Fachgebiet
Regelungstechnik

TECHNISCHE UNIVERSITÄT
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# Conference on Decision and Control - 2022

## *Non-asymptotic Observer Design for Nonlinear Systems Based on Linearization*


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*Observers for Nonlinear Systems*

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# Motivation of particular perspective

| *(i)* | *(ii)* | *(iii)* | *(iv)* | textit(v) |
|---|---|---|---|---|
| Finite time interval & moving horizon | Continuous-time models | Non-asymptotic/ algebraic | Signal projection | `FIR` implementation |

| Dynamic observer | Algebraic observer |
|---|---|
| $\dot{\hat{x}} = f(\hat{x}, u) + L(t)(\hat{y} - y)\,,\ \hat{x}(0) = \hat{x}_0$ | $\hat{x}(t) = \langle H_1, y\rangle + \langle H_2, u\rangle$ |
| $\ominus$ virtual dynamics (*digital inertia*) | $\oplus$ instantaneous result from alg. expr. |
| $\oplus$ efficient implementation | $\oplus$ no stability issues in coupled setup |
| $\oplus$ inherent robustness via feedback | $\ominus$ sensitive to model errors & numerics |
| $[0, t], [0, \infty]$ | $[0, T], [t - T, t]$ |

# Demonstration of modulation mechanism

*Operator structure:*

$$\underbrace{Ly(t)}\;=\;\underbrace{f(t)}$$

Example:

$$\ddot{y}+a_1\dot{y}+a_0y=u(t)$$

**Modulation integral:** MF $\varphi:[0,T]\to\mathbb{R}$

$$\begin{aligned}\mathcal{M}[\ddot{y}]=\langle\varphi,\ddot{y}\rangle:=\int_{t-T}^{t}\varphi(\tau-t+T)\,\ddot{y}(\tau)d\tau&=\varphi\,\dot{y}\,\big|_0^T-\int_{t-T}^{t}\dot{\varphi}(\tau-t+T)\,\dot{y}(\tau)d\tau\\&=\varphi\,\dot{y}\,\big|_0^T-\dot{\varphi}\,y\,\big|_0^T+\int_{t-T}^{t}\ddot{\varphi}(\tau-t+T)\,y(\tau)d\tau\end{aligned}$$

*Idea:* Shift of derivative

$\Rightarrow$ Zero **B**oundary **C**onditions:

$$\begin{cases}\varphi(0)=\varphi(T)=0\\\dot{\varphi}(0)=\dot{\varphi}(T)=0\end{cases}\Rightarrow\boxed{\begin{aligned}&\langle\ddot{\varphi},y\rangle-a_1\langle\dot{\varphi},y\rangle\\&+a_0\langle\varphi,y\rangle=\langle\varphi,u\rangle\end{aligned}}$$

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$$= \varphi\, \dot{y}\,\Big|_0^T - \dot{\varphi}\, y\,\Big|_0^T + \int_{t-T}^{t} \ddot{\varphi}(\tau - t + T)\, y(\tau) d\tau$$

*Idea:* Isolate state

$\Rightarrow$ Non-zero **BC**s:

$$\begin{cases} \varphi(0) = 0, \ \varphi(T) = 1 \\ \dot{\varphi}(0) = 0, \ \dot{\varphi}(T) = a_1 \end{cases} \Rightarrow \boxed{\dot{y}(t) = -\overbrace{\langle L^* \varphi, y \rangle}^{\langle \ddot{\varphi} - a_1 \dot{\varphi} + a_0 \varphi, y \rangle} + \langle \varphi, u \rangle}$$

# Outline

# Agenda overview

1. Modulating Function based state estimation
2. Modulation of nonlinear systems by linearization
3. Implementation & simulation results

# Linear differential equations

Linear output ODE:

$$\overbrace{a_n(t)y^{(n)} + \ldots + a_1(t)\dot{y} + a_0(t)y}^{\sum_{k=0}^{n} a_k(t)y^{(k)} = Ly = a^\top Dy} = \xi(t)$$

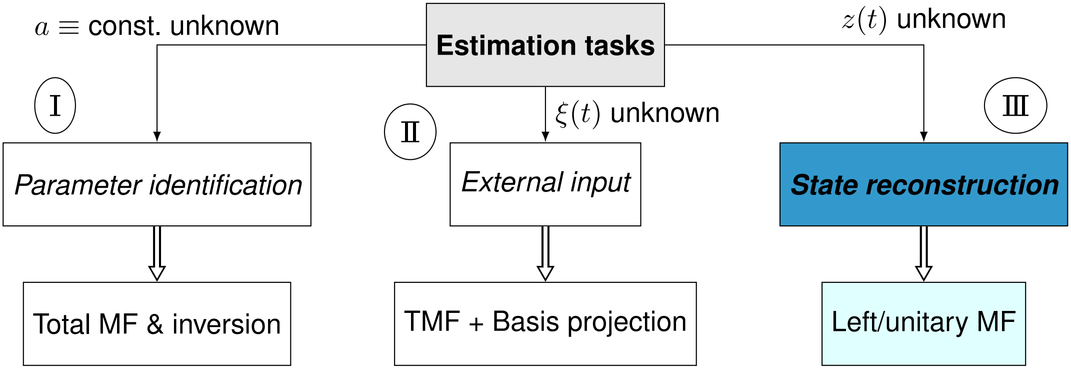
Linear differential operator $L$ and its adjoint $L^*$:

$$L = \sum_{k=0}^{n} a_k(t)\,\frac{\mathrm{d}^\mathrm{k}}{\mathrm{dt}^\mathrm{k}}\,(\,\cdot\,) =: a(t)^\top D \quad \Rightarrow \quad L^* = \sum_{k=0}^{n} (-1)^k \Big(a_k(t)\,(\,\cdot\,)\Big)^{(k)}$$

Modulation w.r.t. kernel $\varphi \in \mathcal{C}^{(n-1)}([0,T],\mathbb{R}^n)$:

$$\langle \varphi, Ly\rangle = \langle L^*\varphi, y\rangle + \left[\, L_1^*\varphi \,|\, \cdots \,|\, L_n^*\varphi \,\right] \overbrace{\begin{pmatrix} y \\ \vdots \\ y^{(n-1)} \end{pmatrix}}^{=z} \Bigg|_0^T = \langle \varphi, \xi\rangle$$

# Estimation problem formulation



# Algebraic observer design

## III. ***State reconstruction*** [JR15]: Left MF

$$\boxed{\varphi^{(i)}(0) = 0,\ \varphi^{(i)}(T) \neq 0}$$

$$\Rightarrow \underbrace{\left[\, L_1^*\varphi(T) \,|\, \cdots \,|\, L_n^*\varphi(T) \,\right]}_{=\Delta(a)} z(t) = \underbrace{\langle \varphi, \xi \rangle - \langle L^*\varphi, y \rangle}_{Q_a[y,\xi]}$$

where

$$L_k(\,\cdot\,) = \sum_{j=0}^{n-k} a_{n-j}\,(\,\cdot\,)^{(n-k-j)} \quad \Rightarrow \quad L_0 = L.$$

$\rightarrow$ **Q:** Invertibility of $\Delta(a)\,\forall a$?

# Choice of boundary conditions

*Unitary MF* (inspired by *generalized MFs* [TWLB19]): allowing $\varphi^{(i)}(T) = c_i \in \mathbb{R}$

$$\varphi_l^{(i)}(T) = \begin{cases} (-1)^{l-n} & : n - l = i \\ 0 & : \text{else} \end{cases}$$

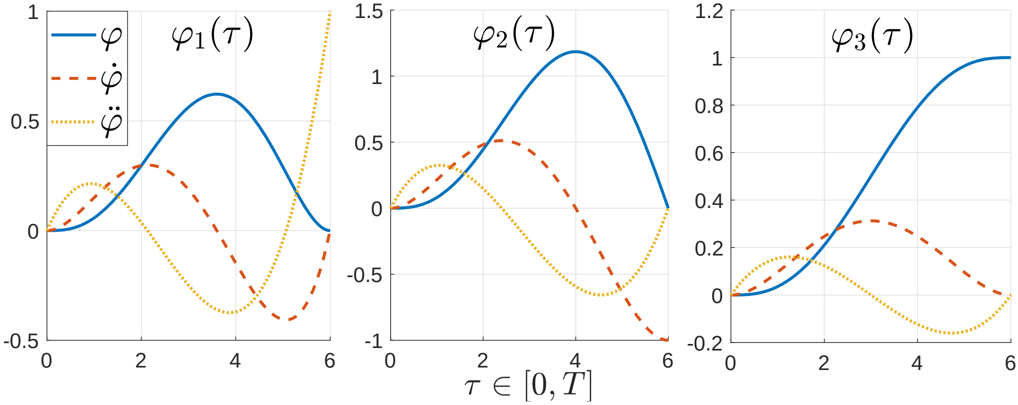
This leads to

$$\Delta(a) = \begin{pmatrix} L_1^*\varphi_1(T) & \cdots & L_n^*\varphi_1(T) \\ \vdots & \ddots & \vdots \\ L_1^*\varphi_n(T) & \cdots & L_n^*\varphi_n(T) \end{pmatrix} = \begin{pmatrix} a_n & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix} \overset{a \equiv \text{const}}{\underset{n=3}{=}} \begin{pmatrix} a_3 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

Usual case: $a_n \equiv 1 \Rightarrow \kappa\big(\Delta(a)\big) = 1 \, \forall a$ and

$$\hat{z}(t) = \Delta(a)^{-1} \left( \langle \varphi, \xi \rangle - \langle L^* \varphi, y \rangle \right).$$

# Kernel construction



# Connection to observability normal form

Observation canonical form (SISO LTI):

$$\underbrace{y^{(n)} + \sum_{i=0}^{n-1} a_i\, y^{(i)}}_{=Ly} = \underbrace{\sum_{j=0}^{n-1} b_j\, u^{(j)}}_{=Ru} \quad \overset{y=x_1}{\Rightarrow} \quad \dot{x} = \begin{pmatrix} -a_{n-1} & 1 & & \\ -a_{n-2} & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ -a_0 & 0 & \cdots & 0 \end{pmatrix} x + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{pmatrix} u$$

Adjustment of operator form with boundaries:

$$\langle \varphi, Ly - Ru \rangle = \langle L^*\varphi, y \rangle - \langle R^*\varphi, u \rangle + \left[ \varphi \,|\, \text{-}\dot{\varphi} \,|\, \cdots \,|\, (\text{-}1)^{n\text{-}1} \varphi^{(n\text{-}1)} \right] \overbrace{\begin{pmatrix} L_n y \\ L_{n-1} y - R_n u \\ \vdots \\ L_1 y - R_2 u \end{pmatrix}}^{=x} \Bigg|_0^T$$

$$\overset{\text{UMF}}{\Rightarrow} \quad \boxed{\hat{x}(t) = -\langle L^*\varphi, y \rangle + \langle R^*\varphi, u \rangle}$$

# Algebraic observer discussion & overview

| *Case 1 (LTI / parameters known)* | *Case 2 (LTI / parameters unknown)* |
|---|---|
| $\hat{z} = \Delta(\theta)^{-1} Q(y,u)$ | $\hat{z} = \Delta(\hat{\theta})^{-1} Q(y,u)$ |
| $\rightarrow$ offline inversion of $\Delta$ | $\rightarrow$ guarantee full rank via UMF |
| *Case 3 (LTV / parameters known)* | *Case 4 (LTV / parameters unknown)* |
| $\hat{z} = \Delta\big(a(t)\big)^{-1} Q(y,u)$ | $\hat{z} = \Delta\big(a(w(t))\big)^{-1} Q(y,u)$ |
| $\rightarrow$ TV terms as part of operator | $\rightarrow$ online update including derivatives! |

**Other notable approaches:**

- *Exact state observers* [Byr03] - input/output error `sensitivity` with resulting DRE
- Mazenc et. al. [MMN22] - *temporal shift* based operation (sampled data systems)
- *Deadbeat observer* [PLFP17] *(F)* - exceeding horizon formulation (exponential MFs)
- Auxiliary Problem approach with extension to *PDEs* [GNRL20]

# Agenda overview

# Nonlinear input-output form

**State transformation** (observability map) [Bes07]:

$$o_k(x) = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(k)} \end{pmatrix} = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x) \\ \vdots \\ \mathcal{L}_f^k h(x) \end{pmatrix} =: z \quad \Rightarrow \quad \mathcal{O}_k(x) := \frac{\partial o_k}{\partial x}(x)$$

General output **transfer behavior**:

$$\begin{aligned} y^{(n)} &= \mathcal{L}_f^n h\big(o_k^{-1}(x)\big) \approx \mathcal{L}_f^n h\Big(\bar{x} + \left[\frac{\partial o_k}{\partial x}\right]_{\bar{x}}^{-1} \big(z - \underbrace{o_k(\bar{x})}_{\bar{z}}\big)\Big) \\ &= F(y, \dot{y}, \ddot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots) \end{aligned}$$

# Classical structuring problem

For a given function $y \in \mathcal{C}^i([0,T],\mathbb{R}^p)$ and parameter $a \in \mathbb{R}$ define:

| | | |
|---|---|---|
| *(i)* | **Linear signal**: | $\xi(t) = a\, y^{(i)}(t)$ |
| *(ii)* | **Integrable signal**: | $\xi(t) = a\, g^{(i)}\big(y(t)\big)$ |
| *(iii)* | **Convolvable signal**: | $\xi(t) = a\, g\big(y(t)\big)\, f^{(i)}\big(y(t)\big)$ |
| *(iv)* | **E-convolvable signal**: | $\xi(t) = a\, g^{(i)}\big(y(t)\big)\, f^{(i)}\big(y(t)\big)$ |

with $g, f \in \mathcal{C}^i(\mathbb{R}^p,\mathbb{R})$ arbitrary functions [CTR14].

Resulting **structuring problem** for suitable representation [Pea92]:

$$F(y,\dot{y},\ddot{y},\ldots,y^{(n-1)}) \stackrel{?}{=} \sum_{j=0}^{n-1} a_j\, g_j^{(j)}(y) + \sum_{i=0}^{n-1} b_i\, g_i(y)\, f^{(i)}(y)$$

# Linearization-based approach

Considered nonlin. output dyn. (ONF):

$$y^{(n)} = F(\underbrace{y, \dot{y}, \ldots, y^{(n-1)}}_{=:z}, u)$$

Linearization around $\bar{z}$:

$$y^{(n)} = F(z,u) \approx F(\bar{z},u) + \overbrace{\frac{\partial F}{\partial z}\bigg|_{\bar{z}}}^{=:\alpha(t)} (z - \bar{z})$$

$$\Rightarrow \underbrace{y^{(n)} - \frac{\partial F}{\partial z}\bigg|_{\bar{z}} z}_{L(a)y} = \underbrace{F(\bar{z},u) - \frac{\partial F}{\partial z}\bigg|_{\bar{z}} \bar{z}}_{\xi(t)}$$

With $a(t) = (-F'(\bar{z}), 1)^\top = (-\alpha(t), 1)^\top$:

$$\Rightarrow \Delta\big((a(t)\big)\, z(t) = \mathcal{M}[\xi] - \big\langle L\big((-F'(\bar{z}),1)\big)^* \varphi, y\big\rangle$$

**Final estimator form:**

$$\hat{z}(t) = \Delta\big(a(t)\big)^{-1}\Big(\mathcal{M}[F(\bar{z},u) - F'(\bar{z})\,\bar{z}] - \big\langle L\big(a(t)\big)^* \varphi, y\big\rangle\Big)$$

# Estimator realization

**Strategy:**

- last estimate $\bar{z}(t) = \hat{z}(t - T_s)$
- UMF and symbolic inversion
- FIR filter implementation

**Challenges:**

- ▷ inverse observability map needed
- ▷ integration over moving horizon filled with past estimates
- ▷ estimation feedback within coefficient $a(t)$ and its derivatives
- ▷ initialization and convergence behavior

Coefficient time derivatives $a^{(i)}$:

$$a^{(0)} = (-F'(\bar{z}), 1)^\top,$$

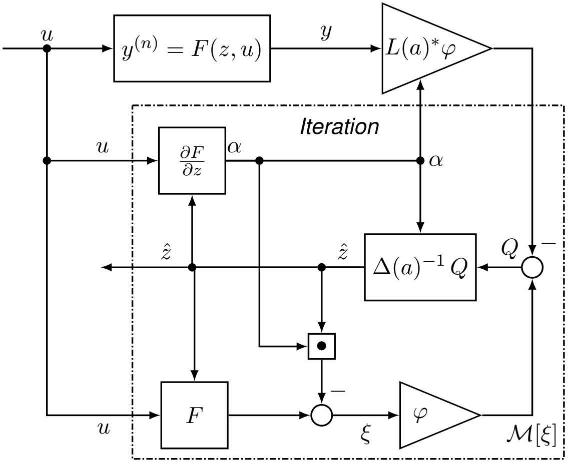
$$a^{(i)} = \frac{\partial a^{(i-1)}}{\partial z} f(\bar{z}, u) \, \forall i \in \{1, \dots, n\}$$

Basic ZoH assumption for *simplification*:

$$a(\tau) = \text{const.} \quad \forall \tau \in [t - T, t]$$

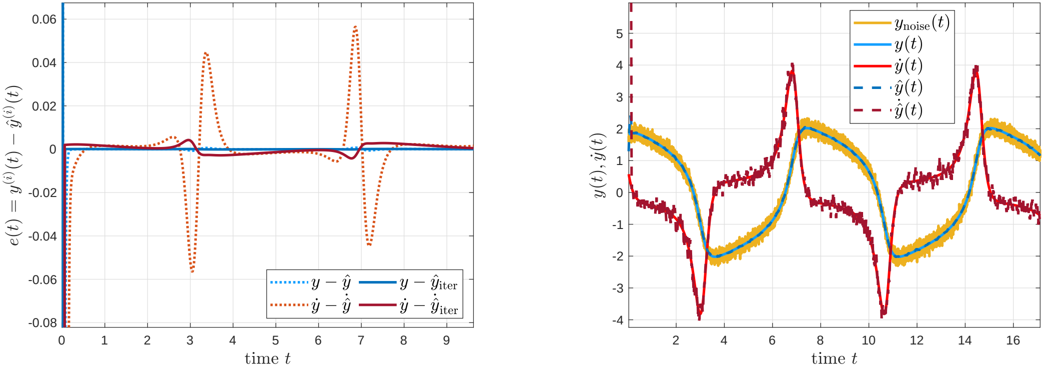
# Agenda overview

# Linearization-based algorithm iteration



# Simulation results - Van-der-Pol oscillator

NL Output dynamics: $\ddot{y} = \mu\,(1 - y^2)\,\dot{y} - y = F(z)$



# Simulation results - Two-Tank system

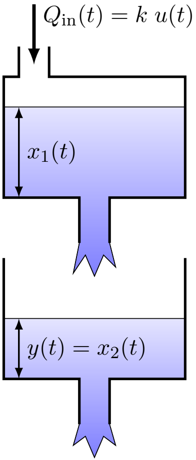
System dynamics:

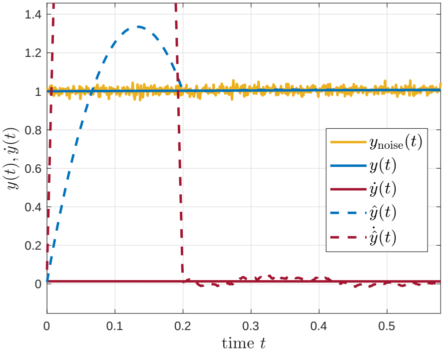
$$\dot{x}_1 = \frac{k\,u - a_1\sqrt{2\,g\,x_1}}{A_1}$$

$$\dot{x}_2 = \frac{a_1\sqrt{2\,g\,x_1} - a_2\sqrt{2\,g\,x_2}}{A_2}$$

Observability transform:

$$y^{(n)} = \mathcal{L}_f^n h(\mathcal{O}(z)^{-1}) = F(z,u)$$



# Final remarks

**Evaluation:**

- MF operator form for LTV systems
- UMF design for guaranteeing $\Delta(a)$ inversion
- linearization of NL input-output dynamics
- efficient realization using ZoH approximation over moving horizon
- iterative processing algorithm for fixed-time estimation
- simulative demonstration of functionality and robustness

**Further work:**

- error analysis and convergence proof
- piece-wise continuous coefficient $a(t)$ approximation instead of ZoH
- enabling larger horizon lengths $T$ for better noise suppression
- advanced and optimal tuning options
- adaptive observation and control

# Research & Innovation Staff Exchange

## Improved Estimation Algorithms for Water Purification and Desalination Systems

https://zenodo.org/communities/pure-water/

This project receives funding from the European Union's Horizon 2020 Research and Innovation Programme grant agreement No 824046.

# Related Literature I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- BYRSKI, Witold:
  The survey for the exact and optimal state observers in Hilbert spaces.
  In: 2003 European Control Conference (ECC) (2003), S. 3548–3553
- CIEZA, Oscar B. ; TAFUR, Julio C. ; REGER, Johann:
  Frequency Domain Modulating Functions for Continuous-Time Identification of Linear and Nonlinear Systems.
  In: Latinamerican Control Conference, 2014, S. 690–695

# Related Literature II


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In: IEEE Conference on Control Applications, 2015, S. 394–399

MAZENC, Frédéric ; MALISOFF, Michael ; NICULESCU, Silviu-Iulian:
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# Related Literature III


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In: [1992] Proceedings of the 31st IEEE Conference on Decision and Control, 1992, S. 3656–3660 vol.4

PIN, Gilberto ; LI, Peng ; FEDELE, Giuseppe ; PARISINI, Thomas:
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# Related Literature IV

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Theory and application of the modulating function method - I. Review and theory of the method and theory of the spline-type modulating functions.
In: Computers & chemical engineering 17 (1993), Nr. 1, S. 1–16

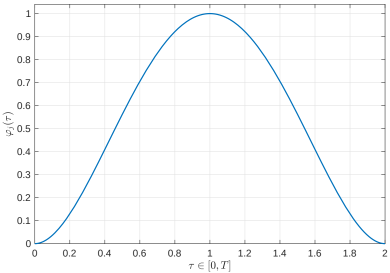
TIAN, Yang ; WEI, Yan-Qiao ; LIU, Da-Yan ; BOUTAT, Driss:
Fast and robust estimation for positions and velocities from noisy accelerations using generalized modulating functions method.
In: Mechanical Systems and Signal Processing 133 (2019), S. 106270

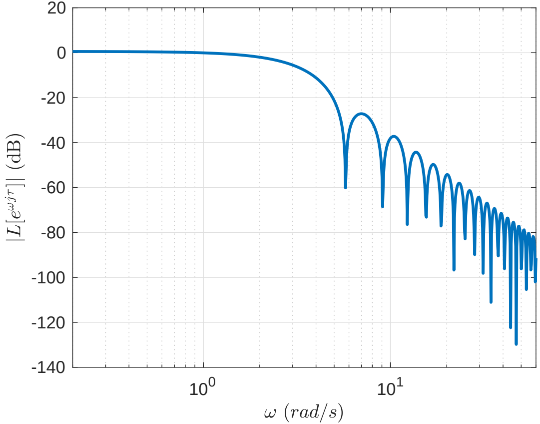
# Filtering characteristics

Frequency sweep: $\omega \in \Omega$ (compare to [PR93])

$$\langle \varphi, e^{j\omega\cdot} \rangle = \int_0^T \varphi(\tau)\, e^{j\omega\tau} d\tau$$

**Bode plot**: *ideal case*



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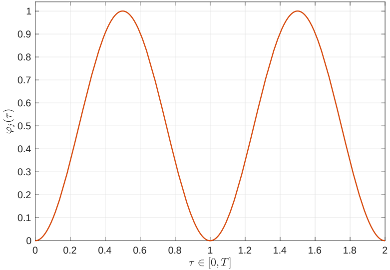


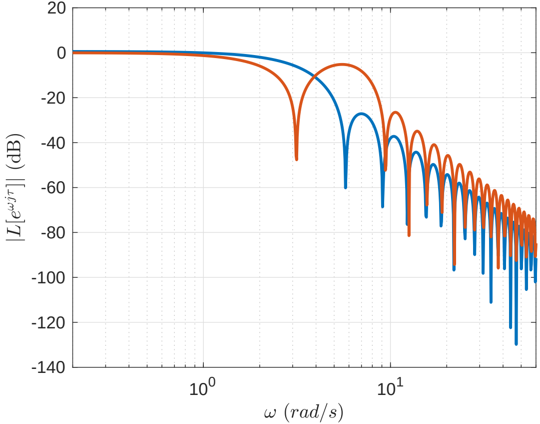

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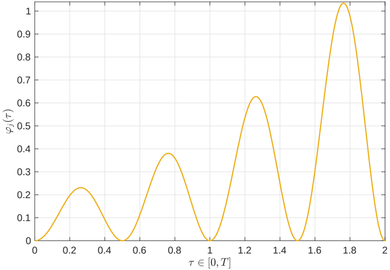


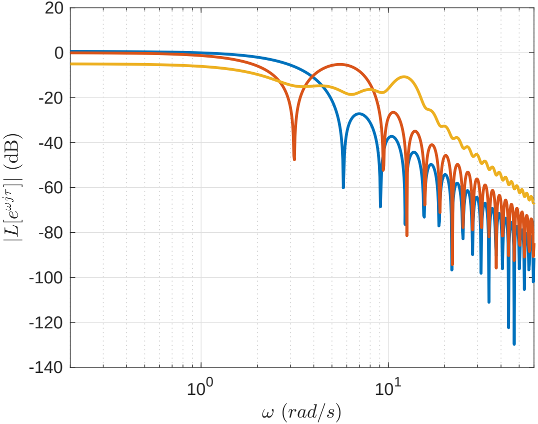

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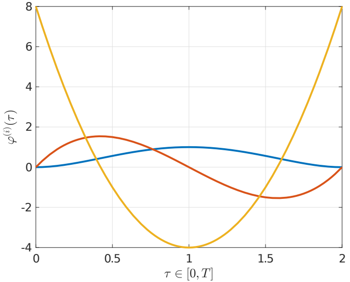


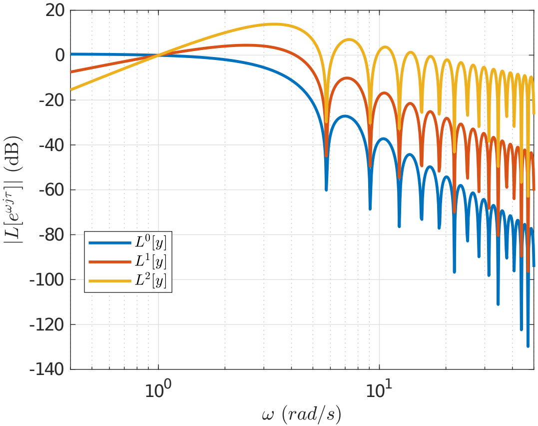

# Note: Numerical realization

*Receding horizon* MF operator realization as an **FIR filter** gain:

$$\mathcal{M}^i[y] = (-1)^i \int_{t-T}^{t} \varphi^{(i)}(\tau - t + T)\, y(\tau) d\tau \approx (-1)^i \sum_{k=0}^{N} W_k\, \varphi^{(i)}(k)\, y\big(l - N + k\big)$$

$$= \begin{pmatrix} K_\varphi^0 & \cdots & K_\varphi^N \end{pmatrix} \begin{pmatrix} y(l-N) \\ \vdots \\ y(l) \end{pmatrix} =: K_\varphi Y$$

**RT implementation** of moving horizon projection gain in Simulink: