

A closed-loop design for scalable high-order consensus

Jonas Hansson and Emma Tegling

Abstract—This paper studies the problem of coordinating a group of n^{th} -order integrator systems. As for the well-studied conventional consensus problem, we consider linear and distributed control with only local and relative measurements. We propose a closed-loop dynamic that we call *serial consensus* and prove it achieves n^{th} order consensus regardless of model order and underlying network graph. This alleviates an important scalability limitation in conventional consensus dynamics of order $n \geq 2$, whereby they may lose stability if the underlying network grows. The distributed control law which achieves the desired closed loop dynamics is shown to be localized and obey the limitation to relative state measurements. Furthermore, through use of the small-gain theorem, the serial consensus system is shown to be robust to both model and feedback uncertainties. We illustrate the theoretical results through examples.

I. INTRODUCTION

Properties of dynamical systems over networks have been a subject of significant research over the last two decades. A problem of interest the coordination of agents in a network through localized feedback, leading to the prototypical distributed consensus dynamics, studied early on by [1]–[3]. Over the years, it has become clear that the structural constraints imposed by the network topology in consensus problems often lead to fundamentally poor dynamic behaviors in large networks. This concerns controllability [4], performance [5], [6] and disturbance propagation [7], [8], but, as recently highlighted in [9], also stability. The poor stability properties characterized in earlier work [9] (which motivate the present work) apply to higher-order consensus, where the local dynamics of each agent is modeled as an n^{th} order integrator, with $n \geq 2$, and the control is a weighted average of neighbors’ relative states. This is a theoretical generalization of first-order consensus [10], but is also relevant in practice. For example, a model where $n = 3$ and thus has consensus in position, velocity and acceleration, can capture flocking behaviors [11].

More specifically, [9] shows that conventional high-order consensus ($n \geq 3$) is not *scalably stable* for many growing graph structures. When the network grows beyond a certain size, stability is lost. The same holds for second-order consensus ($n = 2$) in, for example, directed ring graphs, as also described in [12]. To address this lack of scalable stability we propose an alternative generalization of the first-order consensus dynamics, which we prove achieves scalable

stability for any model order n .

To illustrate our proposed controller, consider the conventional second-order consensus system where the controller $u(t) = -L_1\dot{x}(t) - L_2x(t) + u_{\text{ref}}(t)$, with $L_{1,2}$ being weighted graph Laplacians, is used to achieve the closed loop

$$\ddot{x}(t) = -L_2\dot{x}(t) - L_1x + u_{\text{ref}}(t). \quad (1)$$

While for first-order consensus ($\dot{x} = Lx + u_{\text{ref}}(t)$), a sufficient condition for convergence to consensus is that the graph underlying the graph Laplacian L contains a connected spanning tree. However, this no longer suffices when $n \geq 2$ as in (1). Therefore, we instead propose the following controller $u(t) = -(L_1 + L_2)\dot{x}(t) - L_1L_2x(t) + u_{\text{ref}}(t)$. The reason for this choice of controller is best illustrated by considering the resulting closed loop in the Laplace domain:

$$(sI + L_1)(sI + L_2)X(s) = U_{\text{ref}}. \quad (2)$$

For this system, like for the first-order case, it is sufficient that the graphs underlying L_1 and L_2 contain a connected spanning tree for the system to eventually coordinate in both x and its derivative \dot{x} (regardless of network size!). This closed loop system, which we will call *serial consensus*, thus mimics one core property of the standard consensus protocol, and can also be generalized to any order n .

The main results of this paper are proofs of some key properties of the proposed n^{th} -order serial consensus. The controller is proven to remain localized (within an n -hop neighborhood) and implementable through relative measurements. We also prove that the closed loop will achieve consensus in all n states. Furthermore, we study the robustness of the proposed closed loop and show that the system will still coordinate when subject to unstructured uncertainty. The beneficial properties of the form (2) (generalized to any order n) are thus not contingent on an idealized implementation.

The remainder of this paper is organized as follows. We first introduce the n^{th} order consensus model and define our choice of control structure. Then the serial consensus system is defined and motivated. In Sec. III we provide proofs for the stability and robustness of the serial consensus system. Our main results are then illustrated through examples in Sec. IV. Lastly, we provide our Conclusions in Sec. V.

II. PROBLEM SETUP

We start by introducing some graph theory before introducing the general n^{th} order consensus problem for which we propose the new serial consensus setup. We discuss its properties and then end with some useful definitions.

The authors are with the Department of Automatic Control and the ELLIIT Strategic Research Area at Lund University, Lund, Sweden. Email: {jonas.hansson, emma.tegling}@control.lth.se

This work was partially funded by Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation and the Swedish Research Council through Grant 2019-00691.

A. Network model and definitions

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ denote a graph of size $N = |\mathcal{V}|$. The set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the set of edges. The graph can be equivalently represented by the adjacency matrix $W \in \mathbb{R}^{N \times N}$ where $w_{i,j} > 0 \iff (j, i) \in \mathcal{E}$. The graph is called *undirected* if $W = W^T$. The graph contains a *connected spanning tree* if for some $i \in \mathcal{V}$ there is a path from i to any other vertex $j \in \mathcal{V}$.

Associated with a weighted graph we have the weighted graph Laplacian L defined as

$$[L]_{i,j} = \begin{cases} -w_{i,j}, & \text{if } i \neq j \\ \sum_{k \neq i} w_{i,k}, & \text{if } i = j \end{cases}$$

Under the condition that the graph generating the graph Laplacian contains a connected spanning tree, L will have a simple and unique eigenvalue at 0 and the remaining eigenvalues will lie strictly in the right half plane (RHP).

We will also consider networks with a growing number of nodes. With \mathcal{G}_N we denote a graph in a family $\{\mathcal{G}_N\}$, where N is the size of the growing network.

We will denote the space of all proper, real rational, and stable transfer matrices \mathcal{RH}_∞ and denote the \mathcal{H}_∞ norm as $\|\cdot\|_{\mathcal{H}_\infty}$ following the notation in [13].

B. n^{th} order consensus

Let the system be modeled as N agents with identical n^{th} order integrator dynamics, i.e.

$$\frac{d^n x_i(t)}{dt^n} = u_i(t), \quad (3)$$

for all $i \in \mathcal{V}$. We will use the convention $x_i^{(0)}(t) = x_i(t)$ and $x_i^{(k)}(t) = \frac{d^k}{dt^k} x_i(t)$ to denote time derivatives. When clear we may omit the time argument for brevity.

In this paper we will consider the problem of synchronizing the agents and thus achieve a state of consensus.

Definition 1 (n^{th} order consensus): The multi-agent system (3) is said to achieve (n^{th} order) consensus if $\lim_{t \rightarrow \infty} |x_i^{(k)}(t) - x_j^{(k)}(t)| = 0$, for all $i, j \in \mathcal{V}$ and $k \in \{0, 1, \dots, n-1\}$.

C. Control structure

A linear state feedback controller of (3) can be written as

$$u(t) = u_{\text{ref}}(t) - \sum_{k=0}^{n-1} A_k x^{(k)}(t). \quad (4)$$

Where $u_{\text{ref}}(t) \in \mathbb{R}^N$ is a feedforward term and $A_k \in \mathbb{R}^{N \times N}$ represent the feedback of the k^{th} derivative. We will restrict this class of controllers in three ways. The controllers

- i) can only use *relative* feedback;
- ii) have a limited gain;
- iii) and depend on the local neighborhood of each agent.

The limitation to relative feedback translates to the condition $A_k \mathbf{1}_N = 0$ for all k , while a limited gain can be encoded by demanding that $\|A_k\|_\infty \leq c$. To capture the notion of locality, consider the adjacency matrix W representing the communication and measurement structure, which we here

assume to be the same. That is, if $W_{i,j} = 1$, then agent i can directly receive or measure the relative distance to agent j . Next, consider the non-negative matrix W^q . This matrix has the property that $[W^q]_{i,j} \neq 0$ if and only if there is a path of length q from agent j to agent i . Thus, if we want the controller to only depend on information that is at most q steps away from each agent the following implication should hold: $\sum_{k=0}^q W_{i,j}^k = 0 \implies [A_k]_{i,j} = 0$. Putting all the conditions together gives us a family of controllers that we will consider in this paper:

Definition 2 (q -step implementable relative feedback):

A relative feedback controller of the form (4) is q -step implementable with respect to the adjacency matrix W and gain $c > 0$ if $A_k \in \mathcal{A}^q(W, c)$ for all k , where

$$\mathcal{A}^q(W, c) = \left\{ A \mid \begin{cases} [\sum_{k=0}^q W^k]_{i,j} = 0 \implies A_{i,j} = 0, \\ A \mathbf{1}_N = 0, \|A\|_\infty \leq c \end{cases} \right\}.$$

The conventional controller for achieving n^{th} order consensus can be realized as (4) where each A_k is given by a graph Laplacian, e.g. $A_k = L_k \in \mathcal{A}^1(W, c)$. In many cases these are also assumed to be the same such that $L_k = p_k L$ for some graph Laplacian L and constants $p_k > 0$.

D. A Novel Design: Serial Consensus

We propose the following controller of (3), expressed in the Laplace domain, to achieve n^{th} order consensus

$$U(s) = U_{\text{ref}}(s) + \left(s^n I - \prod_{k=1}^n (sI + L_k) \right) X(s), \quad (5)$$

where L_k are graph Laplacians and U_{ref} is the transformed reference signal. In this case, it is more instructive to consider the closed-loop dynamics, which take the following form:

Definition 3 (n^{th} order serial consensus system): For all $k \in \{1, 2, \dots, n\}$, let L_k be a weighted and directed graph Laplacian. The n^{th} -order serial consensus system is then

$$\left(\prod_{k=1}^n (sI + L_k) \right) X(s) = U_{\text{ref}}(s). \quad (6)$$

We call this form serial consensus because the same closed loop dynamics can also be achieved by interconnecting n first-order consensus systems in a series. The closed-loop dynamics in (6) can also be transformed to state-space form by introducing the alternative variables Ξ_k with the corresponding states ξ_k . These relate to X through $\Xi_1 = X(s)$, $\Xi_k = (sI + L_{k-1})\Xi_{k-1}$ for $k \in \{2, \dots, n-1\}$, and $s\Xi_n = -L_n\Xi_n + U_{\text{ref}}$. This leads to the following continuous-time state-space representation

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 & I & & & \\ & -L_2 & \ddots & & \\ & & \ddots & I & \\ & & & \ddots & -L_n \end{bmatrix}}_A \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_{\text{ref}} \end{bmatrix} \quad (7)$$

The serial consensus form has several advantages, which will be the focus of the paper. First, however, we show that

it satisfies the constraints we impose on the controller, as given by Definition 2. In other words, we will discuss how the closed-loop structure in (6) can be implemented on a network.

When analysing the serial consensus controller of (5) we will make use of the following assumption on the graph structure.

Assumption A1: (Connected spanning tree) All graphs underlying the graph Laplacians L_k contain a connected spanning tree.

E. Implementing Serial consensus

The following proposition ensures that the serial consensus system can be achieved by controlling the n^{th} order integrator system (3) with an n -step implementable relative feedback controller as defined in Definition 2.

Proposition 1: Consider the n^{th} -order serial consensus as defined in (6). If each $L_k \in \mathcal{A}^1(W, c)$ for some constant c and adjacency matrix W , then the controller in (5) is an n -step implementable relative feedback controller with respect to W and a finite gain c' .

To prove this proposition we first need the following two lemmas whose proofs are provided in the appendix.

Lemma 2: If $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}(W, c_2)$ then the sum $(A_1 + A_2) \in \mathcal{A}^{\max(q_1, q_2)}(W, c_1 + c_2)$

Lemma 3: Let $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}(W, c_2)$ then the product $(A_1 A_2) \in \mathcal{A}^{q_1 + q_2}(W, c_1 c_2)$

Now we can prove *Proposition 1*.

Proof: The serial consensus controller can be expanded to the matrix polynomial

$$\begin{aligned} U(s) &= U_{\text{ref}}(s) + \left(s^n I - \prod_{k=1}^n (sI + L_k) \right) X(s) \\ &= U_{\text{ref}}(s) + \left((s^n - s^n)I - \sum_{k=0}^{n-1} s^k A_k \right) X(s), \end{aligned}$$

for some matrices A_k . To show the proposition, we need to show that $A_k \in \mathcal{A}^q(W, c')$ for all $k = 0, \dots, n-1$, with $q \leq n$ and $c' < \infty$. Let

$$\mathcal{I}_k = \{ \alpha \mid |\alpha| = n - k, \alpha \subset \{1, 2, \dots, n\}, \\ i < j \implies \alpha(i) < \alpha(j) \}$$

denote all the ordered subsets of the range $[1, n]$ with size $n - k$. Then

$$A_k = \sum_{\alpha \in \mathcal{I}_k} \prod_{j \in \alpha} L_j, \text{ for all } k \in [0, n-1].$$

Since all $\alpha \in \mathcal{I}_k$ has $n - k$ elements we can show that $\prod_{j \in \alpha} L_j = B_\alpha \in \mathcal{A}^{n-k}(W, c^{n-k})$ by applying Lemma 3 recursively. Now we have a sum

$$A_k = \sum_{\alpha \in \mathcal{I}_k} B_\alpha$$

The number of ordered subsets of the range $[1, n]$ with size $n - k$ is given by the binomial coefficients and therefore the size of $|\mathcal{I}_k| = \binom{n}{n-k}$. Applying Lemma 2 recursively

shows that $A_k \in \mathcal{A}^{n-k}(W, \binom{n}{n-k} c^{n-k})$. Clearly, we have that $n - k \leq n$ and $\binom{n}{n-k} c^{n-k} \leq \binom{n}{\lceil n/2 \rceil} \max(c, c^n) < \infty$ for all k . Let $c' = \binom{n}{\lceil n/2 \rceil} \max(c, c^n)$ and then we have that $A_k \in \mathcal{A}^n(W, c')$ for all k . ■

Example 1: For clarity let us consider the controller for the case $n = 3$. Then the controller is

$$\begin{aligned} U(s) &= U_{\text{ref}}(s) + \left(s^3 I - \prod_{k=1}^3 (sI + L_k) \right) X(s) \\ &= U_{\text{ref}}(s) - \left(s^2(L_1 + L_2 + L_3) + \right. \\ &\quad \left. s(L_1 L_2 + L_1 L_3 + L_2 L_3) + L_1 L_2 L_3 \right) X(s) \end{aligned}$$

Here, $A_0 = L_1 L_2 L_3$, $A_1 = L_1 L_2 + L_1 L_3 + L_2 L_3$, and $A_2 = L_1 L_2 L_3$. The proposition asserts that if L_1 , L_2 , and L_3 share a sparsity pattern and have bounded gains, then the resulting controller gains A_0 , A_1 , and A_2 will be sparse and have bounded gains.

F. Scalable stability

Coordinating a multi-agent system is inherently a decentralized problem where the goal for each agent is to coordinate with its nearest neighbors. However when the controllers only depend on local measurements there is a possibility that controllers that manage to coordinate N agents stop stabilizing as the number of agents increases. In [9] it was shown that for the 3rd and higher order consensus problem with controller $A_k = a_k L_N$ in (4), the closed loop system will become unstable if the algebraic connectivity $\lambda_2(L_N) \rightarrow 0$ as $N \rightarrow \infty$. This motivates the notion of *Scalable stability*

Definition 4 (Scalable stability [9, Def. 2.1]): A consensus control design is scalably stable if the resulting closed-loop system achieves consensus over any graph in the family $\{\mathcal{G}_N\}$.

III. MAIN RESULTS

Our main contribution is two-fold. First we show the serial consensus achieves scalable stability and then we show that the implementation is robust to two classes of perturbations

A. Scalable stability

Theorem 4: Consider the n^{th} order serial consensus system as defined in *Definition 3* under Assumption A1 and with $U_{\text{ref}} \in \mathcal{RH}_\infty$. Then the closed loop dynamics have the following properties:

- (i) The poles of (6) are given by the union of the eigenvalues of $-L_k$.
- (ii) The solution achieves n^{th} order consensus.

Proof: (i) Any square matrix can be unitarily transformed to upper triangular form by the Schur triangularization theorem. Let $U_k L_k U_k^H = T_k$ be upper triangular. Then the block diagonal matrix $U = \text{diag}(U_1, U_2, \dots, U_n)$ is a unitary matrix that upper triangularizes A in (7). For any triangular matrix the eigenvalues lie on the diagonal and this will be the eigenvalues of each $-L_k$.

(ii) First, consider the closed loop dynamics of (6) which will be

$$X(s) = \left(\prod_{k=n}^1 (sI + L_k)^{-1} \right) U_{\text{ref}}(s).$$

Since, U_{ref} is stable, we know that the limit $\lim_{s \rightarrow 0} U_{\text{ref}}(s) = U_{\text{ref}}(0)$ exists. To prove that the system achieves n^{th} order consensus we want to show that

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} C(s)X(s) = 0$$

for some transfer matrix $C(s)$, which encodes the consensus states. But since the reference dependence is only related to $U_{\text{ref}}(0)$, we can simplify the problem to only consider impulse responses. But the impulse response has the same transfer function as the initial value response where $\xi_n(0) = U_{\text{ref}}(0)$. Therefore, WLOG, assume that $U_{\text{ref}}(s) = 0$ and an arbitrary initial condition

$$\xi(0) = [\xi_1(0)^T, \xi_2^T, \dots, \xi_n(0)^T]^T.$$

The solution of (7) is given by $\exp(At)\xi(0) = S \exp(J(A)t)S^{-1}\xi(0)$ where $J(A)$ is the Jordan normal form of A and S is an invertible matrix. From (i) and the diagonal dominance of the graph Laplacians we know that all eigenvalues of A lie in the left half plane. By Assumption A1 it follows that the zero eigenvalue for each L_k is simple. Now we prove that these n zero eigenvalues form a Jordan block of size n . Let e_k denote the k^{th} 1-block vector, e.g. $e_1 = [\mathbf{1}_N^T \ 0_N \ \dots \ 0_N]^T$ and $e_2 = [0_N \ \mathbf{1}_N^T \ 0_N \ \dots \ 0_N]^T$. Then e_1 is an eigenvector since $Ae_1 = 0$. For $k \in \{2, 3, \dots, n\}$ we have $Ae_k = e_{k-1}$ which implies that $A^k e_k = 0$. This shows that there is a Jordan block of size n with an invariant subspace spanned by the vectors e_k . Since all other eigenvectors make up an asymptotically stable invariant subspace, it follows that $\xi(t)$ will converge towards a solution in $\text{span}(e_1, e_2, \dots, e_n)$ and thus $\lim_{t \rightarrow \infty} \xi_k(t) = \alpha_k(t)\mathbf{1}_N$. Now, since $x(t) = \xi_1(t)$, it follows that $\lim_{t \rightarrow \infty} x(t) = \alpha_1(t)\mathbf{1}_N$, and furthermore, since

$$\dot{\xi}_k = -L_k \xi_k + \xi_{k+1} \rightarrow \xi_{k+1} \text{ as } t \rightarrow \infty$$

for $k \in \{1, \dots, n-1\}$, it follows that $\lim_{t \rightarrow \infty} x^{(k)}(t) = \alpha_{k+1}(t)\mathbf{1}_N$ which shows that the system achieves n^{th} order consensus. ■

This proposition shows that the stability of the consensus for the n^{th} order serial consensus can be reduced to verifying that the n first-order consensus systems $\dot{x} = -L_k x$ achieve consensus. This is equivalent to determining whether the graphs underlying each L_k contains a connected spanning tree. This result together with *Proposition 1* shows that n^{th} order consensus can be achieved with a local relative feedback controller with finite gain and thus achieve scalable stability. This result can be compared with [9] where it is shown that the conventional consensus is not scalably stable for any order larger than $n = 3$ if a graph Laplacian with vanishing algebraic connectivity is used. We can summarize this fact in the following corollary:

Corollary 5: For any n , the controller (5) is scalably stable over any graph family $\{\mathcal{G}_N\}$ that underlies L_k , provided each \mathcal{G}_N satisfies Assumption A1.

Remark 1: Note that, by Theorem 4, scalable stability is achieved also with different graph families underlying each L_k , and $\|L_k\|_{\infty}$ are allowed to be arbitrarily small.

B. Robustness of serial consensus

The proposed controller in (5) is a relative state-feedback controller which is designed to ensure that the closed loop system achieves n^{th} order consensus as guaranteed through *Theorem 4*. However, the n^{th} order integrator system may be an idealization of the system and the relative state feedback may need observers to be fully realized, and there could be unmodeled dynamics. These potential sources of errors call for a robust controller. We will now present two theorems, which prove that the serial consensus is robust towards two different types of uncertainties.

1) Additive perturbation: The following theorem asserts that the n^{th} order serial consensus controller can handle additive uncertainties.

Theorem 6: Consider the n^{th} order serial consensus system as defined in *Definition 3*, under Assumption A1, with $L_k = L$ for all k , and $L = L^T$. Then the perturbed system

$$(sI + L)^n X = U_{\text{ref}} + \left(\sum_{k=0}^n \Delta_k s^k L^{n-k} \right) X,$$

where $U_{\text{ref}}, \Delta_k \in \mathcal{RH}_{\infty}$, achieves n^{th} order consensus if

$$\|\Delta_0\|_{\mathcal{H}_{\infty}} + \|\Delta_n\|_{\mathcal{H}_{\infty}} + \sum_{k=1}^{n-1} \|\Delta_k\|_{\mathcal{H}_{\infty}} \sqrt{\frac{k^k}{n^n} (n-k)^{n-k}} < 1$$

Proof: First, note that the closed-loop system can be represented by the block diagram in Fig. 1, which in turn can be simplified to Fig. 2. Since U_{ref} is stable we can apply the small-gain theorem which asserts that $U(s)$ (as defined in the figures) will be stable if

$$\left\| \sum_{k=0}^n \Delta_k s^k L^{n-k} (sI + L)^{-n} \right\|_{\mathcal{H}_{\infty}} < 1.$$

Applying the triangle inequality and submultiplicativity on the left-hand side (LH) yields

$$LH \leq \sum_{k=0}^n \|\Delta_k\|_{\mathcal{H}_{\infty}} \|s^k L^{n-k} (sI + L)^{-n}\|_{\mathcal{H}_{\infty}} \quad (8)$$

Since L is symmetric, it is possible to unitarily diagonalize it. Let $U = U^H$ denote one such unitary matrix. Then $L = U\Lambda U^H$ where Λ is a non-negative real diagonal matrix.

$$\|s^k L^{n-k} (sI + L)^{-n}\|_{\mathcal{H}_{\infty}} = \|s^k \Lambda^{n-k} (sI + \Lambda)^{-n}\|_{\mathcal{H}_{\infty}}.$$

For a diagonal matrix the singular values are given by the absolute value of the diagonal. Let, $\lambda > 0$ be an arbitrary positive constant. The maximum gain for each diagonal can then be calculated through

$$\max_{\omega} \left| \frac{\omega^k \lambda^{n-k}}{(j\omega + \lambda)^n} \right| = \sqrt{\max_{\omega} \frac{\omega^{2k} \lambda^{2n-2k}}{(\omega^2 + \lambda^2)^n}}.$$

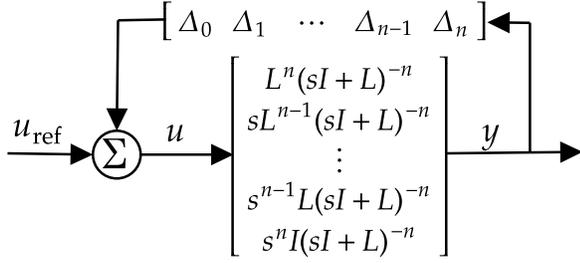


Fig. 1: Block diagram illustrating the perturbation model in proof of Theorem 6.

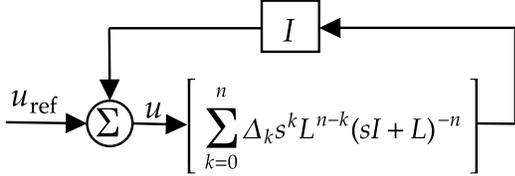


Fig. 2: Block diagram illustrating the perturbation model in proof of Theorem 6.

The latter optimization problem is given by a continuous function and thus the derivative must be 0 at the maximum. Simple calculus shows that the optimum is found at $\omega^2 = \lambda^2 k / (n - k)$ for $k = 0, 1, \dots, n - 1$ and at $\omega = \infty$ for $k = n$. Inserting yields

$$\max_{\omega} \left| \frac{\omega^k \lambda^{n-k}}{(j\omega + \lambda)^n} \right| = \begin{cases} \sqrt{\frac{k}{n^n} (n-k)^{n-k}} & \text{if } 0 < k < n \\ 1 & \text{else} \end{cases}$$

Now for the case where $\lambda = 0$. Then we have for $k = 0, \dots, n - 1$

$$\max_{\omega} \left| \frac{\omega^k 0^{n-k}}{(j\omega + 0)^n} \right| = 0$$

and for $k = n$

$$\max_{\omega} \left| \frac{\omega^n}{(j\omega + 0)^n} \right| = 1.$$

This is less restrictive than for $\lambda > 0$ and thus we can use the result for $\lambda > 0$. Plugging this into the upper bound of the LH (8) results in the sought inequality.

Finally, we must ensure that stability of the closed loop in Fig. 2 implies n^{th} order consensus. Since the transfer matrix from u to y in Fig. 1 is stable it follows that $Y(s)$ will be stable. This means that we have shown the following $\lim_{t \rightarrow \infty} L^{n-k} x^{(k)}(t) = 0$. By Assumption A1 the 0 eigenvalue of L is unique and therefore 0 is a unique eigenvalue of L^{n-k} too. Subsequently, $\lim_{t \rightarrow \infty} x^{(k)}(t) \in \ker(L^{n-k})$. Since $L^{n-k} \mathbf{1}_N = 0$ it follows that $\lim_{t \rightarrow \infty} x^{(k)}(t) \in \text{span}(\mathbf{1}_N)$ and that the agents will reach consensus in all the $n - 1$ first time derivatives and thus achieve n^{th} order consensus. ■

It is worth noting that the norm bound on the uncertainty blocks Δ is independent of the number of agents in the system. Therefore, the serial consensus implementation is *scalably robust* in the sense that it allows equally sized perturbations regardless of network size. This is not the case

for localized conventional consensus, following the results in [9].

2) *Multiplicative perturbation*: It is also possible to see the closed-loop serial consensus system as a series of interconnected first-order systems. Therefore it is also interesting to consider the robustness with respect to the individual factors. The following theorem gives a sufficient condition for the unforced closed loop system to achieve n^{th} order consensus.

Theorem 7: The following perturbed n^{th} order serial consensus system

$$(sI + s\Delta_0 + (I + \Delta_n)L_n) \prod_{k=1}^{n-1} (sI + (I + \Delta_k)L_k) X = U_{\text{ref}}$$

where $U_{\text{ref}}, \Delta_k \in \mathcal{RH}_{\infty}$ and $L_k = L_k^T$ for $k = 1, \dots, n$, achieves n^{th} order consensus if

$$\|\Delta_k\|_{\mathcal{H}_{\infty}} < 1, \text{ for all } k$$

and

$$\|\Delta_0\|_{\mathcal{H}_{\infty}} + \|\Delta_n\|_{\mathcal{H}_{\infty}} < 1.$$

Proof: First, note that we can construct $X(s) = \Xi_1(s)$ and $s\Xi_k = -(I + \Delta_k)L_k\Xi_k + \Xi_{k+1}$ for $k = 1, \dots, n - 1$ and $s(I + \Delta_0)\Xi_n = -(I + \Delta_n)L_n\Xi_n + U_{\text{ref}}$. For Ξ_n we have exactly the 1st order case of Theorem 6 and thus $\lim_{t \rightarrow \infty} \xi_n(t) = \alpha_n(t)\mathbf{1}_N$ if $\|\Delta_0\|_{\mathcal{H}_{\infty}} + \|\Delta_n\|_{\mathcal{H}_{\infty}} < 1$. Consider the following induction hypothesis: if $\Xi_{k+1}(s) = \mathbf{1}_N G_{k+1}(s) + H_{k+1}(s)$ where $H_{k+1}(s) \in \mathcal{RH}_{\infty}$ then $\Xi_k = \mathbf{1}_N G_k(s) + H_k(s)$ for some $H_k(s) \in \mathcal{RH}_{\infty}$. We have

$$s\Xi_k = -(I + \Delta_k)L_k\Xi_k + \Xi_{k+1}$$

which can be represented by the block diagram Fig. 3. Here, note that

$$L_k(sI + L_k)^{-1}\Xi_{k+1} = (sI + L_k)^{-1}L_k(H_{k+1}(s))$$

and the potentially unstable term of Ξ_{k+1} can be ignored. Reusing a result from the previous proof we have $\|L_k(sI + L_k)^{-1}\|_{\mathcal{H}_{\infty}} = 1$ and therefore $L_k\Xi_k \in \mathcal{RH}_{\infty}$ if $\|\Delta_k\|_{\mathcal{H}_{\infty}} < 1$. Since the 0 eigenvalue of L_k is unique, it follows that $\Xi_k(s) = \mathbf{1}_N G_k(s) + H_k(s)$ with $H_k \in \mathcal{RH}_{\infty}$ which proves the induction hypothesis since we have already shown the base case $\Xi_n(s) = \mathbf{1}_N G_n(s) + H_n(s)$. Left is to prove that the system will reach n^{th} order consensus. Note that $L_1 X(s) = L_1 \Xi_1(s)$ is stable and therefore we get through the final value theorem

$$\lim_{t \rightarrow \infty} L_1 x(t) = \lim_{s \rightarrow 0} s L_1 \Xi_1(s) = 0.$$

Furthermore, we have for all k : $\lim_{s \rightarrow 0} s L_k \Xi_k(s) = 0$. This, combined with $s^2 \Xi_k(s) = -(I + \Delta_k)s L_k \Xi_k(s) + s \Xi_{k+1}(s)$ shows that

$$\begin{aligned} \lim_{t \rightarrow \infty} L_{k+1} x^{(k)}(t) &= \lim_{s \rightarrow 0} s (s^k L_{k+1} X(s)) \\ &= \lim_{s \rightarrow 0} s L_{k+1} \Xi_{k+1}(s) = 0 \end{aligned}$$

Finally, since the 0 eigenvalues for each L_k are unique with corresponding eigenvector $\mathbf{1}_N$ we see that n^{th} order consensus will be achieved. ■

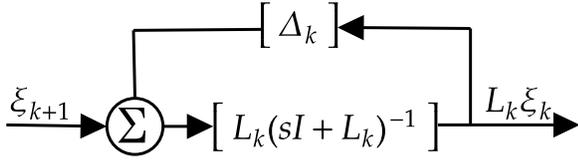


Fig. 3: Block diagram illustrating the perturbation model of a general first-order consensus block which is used in the proof of *Theorem 7*.

This theorem shows that the n^{th} order serial consensus is robust in its construction.

IV. EXAMPLES

A. 2nd order consensus on circular graph

Consider the directed cycle graph, which is represented by the adjacency matrix

$$[W]_{i,j} = 1 \text{ iff } i - j = 1 \pmod{N}.$$

The corresponding graph Laplacian L_c is a circulant matrix and therefore, the eigenvalues are known analytically. In particular, the eigenvalue with the second smallest real part is $\lambda_2(L_c) = 1 - \exp(2\pi/N) = 1 - \cos(2\pi/N) - \mathbf{i} \sin(2\pi/N)$. For large N , this eigenvalue can be approximated with a 1st order Taylor approximation, which yields $\lambda_2(L_c) \approx -\mathbf{i}2\pi/N$. This eigenvalue will cause problems when designing a controller using the conventional consensus. To see this, consider the closed loop dynamics

$$s^2 I + 2p_1 s L_c + p_0 L_c = U_{\text{ref}}.$$

The system can be diagonalized and, in particular, two of the poles are given by the equation

$$s^2 + 2p_1 \lambda_2(L_c) + p_0 \lambda(L_c) = 0.$$

In the case that p_0 and p_1 are designed independently of the network size N , then for sufficiently large N the roots can be approximated as

$$s_p = -p_1 \lambda_2 \pm \sqrt{p_1^2 \lambda_2^2 - p_0 \lambda} \approx \pm(1 + \mathbf{i}) \sqrt{\frac{\pi p_0}{N}}$$

Since one of these poles will eventually lie in the RHP, it follows that the closed loop system will become unstable when N is sufficiently large, regardless of the choice of p_0 and p_1 .

For the serial consensus it suffices to check that all eigenvalues but the unique 0 eigenvalue of L_c lie in the RHP or equivalently if $0 < \text{Re}(\lambda_2(L_c)) = 1 - \cos(2\pi/N)$ which is clearly true for any finite N . Alternatively, it is also sufficient that the underlying graph contains a connected spanning tree.

B. 3rd order consensus

It has been shown that for $n \geq 3$ it is not possible to achieve scalable stability for any graph family $\{G_N\}$ where the corresponding graph Laplacian L_N has an eigenvalue with vanishing real part as the graph is growing, i.e. if $\lim_{N \rightarrow \infty} \text{Re}(\lambda_2(L_N)) = 0$. At least, this is not possible with

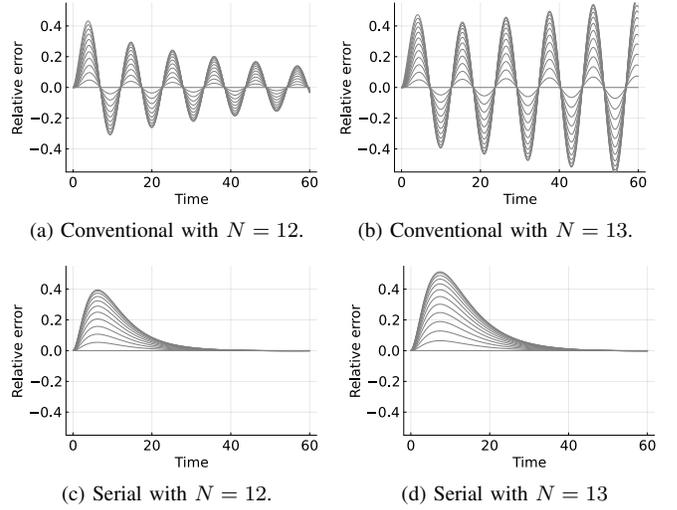


Fig. 4: 3rd order consensus in a chain of vehicles is considered. The plots show the intervehicle relative errors over time when the lead vehicle moves at constant acceleration. Panels (a) and (b) show that the addition of one agent destabilizes the closed loop for the conventional consensus. Panels (c) and (d) illustrate the fact that the serial consensus will remain stable under such agent additions.

the conventional consensus control. On the serial consensus form this is no longer a problem. The controller

$$U(s) = U_{\text{ref}} + \left(s^3 I - \prod_{k=1}^3 (sI + L_N) \right) X(s)$$

will achieve consensus as long as the underlying graphs $\{G_N\}$ all contains a connected spanning tree. To illustrate this, consider the graph defined by $W \in \mathbb{R}^{N \times N}$, the adjacency matrix

$$W_{i,j} = \begin{cases} 1 & \text{if } |i - j| = 1 \text{ and } i \neq 1 \\ 0 & \text{else} \end{cases}.$$

This corresponds to a bidirectional string with a leader (Agent 1). Let L be the associated graph Laplacian. It is true that $\lim_{N \rightarrow \infty} \lambda_2(L) = 0$ and thus any conventional control design with L will eventually lead to an unstable closed loop. For this example, let the conventional control law be $u(t) = u_{\text{ref}}(t) - 6L\ddot{x} - 4L\dot{x} - 2Lx$ and the serial consensus controller (5) be defined with the same graph Laplacians $L_k = 2kL$. The response to a constant acceleration of the leader is shown in Fig. 4. Here we see that the addition of a 13th agent to the system destabilizes the closed loop for the conventional consensus while the serial consensus only loses some performance.

C. Robustness of the 2nd order serial Consensus.

Theorems 6 and 7 show that the serial consensus can be perturbed and still achieve n^{th} order consensus. Now we want to illustrate what the block Δ_k can be. Consider the perturbed 2nd order consensus system in Theorem 6. Writing out all terms we get

$$s^2(I + \Delta_2)X = U_{\text{ref}} - (s(2I + \Delta_1)LX + (I + \Delta_0)L^2X).$$

In this form, the Δ_2 block can be thought of as representing model errors; we may control a system which we model as being N identical double integrator systems but in reality they may differ. This is obviously the case for vehicle platoons, which are often modeled as chains of identical double integrators. Through our theorem we can for instance allow Δ_2 to be a diagonal transfer matrix with elements $[\Delta_2]_{i,i} = \frac{k_i}{T_i s + 1}$ where $|k_i| < 1$ and $T_i > 0$ for all i . Then, the closed loop system would remain stable despite the heterogeneous agents. The blocks Δ_1 and Δ_0 are also important. For instance the signals $L^2 x(t)$ and $L\dot{x}(t)$ may not be directly measured but estimated through linear filters. This could be thought of as unmodeled dynamics which these blocks can capture.

If we focus on Theorem 7, then the perturbed model is

$$(s(\Delta_0 + I) + (\Delta_1 + I)L_1)(sI + (\Delta_2 + I)L_2)X = U_{\text{ref}}$$

The theorem only asserts robustness for symmetrical graph Laplacians L_k . However, since Δ_k can also be constant matrices, it is also possible to construct new (asymmetric) graph Laplacians $L'_k = (I + \Delta_k)L_k$ by designing the Δ_k blocks.

V. CONCLUSION

This work has introduced the n^{th} order serial consensus system which can be seen as a natural generalization of the well-known consensus protocols. The stability of the introduced system can be analysed by considering n regular first order consensus protocol. The proposed controller to achieve n^{th} order serial consensus has been shown to be implementable using relative measurements confined to a local neighborhood of each agent and can therefore be considered a decentralized control scheme. Robustness of the proposed system has also been analyzed. This has been addressed in terms of additive and model perturbations. The analysis showed that the size, measured in the \mathcal{H}_∞ norm, of the allowable uncertainties were independent of the number of agents.

Future and ongoing work includes looking into the performance of the serial consensus and how this relates to string stability. It would also be interesting to look into an implementation where each agent implements an observer to compute their control action.

APPENDIX

Here we prove Lemmas 2, 3 which describe how the sparsity pattern of two matrices changes through addition and multiplication. The Lemmas are restated for convenience.

Lemma 2: If $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}(W, c_2)$ then the sum $(A_1 + A_2) \in \mathcal{A}^{\max(q_1, q_2)}(W, c_1 + c_2)$

Proof: First, we have $\|A_1 + A_2\|_\infty \leq \|A_1\|_\infty + \|A_2\|_\infty \leq c_1 + c_2$ which follows from the triangle inequality.

For the second part we have $(A_1 + A_2)\mathbf{1}_N = 0 + 0 = 0$.

For the last part, WLOG, suppose that $q_1 \leq q_2 = \max(q_1, q_2)$. Since W is a positive matrix, we get

$$0 \leq \left(\sum_{k=0}^{q_1} W^k \right)_{i,j} \leq \left(\sum_{k=0}^{q_2} W^k \right)_{i,j}.$$

In particular, the following implication follows

$$\left(\sum_{k=0}^{q_2} W^k \right)_{i,j} = 0 \implies \left(\sum_{k=0}^{q_1} W^k \right)_{i,j} \implies [A_1 + A_2]_{i,j} = 0$$

■

To prove the result on the product of two matrices, Lemma 3, we need the following three lemmas:

Lemma 8: Let $A, B \in \mathbb{C}^{N \times N}$ and define $\hat{A}_{i,j} = |A|_{i,j}$ and $\hat{B}_{i,j} = |B|_{i,j}$. If $(AB)_{i,j} \neq 0$ then $(\hat{A}\hat{B})_{i,j} \neq 0$.

Proof: Suppose the statement is false, i.e. $(\hat{A}\hat{B})_{i,j} = 0$ but $(AB)_{i,j} \neq 0$. Then we know that

$$(\hat{A}\hat{B})_{i,j} = \sum_{k=1}^N |A_{i,k}| |B_{k,j}| = 0,$$

but this implies that at least one of $A_{i,k}$ and $B_{k,j}$ is equal to 0 for all k . But from this it follows that

$$(AB)_{i,j} = \sum_{k=1}^N A_{i,k} B_{k,j} = \sum_{k=1}^N 0 = 0.$$

This is a contradiction and concludes the proof. ■

Lemma 9: Let $A, A_1, B, B_1 \in \mathbb{R}_+^{N \times N}$. If $(AB)_{i,j} \neq 0$ then $((A + A_1)(B + B_1))_{i,j} \neq 0$.

Proof: Expand the product to get

$$\begin{aligned} ((A + A_1)(B + B_1))_{i,j} &= (AB)_{i,j} + (AB_1)_{i,j} \\ &\quad + (A_1B)_{i,j} + (A_1B_1)_{i,j} \geq (AB)_{i,j} \end{aligned}$$

which followed from the fact that the product of 2 nonnegative matrices is also nonnegative. ■

Lemma 10: Let $A, A_1, B, B_1 \in \mathbb{R}_+^{N \times N}$ be such that $A_{i,j} = 0$ if and only if $A_{1i,j} = 0$, and $B_{i,j} = 0$ if and only if $B_{1i,j} = 0$. Then, $(AB)_{i,j} = 0$ if and only if $(A_1B_1)_{i,j} = 0$.

Proof: The statement is clearly symmetrical and it is enough to prove sufficiency. Now, if $(AB)_{i,j} = 0$ then we know that

$$\sum_k A_{i,k} B_{k,j} = 0, \implies A_{i,k} B_{k,j} = 0, \forall k$$

But this implies that either $A_{i,k} = 0$ or $B_{k,j} = 0$. In turn, this implies that either $(A_1)_{i,k} = 0$ or $(B_1)_{k,j} = 0$. And this leads to

$$(A_1B_1)_{i,j} = \sum_k (A_1)_{i,k} (B_1)_{k,j} = 0$$

■

Now we can prove Lemma 3:

Lemma 3: Let $A_1 \in \mathcal{A}^{q_1}(W, c_1)$ and $A_2 \in \mathcal{A}^{q_2}(W, c_2)$ then the product $(A_1A_2) \in \mathcal{A}^{q_1+q_2}(W, c_1c_2)$

Proof: First, the gain can be bounded as $\|A_1A_2\|_\infty \leq \|A_1\|_\infty \|A_2\|_\infty \leq c_1c_2$ which followed from submultiplicity of the induced norm and from the definition of the sets.

For the second part we have $A_1A_2\mathbf{1}_N = A_10 = 0$.

For the last part we have to do slightly more. First replace each element in A_1 and A_2 with its absolute value and denote these B_1 and B_2 . Now introduce two non-negative matrices

C_1 and C_2 such that $B_1 + C_1 = 0 \iff \sum_{k=0}^{q_1} W^k$ and $B_2 + C_2 = 0 \iff \sum_{k=0}^{q_2} W^k$. Finally note that

$$\left(\sum_k^{q_1} W^k\right)\left(\sum_j^{q_2} W^j\right) = \sum_k^{q_1+q_2} w_k W^k$$

for some $w_k > 0$. By applying Lemma 10 two times we get that

$$\left[\sum_k^{q_1+q_2} W^k\right]_{i,j} = 0 \implies [(B_1 + C_1)(B_2 + C_2)]_{i,j} = 0$$

Through Lemma 9 we get

$$[(B_1 + C_1)(B_2 + C_2)]_{i,j} = 0 \implies [B_1 B_2]_{i,j} = 0$$

And finally applying Lemma 8 results in

$$[B_1 B_2]_{i,j} = 0 \implies [A_1 A_2]_{i,j} = 0$$

■

ACKNOWLEDGEMENT

We want to thank Richard Pates for useful discussions regarding the robustness results.

REFERENCES

- [1] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep 2004.
- [2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sept 2004.
- [3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, June 2003.
- [4] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics, limitations and algorithms for complex networks," in *2014 American Control Conf.*, 2014, pp. 3287–3292.
- [5] B. Bamieh, M. R. Jovanović, P. Mitra, and S. Patterson, "Coherence in large-scale networks: Dimension-dependent limitations of local feedback," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2235–2249, Sep. 2012.
- [6] M. Siami and N. Motee, "Fundamental limits and tradeoffs on disturbance propagation in large-scale dynamical networks," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4055–4062, 2016.
- [7] D. Swaroop and J. Hedrick, "String stability of interconnected systems," *IEEE Trans. Autom. Control*, vol. 41, no. 3, pp. 349–357, March 1996.
- [8] P. Seiler, A. Pant, and K. Hedrick, "Disturbance propagation in vehicle strings," *IEEE Trans. Autom. Control*, vol. 49, no. 10, pp. 1835–1842, Oct 2004.
- [9] E. Tegling, B. Bamieh, and H. Sandberg, "Scale fragilities in localized consensus dynamics," *Automatica*, 2023, to appear. Available: <https://arxiv.org/abs/2203.11708>.
- [10] F. Jiang, L. Wang, and Y. Jia, "Consensus in leaderless networks of high-order-integrator agents," *American Control Conf.*, pp. 4458–4463, June 2009.
- [11] W. Ren, K. Moore, and Y. Chen, "High-order consensus algorithms in cooperative vehicle systems," *IEEE International Conf. on Networking, Sensing and Control*, pp. 457–462, 2006.
- [12] S. Stüdli, M. M. Seron, and R. H. Middleton, "Vehicular platoons in cyclic interconnections with constant inter-vehicle spacing," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 2511–2516, 2017, 20th IFAC World Congress.
- [13] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice hall Upper Saddle River, NJ, 1998, vol. 104.