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Towards Provably Secure Encrypted Control Using Homomorphic Encryption*

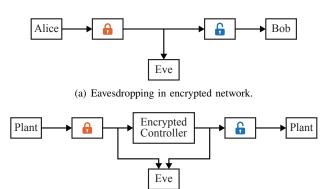
Kaoru Teranishi^{1,2} and Kiminao Kogiso¹

Abstract—Encrypted control is a promising method for the secure outsourcing of controller computation to a public cloud. However, a feasible method for security proofs of control has not yet been developed in the field of encrypted control systems. Additionally, cryptography does not consider certain types of attacks on encrypted control systems; therefore, the security of such a system cannot be guaranteed using a secure cryptosystem. This study proposes a novel security definition for encrypted control under attack for control systems using cryptography. It applies the concept of provable security, which is the security of cryptosystems based on mathematical proofs, to encrypted control systems. Furthermore, this study analyzes the relation between the proposed security and the conventional security of cryptosystems. The results of the analysis demonstrated that the security of an encrypted control system can be enhanced by employing secure homomorphic encryption.

I. INTRODUCTION

The encrypted control method is based on the homomorphism of cryptosystems that enables implementation of controller computation in an encrypted manner [1]–[4]. The controller parameters and inputs/outputs of an encrypted controller are encrypted against attackers originating outside the closed-loop system. This control method can be implemented for the secure outsourcing computation of controllers to a public cloud because the control inputs are computed without using a decryption key.

Although encrypted control is a promising method for secure control, research on the security of encrypted control systems remains limited. In most of the previous studies. an encrypted control system was assumed to be secure if the underlying cryptosystem was secure. However, certain attacks against encrypted control systems are not considered in cryptography. Fig. 1(a) depicts a typical situation in standard signal encryption. Alice encrypts a message and transmits the encrypted message to Bob, who decrypts it. Eve, an attacker, intercepts the encrypted data and tries to recover it. Eve eavesdrops on the input and output of the encrypted controller while attacking the encrypted control system, as shown in Fig. 1(b). This is because control systems are typically closed-loop systems. In this case, Eve's objective is not necessary to obtain the original controller input and output. For example, Eve may attempt to recover an encrypted controller parameter from encrypted messages.



(b) Eavesdropping in encrypted control.

Fig. 1. Eavesdropping in encrypted network and encrypted control.

This objective may be achievable even if Eve fails to decrypt the encrypted input and output. Although this type of attack on encrypted control systems is fairly common, it is not considered in standard cryptosystems, including the ElGamal encryption [5] and Paillier encryption [6], which are widely used for encrypted control. Therefore, using a secure cryptosystem does not necessarily mean that an encrypted control system is secure.

Furthermore, the definition of security for encrypted control systems is ambiguous, and the types of attacks against the systems are not clearly defined. Meanwhile, the security of cryptosystems has been mathematically defined and analyzed in the field of cryptography [7]. One of the security approaches, called provable security, is formulated based on the objectives and capabilities of the attackers. Additionally, it is proved under some assumptions of computational complexity. More precisely, the security is usually demonstrated using proof by contradiction, namely if there exists an algorithm to break a cryptosystem, then a practically difficult computational problem can be efficiently solved by using the algorithm. This proof construction is called the reductionist approach and is fundamental in proving the security of modern cryptosystems. The security of encrypted control systems can be clarified and formally analyzed by applying a similar security definition and proof construction. Consequently, the standard security definition of cryptosystems must be expanded to encompass the specific attacks for encrypted control systems because the attack scenarios considered in cryptography and control systems vary significantly. Some studies have already applied cryptographic security to encrypted control protocols. The privacy of cloud-based optimization, model predictive control, and state estimation is achieved through the computational indistinguishability of two random ensembles [8]-[10], and that of distributed and

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¹Department of Mechanical and Intelligent Systems Engineering, The University of Electro-Communications, 1-5-1 Chofugaoka, Chofu, Tokyo 1828585, Japan {teranishi, kogiso}@uec.ac.jp

²Research Fellow of Japan Society for the Promotion of Science

cooperative controls is achieved based on a cryptographic game [11], [12]. However, the guaranteed privacy of the aforementioned studies is based on the input/output privacy of the systems rather than a controller parameter and is provided for specific protocols.

This study proposes a novel security definition of encrypted control systems under a parameter estimation attack via a game based on a cryptographic security notion. In a parameter estimation attack, an attacker attempts to disclose the original controller parameter of an encrypted controller from multiple data of encrypted input and output. The proposed security is defined as an attacker obtaining no information about a controller parameter by the attack. Additionally, we analyze the relation of the proposed security to the standard security of cryptosystems. Contrary to the existing studies [8]-[12], this study aims to expand the application of security notion in cryptography to encrypted control systems. Moreover, it contributes to the establishment of a methodology of security proofs for general encrypted control. The proposed security definition allows for the analysis of the security of a broader encrypted control system under a parameter estimation attack. The analysis of this study clarifies the strength of the proposed security. It is shown that the proposed security is weaker than or equal to the standard security of cryptosystems. This implies that a secure encrypted control system under the attack can be realized using a secure cryptosystem that satisfies the standard cryptographic security. In addition, this study discusses a condition in which the proposed security is as secure as the standard cryptosystem security. The condition suggests that the proposed security for most controls may be achievable by a simple and lightweight scheme rather than a cryptosystem that satisfies the standard security.

The remainder of this paper is organized as follows. Section II introduces homomorphic encryption and encrypted control employed in this study. In addition, the concept of provable security in cryptography is provided. Section III proposes a novel security definition for encrypted control and presents the analysis results. Section IV presents the scope for further development of the proposed security definition. Section V presents the conclusions and future work.

II. PRELIMINARIES

A. Notation

The set of real numbers is denoted by \mathbb{R} . The sets of n-dimensional column-vectors and m-by-n matrices of which elements belong to a set X are denoted by X^n and $X^{m \times n}$, respectively. An algorithm \mathcal{A} is polynomial-time algorithm if there exists a polynomial p such that, for every k-bit length input x, $\mathcal{A}(x)$ terminates within at most p(k) steps [7]. We say a function $\epsilon:\{1,2,3,\ldots\}\to\mathbb{R}$ is negligible if, for every positive integer c>0, there exists an integer N such that $|\epsilon(n)|< n^{-c}$ holds for all n>N [7].

B. Homomorphic encryption

This section summarizes the basics of homomorphic encryption. Homomorphic encryption is a cryptosystem that

enables the application of arithmetic over a ciphertext space. The definition of a cryptosystem is as follows.

Definition 1 (Cryptosystem [13]): A cryptosystem is defined as a tuple $\Pi = (KeyGen, Enc, Dec)$, which satisfies the following properties:

- KeyGen : $S \to \mathcal{K}$ is a key-generation function that determines keys $(pk, sk) \in \mathcal{K}$ from a security parameter $\lambda \in S$, where \mathcal{K} is a key space, and S is a set of security parameters.
- Enc_{pk}: M → C is an encryption function that outputs a ciphertext c∈ C from a plaintext m∈ M, where M is a plaintext space, and C is a ciphertext space.
- $\mathsf{Dec}_{\mathsf{sk}}: \mathcal{C} \to \mathcal{M}$ is a decryption function that outputs a plaintext $m \in \mathcal{M}$ from a ciphertext $c \in \mathcal{C}$.
- For any pk, there exists sk such that $\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m)) = m$ for all $m \in \mathcal{M}$.

Further, we often say Π is symmetric-key encryption if pk = sk; otherwise it is public-key encryption.

Homomorphic encryption is defined as follows.

Definition 2 (Homomorphic encryption): A cryptosystem Π in Definition 1 is called homomorphic encryption if there exist binary operations $\bullet: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $\circ: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that $\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m) \circ \mathsf{Enc}_{\mathsf{pk}}(m')) = m \bullet m'$ for all $m, m' \in \mathcal{M}$.

Homomorphic encryption is classified based on the allowed arithmetic, into additive, multiplicative, somewhat, and fully homomorphic encryption. The additive and multiplicative homomorphic encryption satisfy $\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m) \oplus \mathsf{Enc}_{\mathsf{pk}}(m')) = m + m'$ and $\mathsf{Dec}_{\mathsf{sk}}(\mathsf{Enc}_{\mathsf{pk}}(m) \otimes \mathsf{Enc}_{\mathsf{pk}}(m')) = mm'$, respectively. Somewhat homomorphic encryption can perform the arithmetic \oplus and \otimes only a limited number of times. Fully homomorphic encryption allows the computation of any number of arbitrary arithmetic.

C. Encrypted control

Various encrypted control frameworks have been proposed in conventional works according to the controller type, such as static output feedback [2], cooperative control [12], [14], and model predictive control [15], [16]. However, this study does not assume a specific type of controller to be encrypted. A controller is considered as a map from a plant output to the control input to analyze the cryptographic security for as broad a class of encrypted controllers as possible.

Definition 3 (Encrypted control): Given a control law

$$\boldsymbol{u} = f(\boldsymbol{y}; \boldsymbol{K}), \tag{1}$$

where $u \in \mathcal{M}^q$ is a control input, $y \in \mathcal{M}^\ell$ is a plant output, and $K \in \mathcal{M}^r$ is a controller parameter. Let Π be homomorphic encryption in Definition 2. We say a map f_{Π} is an encrypted control law of f with Π if

$$\mathsf{Dec}_{\mathsf{sk}}(f_{\Pi}(\mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{y});\mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{K})) = f(\boldsymbol{y};\boldsymbol{K})$$

holds, where for a plaintext vector $\boldsymbol{m} = [m_1 \cdots m_k]^{\top} \in \mathcal{M}^k$ and ciphertext vector $\boldsymbol{c} = [c_1 \cdots c_n]^{\top} \in \mathcal{C}^n$ the encryption and decryption functions perform each element of the vectors, namely $\mathsf{Enc_{pk}}(\boldsymbol{m}) = [\mathsf{Enc_{pk}}(m_1) \cdots \mathsf{Enc_{pk}}(m_k)]^{\top}$ and $\mathsf{Dec_{sk}}(\boldsymbol{c}) = [\mathsf{Dec_{sk}}(c_1) \cdots \mathsf{Dec_{sk}}(c_n)]^{\top}$.

Remark 1: In practice, y and K are given by real-valued vectors, and \mathcal{M} is not necessarily the same as \mathbb{R} . Therefore, the elements of y and K must be encoded into plaintext before encryption [2], [17], [18]. This encoding process is omitted in this study because it is focused on the cryptographic properties of the encrypted controls.

D. Provable Security

The security of modern cryptosystems is demonstrated through mathematical proofs using the model of an attacker, and the security proved by such formal procedures is called *provable security*. A popular method to define the security is a *game* between an attacker and a challenger. The game simulates a situation in which an attacker attempts to break a cryptosystem using an *oracle*, which is a black box that outputs an ideal response for an input. The oracle represents the capabilities of the attacker. It should be noted that specific attack methodologies for cryptosystems, such as brute-force and side-channel attacks, are not considered in provable security. In the framework of provable security, attackers are considered as probabilistic polynomial-time algorithms.

This section introduces *indistinguishability under chosen* plaintext attack (IND-CPA), which is a traditional security concept for public-key encryption, as an example of provable security.

Definition 4 (IND-CPA [7]): Let $\Pi = (\text{KeyGen}, \text{Enc}, \text{Dec})$ be public-key encryption. Define a game between an attacker $A = (A_1, A_2)$ and a challenger as follows.

```
Game_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}}:

1. (\mathsf{pk},\mathsf{sk}) \leftarrow \mathsf{KeyGen}(\lambda)

2. (m_0,m_1,\sigma) \leftarrow \mathcal{A}_1(\mathsf{pk})

3. b \overset{\$}{\leftarrow} \{0,1\}

4. c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_b)

5. \hat{b} \leftarrow \mathcal{A}_2(c,\sigma)
```

- **Setup**: The challenger computes keys pk and sk from a security parameter λ using the key-generation function KeyGen and gives pk to the attacker (Line 1).
- Challenge: The attacker computes two different plaintexts $m_0, m_1 \in \mathcal{M}$, which are of identical size, and an intermediate state σ using the polynomial-time algorithm \mathcal{A}_1 (Line 2). The attacker gives m_0 and m_1 to the challenger. The challenger randomly selects a bit $b \in \{0,1\}$ and computes a ciphertext c of m_b using the encryption function Enc with pk (Lines 3–4). The challenger gives c to the attacker.
- Guess: The attacker guesses the plaintext that has been encrypted from the given ciphertext c using the polynomial-time algorithm A₂ and outputs a bit b̂ ∈ {0,1} (Line 5). If b̂ = b, the attacker wins the game; otherwise, the challenger wins.

The game is called the IND-CPA game.

Define the advantage of the attacker in the IND-CPA game as $\mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} := \left| \Pr\left[\hat{b} = b \mid \mathsf{Game}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} \right] - 1/2 \right|$. We say Π is IND-CPA secure, or Π satisfies the IND-CPA security if $\mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}}$ is negligible, that is, there exists a negligible function ϵ such that $\mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} \leq \epsilon$.

Remark 2: Public-key encryption must at least satisfy the IND-CPA security to be used for secure communication. If the advantage of the attacker is not negligible, he/she can guess the encrypted message with a significant probability.

III. PROVABLE SECURITY FOR ENCRYPTED CONTROL

This section proposes a novel security notion of encrypted control based on the traditional provable-security notion in cryptography and reveals the relationship between the notions.

A. Indistinguishability against parameter estimation attack

This study considers an attack scenario in which an attacker attempts to estimate the controller parameters from the ciphertexts of the controller inputs and outputs. The indistinguishability for encrypted control against the attack can be defined as follows.

Definition 5 (IND-PEA): Let $\Pi = (\text{KeyGen, Enc, Dec})$ be homomorphic encryption, and let f_{Π} be an encrypted control law of (1) with Π . Define a game between an attacker $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ and a challenger as follows.

```
\begin{aligned} \mathsf{Game}^{\mathsf{IND-PEA}}_{\Pi,f,\mathcal{B}}: \\ 1. \ \ (\mathsf{pk},\mathsf{sk}) &\leftarrow \mathsf{KeyGen}(\lambda) \\ 2. \ \ (\boldsymbol{K}_0,\boldsymbol{K}_1,\sigma) &\leftarrow \mathcal{B}_1(\mathsf{pk}) \\ 3. \ \ b &\stackrel{\$}{\leftarrow} \{0,1\} \\ 4. \ \ \boldsymbol{c_K} &\leftarrow \mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{K}_b) \\ 5. \ \ \hat{b} &\leftarrow \mathcal{B}^{\mathcal{O}}_{\mathcal{Q}}(\sigma) \end{aligned}
```

- **Setup**: The challenger computes keys pk and sk from a security parameter *λ* using the key-generation function KeyGen and gives pk to the attacker (Line 1).
- Challenge: The attacker computes two different parameters $K_0, K_1 \in \mathcal{M}^r$, which are of identical size, and an intermediate state σ using the polynomial-time algorithm \mathcal{B}_1 (Line 2). The attacker gives K_0 and K_1 to the challenger. The challenger randomly selects a bit $b \in \{0,1\}$ and computes a ciphertext c_K of K_b using the encryption function Enc with pk (Lines 3–4). The challenger sets c_K to the encrypted controller.
- Guess: The attacker guesses the selected parameter using the polynomial-time algorithm \mathcal{B}_2 and by querying an encrypted control oracle \mathcal{O} polynomial times of λ and outputs a bit $\hat{b} \in \{0,1\}$ (Line 5). The oracle receives a query $c_y \in \mathcal{C}^\ell$ and returns an output $c_u = f_\Pi(c_y; c_K)$. If $\hat{b} = b$, the attacker wins the game; otherwise, the challenger wins.

The game is called the *indistinguishability under parameter* estimation attack (IND-PEA) game.

Define the advantage of the attacker in the IND-PEA game as $\mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} \coloneqq \left| \Pr \left[\hat{b} = b \mid \mathsf{Game}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} \right] - 1/2 \right|$. We say f_Π is IND-PEA secure, or f_Π satisfies the IND-PEA security if $\mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}}$ is negligible.

Fig. 2 presents the schematic diagrams of the IND-CPA and IND-PEA games. In the IND-CPA game of Fig. 2(a), an attacker guesses whether the plaintext, m_0 or m_1 , has been encrypted from a given ciphertext c of m_b . Note that an attacker in this game can obtain the ciphertext of the target

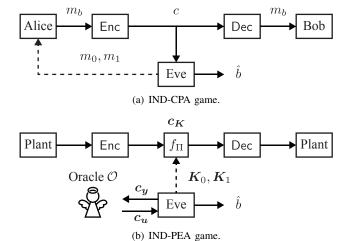


Fig. 2. Schematic pictures of IND-CPA and IND-PEA games.

plaintext. Conversely, an attacker in the IND-PEA game of Fig. 2(b) cannot obtain a ciphertext of the target parameter, K_b . An attacker can use the encrypted control oracle $\mathcal O$ with the encrypted parameter $c_K = \mathsf{Enc}_{\mathsf{pk}}(K_b)$ to collect the encrypted controller outputs c_y for any input c_u . Therefore, the IND-PEA game simulates different situations within the IND-CPA game.

It should be noted that the oracle to be used by an attacker indicates that encrypted control is provided as a cloud service. Most studies conducted on encrypted control consider cloud-based control systems [8], [15], [16], [19]–[21], and in such cases, an attacker can freely use an encrypted control algorithm. Therefore, the use of the oracle in the IND-PEA game is reasonable when considering the security for the encrypted control.

B. Relationship between the security notions

The security of encrypted control can be demonstrated using mathematical proofs based on the definition of IND-PEA security. Furthermore, it gives the other benefit of discussing how much security strength is required to achieve secure encrypted control. The following theorems disclose the relation between the IND-PEA security and IND-CPA security and clarify the security strength of the IND-PEA security.

Theorem 1: Let Π be homomorphic encryption and let f_{Π} be an encrypted control law of (1) with Π . If Π is IND-CPA secure, f_{Π} is IND-PEA secure.

Proof: It is proved that if f_{Π} is not IND-PEA secure, Π is not IND-CPA secure. Define polynomial-time algorithms A_1 and A_2 as follows.

$$\mathcal{A}_{1}(\mathsf{pk}):$$

$$1. \ (m_{0}, m_{1}, \sigma) \leftarrow \mathcal{B}_{1}(\mathsf{pk})$$

$$2. \ \mathbf{return} \ (m_{0}, m_{1}, \sigma)$$

$$\mathcal{A}_{2}(c, \sigma):$$

$$1. \ \mathbf{return} \ \mathcal{B}_{2}^{\mathcal{O}}(\sigma)$$

Substituting the algorithms into $\mathsf{Game}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}}$ in Definition 4, we obtain the following game.

 $\mathsf{Game}_{\Pi,f,\mathcal{B}}$:

- 1. $(pk, sk) \leftarrow KeyGen(\lambda)$
- 2. $(m_0, m_1, \sigma) \leftarrow \mathcal{B}_1(\mathsf{pk})$
- 3. $b \stackrel{\$}{\leftarrow} \{0, 1\}$
- 4. $c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m_b)$
- 5. $\hat{b} \leftarrow \mathcal{B}_2^{\mathcal{O}}(\sigma)$
- **Setup**: This phase is identical to the setup phase in Definition 4.
- Challenge: The attacker computes two different parameters $m_0, m_1 \in \mathcal{M}$, which are of identical size, and an intermediate state σ using the polynomial-time algorithm \mathcal{B}_1 (Line 2). The attacker gives m_0 and m_1 to the challenger. The challenger randomly selects a bit $b \in \{0,1\}$ and computes a ciphertext c of m_b using the encryption function Enc with pk (Lines 3–4). The challenger sets c to the encrypted controller.
- Guess: This phase is identical to the guess phase in Definition 5.

Define the advantage of the attacker in the game $\mathsf{Game}_{\Pi,f,\mathcal{B}}$ as $\mathsf{Adv}_{\Pi,f,\mathcal{B}} \coloneqq \left| \Pr \left[\hat{b} = b \mid \mathsf{Game}_{\Pi,f,\mathcal{B}} \right] - 1/2 \right|$. Here, we consider the following lemmas to prove the theorem.

Lemma 1:
$$\left| \mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} - \mathsf{Adv}_{\Pi,f,\mathcal{B}} \right|$$
 is negligible.

Proof: The substitution to obtain the game $\mathsf{Game}_{\Pi,f,\mathcal{B}}$ does not change the probability of the game between the challenger and the attacker.

Lemma 2:
$$\left| \mathsf{Adv}_{\Pi,f,\mathcal{B}} - \mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} \right|$$
 is negligible.

Proof: The game $\mathsf{Game}_{\Pi,f,\mathcal{B}}$ corresponds to the IND-PEA game $\mathsf{Game}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}}$ with r=1, which is the dimension of the controller parameter. The advantage of the attacker does not change between these games.

From Lemma 1 and Lemma 2, it follows that the difference of advantages, $\left| \mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} - \mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} \right|$, is negligible. Furthermore, $\mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}}$ is not negligible because f_Π is not IND-PEA secure, based on the assumption of this proof. Therefore, $\mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}}$ is not negligible, and so Π is not IND-CPA secure.

The theorem demonstrates that the IND-PEA security is weaker than or equal to the IND-CPA security. This implies that secure encrypted control against the parameter estimation attack is achievable by using secure homomorphic encryption. The following theorem presents one of the conditions so that the IND-PEA security is equivalent to the IND-CPA security.

Theorem 2: Let Π be homomorphic encryption and let f_{Π} be an encrypted control law of (1) with Π . Suppose f in (1) is bijective for a fixed $\mathbf{y} \in \mathcal{M}^{\ell}$. If f_{Π} is IND-PEA secure and if the attacker can compute an encrypted control law of f^{-1} with Π , then Π is IND-CPA secure.

Proof: Under the assumptions, we prove that if Π is not IND-CPA secure, f_{Π} is not IND-PEA secure. Let f_{Π}^{-1} be an encrypted control law of f^{-1} with Π . Since f in (1) is bijective for some fixed $\boldsymbol{y} \in \mathcal{M}^{\ell}$, it follows that

$$f^{-1}(\boldsymbol{y}; f(\boldsymbol{y}; \boldsymbol{K})) = \boldsymbol{K} \tag{2}$$

for all $K \in \mathcal{M}^r$. Moreover,

$$f_{\Pi}^{-1}(\boldsymbol{c}_{\boldsymbol{y}}; f_{\Pi}(\boldsymbol{c}_{\boldsymbol{y}}; \boldsymbol{c}_{\boldsymbol{K}})) = \boldsymbol{c}_{\boldsymbol{K}}$$
(3)

holds for $c_{\boldsymbol{u}} = \mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{y})$ and $c_{\boldsymbol{K}} = \mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{K})$.

Define polynomial-time algorithms \mathcal{B}_1 and \mathcal{B}_2 as follows.

```
\mathcal{B}_1(\mathsf{pk}):
      1. (K_0, K_1, \sigma) \leftarrow \mathcal{A}_1(\mathsf{pk})
     2. \mathbf{K}_0 \leftarrow [K_0 \cdots K_0]^{\top}, \mathbf{K}_1 \leftarrow [K_1 \cdots K_1]^{\top}
     3. Fix y \in \mathcal{M}^{\ell} to satisfy (2)
      4. \sigma \leftarrow (\sigma, \boldsymbol{y})
     5. return (\mathbf{K}_0, \mathbf{K}_1, \sigma)
```

```
\mathcal{B}_2^{\mathcal{O}}(\sigma):
        1. (\sigma, \boldsymbol{y}) \leftarrow \sigma
       2. c_{\boldsymbol{y}} \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{y})
       3. c_u \leftarrow \mathcal{O}(c_y)
4. \hat{c}_K \leftarrow f_{\Pi}^{-1}(c_y; c_u), c_K \leftarrow \hat{c}_{K,1}
        5. return A_2(c_K, \sigma)
```

Substituting the algorithms into $\mathsf{Game}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}}$ in Definition 5, we obtain the following game.

```
\mathsf{Game}^1_{\Pi,f,\mathcal{A}}:
       1. (pk, sk) \leftarrow KevGen(\lambda)
      2. (K_0, K_1, \sigma) \leftarrow \mathcal{A}_1(\mathsf{pk})
      3. \mathbf{K}_0 \leftarrow [K_0 \ \cdots \ K_0]^{\top}, \ \mathbf{K}_1 \leftarrow [K_1 \ \cdots \ K_1]^{\top}
      4. Fix y \in \mathcal{M}^{\ell} to satisfy (2)
      5. \sigma \leftarrow (\sigma, \boldsymbol{y})
      6. b \stackrel{\$}{\leftarrow} \{0, 1\}
      7. c_{\boldsymbol{K}} \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\boldsymbol{K}_b)
      8. (\sigma, \boldsymbol{y}) \leftarrow \sigma
      9. c_{m{y}} \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m{y})
   10. c_{\boldsymbol{u}} \leftarrow \mathcal{O}(c_{\boldsymbol{y}})
11. \hat{c}_{\boldsymbol{K}} \leftarrow f_{\Pi}^{-1}(c_{\boldsymbol{y}}; c_{\boldsymbol{u}}), c_{K} \leftarrow \hat{c}_{\boldsymbol{K},1}
    12. \hat{b} \leftarrow \mathcal{A}_2(c_K, \sigma)
```

- **Setup**: This phase is identical to the setup phase in Definition 5.
- **Challenge:** The attacker computes two different plaintexts $K_0, K_1 \in \mathcal{M}$, which are of identical size, and an intermediate state σ using the polynomial-time algorithm A_1 (Line 2). Subsequently, the attacker constructs vectors $K_0, K_1 \in \mathcal{M}^r$ from K_0, K_1 and gives them to the challenger (Line 3). Additionally, the attacker fixes $y \in \mathcal{M}^{\ell}$ to satisfy (2) and updates σ to (σ, y) (Lines 4– 5). The challenger randomly selects a bit $b \in \{0, 1\}$ and computes a ciphertext c_K of K_b using the encryption function Enc with pk (Lines 6-7). The challenger sets c_K to the encrypted controller.
- Guess: The attacker takes out the intermediate state and y from σ and computes the ciphertext c_y of y (Lines 8-9). Subsequently, the attacker queries the encrypted control oracle \mathcal{O} to obtain a controller output c_u and computes a ciphertext \hat{c}_K by using the encrypted controller of f^{-1} with Π (Line 10–11). The attacker guesses the parameter that has been encrypted from

the first element c_K of the ciphertext vector $\hat{\boldsymbol{c}}_K$ using the polynomial-time algorithm A_2 and outputs a bit $\hat{b} \in \{0,1\}$ (Line 11–12). If $\hat{b} = b$, the attacker wins the game; otherwise, the challenger wins.

Define the advantage of the attacker in the game $\mathsf{Game}_{\Pi,f,\mathcal{A}}^1$ as $\mathsf{Adv}^1_{\Pi,f,\mathcal{A}} := \left| \Pr \left[\hat{b} = b \mid \mathsf{Game}^1_{\Pi,f,\mathcal{A}} \right] - 1/2 \right|$. Furthermore, we modify lines 3–5 and 7–11 of $Game_{\Pi,f,A}^1$ to obtain the following game.

```
\mathsf{Game}^2_{\Pi} _{\Delta}:
      1. (pk, sk) \leftarrow KeyGen(\lambda)
     2. (K_0, K_1, \sigma) \leftarrow \mathcal{A}_1(\mathsf{pk})
     3. b \stackrel{\$}{\leftarrow} \{0, 1\}
     4. c_K \leftarrow \mathsf{Enc}_{\mathsf{pk}}(K_b)
     5. b \leftarrow \mathcal{A}_2(c_K, \sigma)
```

- Setup: This phase is identical to the setup phase in Definition 5.
- Challenge: The attacker computes two different plaintexts $K_0, K_1 \in \mathcal{M}$, which are of identical size, and an intermediate state σ using the polynomial-time algorithm A_1 (Line 2). The attacker gives K_0 and K_1 to the challenger. The challenger randomly selects a bit $b \in \{0,1\}$ and computes a ciphertext c_K of K_b using the encryption function Enc with pk (Lines 3-4). The challenger gives c_K to the attacker.
- Guess: This phase is identical to the guess phase in Definition 4.

Define the advantage of the attacker in the game $\mathsf{Game}_{\Pi,\mathcal{A}}^2$ as $\operatorname{Adv}_{\Pi,\mathcal{A}}^2 := \left| \operatorname{Pr} \left[\hat{b} = b \mid \operatorname{\mathsf{Game}}_{\Pi,\mathcal{A}}^2 \right] - 1/2 \right|$. Here, we consider the following lemmas to prove the theorem. $\operatorname{\mathit{Lemma 3:}} \left| \operatorname{\mathsf{Adv}}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} - \operatorname{\mathsf{Adv}}_{\Pi,f,A}^1 \right|$ is negligible.

Lemma 3:
$$\left| \mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} - \mathsf{Adv}_{\Pi,f,A}^{1} \right|$$
 is negligible

Proof: The substitution for obtaining $Game_{\Pi,f,A}^1$ does not change the probability of the game between the challenger and the attacker.

Lemma 4: $\left| \mathsf{Adv}_{\Pi,f,A}^1 - \mathsf{Adv}_{\Pi,A}^2 \right|$ is negligible.

Proof: It follows from (3) that $\hat{c}_{K} = c_{K} =$ $\mathsf{Enc}_{\mathsf{pk}}(K_b) = [\mathsf{Enc}_{\mathsf{pk}}(K_b) \cdots \mathsf{Enc}_{\mathsf{pk}}(K_b)]^{\mathsf{T}}$. Therefore, we can rewrite line 11 in $\mathsf{Game}_{\Pi,f,\mathcal{A}}^1$ as line 4 in $\mathsf{Game}_{\Pi,\mathcal{A}}^2$ without changing the probability of the games. The lines 3-5 and 7-10 of $\mathsf{Game}_{\Pi,f,\mathcal{A}}^1$ do not affect the advantage of the attacker in the game, and so we can remove the lines.
Lemma 5: $\left| Adv_{\Pi,A}^2 - Adv_{\Pi,A}^{IND-CPA} \right|$ is negligible.

Lemma 5:
$$\left| Adv_{\Pi,A}^2 - Adv_{\Pi,A}^{\mathsf{IND-CPA}} \right|$$
 is negligible.

Proof: By definition, the game $\operatorname{\mathsf{Game}}^2_{\Pi,A}$ is the same as the game $Game_{\Pi,A}^{\mathsf{IND-CPA}}$.

It follows from Lemma 3 to Lemma 5 that the difference of advantages, $\left| \mathsf{Adv}_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}} - \mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}} \right|$, is negligible. Furthermore, $\mathsf{Adv}_{\Pi,\mathcal{A}}^{\mathsf{IND-CPA}}$ is not negligible since Π is not IND-CPA secure based on the assumption of this proof. Therefore, $Adv_{\Pi,f,\mathcal{B}}^{\mathsf{IND-PEA}}$ is not negligible, and so f_{Π} is not IND-PEA secure.

Note that few controllers can satisfy the assumption in Theorem 2 in practice. For example, in a static output feedback controller u = Fy, where $F \in \mathcal{M}^{q \times \ell}$ is a feedback gain, the existence of inverse mapping f^{-1} in Theorem 2 is equivalent to the condition that q=r, namely $\ell=1$. Such a controller is a particular case of this control, and hence, it seems that the IND-PEA security is weaker than the IND-CPA security for most of the controllers formulated by (1). Although one may think that this is a negative result, it suggests that homomorphic encryption satisfying IND-CPA security may not necessarily be required to prevent the parameter estimation attack. In other words, the IND-PEA security might be achieved using more lightweight homomorphic encryption schemes than the conventional schemes to reduce the computation costs. This implication is significant because encrypted control generally increases computational costs, and control systems require real-time computation.

IV. FURTHER DEVELOPMENT

The security definition and analysis presented in this study were based on the assumption that the control law (1) was employed. Although the control law represents all static controllers with a controller parameter K, there is scope for further expansion. The results of this study can be expanded upon using a dynamical controller $x_{t+1} = f(x_t, y_t; K)$, $u_t = g(x_t, y_t; K)$, a time-varying controller $u_t = f(y_t; K_t)$ instead of (1), where x is a controller state, and t is a time step. According to this change, the encrypted control oracle must be modified.

For example, the modified oracle for the dynamical controller must include a state updated by each query of an attacker and return an output based on the current state and input. Note that the oracle state is hidden against the attacker. The security of dynamical controllers under a parameter estimation attack requires to be analyzed while considering the state trajectory of the controller because the controller output depends on both the controller state and the parameter. Such analysis is challenging because it needs to consider the effect of fundamental properties of dynamical systems, such as stability, controllability, and observability, on the provable security for encrypted control. Moreover, an attacker may aim to disclose the (initial) state of an encrypted dynamical controller. In this case, the security is defined through a game that differs from the IND-PEA game.

Furthermore, the security of controllers formulated by stochastic processes such as a Markov decision process can also be considered. An example of this type of controller is reinforcement learning, which is an application of encrypted control [21]. In this case, the state of the environment randomly moves in each time step, and an action, which is an input for the environment, is determined by a policy that maximizes the expected cumulative reward. An attacker for an agent of reinforcement learning may be interested in the policy rather than the controller parameter. Further research on provably secure encrypted control involves considering such stochastic controllers.

V. CONCLUSIONS

This study defined a novel security for encrypted control under a parameter estimation attack in terms of provable security in cryptography and analyzed its security strength. The definition enables us to prove the security for encrypted control by mathematical procedures. The analysis revealed that encrypted control is secure if homomorphic encryption used for the control is secure. This result means that most existing encrypted controls are secure under a parameter estimation attack. The proposed security can be extended in the future to encrypted control using dynamical controllers.

REFERENCES

- K. Kogiso and T. Fujita, "Cyber-security enhancement of networked control systems using homomorphic encryption," in *IEEE Conference* on Decision and Control, 2015, pp. 6836–6843.
- [2] F. Farokhi, I. Shames, and N. Batterham, "Secure and private control using semi-homomorphic encryption," *Control Engineering Practice*, vol. 67, pp. 13–20, 2017.
- [3] J. Kim, C. Lee, H. Shim, J. H. Cheon, A. Kim, M. Kim, and Y. Song, "Encrypting controller using fully homomorphic encryption for security of cyber-physical systems," *IFAC-PapersOnLine*, vol. 49, no. 22, pp. 175–180, 2016.
- [4] M. S. Darup, A. B. Alexandru, D. E. Quevedo, and G. J. Pappas, "Encrypted control for networked systems – An illustrative introduction and current challenges," *IEEE Control Systems Magazine*, vol. 41, no. 3, pp. 58–78, 2021.
- [5] T. Elgamal, "A public key cryptosystem and a signature scheme based on discrete logarithms," *IEEE Transactions on Information Theory*, vol. 31, no. 4, pp. 469–472, 1985.
- [6] P. Paillier, "Public-key cryptosystems based on composite degree residuosity classes," in *Proceedings of the 17th International Conference on Theory and Application of Cryptographic Techniques*, 1999, pp. 223–238.
- [7] J. Katz and Y. Lindell, Introduction to Modern Cryptography, D. R. Stinson, Ed. Boca Raton: CRC Press, 2021.
- [8] A. B. Alexandru, K. Gatsis, Y. Shoukry, S. A. Seshia, P. Tabuada, and G. J. Pappas, "Cloud-based quadratic optimization with partially homomorphic encryption," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2357–2364, 2021.
- [9] A. B. Alexandru, A. Tsiamis, and G. J. Pappas, "Data-driven control on encrypted data," arXiv:2008.12671, Aug 2020.
- [10] M. Ristic, B. Noack, and U. D. Hanebeck, "Cryptographically privileged state estimation with Gaussian keystreams," in *IEEE Conference on Decision and Control*, 2021, pp. 602–607.
- [11] A. B. Alexandru and G. J. Pappas, "Private weighted sum aggregation for distributed control systems," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 11081–11088, 2020.
- [12] A. B. Alexandru, M. S. Darup, and G. J. Pappas, "Encrypted cooperative control revisited," in *IEEE Conference on Decision and Control*, 2019, pp. 7196–7202.
- [13] J. Buchmann, Introduction to Cryptography. New York: Springer-Verlag, 2001.
- [14] M. S. Darup, A. Redder, and D. E. Quevedo, "Encrypted cooperative control based on structured feedback," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 37–42, 2019.
- [15] M. S. Darup, A. Redder, I. Shames, F. Farokhi, and D. E. Quevedo, "Towards encrypted MPC for linear constrained systems," *IEEE Control Systems Letters*, vol. 2, no. 2, pp. 195–200, 2018.
- [16] A. B. Alexandru, M. Morari, and G. J. Pappas, "Cloud-based MPC with encrypted data," in *IEEE Conference on Decision and Control*, 2018, pp. 5014–5019.
- [17] K. Teranishi, N. Shimada, and K. Kogiso, "Stability analysis and dynamic quantizer for controller encryption," in *IEEE Conference on Decision and Control*, 2019, pp. 7184–7189.
- [18] M. S. Darup, "Encrypted polynomial control based on tailored twoparty computation," *International Journal of Robust and Nonlinear Control*, vol. 30, no. 11, pp. 4168–4187, 2020.
- [19] N. Schlüter and M. S. Darup, "Encrypted explicit MPC based on two-party computation and convex controller decomposition," in *IEEE Conference on Decision and Control*, 2020, pp. 5469–5476.
- [20] K. Teranishi, T. Sadamoto, and K. Kogiso, "Input-output history feed-back controller for encrypted control with leveled fully homomorphic encryption," arXiv:2109.10718, Sep 2021.
- [21] J. Suh and T. Tanaka, "Encrypted value iteration and temporal difference learning over leveled homomorphic encryption," in *American Control Conference*, 2021, pp. 2555–2561.