# Multiconsensus control of homogeneous LTI hybrid systems under time-driven jumps

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*Abstract*— **In this paper, we consider a network of homogeneous LTI hybrid dynamics under time-driven aperiodic jumps and exchanging information over a fixed communication graph. Based on the notion of almost equitable partitions, we explicitly characterize the clusters induced by the network over the nodes and, consequently, the corresponding multi-consensus trajectories. Then, we design a decentralized control ensuring convergence of all agents to the corresponding multi-consensus trajectory. Simulations over an academic example illustrate the results.**

*Index Terms*— **Linear systems; Agents-based systems; Decentralized control**

## I. INTRODUCTION

Networked systems are nowadays well-considered a bridging paradigm among several disciplines spanning, among many others, from physics to engineering, psychology to medicine, biology to computer science. As typical in control theory [1], we refer to a network (or multi-agent) system as composed of several dynamical units (agents) interconnected through a *communication graph*. In this scenario, most control problems are related to driving all systems composing the network toward a *consensus* behavior that might be common to the all agents or only to subgroups (e.g., [2]–[6]). Despite numerous results for either continuous or discrete-time networks, very few are available when the dynamics are hybrid (i.e., when agents are characterized by both continuous and discrete components) with most of them involving single-consensus only [7]–[9] and the case of scalar units. In the latter case, a first characterization of multi-consensus in hybrid networks has been proposed in [10] when considering distinct topology for the jump and flow behaviors.

The scope of this paper is hence to make a step farther in this direction by considering a network of homogeneous linear impulsive agents under a fixed communication graph and aperiodic jump instants. The interest for this class of dynamics is motivated by their involvement in several practical scenarios as, for instance, cybersecurity and sampled-data networks (e.g., [11]–[13]). In doing so, we adopt the general modeling framework for hybrid control systems proposed in [14] and investigate the effects of the hybrid coupling functions and the network topology on the agents' trajectories. In particular, we are interested in the multi-consensus problem: the definition of a decentralized control (i.e., local to each agent and exploiting only information available from the neighbors) making all agents cluster into several subgroups induced by the hybrid network with the property that nodes in the same subgroup exhibit the same steady-state. The contribution of this paper is hence twofold: first, we characterize the consensuses dynamics over the hybrid multi-agent systems depending on the structure of the communication digraphs with no assumption on the corresponding connectivity properties; then, the coupling control laws are designed, locally to all agents, to guarantee convergence to the multi-consensus. We show that agents cluster into a precise number of groups. Nodes in the same cluster are governed by the so-called mean-field dynamics being a convex combination of all agents trajectories and defining, under suitable conditions, the consensus dynamics. Such clusters are uniquely dictated by the so-called coarsest Almost Equitable Partition (AEPs, [15]) of the graph.

The remainder of the paper is organized as follows. In Section II the problem is stated and contextualized with a few recalls on graph theory. The main results are in Section III with an illustrative example in Section IV. Section V concludes the paper.

*Notations:* $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the right and left hand side of the complex plane. $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Given a set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. We denote by $0$ either the zero scalar or the zero matrix of suitable dimensions. $\mathbb{1}_c$ denotes the $c$-dimensional column vector whose elements are all ones while $I_n$ is the identity matrix of dimension $n \geq 1$. Given a matrix $A \in \mathbb{R}^{n\times n}$ $\sigma\{A\} \subset \mathbb{C}$ denotes its spectrum. A positive definite (semi-definite) matrix $A = A^\top$ is denoted by $A \succ 0$ ($A \succeq 0$); a negative definite (semi-definite) matrix $A = A^\top$ is denoted by $A \prec 0$ ($A \preceq 0$).

## II. RECALLS AND PROBLEM STATEMENT

### A. Directed graphs and algebraic properties

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph (or digraph for short) with $|\mathcal{V}| = N$, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$. The set of neighbors to a node $i \in \mathcal{V}$ is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \text{ s.t. } (j,i) \in \mathcal{E}\}$. Denoting by $\mathcal{R}_i$ for $i = 1,\dots,j$, the reaches of $\mathcal{G}$, the exclusive part of $\mathcal{R}_i$ is defined as $\mathcal{H}_i = \mathcal{R}_i / \cup_{\ell=1,\ell\neq i}^{j} \mathcal{R}_\ell$ with cardinality $h_i = |\mathcal{H}_i|$. Finally, the common part of $\mathcal{G}$ is given by $\mathcal{C} = \mathcal{V}/\cup_{i=1}^{j} \mathcal{H}_i$ with cardinality $c = |\mathcal{C}|$.

The Laplacian matrix of $\mathcal{G}$ is given by $L = D - A$ with $D \in \mathbb{R}^{N\times N}$ and $A \in \mathbb{R}^{N\times N}$ being respectively the in-degree

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and the adjacency matrices. $L$ possesses one eigenvalue $\lambda = 0$ with multiplicity coinciding with $j$, the number of reaches of $\mathcal{G}$, and the remaining $N - j$ with positive real part [16]. Hence, after a suitable re-labeling of nodes, the Laplacian always admits the lower triangular form

$$L = \begin{pmatrix} L_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & L_\mu & 0 \\ M_1 & \dots & M_\mu & M \end{pmatrix} \tag{1}$$

where: $L_i \in \mathbb{R}^{h_i \times h_i}$ $(i = 1, \dots, j)$ is the Laplacian associated with the subgraph $\mathcal{H}_i$ and possessing one eigenvalue in zero with single multiplicity; $M \in \mathbb{R}^{c \times c}$ verifying $\sigma(M) \subset \mathbb{C}^+$ corresponds to the common component $\mathcal{C}$. Thus, the eigenspace associated with $\lambda = 0$ for $L$ is spanned by the right eigenvectors

$$z_1 = \begin{pmatrix} \mathbb{1}_{h_1} \\ \vdots \\ 0 \\ \gamma^1 \end{pmatrix} \quad \dots \quad z_\mu = \begin{pmatrix} 0 \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} \tag{2}$$

with $\sum_{i=1}^{j} \gamma^i = \mathbb{1}_c$ and $M_i \mathbb{1}_{h_i} + M\gamma^i = 0$ for all $i = 1, \dots, j$. In addition, the left eigenvectors associated with the zero eigenvalues are given by

$$\tilde{w}_1^\top = \begin{pmatrix} w_1^\top & \dots & 0 & 0 \end{pmatrix} \dots \tilde{w}_\mu^\top = \begin{pmatrix} 0 & \dots & w_\mu^\top & 0 \end{pmatrix} \tag{3}$$

with $w_i^\top = \begin{pmatrix} w_i^1 & \dots & w_i^{h_i} \end{pmatrix} \in \mathbb{R}^{1 \times h_i}$, $w_i^s > 0$ if the corresponding node is root or zero otherwise. A partition $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$ is a collection of cells $\rho_i \subseteq \mathcal{V}$ verifying $\rho_i \cap \rho_\mu = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{r} \rho_i = \mathcal{V}$. The characteristic vector of $\rho \subseteq \mathcal{V}$ is given by $p(\rho) = (p_1(\rho) \ \dots p_N(\rho))^\top \in \mathbb{R}^N$ with for $i = 1, \dots, N$

$$p_i(\rho) = \begin{cases} 1 & \text{if } i \in \rho \\ 0 & \text{otherwise.} \end{cases}$$

For a partition $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$, the characteristic matrix of $\pi$ is $P(\pi) = \begin{pmatrix} p(\rho_1) & \dots & p(\rho_r) \end{pmatrix}$ with $\mathcal{P} = \text{Im}P(\pi)$ with, by definition of partition, each row of $P(\pi)$ possessing only one element equal to one and all other being zero. Given two partitions $\pi_1$ and $\pi_2$, $\pi_1$ is said to be finer than $\pi_2$ ($\pi_1 \preceq \pi_2$) if all cells of $\pi_1$ are a subset of some cell of $\pi_2$ so implying $\text{Im}P(\pi_2) \subseteq \text{Im}P(\pi_1)$; equivalently, we say that $\pi_2$ is coarser than $\pi_1$ ($\pi_2 \succeq \pi_1$), with $\text{Im}P(\pi_1) \subseteq \text{Im}P(\pi_2)$. We name $\pi = \mathcal{V}$ the trivial partition as composed of a unique cell with all nodes. Given a cell $\rho \in \mathcal{V}$ and a node $i \notin \rho$, we denote by $\mathcal{N}(i, \rho) = \{i \in \rho \text{ s.t } (i, i) \in \mathcal{E}\}$ the set of neighbors of $i$ in the cell $\rho$.

*Definition 2.1:* A partition $\pi^\star = \{\rho_1, \rho_2, \dots, \rho_k\}$ is said to be an *almost equitable partition (AEP)* of $\mathcal{G}$ if, for each $i, j \in \{1, 2, \dots, k\}$, with $i \neq j$, there exists an integer $d_{ij}$ such that $|\mathcal{N}(i, \rho_j)| = d_{ij}$ for all $i \in \rho_i$.

In other words, an AEP verifies that each node in $\rho_i$ has the same number of neighbors in $\rho_j$, for all $i, j$ with $i \neq j$. The property of almost equitability is equivalent to the invariance of the subspaces generated by the characteristic vectors of its cells. In particular, we can give the following equivalent characterization of an AEP $\pi^\star$ [17], [18].

*Proposition 2.1:* Consider a graph $\mathcal{G}$ and a partition $\pi^\star = \{\rho_1, \rho_2, \dots, \rho_k\}$ with $\mathcal{P}^\star = \text{Im}P(\pi^\star)$. $\pi^\star$ is an AEP if and only if $\mathcal{P}^\star$ is $L$-invariant, that is $L\mathcal{P}^\star \subseteq \mathcal{P}^\star$.

We say that a non trivial partition $\pi^\star$ is the coarsest AEP of $\mathcal{G}$ if $\pi^\star \succeq \pi$ for all non trivial $\pi$ AEP of $\mathcal{G}$ and, equivalently, $\text{Im}P(\pi^\star) \subseteq \text{Im}P(\pi)$. Algorithms for computing almost equitable partitions are available for arbitrary unweighted digraphs such as, among several others, the one in [18].

### *B. The hybrid multi-consensus problem*

We consider a group of $N \in \mathbb{N}$ identical agents, with state $x_i \in \mathbb{R}^n$ for any $i = 1, ..., N$. The evolution of each agent state is assumed to be governed by a hybrid dynamics, i.e, characterized by the interplay of continuous-time and discrete-time behaviours [14]. The alternate selection of continuous and discrete dynamics can be either driven by specific time patterns or triggered by conditions on the state. In this paper we consider agents whose dynamics is given by a hybrid integrator with time-driven jumps. In particular, all agents are assumed impulsive and of the form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{J} \tag{4a}$$

$$x_i^+(t) = Ex_i(t) + Fv_i(t), \quad t \in \mathcal{J} \tag{4b}$$

where $u_i, v_i \in \mathbb{R}^m$, $i = 1, ..., N$ and

$$\mathcal{J} = \{t_j \in \mathbb{R}^+, j = 1, ..., \aleph_{\mathcal{J}} : t_j < t_{j+1}, \ \aleph_{\mathcal{J}} \in \mathbb{N} \cup \infty\}$$
$$\tau_{\text{min}} < t_{j+1} - t_j < \tau_{\text{max}} \qquad \forall j \in \mathcal{J}.$$

The differential equation in (4a) is referred to as *flow dynamics*, whereas the difference equation (4b) corresponds to the *jump dynamics*. To keep track of the jumps, it is convenient to introduce the notion of hybrid time domain as a special case of [14, Definition 2.3].

*Definition 2.2:* A hybrid time domain is a set $\mathcal{T}$ in $[0, \infty) \times \mathbb{N}$ defined as the union of *indexed intervals*

$$\mathcal{T} := \bigcup_{j \in \mathbb{N}} \{ [t_j, t_{j+1}] \times \{j\}\} \tag{5}$$

Given an hybrid time domain, its length is defined as $length(\mathcal{T}) = \sup_t \mathcal{T} + \sup_j \mathcal{T}$. A hybrid time domain is said $\tau$-periodic, for some constant $\tau > 0$, if $t_{j+1} - t_j = \tau$ for any $j \in \mathbb{N}$. $\triangle$

In view of the latter definition, we can enhance the notation for the state of the agents as follows

$$x_i(t, j) \text{ with } (t, j) \in \mathcal{T}, \ i = 1, ..., N.$$

As previously mentioned, the agents are supposed to be connected through a suitable communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Accordingly, we aim at designing the control inputs in a decentralized way of the form

$$u_i = -\kappa \sum_{k \in \mathcal{N}_i} K(x_i - x_k) \tag{6a}$$

$$v_i = -\alpha \sum_{k \in \mathcal{N}_i} H(x_i - x_k) \tag{6b}$$

where $\kappa,\ \alpha > 0$ and $K, H \in \mathbb{R}^{m\times n}$ are suitable scalar and matrix gains (referred to as coupling gains and coupling matrices).

Defining the agglomerate states, for $i = 1, \dots, \mu$, as

$$\boldsymbol{x}_i = \text{col}\{x_j\}_{j\in\mathcal{H}_i} \in \mathbb{R}^{h_i n},\ \boldsymbol{x}_\mathcal{C} = \text{col}\{x_j\}_{j\in\mathcal{C}} \in \mathbb{R}^{cn}$$
$$\boldsymbol{u}_i = \text{col}\{u_j\}_{j\in\mathcal{H}_i} \in \mathbb{R}^{h_i m},\ \boldsymbol{u}_\mathcal{C} = \text{col}\{u_j\}_{j\in\mathcal{C}} \in \mathbb{R}^{cm}$$
$$\boldsymbol{v}_i = \text{col}\{v_j\}_{j\in\mathcal{H}_i} \in \mathbb{R}^{h_i m},\ \boldsymbol{v}_\mathcal{C} = \text{col}\{v_j\}_{j\in\mathcal{C}} \in \mathbb{R}^{cm}$$

and assuming $L$ of the form (1), one gets

$$\boldsymbol{u}_i = -\kappa(L_i \otimes K)\boldsymbol{x}_i,\ \boldsymbol{v}_i = -\alpha(L_i \otimes H)\boldsymbol{x}_i$$
$$\boldsymbol{u}_\mathcal{C} = -\kappa\sum_{i=1}^{\mu}(M_i \otimes K)\boldsymbol{x}_i - \kappa(M \otimes K)\boldsymbol{x}_\mathcal{C}$$
$$\boldsymbol{v}_\mathcal{C} = -\alpha\sum_{i=1}^{\mu}(M_i \otimes H)\boldsymbol{x}_i - \alpha(M \otimes H)\boldsymbol{x}_\mathcal{C}$$

so that the overall network dynamics gets the form (7)

$$\dot{\boldsymbol{x}}_i = \Big(\big(I_{h_i} \otimes A\big) - \kappa\big(L_i \otimes BK\big)\Big)\boldsymbol{x}_i \tag{7a}$$
$$\boldsymbol{x}_i^+ = \Big(\big(I_{h_i} \otimes E\big) - \alpha\big(L_i \otimes FH\big)\Big)\boldsymbol{x}_i \tag{7b}$$
$$\dot{\boldsymbol{x}}_\mathcal{C} = -\kappa\sum_{i=1}^{\mu}\Big(M_i \otimes BK\Big)\boldsymbol{x}_i + \Big(\big(I_c \otimes A\big) - \kappa\big(M \otimes BK\big)\Big)\boldsymbol{x}_\mathcal{C} \tag{7c}$$
$$\boldsymbol{x}_\mathcal{C}^+ = -\alpha\sum_{i=1}^{\mu}\Big(M_i \otimes FH\Big)\boldsymbol{x}_i + \Big(\big(I_c \otimes E\big) - \alpha\big(M \otimes FH\big)\Big)\boldsymbol{x}_\mathcal{C} \tag{7d}$$

We investigate the asymptotic cluster properties of the dynamics (7) induced by the network interconnection and design the coupling gains and matrices in (6) so that nodes asymptotically converge to suitable multi-consensus trajectories.

*Remark 2.1:* In the following, for the sake of simplicity and without loss of generality, we assume $\sigma\{L\} \subset \mathbb{R}_{\geq 0}$. All results to come hold true when considering the real part $\nu \in \mathbb{R}_{\geq 0}$ (flow dynamics) or the modulus $\sqrt{\nu^2 + \omega^2}$ (jump dynamics) of any eigenvalue $\lambda = \nu \pm j\omega \in \sigma\{L\}$ associated to a Jordan block

$$J_\lambda = \begin{pmatrix} \nu & \omega \\ -\omega & \nu \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

## III. The Hybrid Multi-Consensus Dynamics

For the sake of clarity, we first characterize the multi-consensus rising in the hybrid network (7) independently on both the sequence of the jump instants and the flow period. Then, we provide a constructive design for the inputs (6) making it attractive when assuming, for the sake of simplicity, the case of periodic jumps.

### *A. The characterization of the consensus dynamics*

The next results highlights the consensus dynamics over each exclusive reach via the definition of the so-called mean-field dynamics [19], [20]. In doing so, we extend the results in [1, Chapter 4] for (single) consensus of continuous-time systems to the hybrid multi-consensus case.

*Proposition 3.1:* Consider the multi-agent system composed of $N$ units evolving as (4) over communication digraph $\mathcal{G}$ with Laplacian of the form (1). Let the right and left eigenvectors associated to the zero eigenvalue of $L$ be of the form (2)-(3) with $w_i^\top \mathbb{1}_{h_i} = h_i$ for $i = 1, \dots, \mu$. Denote by $\sigma\{L_i\} = \{0, \lambda_i^1, \dots, \lambda_i^{h_i-1}\} \subset \mathbb{C}^+$. Then, all nodes in $\mathcal{H}_i$ ($i = 1, \dots, \mu$) converge to a common consensus provided that the hybrid dynamics

$$\dot{\varepsilon}_{i,\ell} = (A - \kappa\lambda_i^\ell BK)\varepsilon_{i,\ell} \tag{8a}$$
$$\varepsilon_{i,\ell}^+ = (E - \alpha\lambda_i^\ell FH)\varepsilon_{i,\ell} \tag{8b}$$

are asymptotically stable for all $\lambda_i^\ell \in \sigma\{L_i\} \setminus \{0\}$, $\ell = 1, \dots, h_i - 1$ and $i = 1, \dots, \mu$; i.e., $\boldsymbol{x}_i \to \mathbb{1}_{h_i} \otimes x_{s,i}$ with

$$x_{s,i} = \big(w_i^\top \otimes I_n\big)\boldsymbol{x}_i \in \mathbb{R}^n. \tag{9}$$

the mean-field unit evolving with mean-field dynamics

$$\dot{x}_{s,i} = Ax_{s,i} \tag{10a}$$
$$x_{s,i}^+ = Ex_{s,i}. \tag{10b}$$

*Proof:* For proving the result one must show that: $(i)$ the mean-field dynamic (10) with the mean-field unit (9) generates the consensus over the reach $\mathcal{H}_i$; $(ii)$ for each $\mathcal{H}_i$ the consensus error defined as

$$e_i = \text{col}\{e_{i,1}, \dots, e_{i,h_i}\} = \boldsymbol{x}_i - \big(\mathbb{1}_{h_i} \otimes I_n\big)x_{s,i}$$

vanishes whenever (8) are asymptotically stable. Let us note, first, that $w_i^\top \in \mathbb{R}^{1\times h_i}$ is the unique left eigenvector of $L_i$ associated to 0 verifying $w_i^\top \mathbb{1}_{h_i} = 1$. Accordingly, because $w_i^\top \otimes e_i = 0$, only $(h_i - 1)n$ components of $e_i$ are linearly independent; thus, $e_i \to 0$ if and only if only a suitably defined subset of $(h_i - i)n$ component do. With this in mind, the consensus-error dynamics over $\mathcal{H}_i$ is given by

$$\dot{e}_i = \Big(\big(I_{h_i} \otimes A\big) - \kappa\big(L_i \otimes BK\big)\Big)e_i$$
$$e_i^+ = \Big(\big(I_{h_i} \otimes E\big) - \alpha\big(L_i \otimes GH\big)\Big)e_i.$$

Fixing the non singular matrix $T_i \in \mathbb{R}^{h_i\times h_i}$ such that

$$\Lambda_i = TL_iT_i^{-1} = \text{diag}\{0, \lambda_i^1, \dots, \lambda_i^{h_i-1}\}$$

with $\lambda_i^\ell > 0$ for all $\ell = 1, \dots, h_i - 1$, the independent components of the consensus error are given by

$$\begin{aligned}\varepsilon_i =& \text{col}\{0, \varepsilon_{i,1}, \dots, \varepsilon_{i,h_i-1}\}\\ =& (T_i \otimes I_n)\Big(\boldsymbol{x}_i - \big(\mathbb{1}_{h_i} \otimes I_n\big)x_{s,i}\Big)\end{aligned}$$

because, by definition, $T_i\mathbb{1}_{h_i} = \begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}^\top$. Each of those components evolves as (8). By these arguments, $e_i \to 0$ if and only if $\varepsilon_{i,\ell} \to 0$ for $\ell = 1, \dots, h_i - 1$ that is, if and only if the corresponding dynamics (8) are asymptotically stable so that $(ii)$ is proved. $(i)$ follows by noticing that, by definition, the set $\{e_i = 0\} \equiv \{\boldsymbol{x}_i = I_{h_i} \otimes x_{s,i}\}$ is an invariant subspace for (7). The residual dynamics can be easily computed; for the flow, one gets

$$\begin{aligned}\dot{x}_{s,i} =&\big(w_i^\top \otimes I_n\big)\Big(\big(I_{h_i}\otimes A\big)-\kappa\big(L_i\otimes BK\big)\Big)\boldsymbol{x}_i\\ =&\Big(\big(w_i^\top\otimes A\big)-\kappa\big(w_i^\top L_i\otimes BK\big)\Big)\boldsymbol{x}_i\\ =&A\big(w_i^\top\otimes I_n\big)\boldsymbol{x}_i = Ax_{s,i}\end{aligned}$$

whereas the jump component is deduced computing

$$\begin{aligned}x^+_{s,i} =&\big(w_i^\top \otimes I_n\big)\Big(\big(I_{h_i}\otimes E\big)-\alpha\big(L_i\otimes FH\big)\Big)\boldsymbol{x}_i\\ =&\Big(\big(w_i^\top\otimes E\big)-\alpha\big(w_i^\top L_i\otimes FH\big)\Big)\boldsymbol{x}_i\\ =&E\big(w_i^\top\otimes I_n\big)\boldsymbol{x}_i = Ex_{s,i}.\end{aligned}$$

■

We can now characterize the consensus arising in the whole network with necessary and sufficient conditions making it asymptotically stable for all nodes. We first show that, as in the continuous-time case [17], nodes in the common regroup into as many clusters as the number (say $p \geq 1$) of the distinct components of the vectors $\gamma^i$ in (2). To this end, let us assume nodes in $\mathcal{C}$ are sorted in such a way that all vectors $\gamma^i$ in (2) get the form

$$\gamma^i = \begin{pmatrix}\gamma_1^i \mathbb{1}^\top_{c_1} & \dots & \gamma_p^i\mathbb{1}^\top_{c_p}\end{pmatrix}^\top \tag{11}$$

with $\gamma^i_r \in \mathbb{R}$ for $r = 1,\dots,p$ and verifying $\sum_{r=1}^p \gamma_r^i = 1$ and $\sum_{r=1}^p c_i = c = |\mathcal{C}|$. In this way, one can denote by $\mathcal{C}_{\mu+r}\subseteq\mathcal{C}$ with $c_p = |\mathcal{C}_{\mu+r}|$ the cell composed of all nodes $j\in\mathcal{C}$ corresponding to the same component $\gamma^i_r$ of the vector (11). Consequently, we denote $\boldsymbol{x}_\mathcal{C} = \{\boldsymbol{x}_{\mu+1},\dots,\boldsymbol{x}_{\mu+p}\}$ with

$$\boldsymbol{x}_{\mu+r} = \text{col}\{x_j\}_{j\in\mathcal{C}_{\mu+r}} \in \mathbb{R}^{c_i n}.$$

*Remark 3.1:* We note that $\mathcal{C} = \cup_{r=1}^p \mathcal{C}_{\mu+r}$ with, for all $r_1, r_2 = 1,\dots,p$ and $r_1 \neq r_2$, $\mathcal{C}_{\mu+r_1}\cap\mathcal{C}_{\mu+r_2} = \emptyset$. In addition, as proved in [17], the coarsest nontrivial AEP of a digraph $\mathcal{G}$ is

$$\pi^\star = \{\mathcal{H}_1,\dots,\mathcal{H}_\mu,\mathcal{C}_{\mu+1},\dots,\mathcal{C}_{\mu+p}\} \tag{12}$$

At this point, the following result can be given proving that, under suitable conditions on the coupling gains and matrices, the network induces as many consensuses as the number of cells of the AEP (12) associated to $\mathcal{G}$.

*Proposition 3.2:* Consider the multi-agent system composed of $N$ units evolving as (4) under the hypotheses of Proposition 3.1 and consider $\pi^\star$ as in (12) be an AEP for $\mathcal{G}$. Let (3) be the right and left eigenvectors associated to the zero eigenvalue of $L$ verifying $w_i^\top \mathbb{1}_{h_i} = h_i$ with each $\gamma^i\in\mathbb{R}^c$ of the form (11). Then, the trajectories of the nodes belonging to $\mathcal{C}_{\mu+r}\subseteq\mathcal{C}$ converge to the consensus trajectory

$$x_{s,\mu+r} = \sum_{i=1}^{\mu}\gamma_r^i x_{s,i}\in\mathbb{R}^n \tag{13}$$

for $r = 1,\dots,p$ and $x_{s,i}$ as in (9) if and only if the dynamics

$$\dot{\varepsilon}_{\mathcal{C},q} = (A-\kappa\lambda_M^q BK)\varepsilon_{\mathcal{C},q} \tag{14a}$$
$$\varepsilon^+_{\mathcal{C},q} = (E-\alpha\lambda_M^q FH)\varepsilon_{\mathcal{C},q} \tag{14b}$$

are asymptotically stable for all $\lambda_M^q \in \sigma\{M\}$ , $q = 1,\dots,c$.

*Proof:* By the structure (1) one gets that $\lambda_M^q > 0$ for all $q = 1,\dots,\mu$. To prove the result, under the hypotheses of Proposition 3.1, one must show that the multi-consensus error over $\mathcal{C}$ defined as

$$e_\mathcal{C} = \boldsymbol{x}_\mathcal{C} - \sum_{i=1}^{\mu}\gamma^i\otimes x_{s,i}$$

vanishes asymptotically provided that (14) are asymptotically stable. To this end, we first note that all components of $e_\mathcal{C}$ are independent. In addition, by Proposition 3.1, the consensus error over the reaches $e_i \to 0$ (for $i = 1,\dots,\mu$) asymptotically. Accordingly, $e_\mathcal{C}$ asymptotically converges to zero if and only if

$$\varepsilon_\mathcal{C} = \text{col}\{\varepsilon_{\mathcal{C},1},\dots,\varepsilon_{\mathcal{C},c}\} = \big(T_\mathcal{C}\otimes I_n\big)e\to 0$$

with $e = \text{col}\{e_1,\dots,e_\mu,e_\mathcal{C}\}$, $T_\mathcal{C}\in\mathbb{R}^{c\times N}$ such that

$$\begin{aligned}T &= \begin{pmatrix}\text{diag}\{T_1,\dots,T_\mu\}\\ T_\mathcal{C}\end{pmatrix}\\ \Lambda &= TLT^{-1} = \text{diag}\{\Lambda_1,\dots,\Lambda_\mu,\Lambda_\mathcal{C}\}\\ \Lambda_\mathcal{C} &= \text{diag}\{\lambda_M^1,\dots,\lambda_M^c\}\in\mathbb{R}^{c\times c}.\end{aligned}$$

It is a matter of computations to check that the dynamics of all components $\varepsilon_{\mathcal{C},q}\in\mathbb{R}^n$ gets the form (14). Thus, $e_\mathcal{C}\to 0$ if and only if such evolutions are asymptotically stable for all $q = 1,\dots,c$. Rewriting block component-wise

$$e_\mathcal{C} = \begin{pmatrix}\boldsymbol{x}_{\mu+1} - \sum_{i=1}^{\mu}(\gamma_1^i\mathbb{1}_{c_1}\otimes I_n)x_{s,i}\\ \vdots\\ \boldsymbol{x}_{\mu+p} - \sum_{i=1}^{\mu}(\gamma_1^i\mathbb{1}_{c_p}\otimes I_n)x_{s,i}\end{pmatrix}$$

one gets the result. ■

*Remark 3.2:* By the result above, the subspace

$$\mathcal{P}^\star = \text{Im}\{P(\pi^\star)\otimes I_n\} = \ker\{L\otimes I_n\}$$

associated to the AEP (12) defines the consensus subspace of the hybrid network (7).

*Remark 3.3:* It can be proved that, by Propositions 3.1 and 3.2, asymptotic stability of the uncontrolled hybrid agents (4) ensure convergence to an asymptotically stable consensus trajectory. However, as the intuition suggests, the reverse is not true: when gains are suitably designed, unstable agents converge to a consensus trajectory which is, thus, unbounded.

### *B. The hybrid coupling design*

By Propositions 3.1 and 3.2, the hybrid network induces $\mu+k$ consensus trajectories: $\mu\geq 1$ independent consensuses arise over each reach $\mathcal{H}_i$; $p\geq 1$ further consensuses are exhibited over the common $\mathcal{C}$. Convergence to such behaviors are guaranteed provided that the coupling gains $\kappa,\alpha$ and coupling matrices $K, H$ in (6) are chosen so to make all dynamics (8)-(14) asymptotically stable. Those conditions strictly depend on the non-zero eigenvalues of the Laplacian $L$ so that a qualitative design assigning the eigenvalues for all possible $\lambda\in\sigma\{L\}$ might not be feasible in a decentralized way. In the sequel, constructive conditions over (6) are given to enforce this property in a robust way. To this end, let us define the quantities

$$\lambda^{\dagger} = \min_{\substack{i=1,\dots,\mu \\ \ell=1,\dots,h_i-1 \\ q=1,\dots,c}} \{\lambda_i^{\ell}, \lambda_M^q\}$$

$$\lambda^{\circ} = \max_{\substack{i=1,\dots,\mu \\ \ell=1,\dots,h_i-1 \\ q=1,\dots,c}} \{\lambda_i^{\ell}, \lambda_M^q\}$$

*Theorem 3.1:* Consider the hybrid system (4) driven by a control law of the type (6) under the condition of $\tau$-periodic jumps. Suppose that there exists a common solution $\hat{Q} = \hat{Q}^{\top} \succ 0$ to the continuous-time algebraic Riccati inequality

$$A^{\top}\hat{Q} + \hat{Q}A - \chi_1 \hat{Q}BB^{\top}\hat{Q} \prec -\upsilon_1 I \tag{15}$$

and to the discrete-time algebraic Riccati inequality

$$E^{\top}\hat{Q}E - \hat{Q} - \chi_2 E^{\top}\hat{Q}F(I + F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E < -\upsilon_2 I \tag{16}$$

for some constants $\upsilon_1, \upsilon_2, \chi_1, > 0$ and $\chi_2 \in (0,1)$ with

$$\frac{1-\sqrt{1-\chi_2}}{1+\sqrt{1-\chi_2}} < \frac{\lambda^{\dagger}}{\lambda^{\circ}} \tag{17}$$

Then, if the feedback gains in (6) are selected as

$$K = B^{\top}\hat{Q}, \quad H = (I + F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E$$

and

$$\kappa \geq \kappa^{\star} := \frac{\chi_1}{2\lambda^{\dagger}},\ \alpha \in \mathcal{I}_{\alpha}^{\star} := \left(\frac{1-\sqrt{1-\chi_2}}{\lambda^{\dagger}}, \frac{1+\sqrt{1-\chi_2}}{\lambda^{\circ}}\right),$$

the hybrid systems in (8)-(14) are asymptotically stable for any $i, \ell, q$, and therefore consensus in (4) is reached.

*Proof:* Let $\hat{Q}$ be the solution of (15), pick $K = B^{\top}\hat{Q}$ and consider the Lyapunov-like condition

$$(A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q})^{\top}\hat{Q} + \hat{Q}(A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q})$$

where $\lambda_{\sharp}$ is any of the eigenvalues appearing in either (8a) or (14a). Expanding the products and picking $\kappa \geq \kappa^{\star} \geq \chi_1/2\lambda_{\sharp}$ one gets

$$\begin{array}{cl} & (A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q})^{\top}\hat{Q} + \hat{Q}(A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q}) \\ = & A^{\top}\hat{Q} + \hat{Q}A - 2\kappa\lambda_{\sharp}\hat{Q}BB^{\top}\hat{Q} \\ \prec & A^{\top}\hat{Q} + \hat{Q}A - \chi_1\hat{Q}BB^{\top}\hat{Q} \prec -\upsilon_1 I, \end{array},$$

this proving that $V(e) = e^{\top}\hat{Q}e$ is a common Lyapunov function for any of the flow dynamics appearing in either (8a) or (14a). We can also observe that, by using the properties of the transition map, one has

$$\Sigma_{\sharp}^{\top}\hat{Q}\Sigma_{\sharp} - \hat{Q} \prec 0 \tag{18}$$

where

$$\Sigma_{\sharp} = \exp\{(A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q})\tau\}$$

Let us now analyze the jump dynamics (8b) and (14b), by considering the corresponding monodromy matrix

$$Y_{\sharp} = \Sigma_{\sharp}(E - \alpha\lambda_{\sharp}FH).$$

and the discrete-time Lyapunov condition

$$V(e^+) - V(e) = e^{\top}(Y_{\sharp}^{\top}\hat{Q}Y_{\sharp} - \hat{Q})e.$$

Choosing the gain $H = (I + F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E$ and using inequality (18), one gets

$$\begin{array}{l} Y_{\sharp}^{\top}\hat{Q}Y_{\sharp} - Q \\ \prec (E - \alpha\lambda_{\sharp}FH)^{\top}\hat{Q}(E - \alpha\lambda_{\sharp}FH) - Q \\ \prec E^{\top}\hat{Q}E - \hat{Q} - 2\alpha\lambda_{\sharp}E^{\top}\hat{Q}F(I + F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E \\ + \alpha^2\lambda_{\sharp}E^{\top}\hat{Q}F(I+F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}F(I+F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E \\ \prec E^{\top}\hat{Q}E - \hat{Q} + (\alpha^2\lambda_{\sharp}^2 - 2\alpha\lambda_{\sharp})E^{\top}\hat{Q}F(I+F^{\top}\hat{Q}F)^{-1}F^{\top}\hat{Q}E \end{array}$$

Now, picking $\alpha \in \mathcal{I}_{\alpha}^{\star}$, one is guaranteed to have $\alpha^2\lambda_{\sharp}^2 - 2\alpha\lambda_{\sharp} < -\chi_2$ and thus, thanks to (16)-(17), the condition

$$Y_{\sharp}^{\top}\hat{Q}Y_{\sharp} - Q \prec -\upsilon_2 I$$

is satisfied for any of the jump dynamics appearing in either (8b) or (14b). The stability of the hybrid sub-systems (8)-(14), thereby enforcing consensus for the hybrid multi-agent system (4), follows then by invoking [21, Theorem 2.1, c) ⇒ a)], which holds uniformly in $i, \ell, q$ with $V(e) = e^{\top}\hat{Q}e$. ∎

*Remark 3.4:* It is worth stressing that the conditions provided by Theorem 3.1 are independent on the time-domain but only sufficient. Milder design conditions can be easily found by direct inspection as illustrated, for instance, in the simulation study presented in the next section.

*Remark 3.5:* The sufficient conditions of Theorem 3.1 can be readily extended to the non-periodic case along those lines and using the arguments in [14, Example 3.22]. In fact, inequality (18) still holds true if one replace $\Sigma_{\sharp}$ with $\exp\{(A - \kappa\lambda_{\sharp}BB^{\top}\hat{Q})\hat{t}\}$, whatever $\hat{t} \in [\tau_{\min}, \tau_{\max}]$ is, and so the conditions of [21, Theorem 2.2] apply.

## IV. Illustrative examples

In the following, we consider a network of $N = 8$ hybrid agents of the form (4) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ E = \frac{1}{3}\begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 2\sqrt{2} \end{pmatrix},\ F = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

under communication graph

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{19}$$

with two exclusive reaches $\mathcal{H}_1 = \{1,2,3\}$, $\mathcal{H}_2 = \{4,5\}$ and the common $\mathcal{C} = \{6,7,8\}$. Thus, one gets $\mu = 2$ and the eigenvectors of the form (3)-(2) with

$$w_1^{\top} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\quad w_2^{\top} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} \frac{1}{2}\mathbb{1}_2^{\top} & \frac{3}{4} \end{pmatrix}^{\top},\quad \gamma^2 = \begin{pmatrix} \frac{1}{2}\mathbb{1}_2^{\top}\frac{1}{4} \end{pmatrix}^{\top}.$$

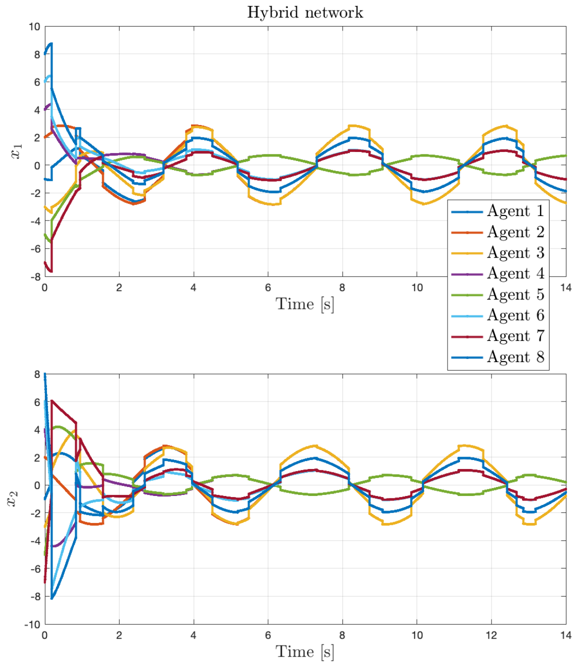


Fig. 1. A network of 8 hybrid agents under aperiodic jumps and (19).

As a consequence, the AEP gets the form (12) with further partitioning of the common as $\mathcal{C}_{\mu+1} = \{6,7\}$, $\mathcal{C}_{\mu+2} = \{6,7\}$.å Accordingly, the mean-field units (9) are given by

$$x_{s,1} = (w_1^\top \otimes I_2)\boldsymbol{x}_1 = x_2, \; x_{s,2} = (w_2^\top \otimes I_2)\boldsymbol{x}_1 = \frac{\sqrt{2}}{2}(x_4 + x_5)$$

defining, by Proposition 3.1, the consensus trajectory of all nodes in the same reach. In addition, by Proposition 3.2, the mean-field units (13) within the cells of $\mathcal{C}$ get the form

$$\begin{aligned} x_{s,\mu+1} &= \gamma_1^1 x_{s,1} + \gamma_1^2 x_{s,2} = \frac{1}{2}(x_{s,1} + x_{s,2}) \\ x_{s,\mu+1} &= \gamma_2^1 x_{s,1} + \gamma_2^2 x_{s,2} = \frac{1}{4}(3x_{s,1} + x_{s,2}). \end{aligned}$$

By the structure of the dynamical matrices $A$ and $E$, those units are periodic trajectories identifying 4 consensuses generated by the corresponding mean-field dynamics (10). Accordingly, all nodes converge to the corresponding trajectory of the mean-field units above under the couplings

$$\kappa = 1, \quad \alpha = \frac{1}{4}, \quad K = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

guaranteeing stability of all error dynamics (8)-(14).

For completeness, simulations are reported in Figure 1 when fixing the initial conditions as $x_i(0) = (-1)^i i \mathbb{1}_2$ (with $i = 1, \ldots, 8$) and jumps occurring at random time instants $t_{\mathrm{j}}$ under the bound $t_{\mathrm{j}+1} - t_{\mathrm{j}} \in [0.1, 1]$.

## V. Conclusions and Perspectives

In this paper, we have provided an explicit characterization of the multi-consensus induced over a network of hybrid linear agents under time-driven jumps. In particular, agents behave in clusters depending on the coarsest AEP of the underlying communication graphs. Then, we provide sufficient conditions on the coupling gains ensuring that the network reaches the corresponding multi-consensus. Future works include the case of heterogeneous agents with possibly asynchronous jump sets and general hybrid systems under state-driven jumps and switching topologies as well.

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