.43)

Then by using (III.24), need to determine the matrix $\Sigma_{(E_{1,t}^{-},E_{2,t}^{-})}$. Suppose its diagonal entries are α_t,β_t and its non-diagonal entry is γ_t . Then the equations for $\Sigma_{(E_{1,t}^-, E_{2,t}^-)}$, for $t = 2, \ldots, n$, are given by

$$\alpha_t = a_{11}^2 \Sigma_{E_1,t-1} + a_{12}^2 \Sigma_{E_2,t-1} + q_1, \qquad \text{(III.44)}$$

$$\beta_t = a_{21}^2 \Sigma_{E_1, t-1} + a_{22}^2 \Sigma_{E_2, t-1} + q_2, \qquad (III.45)$$

$$\gamma_t = a_{11}a_{21}\Sigma_{E_1,t-1} + a_{22}a_{12}\Sigma_{E_2,t-1}$$
(III.46)

For $t = 1, \alpha_1 = \sigma_1^2, \beta_1 = \sigma_2^2, \gamma_1 = 0$. Therefore, the joint NRDF is given by,

$$R_{X_{1}^{n},X_{2}^{n}}(\Delta_{1},\Delta_{2}) = \frac{1}{2} \sum_{t=1}^{n} \log\left(\frac{\alpha_{t}\beta_{t} - \gamma_{t}^{2}}{\Sigma_{E_{1},t}\Sigma_{E_{2},t}}\right) \quad \text{(III.47)}$$

$$= \frac{1}{2} \log\left(\frac{\alpha_{1}\beta_{1} - \gamma_{1}^{2}}{\Sigma_{E_{1},1}\Sigma_{E_{2},1}}\right) + \frac{(n-2)}{2} \log\left(\frac{\alpha_{t}\beta_{t} - \gamma_{t}^{2}}{\Sigma_{E_{1},t}\Sigma_{E_{2},t}}\right)$$

$$+ \frac{1}{2} \log\left(\frac{\alpha_{n}\beta_{n} - \gamma_{n}^{2}}{\Sigma_{E_{1},n}\Sigma_{E_{2},n}}\right) \quad \text{(III.48)}$$

and (III.47) holds for (Δ_1, Δ_2) such that $\Sigma_{(E_{1,t}^-, E_{2,t}^-)} - \Sigma_{(E_{1,t}, E_{2,t})} \succ 0$ for all $t = 1, \ldots, n$. The closed form calculations of this region as a function of $(\lambda_1, \lambda_2, A, \overline{Q}_t)$ are lengthy hence omitted due to space limitation. The per unit time limit is, then obtain from the solution of (III.44)-(III.46), at any $t \in \{2, \ldots, n-1\}$, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} R_{X_1^n, X_2^n}(\Delta_1, \Delta_2) = \frac{1}{2} \log \left(\frac{\alpha_t \beta_t - \gamma_t^2}{\Sigma_{E_1, t} \Sigma_{E_2, t}} \right).$$
(III.49)

Remark 2: For the sub-set of the rate region for which Theorem 5.(b) holds, it is possible to compute the closedform expression of the joint NRDF $R_{X_1^n,X_2^n}(\Delta_1, \Delta_2)$, by using (III.37), (III.38), with $\Theta_t = 0, t = 1, ..., n$. This will lead to a generalization of the classical joint RDF given in [18, Theorem III.3], for the tuple of jointly independent and identically (IID) distributed Gaussian process (X_1^n, X_2^n) , i.e., $\mathbf{P}_{X_{1,t},X_{2,t}} = \mathbf{P}_{X_1,X_2}, t = 1, ..., n$, with individual distortion criteria [18, Theorem III.3]. The case $\Sigma_{(E_{1,t},E_{2,t})} - \Sigma_{(E_{1,t}^-,E_{2,t}^-)} \succeq 0$ but not $\Sigma_{(E_{1,t},E_{2,t})} - \Sigma_{(E_{1,t}^-,E_{2,t}^-)} \succ 0$ is to our experience, a challenging problem, even for the simplest application example of a tuple of IID process.

IV. CONCLUSIONS AND OPEN PROBLEMS

The joint nonanticpative RDF is analyzed for a tuple of random process with individual fidelity criteria. Achievable lower bound and structural properties of the test channels are derived. The application example of a tuple of jointly multivariate Gaussian Markov process with two square-error fidelity criteria is analyzed. A fundamental open problem which is not addressed in this paper is the computation of the joint nonanticipative RDF of Theorem 4. Although, for a tuple of multivariate Gaussian Markov process this is challenging problem (i.e., currently the only known solution is, for a tuple of scalar, IID Gaussian random variables [17]), some progress is expected.

References

- A. K. Gorbunov and M. S. Pinsker, "Nonanticipatory and prognostic epsilon entropies and message generation rates," *Problems of Information Transmission*, vol. 9, no. 3, pp. 184–191, 1973.
- [2] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1549–1561, 2004.
- [3] M. S. Derpich and J. Østergaard, "Improved upper bounds to the causal quadratic rate-distortion function for Gaussian stationary sources," *IEEE Transactions on Information Theory*, vol. 58, pp. 3131–3152, May 2012.
- [4] C. D. Charalambous, P. A. Stavrou, and N. U. Ahmed, "Nonanticipative rate distortion function and relations to filtering theory," *IEEE Transactions on Automatic Control*, vol. 59, pp. 937–952, April 2014.
- [5] T. Tanaka, K. K. K. Kim, P. A. Parrilo, and S. K. Mitter, "Semidefinite programming approach to Gaussian sequential rate-distortion tradeoffs," *IEEE Transactions on Automatic Control*, vol. 62, pp. 1896– 1910, April 2017.
- [6] P. A. Stavrou, T. Charalambous, C. D. Charalambous, and S. Loyka, "Optimal estimation via nonanticipative rate distortion function for time-varying Gauss-Markov processes," *SIAM Journal on Control and Optimization (SICON)*, vol. 56, no. 5, pp. 3731–3765, 2018.
- [7] C. E. Shannon, "Coding theorems for a discrete source with a fidelity criterion," *IRE Nat. Conv. Rec*, vol. 4, no. 3, pp. 142–163, 1959.
- [8] R. T. Gallager, Information Theory and Reliable Communication. John Wiley & Sons, Inc., New York, 1968.

- [9] D. Neuhoff and R. Gilbert, "Causal source codes," *IEEE Transactions* on Information Theory, vol. 28, pp. 701–713, Sep 1982.
- [10] N. Gaarder and D. Slepian, "On optimal finite-state digital transmission systems," *IEEE Transactions on Information Theory*, vol. 28, no. 2, pp. 167–186, 1982.
- [11] T. Linder and G. Lagosi, "A zero-delay sequential scheme for lossy coding of individual sequences," *IEEE Transactions on Information Theory*, vol. 47, no. 6, pp. 2533–2538, 2001.
- [12] P. A. Stavrou, J. Ostergaard, and C. D. Charalambous, "Zero-delay rate distortion via filtering for vector-valued Gaussian sources," *IEEE Journal of Selected Topics in Signal Processing*, vol. 12, no. 5, pp. 841–856, 2018.
- [13] R. Zamir and M. Feder, "On lattice quantization noise," *IEEE Transactions on Information Theory*, vol. 42, pp. 1152–1159, July 1996.
- [14] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control* and Optimization (SICON), vol. 43, no. 2, pp. 413–436, 2004.
- [15] C. D. Charalambous, T. Charalambous, C. Kourtellaris, and J. H. Van Schuppen, "Structural properties of nonanticipatory epsilon entropy of multivariate gaussian sources," in 2020 IEEE International Symposium on Information Theory (ISIT), pp. 2867–2872, IEEE, 2020.
- [16] A. K. Gorbunov and M. S. Pinsker, "Prognostic epsilon entropy of a Gaussian message and a Gaussian source," *Problems of Information Transmission*, vol. 10, no. 2, pp. 93–109, 1974.
- [17] J.-J. Xiao and Z.-Q. Luo, "Compression of correlated gaussian sources under individual distortion criteria," in *43rd Allerton Conference on Communication, Control, and Computing*, pp. 438–447, 2005.
- [18] E. Stylianou, C. D. Charalambous, and T. Charalambous, "Joint rate distortion function of a tuple of correlated multivariate gaussian sources with individual fidelity criteria," *arXiv preprint arXiv:2102.07236*, 2021.
- [19] R. Gray and A. Wyner, "Source coding for a simple network," *Bell System Technical Journal*, vol. 53, no. 9, pp. 1681–1721, 1974.
- [20] C. D. Charalambous and J. H. van Schuppen, "Characterization of conditional independence and weak realizations of multivariate gaussian random variables: Applications to networks," in 2020 IEEE International Symposium on Information Theory (ISIT), pp. 2444– 2449, IEEE, 2020.
- [21] T. Berger, Rate Distortion Theory: A Mathematical Basis for Data Compression. Englewood Cliffs, NJ: Prentice-Hall, 1971.
- [22] C. D. Charalambous and P. A. Stavrou, "Optimization of directed information and relations to filtering theory," in *European Control Conference (ECC)*, pp. 1385–1390, June 2014.
- [23] M. Pinsker, Information and Information Stability of Random Variables and Processes. Holden-Day Inc, San Francisco, 1964. Translated by Amiel Feinstein.